Forcing in nonstandard analysis

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Abstract

A nonstandard universe is constructed from a superstructure in a Boolean-valued model of set theory. This provides a new framework of nonstandard analysis with which methods of forcing are incorporated naturally. Various new principles in this framework are provided together with the following applications: (1) An example of an \( \mathcal{N}_1 \)-saturated Boolean ultrapower of the real number field which is not Scott complete is constructed. (2) Infinitesimal analysis based on the generic extension of the hyperreal numbers is provided, and the hull completeness theorem and the Loeb measure construction are extended to objects in the generic extension of the internal universe. (3) The reduction theory of the Boolean-valued complex numbers are developed as a prototype of the applications to the topological reduction theory of Boolean sheaves or operator algebras.

1. Introduction

In 1960 Robinson found that the methods of nonstandard models in mathematical logic provide a suitable framework of calculus with infinitesimal and infinite numbers [30]. His method, nonstandard analysis, has been developed nowadays in various fields of mathematics [34,6,10], in particular, in probability theory and stochastic analysis [16,33,1]. By the way, in 1963 Cohen found the notion of forcing and proved the independence of continuum hypothesis from the ZFC set theory [5]. In 1966 Scott and Solovay invented Boolean-valued models of set theory and reformulated Cohen’s forcing in a simpler framework [2]; a similar idea was developed also by Vopěnka [41,42]. Although the original aim of Boolean-valued models is metamathematics such as independence proofs, its usefulness for ordinary mathematics has been pointed out by

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This idea was realized in 1978 when Takeuti [35] instituted Boolean-valued analysis, systematic applications of Boolean-valued models to mathematical problems, and in 1983 an application of forcing to a problem of analysis was realized by Ozawa [24–26] when the Kaplansky conjecture in the structure theory of algebras of type I was proved by Boolean-valued analysis. The Boolean-valued analysis has been developed in algebra [7,8,32,20,39], harmonic analysis [36,21,22], and operator algebras [37,38,23,12,27–29].

It has been pointed out for some time that nonstandard analysis and forcing, or Boolean-valued analysis, may be developed in a unified basis. In view of sheaf theory, forcing can be regarded as a method of constructing a stalk of a Boolean sheaf of models of set theory, while a nonstandard universe constructed by an ultrapower is viewed as a stalk of a constant sheaf of a superstructure, in a certain generalized or idealized sense. The purpose of this paper is to carry out the unification of nonstandard analysis and Boolean-valued analysis from this point of view. A nonstandard universe is a triple \((V(X), V(Y), \ast)\) such that \(V(X)\) and \(V(Y)\) are superstructures and that \(\ast : V(X) \rightarrow V(Y)\) is a bounded elementary embedding satisfying certain nontriviality conditions; a precise definition will appear in Section 2. Two sorts of constructions of nonstandard universes have been known so far; bounded ultrapowers and the construction by the compactness theorem. In this paper, we will present the third construction. Let \(X\) be a base set, \(B\) a complete Boolean algebra in \(V(X)\), and \(U\) an ultrafilter of \(B\). We construct the \(B\)-valued superstructure \(\hat{V}(X)\) which is a superstructure over \(\tilde{X}\) in \(V(B)\) restricted to the elements with bounded rank relative to \(\tilde{X}\). Then we will show that the quotient of \(\hat{V}(X)\) by the ultrafilter \(U\) yields a nonstandard universe. A special feature of this nonstandard universe is that it possesses external sets with names in \(\hat{V}(X)\). We call the set of elements in \(V(Y)\) with names in \(\hat{V}(X)\) the generic universe. The generic universe can be viewed as a generic extension of the internal universe consisting of the internal elements. A typical element in the generic universe outside the internal universe is the canonical generic filter of \(*B\), which will play in nonstandard analysis a similar role to the generic set constructed by forcing. We will investigate the role of the canonical generic filter played in a nonstandard universe in several aspects.

In Section 2, we give preliminaries on nonstandard universes and introduce the notion of strictly bounded formulas. The strictly bounded formulas are variants of formulas of a language of set theory relativized to a superstructure, but appear to be more tractable in formulating the Łoś type theorem for Boolean-valued superstructures than the relativized formulas. In Section 3, we review basic principles of Boolean-valued set theory. In Section 4, for a given base set and a complete Boolean algebra we construct a Boolean-valued superstructure, and for a given ultrafilter we construct a nonstandard universe, which we call a bounded ultrasheaf of a superstructure. The bounded ultrasheaves are generalization of bounded ultrapowers. When the given complete Boolean algebra is totally atomic, a bounded ultrasheaf is regarded as a bounded ultrapower. An internal or external set with a name in the Boolean-valued superstructure is called an ingeneric set; the set of all ingeneric sets is the generic universe. The most important feature of a bounded ultrasheaf is that the Łoś type theorem holds not only for the internal universe but also for the generic universe. In Section 5, we show that if the given ultrafilter is countably incomplete then the bounded ultrasheaf has the saturation property even for the
generic universe. In Section 6, we show that any Boolean ultrapower of an elementary structure has an isomorphic embedding in a Boolean ultrasheaf of a superstructure. This extends a well-known relation between ultrapowers and bounded ultrapowers of superstructures. In Section 7, we consider the existence of external ingeneric sets and, as immediate applications of forcing, show some particular examples of such external sets. One of them settles affirmatively within ZFC the problem to existence of an $\aleph_1$-saturated elementary extension of the real number field which is not Scott complete. In Section 8, we develop infinitesimal analysis based on a bounded ultrasheaf. This extends the ordinary framework based on the internal universe to the generic universe. We establish the hull completeness theorem for normed linear spaces in the generic universe and the Loeb measure construction for measure spaces in the generic universe. In Section 9, we study the structure of the complex number field in a Boolean-valued universe, and give some new expressions for Boolean complex numbers using Boolean ultrasheaves. The argument in this section will be a prototype of applications of our machinery to the topological reduction theory of Boolean sheaves including operator algebras, which will be discussed elsewhere.

2. Nonstandard universes

We denote by $V$ the universe of sets which satisfies the ZFC set theory. Throughout this paper, we fix the language $\mathcal{L}_e = \{ \in \}$ for first-order theory with equality having a binary relation symbol $\in$ and no constant symbols. For a class $U$, the language $\mathcal{L}_{\overline{e}}(U) = \{ \in \} \cup U$ is the one obtained by adding a name for each element of $U$. For convenience, we use the same symbol for an element of $U$ and its name in $\mathcal{L}_{\overline{e}}(U)$ as well as for the membership relation and the symbol $\in$.

To each sentence $\phi$ of $\mathcal{L}_{\overline{e}}(U)$, the satisfaction relation $\langle U, \in \rangle \models \phi$ is defined by the following recursive rules:

1. $\langle U, \in \rangle \models \psi \iff \psi$.
2. $\langle U, \in \rangle \models \neg \phi \iff \langle U, \in \rangle \not\models \phi$.
3. $\langle U, \in \rangle \models \phi \land \psi \iff \langle U, \in \rangle \models \phi$ and $\langle U, \in \rangle \models \psi$.
4. $\langle U, \in \rangle \models \exists x \phi(x) \iff \exists u \in U$ such that $\langle U, \in \rangle \models \phi(u)$.

We regard the other logical connectives and quantifiers as defined symbols but not primitive ones. Our assumption that $V$ satisfies the ZFC means that if $ZFC \vdash \phi(x_1, \ldots, x_n)$ then $\langle \emptyset \rangle \models \phi(u_1, \ldots, u_n)$ for any formula $\phi(x_1, \ldots, x_n)$ of $\mathcal{L}_e$ and all $u_1, \ldots, u_n \in V$.

Let $X$ be a set. The power set $\mathcal{P}(X)$ is the set of all subsets of $X$. The superstructure over $X$, denoted by $V(X)$, is defined by the following recursion:

$V_0(X) = X, \quad V_{\alpha+1}(X) = V_\alpha(X) \cup \mathcal{P}(V_\alpha(X)), \quad V(X) = \bigcup_{\alpha \in \mathbb{N}} V_\alpha(X),$

where $\mathbb{N}$ is the set of natural numbers. The set $X$ is called a base set if $\emptyset \notin X$ and
$x \cap V(X) = \emptyset$ for all $x \in X$. For an infinite ordinal $\alpha$, any set $X$ such that every element of an element of $X$ has rank $\alpha$ is a base set [4, p. 287] and hence any set can be easily replaced by a base set with the same size.

From now on, we will always assume that $X$ is a base set. An element of $X$ is called an atom relative to $V(X)$, and an element of $V(X) \setminus X$ a set relative to $V(X)$. The atoms can be characterized as those elements $x \in V(X)$ such that $x \neq \emptyset$ but $x \cap V(X) = \emptyset$.

The following lemmas are useful in the later arguments [4, pp. 264–265].

**Lemma 2.1.** For each $n \in \mathbb{N}$, $V_{n+1}(X) = X \cup \mathcal{P}(V_n(X))$ and $V_{n+1}(X) \setminus X = \mathcal{P}(V_n(X))$.

**Lemma 2.2.** If each $n \in \mathbb{N}$, $V_{n+1}(X) = X \cup \mathcal{P}(V_n(X))$ and $V_{n+1}(X) \setminus X = \mathcal{P}(V_n(X))$.

**Lemma 2.3.** Let $V(X)$ be a superstructure over a base set $X$.

1. If $u_1, \ldots, u_m \in V_n(X)$ then $\{u_1, \ldots, u_m\} \in V_{n+1}(X) \setminus X$.
2. If $u_1, \ldots, u_m \in V_n(X)$ then $\langle u_1, \ldots, u_m \rangle \in V_{n+2(m-1)}(X) \setminus X$.
3. If $u \in V_n(X) \setminus X$ and $u \subseteq v$ then $v \in V_n(X) \setminus X$.
4. If $u \in V_n(X) \setminus X$ then $u \times v \in V_{n+1}(X) \setminus X$.
5. If $u \in V_n(X) \setminus X$ then $\mathcal{P}(u) \in V_{n+1}(X) \setminus X$.

We will use the following abbreviations, called bounded quantifiers: $(\forall x \in y) \phi$ means $(\forall x)[x \in y \Rightarrow \phi]$, $(\exists x \in y) \phi$ means $(\exists x)[x \in y \land \phi]$. A $\Delta_0$-formula is a formula constructed from atomic formulas using connectives and bounded quantifiers. In applications, more restricted formulas play an important role. Strictly bounded quantifiers in a free variable $z$ are the following abbreviations:

$$(\forall x \in y \notin z) \phi \quad \text{means} \quad (\forall x)[(x \in y \land y \notin z) \Rightarrow \phi],$$

$$(\exists x \in y \notin z) \phi \quad \text{means} \quad (\exists x)[(x \in y \land y \notin z) \land \phi].$$

A $\Delta_0$-formula $\phi$ of $L_\in$ with $z$ free is called strictly bounded in $z$ if it is constructed from atomic formulas using connectives and strictly bounded quantifiers in $z$. The following three lemmas give absoluteness results necessary for the later arguments.

**Lemma 2.4.** For any $\Delta_0$-formula $\phi(x_0, x_1, \ldots, x_n)$ of $L_\in$ strictly bounded in $x_0$ and all $u_1, \ldots, u_n \in V(X)$.

$$\langle V(X), \in \rangle \models \phi(x_0, u_1, \ldots, u_n) \quad \text{iff} \quad \langle V, \in \rangle \models \phi(x_0, u_1, \ldots, u_n).$$

**Proof.** The proof is by induction on the complexity of strictly bounded formulas. For atomic formulas, the assertion holds obviously. The only nontrivial induction step is the case that $\phi(x_0, x_1, \ldots, x_n)$ is of the form

$$(\exists x)[(x \in y \land y \notin x_0) \land \psi(x_0, x_1, \ldots, x_n, x)],$$

where $y$ is one of $x_0, \ldots, x_n$. Let $u_1, \ldots, u_n \in V(X)$. It is obvious that $\langle V(X), \in \rangle \models \phi(x_0, u_1, \ldots, u_n)$ implies $\langle V, \in \rangle \models \phi(x_0, u_1, \ldots, u_n)$. To prove the converse, suppose $\langle V, \in \rangle \models \phi(x_0, u_1, \ldots, u_n)$. First consider the case where $y$ is $x_0$. In this case, the part
y \not\in x_0 is always true and can be neglected, and then there is a set \( v \in X \) such that \( \langle V, \in \rangle \models \psi(x, u_1, \ldots, u_n, v) \). Since \( X \subseteq V(X) \), \( \langle V(X), \in \rangle \models \phi(x, u_1, \ldots, u_n) \) follows immediately from the induction hypothesis. When \( y \) is not \( x_0 \), we can suppose \( y \) is \( x_1 \) without any loss of generality. Then there is a set \( v \in V \) such that \( v \in u_1, \ u_1 \not\subseteq X \), and \( \langle V, \in \rangle \models \psi(X, u_1, \ldots, u_n, v) \). Since \( u_1 \in V(X) \setminus X \), we have \( u_1 \not\subseteq V_\infty(X) \) for some \( n \) by Lemma 2.1 so that \( v \in V(X) \). Therefore, \( \langle V(X), \in \rangle \models \phi(X, u_1, \ldots, u_n) \) by the induction hypothesis. This completes the induction. \( \square \)

For any \( \Delta_0 \)-formula \( \phi(x_0, \ldots, x_n) \) of \( \mathcal{L}_E \), define a formula \( (\phi(x_0, \ldots, x_n))' \) of \( \mathcal{L}_E \) strictly bounded in \( x_0 \) by the following recursion.

1. \( (x = y)' \) is \( x = y \).
2. \( (x \in y)' \) is \( x \in y \).
3. \( (\neg \phi(x_0, \ldots, x_n))' \) is \( \neg(\phi(x_0, \ldots, x_n))' \).
4. \( (\phi_1(x_0, \ldots, x_n) \lor \phi_2(x_0, \ldots, x_n))' \) is \( (\phi_1(x_0, \ldots, x_n))' \lor (\phi_2(x_0, \ldots, x_n))' \).
5. \( (\forall x)(x \in y \land \phi(x_0, \ldots, x_n, x))' \) is \( (\forall x)[(x \in y \land y \not\subseteq x_0) \land (\phi(x_0, \ldots, x_n, x))'] \).

Then we have the following.

**Lemma 2.5.** For any \( \Delta_0 \)-formula \( \phi(x_0, x_1, \ldots, x_n) \) of \( \mathcal{L}_E \) and all \( u_1, \ldots, u_n \in V(X) \),

\( \langle V(X), \in \rangle \models \phi(x_0, u_1, \ldots, u_n) \) iff \( \langle V(X), \in \rangle \models (\phi(x_0, u_1, \ldots, u_n))' \).

**Proof.** The proof is by induction on the complexity of \( \Delta_0 \)-formulas. The assertion holds for atomic formulas trivially. The only nontrivial induction step is the case that \( \phi(x_0, x_1, \ldots, x_n) \) is of the form

\( (\exists x)[x \in y \land \psi(x_0, x_1, \ldots, x_n, x)] \),

where \( y \) is one of \( x_0, \ldots, x_n \). Then, \( (\phi(x_0, x_1, \ldots, x_n))' \) is

\( (\exists x)[(x \in y \land y \not\subseteq x_0) \land (\psi(x_0, x_1, \ldots, x_n, x))'] \).

If \( y \) is interpreted as an element of \( V(X) \) and \( x_0 \) as \( X \), the part \( y \not\subseteq x_0 \) is always true, and hence can be eliminated in the interpretation. Thus the assertion follows from the induction hypothesis. This completes the induction. \( \square \)

The following two lemmas are adapted from [4, Lemma 4.4.4, p. 265] and the proofs are minor modifications.

**Lemma 2.6.** For each natural number \( n \), there is a \( \Delta_0 \)-formula \( \nu_n(x_0, x_1) \) of \( \mathcal{L}_E \) strictly bounded in \( x_0 \) such that, for any \( u, v \in V \), \( u \in V_n(v) \) iff \( \langle V, \in \rangle \models \nu_n(v, u) \).

**Lemma 2.7.** There are formulas \( \phi_i(x_0, \ldots, x_n) \) \( (i = 1, \ldots, 6) \) of \( \mathcal{L}_E \) strictly bounded in \( x_0 \) such that, for every superstructure \( V(X) \) over a base set \( X \), and all \( u, u_1, \ldots, u_n \in V(X) \), the following hold.
The practical significance of strictly bounded formulas is as follows. In mathematics formalized in a superstructure, the predicate that \( u \) is an element of \( v \) always implies \( v \) is a set relative to the superstructure. This interpretation is different, in general, from the relativization of a formula of set theory to the superstructure. However, for strictly bounded formulas the above practical interpretation is equivalent to the relativization as well as to the nonrelativized interpretation in the universe.

A nonstandard universe is a triple \( \langle V(X), V(Y), \ast \rangle \) consisting of superstructures \( V(X), V(Y) \), and a map \( \ast : V(X) \to V(Y) \) satisfying the following conditions:

1. \( X \) and \( Y \) are infinite base sets.
2. (Transfer Principle) The map \( \ast : a \mapsto \ast a \) is an injective mapping from \( V(X) \) into \( V(Y) \), and for any \( \Delta_0 \)-formula \( \phi(x_1, \ldots, x_n) \) of \( \mathcal{L}_\in \) and all \( u_1, \ldots, u_n \in V(X) \),
   \[ \langle V(X), \in \rangle \models \phi(u_1, \ldots, u_n) \iff \langle V(Y), \in \rangle \models \phi(\ast u_1, \ldots, \ast u_n). \]
3. \( \ast X = Y \).
4. For every infinite subset \( A \) of \( X \), \( \{ \ast a \mid a \in A \} \) is a proper subset of \( \ast A \).

A triple \( \langle V(X), V(Y), \ast \rangle \) satisfying conditions (1)–(3) is called a pre-nonstandard universe.

Assume hereafter that \( \langle V(X), V(Y), \ast \rangle \) is a pre-nonstandard universe. For \( A \in V(X) \setminus X \), we write \( \ast A = \{ \ast a \mid a \in A \} \). An element \( u \in V(Y) \) is said to be standard iff \( u = \ast v \) for some \( v \in V(X) \), or equivalently, \( u \in \ast V_n(X) \) for some \( n \). We denote by \( \ast V(X) \) the set of all standard elements. An element \( u \subset V(Y) \) is said to be internal iff \( u \in \ast v \) for some \( v \in V(X) \setminus X \), or equivalently, \( u \in \ast V_n(X) \) for some \( n \). The set of all internal elements of \( V(Y) \) is denoted by \( \ast V(X) \) and the set \( \ast V(X) \) is called the internal universe over \( X \). Sets relative to \( V(Y) \) which are not internal are called external. The following principle is the usual criterion for internal sets [4, Proposition 4.4.14, p. 277].

**Theorem 2.8** (Internal Definition Principle). Let \( \phi(x_0, \ldots, x_n, x) \) be a \( \Delta_0 \)-formula of \( \mathcal{L}_\in \). If \( u_1, \ldots, u_n, u \in V(Y) \) are internal, then the set

\[ \{ x \in u \mid \langle V(Y), \in \rangle \models \phi(u_1, \ldots, u_n, x) \} \]

is internal.

For the theory of nonstandard universes, we refer to Chang–Keisler [4].
3. Boolean-valued universes

In what follows, \( \mathcal{B} \) denotes a complete Boolean algebra. For \( b, c \in \mathcal{B} \), \( b \lor c \) denotes the supremum of \( \{b, c\} \), \( b \land c \) denotes the infimum of \( \{b, c\} \), \( \neg b \) denotes the complement of \( b \), and \( b \Rightarrow c \) abbreviates \( (\neg b) \lor c \). For the theory of Boolean algebras, we refer to Halmos [9].

For each ordinal \( \alpha \), let

\[
V^\alpha_{(\mathcal{B})} = \left\{ u \mid u : \text{dom}(u) \to \mathcal{B} \text{ and dom}(u) \subseteq \bigcup_{\beta < \alpha} V^\beta_{(\mathcal{B})} \right\}.
\]

The \( \mathcal{B} \)-valued universe \( V^\alpha_{(\mathcal{B})} \) is defined by

\[
V^\alpha_{(\mathcal{B})} = \bigcup_{\alpha \in \text{On}} V^\alpha_{(\mathcal{B})},
\]

where \( \text{On} \) is the class of all ordinals. To each sentence \( \phi \) of \( \mathcal{L}_\mathcal{E}(V^\alpha_{(\mathcal{B})}) \) we assign a \( \mathcal{B} \)-valued truth value \( [\phi] \) by the following recursive rules:

1. \( [u = v] = \inf_{x \in \text{dom}(u)} (u(x) \Rightarrow [x \in v]) \land \inf_{y \in \text{dom}(v)} (v(y) \Rightarrow [y \in u]). \)
2. \( [u \in v] = \sup_{y \in \text{dom}(v)} (v(y) \land [u = y]). \)
3. \( [-\phi] = -[\phi]. \)
4. \( [\phi_1 \lor \phi_2] = [\phi_1] \lor [\phi_2]. \)
5. \( [\exists x \phi(x)] = \sup\{ [\phi(u)] \mid u \in V^\alpha_{(\mathcal{B})} \}. \)

If a sentence \( \phi \) is provable from sentences \( \phi_1, \ldots, \phi_n \) in the first-order predicate calculus, then \( [\phi_1] \land \cdots \land [\phi_n] \leq [\phi] \). We say that a sentence \( \phi \) of \( \mathcal{L}_\mathcal{E}(V^\alpha_{(\mathcal{B})}) \) holds in \( V^\alpha_{(\mathcal{B})} \), if \( [\phi] = 1 \).

The universe \( V \) can be embedded in \( V^\alpha_{(\mathcal{B})} \) by the following operation \( \hat{\cdot} : v \mapsto \hat{v} \) defined by the \( \in \)-recursion: for each \( v \in V \), \( \hat{\cdot} = \{ \hat{u} \mid u \in v \} \times \{1\} \). Then we have the following theorem [2, Theorem 1.23].

**Theorem 3.1** (\( \Delta_0 \)-Absoluteness Principle). For any \( \Delta_0 \)-formula \( \phi(x_1, \ldots, x_n) \) of \( \mathcal{L}_\mathcal{E} \) and all \( u_1, \ldots, u_n \in V \),

\[
\langle V \in \mid = \phi(u_1, \ldots, u_n) \iff [\phi(\hat{u}_1, \ldots, \hat{u}_n)] = 1.
\]

We say that an element \( u \in V^{(B)} \) satisfying some property exists essentially uniquely if there is another \( u' \in V^{(B)} \) satisfying the same property then \( [u = u'] = 1 \). A partition of unity in \( \mathcal{B} \) is a family \( \{b_i \mid i \in I\} \) in \( \mathcal{B} \) such that \( b_i \land b_j = 0 \) whenever \( i \neq j \) and that \( \sup_{i \in I} b_i = 1 \). For any partition \( \{b_i \mid i \in I\} \) of unity and a family \( \{u_i \mid i \in I\} \) of \( \mathcal{B} \)-valued sets, there is an essentially unique element \( u \in V^{(B)} \) such that \( [u = u_i] \geq b_i \) for any \( i \in I \). We denote this \( u \) by \( \sum_{i \in I} u_i b_i \), or \( u_1 b_1 \oplus \cdots \oplus u_n b_n \) if \( I = \{1, \ldots, n\} \).

The following theorem is an important consequence of the axiom of choice [2, Lemma 1.27].
Theorem 3.2 (Maximum Principle). For any formula \( \phi(x) \) of \( \mathcal{L}_c(V^{(B)}) \), there is some \( u \in V^{(B)} \) such that

\[ [\phi(u)] = [(\exists x) \phi(x)]. \]

The basic theorem on the Boolean-valued universe is the following [2, Theorem 1.33].

Theorem 3.3 (ZFC Transfer Principle). For any formula \( \phi(x_1, \ldots, x_n) \) of \( \mathcal{L}_c \) and all \( u_1, \ldots, u_n \in V^{(B)} \),

if \( \text{ZFC} \vdash \phi(x_1, \ldots, x_n) \) then \( [\phi(u_1, \ldots, u_n)] = 1. \)

The \( B \)-valued hull \( u^{(B)} \) of a \( B \)-valued set \( u \in V^{(B)} \) is defined by

\[ u^{(B)} = \{ x \in V^{(B)} | [x \in u] = 1 \}, \]

where \( x \) is a certain representative from the equivalence class \( \{ y \in V^{(B)} | [x = y] = 1 \} \); for convenience we will omit the symbol in such a formula as \( x \in u^{(B)} \) when \( [x \in u] = 1 \). The choice of the representative \( x \) from the proper class \( \{ y \in V^{(B)} | [x = y] = 1 \} \) can be technically carried out within ZFC by at least two ways: in one way we restrict \( y \) under certain rank [35, p. 14], and in the other way we redefine \( y \) as \( y' = \{(z,x(z)) \wedge [z = y]\} | z \in \text{dom}(x) \) [2, Lemma 1.31].

A \( B \)-valued set \( u \in V^{(B)} \) is said to be separated if \( [x = y] = 1 \) implies \( x = y \) for all \( x, y \in \text{dom}(u) \). The \( B \)-valued hull \( u^{(B)} \) of \( u \in V^{(B)} \) is always separated by the above definition.

A \( B \)-valued set \( u \in V^{(B)} \) is said to be definite if \( u(x) = 1 \) for all \( x \in \text{dom}(u) \). If \( u \in V^{(B)} \) is definite then \( [x \in u] = 1 \) for all \( x \in \text{dom}(u) \). If \( u \in V^{(B)} \) is nonempty in \( V^{(B)} \), i.e., \( [u \neq 0] = 1 \), then

\[ [u = u^{(B)} \times \{1\}] = 1. \]

If \( u \in V^{(B)} \) is definite, then

\[ u^{(B)} = \{ \sum_{i \in I} u_i b_i | \{ u_i | i \in I \} \text{ is a family in dom}(u) \text{ and } \{ b_i | i \in I \} \text{ is a partition of unity of } B \}. \]

The following principle is the usual way to manipulate Boolean values of bounded formulas [35, p. 14].

Theorem 3.4 (Bounded Evaluation Principle). For any formula \( \phi(x) \) of \( \mathcal{L}_c(V^{(B)}) \) and definite \( u \in V^{(B)} \), we have:

1. \( [\forall x \in u) \phi(x)] = 1 \) iff \( \forall x \in \text{dom}(u) : [\phi(x)] = 1 \).
2. \( [\exists x \in u) \phi(x)] = 1 \) iff there is some \( x \in u^{(B)} \) such that \( [\phi(x)] = 1. \)
Let $D$ be a subset of $V^{(B)}$, a function $f : D \rightarrow V^{(B)}$ is called extensional if $[[x = y]] \Rightarrow [[f(x) = f(y)]]$ for all $x, y \in D$. Functions in $V^{(B)}$ are characterized as follows [35, Proposition 4.2, p. 22].

**Theorem 3.5.** Let $u, v \in V^{(B)}$ be definite. The relation

$$[[f(x) = g(x)]] = 1$$

for all $x \in \text{dom}(u)$ sets up a one-to-one correspondence between all “functions $f : u \rightarrow v$” in $V^{(B)}$ and all extensional functions $g : \text{dom}(u) \rightarrow v^{(B)}$.

For any $u \in V$ every function on $\text{dom}(\bar{u})$ is extensional, and hence the following theorem is an immediate consequence of the above theorem.

**Theorem 3.6** (Boolean Comprehensiveness Theorem). Let $f, u, v \in V$ be such that $f : u \rightarrow v^{(B)}$. Then there exists a function $g$ in $V^{(B)}$ satisfying $[[g : \bar{u} \rightarrow \bar{v}]] = 1$ and $[[g(x) = f(x)]] = 1$ for all $x \in u$.

For any $u, v \in V^{(B)}$, define $\{u, v\} \in V^{(B)}$ and $\langle u, v \rangle \in V^{(B)}$ as follows: $\{u, v\} = \{u, v\} \times \{1\}$, $\langle u, v \rangle = \{\langle u, u \rangle, \langle u, v \rangle\}_{B}$. Then $[[\{u, v\} = \langle u, v \rangle]] = 1$ and $[[\langle u, v \rangle]] = 1$ [2, p. 45]. For any $u, v \in V^{(B)}$, define $(u \times v) \in V^{(B)}$ as follows:

$$\text{dom}(u \times v) = \{\langle x, y \rangle \mid \langle x, y \rangle \in \text{dom}(u) \times \text{dom}(v)\},$$

$$(u \times v)(\langle x, y \rangle) = [[x \in u]] \wedge [[y \in v]],$$

for all $\langle x, y \rangle \in \text{dom}(u) \times \text{dom}(v)$. Then $[[u \times v]] = 1$ [32, p. 285]. Let $\phi(x_0, x_1, \ldots, x_n)$ be a formula of $\mathcal{L}_{\bar{u}}$ and let $u, u_1, \ldots, u_n \in V^{(B)}$. Define $v \in V^{(B)}$ by $\text{dom}(v) = \text{dom}(u)$ and

$$v(x) = u(x) \wedge [[\phi(x, u_1, \ldots, u_n)]]$$

for all $x \in \text{dom}(v)$. Then, we have [2, Lemma 1.35]

$$[[v = \{x \in u \mid \phi(x, u_1, \ldots, u_n)\}]] = 1.$$

We shall denote this $v$ by $\{x \in u \mid \phi(x, u_1, \ldots, u_n)\}_{B}$.

4. Boolean ultrasheaves

From Lemma 2.6, there is a $\Delta_0$-formula $\nu_{\bar{v}}(x_0, x_1)$ of $\mathcal{L}_{\bar{u}}$ strictly bounded in $x_0$ such that $\langle \bar{v} \in \nu_{\bar{v}}(v, u) \rangle$ iff $u \in \nu_{\bar{v}}(v)$ for all $u, v \in V$. Thus the set-theoretic definition of superstructures is expressible by strictly bounded formulas, and hence the construction is carried out in the Boolean-valued universe by the same formula according to the $\Delta_0$-Absoluteness Principle. For any $v \in V^{(B)}$ and each natural number $n$, define an essentially unique $B$-valued set $V_{\bar{v}}(v)_{B}$ by

$$[[V_{\bar{v}}(v)_{B} = \{x_1 \mid \nu_{\bar{v}}(v, x_1)\}]] = 1.$$
The $\mathcal{B}$-valued hull of $V_\alpha(v)_B$ is denoted by $V_\alpha(v)_B^{(B)}$.

Let $X$ be a base set. Then it follows from the $\Delta_0$-Absoluteness Principle that $\bar{X} \in V^{(B)}$ is a base set in $V^{(B)}$ and that $u \in V_\alpha(X)$ iff $[\bar{u} \in V_\alpha(\bar{X})_B] = 1$ iff $\bar{u} \in V_\alpha(\bar{X})_B$. The $\mathcal{B}$-valued superstructure $\hat{V}(X)$ over $X$ is defined by

$$\hat{V}(X) = \bigcup_{n \in \mathbb{N}} V_n(\bar{X})_B^{(B)}.$$ 

Let $\mathcal{U}$ be an ultrafilter of $\mathcal{B}$, which will be fixed throughout this section. We define an equivalence relation $\equiv_\mathcal{U}$ over $\hat{V}(X)$ by

$$x \equiv_\mathcal{U} y \iff [x = y] \in \mathcal{U}.$$ 

Denote by $\hat{V}(X)/\mathcal{U}$ the set of equivalence classes $[u]$ of all $u \in \hat{V}(X)$, and denote by $\bar{X}^{(B)}/\mathcal{U}$ the set of equivalence classes $[u] \cap \bar{X}^{(B)}$ in $\bar{X}^{(B)}$ of all $u \in \bar{X}^{(B)}$.

For each natural number $n$, define $W_n(X)$ by the relations

$$W_0(X) = \{u \in \hat{V}(X) \mid [u \in \bar{X}] \in \mathcal{U}\},$$ 

$$W_{n+1}(X) = \{u \in \hat{V}(X) \mid [u \in V_{n+1}(\bar{X})_B \setminus V_n(\bar{X})_B] \in \mathcal{U}\}.$$ 

Lemma 4.1. The following statements hold:

1. For any $u \in \hat{V}(X)$, there is some $k \in \mathbb{N}$ uniquely such that $u \in W_k(X)$.
2. For any $u \in V_{k+1}(X) \setminus V_k(X)$, we have $\bar{u} \in W_{k+1}(X)$.
3. For any $u$, $v \in \hat{V}(X)$, if $[u = v] \in \mathcal{U}$ and $v \in W_n(X)$, then $n > 0$ and $u \in W_k(X)$ for some $k < n$.
4. For any $u$, $v \in \hat{V}(X)$, if $[u = v] \in \mathcal{U}$ and $v \in W_n(X)$, then $u \in W_n(X)$.
5. For any $u \in W_0(X)$, there is some $u \in \bar{X}^{(B)}$ such that $[u = u] \in \mathcal{U}$. Moreover, for any $u \in W_{n+1}(X)$, there is some $v \in V_{n+1}(X)_B \setminus V_n(X)_B$ such that $[u = v] \in \mathcal{U}$.

Proof. (1) Let $u \in \hat{V}(X)$. Let $U_0 = \bar{X}$ and, for any $n$, let $U_{n+1} \in V^{(B)}$ be such that $[U_{n+1} = V_{n+1}(\bar{X})_B \setminus V_n(\bar{X})_B] = 1$. Since for any $m$, $n \in \mathbb{N}$ with $m \neq n$,

$$[u \in U_m] \wedge [u \in U_n] = 0,$$

the relation $[u \in U_n] \in \mathcal{U}$ can be satisfied by at most one $n$. By assumption, there is some $n \in \mathbb{N}$ such that $u \in V_n(\bar{X})_B$, and hence

$$[u \in U_0] \vee [u \in U_1] \vee \cdots \vee [u \in U_n] = [u \in V_n(\bar{X})_B] = 1.$$ 

Thus, there is a natural number $k \leq n$ such that $[u \in U_k] \in \mathcal{U}$.

(2) Let $u \in V_{k+1}(X) \setminus V_k(X)$. Then $(\forall v \in X) \models \nu_{k+1}(u, X) \wedge \neg \nu_k(u, X)$. and hence $[\bar{u} \in U_{k+1}] = 1$ by the $\Delta_0$-Absoluteness Principle, so that $\bar{u} \in W_{k+1}(X)$.

(3) Suppose $[u \in v] \in \mathcal{U}$ and $v \in W_n(X)$. By applying the ZFC Transfer Principle to Lemma 2.2, we have $n > 0$ and

$$[u \in V_{n-1}(\bar{X})_B] \geq [u \in v] \in \mathcal{U},$$

and hence $u \in \bigcup_{k=0}^{n-1} W_k(X)$. 


If \( [u = v] \subset \mathcal{U} \) and \( u \in W_{n+1}(X) \) then
\[
[u \in V_{n+1}(\hat{X})_B \setminus V_n(\hat{X})_B] \geq [u = v] \land [u \in V_{n+1}(\hat{X})_B \setminus V_n(\hat{X})_B] \in \mathcal{U},
\]
and hence \( v \in W_{n+1}(X) \).

(5) If \( X = \emptyset \), then \( W_0 = \emptyset \) so that the assertion holds trivially. Suppose \( X \neq \emptyset \), say \( w \in X \), and \( u \in W_0(X) \). Let \( v = ub \oplus w(-b) \) where \( b = [u \in \hat{X}] \). Then \( [u = v] \geq b \in \mathcal{U} \) and \( v \in \hat{X}^B \). The proof of the rest of the assertion is similar. \( \square \)

We define \( Y \) as a base set with the same size as \( \hat{X}^B / \mathcal{U} \) with a fixed bijection \( j : \hat{X}^B / \mathcal{U} \rightarrow Y \). Note that, from Lemma 4.1 (5), \( [u] \cap \hat{X}^B \in \hat{X}^B / \mathcal{U} \) for any \( u \in W_0(X) \). Define a map \( \gamma : \hat{V}(X) \rightarrow V(Y) \) recursively by
\[
\gamma(u) = j([u] \cap \hat{X}^B) \quad \text{for } u \in W_0(X),
\]
\[
\gamma(u) = \left\{ \gamma(x) \mid [x \in u] \subseteq \mathcal{U} \text{ and } x \in \bigcup_{k=0}^n W_k(X) \right\} \quad \text{for } u \in W_{n+1}(X).
\]

We call the map \( \gamma \) the \( \mathcal{U} \)-interpretation, and we say that \( u \in \hat{V}(X) \) is a name of \( a \in V(Y) \) if \( \gamma(u) = a \). Define a map \( * : V(X) \rightarrow V(Y) \) by the relation \( *x = \gamma(\hat{x}) \) for all \( x \in V(X) \).

The following lemma is immediate from Lemma 4.1 (3), (4).

**Lemma 4.2.** Let \( u, v \in \hat{V}(X) \).

(1) \( \gamma(u) \in \gamma(v) \) iff \( [u \in v] \in \mathcal{U} \).

(2) \( \gamma(u) = \gamma(v) \) iff \( [u = v] \in \mathcal{U} \).

We are now ready to show that \( \langle V(X), V(Y), * \rangle \) is a pre-nonstandard universe.

**Lemma 4.3.** We have \( *X = Y \).

**Proof.** Obviously, \( \hat{X} \in W_1(X) \). By Lemma 4.1(5), we have
\[
*X = \gamma(\hat{X})
\]
\[
= \{ \gamma(u) \mid [u \in \hat{X}] \subseteq \mathcal{U} \text{ and } u \in W_0(X) \}
\]
\[
= \{ j([u] \cap \hat{X}^B) \mid u \in W_0(X) \}
\]
\[
= \{ j([v] \cap \hat{X}^B) \mid v \in \hat{X}^B \} = Y \quad \square
\]

The following theorem is the counterpart of the one obtained by the \( \text{\c{L}o\c{s}} \) Theorem and the Mostowski Collapsing Theorem in the ultrapower approach. An advantage of this new theorem is that the constants \( \gamma(u_k) \) appearing in the formula are not necessarily internal elements in \( V(X) \).

**Theorem 4.4 (\( \text{\c{L}o\c{s}} \)-Mostowski Principle).** For any \( \Delta_0 \)-formula \( \phi(x_0, x_1, \ldots, x_n) \) of \( L_\infty \) strictly bounded in \( x_0 \) and all \( u_1, \ldots, u_n \in \hat{V}(X) \),
Proof. The proof is by induction on the complexity of strictly bounded formulas. From Lemma 4.2, the assertion holds for atomic formulas. The only nontrivial induction step is the case where \( \phi(x_0, x_1, \ldots, x_n) \) is of the form

\[
(\exists x) \left[ (x \in y \land y \not\in x_0) \land \psi(x_0, x_1, \ldots, x_n, x) \right],
\]

where \( y \) is one of \( x_0, \ldots, x_n \). The proof for the case where \( y \) is \( x_0 \) is similar to the other cases but slightly easier, so that we may assume without any loss of generality that \( y \) is \( x_1 \). Suppose \( \llbracket \phi(X, u_1, \ldots, u_n) \rrbracket \in \mathcal{U} \). Then, there is some \( u \in V^{(B)} \) such that \( \llbracket u \in u_1 \rrbracket \in \mathcal{U} \), \( \llbracket u_1 \not\in X \rrbracket \in \mathcal{U} \), and \( \llbracket \psi(X, u_1, \ldots, u_n, u) \rrbracket \in \mathcal{U} \). Thus, by assumption, there is some \( k \in \mathbb{N} \) such that \( \llbracket u_1 \in V_k(X) \backslash X \rrbracket \in \mathcal{U} \). From the ZFC Transfer Principle, \( \llbracket u \in V_k(X) \rrbracket \in \mathcal{U} \). It follows from Lemma 4.2 that \( \gamma(u) \in \gamma(u_1) \) and \( \gamma(u_1) \not\in Y \). By the induction hypothesis, \( \llbracket (\forall Y, \in) \models \psi(Y, \gamma(u_1), \ldots, \gamma(u_n), \gamma(u)) \rrbracket \). Thus we have \( \llbracket (\forall Y, \in) \models \phi(Y, \gamma(u_1), \ldots, \gamma(u_n)) \rrbracket \). Conversely, suppose \( \llbracket (\forall Y, \in) \models \phi(Y, \gamma(u_1), \ldots, \gamma(u_n)) \rrbracket \). Then there is some \( v \in V(Y) \) such that \( v \in \gamma(u_1) \) and \( \llbracket (\forall Y, \in) \models \psi(Y, \gamma(u_1), \ldots, \gamma(u_n), v) \rrbracket \) and that \( \gamma(u_1) \not\in Y \), whence \( u_1 \in W_k(X) \) with \( k > 0 \). From Lemma 4.1(1), \( \llbracket u_1 \not\in X \rrbracket \in \mathcal{U} \). From the definition of the \( \mathcal{U} \)-interpretation \( \gamma \), there is some \( u \in V_{k-1}(X) \) such that \( v = \gamma(u) \). Thus, from Lemma 4.2, \( \llbracket u \in u_1 \rrbracket \in \mathcal{U} \), and from the induction hypothesis, \( \llbracket \psi(X, u_1, \ldots, u_n, u) \rrbracket \in \mathcal{U} \). Therefore, we can conclude \( \llbracket \phi(X, u_1, \ldots, u_n) \rrbracket \in \mathcal{U} \). This completes the induction. \( \square \)

Theorem 4.5 (Generic Transfer Principle). For any \( \Delta_0 \)-formula \( \phi(x_0, x_1, \ldots, x_n) \) of \( \mathcal{L}_\in \) strictly bounded in \( x_0 \) and all \( u_1, \ldots, u_n \in \bar{V}(X) \),

\[
\text{if ZFC} \vdash \phi(x_0, x_1, \ldots, x_n) \text{ then } \llbracket (\forall Y, \in) \models \phi(Y, \gamma(u_1), \ldots, \gamma(u_n)) \rrbracket.
\]

Proof. By the ZFC Transfer Principle, \( \llbracket \phi(X, u_1, \ldots, u_n) \rrbracket = 1 \in \mathcal{U} \). Thus the assertion follows immediately from the Los–Mostowski Principle. \( \square \)

Lemma 4.6 (Transfer Principle). For any \( \Delta_0 \)-formula \( \phi(x_1, \ldots, x_n) \) of \( \mathcal{L}_\in \) and all \( u_1, \ldots, u_n \in V(X) \),

\[
\llbracket (\forall X, \in) \models \phi(u_1, \ldots, u_n) \rrbracket \iff \llbracket (\forall Y, \in) \models \phi(\gamma(u_1), \ldots, \gamma(u_n)) \rrbracket.
\]

Proof. Let \( \phi(x_0, x_1, \ldots, x_n) \) be \( \phi(x_1, \ldots, x_n) \), where \( x_0 \) is a variable other than \( x_1, \ldots, x_n \), and consider the formula \( (\phi(x_0, x_1, \ldots, x_n))' \) strictly bounded in \( x_0 \). For all \( u_1, \ldots, u_n \in V(X) \), we have

\[
\llbracket (\forall X, \in) \models \phi(u_1, \ldots, u_n) \rrbracket \iff \llbracket (\forall X, \in) \models \phi(X, u_1, \ldots, u_n) \rrbracket \text{ by the reducency of } x_0
\]

\[
\llbracket (\forall X, \in) \models (\phi(X, u_1, \ldots, u_n))' \rrbracket \text{ by Lemma 2.5}
\]

\[
\llbracket (\forall \in) \models (\phi(u_1, \ldots, u_n))' \rrbracket \text{ by Lemma 2.4}
\]

\[
\llbracket (\phi(X, u_1, \ldots, u_n))' \rrbracket \in \mathcal{U} \text{ by the } \Delta_0 \text{-Absoluteness Principle}
\]
iff \( \langle V(Y), \in \rangle \models \bar{\phi}(Y, *u_1, \ldots, *u_n) \)' by the Łoś–Mostowski Principle
iff \( \langle V(Y), \in \rangle \models \bar{\phi}(X, *u_1, \ldots, *u_n) \) by Lemma 2.5
iff \( \langle V(Y), \in \rangle \models \phi(*u_1, \ldots, *u_n) \) by the redundancy of \( x_0. \)

This completes the proof. \( \square \)

The 4-tuple \( \langle V(X), V(Y), *, \gamma \rangle \) constructed in this section is called the bounded ultrasheaf of a superstructure \( V(X) \mod \mathcal{U} \) of a complete Boolean algebra \( \mathcal{B}. \) We have just proved the following theorem.

**Theorem 4.7.** Let \( X \) be a base set, \( \mathcal{B} \) a complete Boolean algebra, and \( \mathcal{U} \) an ultrafilter of \( \mathcal{B}. \) Let \( \langle V(X), V(Y), *, \gamma \rangle \) be the bounded ultrasheaf of \( V(X) \mod \mathcal{U}. \) Then the triple \( \langle V(X), V(Y), * \rangle \) is a pre-nonstandard universe.

Thus the notions of being standard, internal, and external are applied to bounded ultrasheaves. An element \( u \in V(Y) \) is said to be ingeneric iff it is an element of \( \gamma(V_n(\bar{X})_\mathcal{B}) \) for some \( n, \) or equivalently, \( u \) is an element of \( \gamma(v) \) for some \( v \in \hat{V}(X) \setminus W_0(X), \) or equivalently, \( u = \gamma(v) \) for some \( v \in \hat{V}(X). \) The set of all ingeneric elements is denoted by \( \gamma V(X) \) and the set \( \gamma V(X) \) is called the generic universe over \( X. \) Obviously, we have the following inclusions:

\[ \gamma V(X) \subseteq \gamma V(Y) \subseteq \gamma (V(Y)) \subseteq V(Y). \]

A subset \( A \) of \( V(Y) \) is said to be a transitive submodel iff whenever \( a \in A, b \in V(Y), \) and \( b \in a, \) we have \( b \in A. \) It is easy to see that \( \gamma V(X) \) and \( \gamma V(Y) \) are transitive submodels, and hence they are bounded elementary submodels of \( V(Y) \) [4, Lemma 4.4.7, p. 270].

We say that two pre-nonstandard universes \( \langle V(X), V(Y), *, i \rangle \) \( (i = 1, 2) \) with base set \( X \) are isomorphic iff there is a bijection \( h : V(Y_1) \to V(Y_2) \) such that \( h(*_1(u)) = *_2(u) \) for all \( u \in V(X). \) Let \( I \) be an index set and \( \mathcal{U} \) an ultrafilter of the complete Boolean algebra \( \mathcal{P}(I) \) of subsets of \( I. \) Then the bounded ultrapower \( \langle V(X), V(Y), * \rangle \) of \( V(X) \mod \mathcal{U} \) is constructed in [4, p. 269] and shown to be a pre-nonstandard universe. The following theorem is verified easily by comparing the two constructions.

**Theorem 4.8.** Let \( X \) be a base set, \( I \) an index set, and \( \mathcal{U} \) an ultrafilter on \( \mathcal{P}(I). \) The pre-nonstandard universe constructed as the bounded ultrasheaf of \( V(X) \mod \mathcal{U} \) is isomorphic with the bounded ultrapower of \( V(X) \mod \mathcal{U}. \) In this case, every ingeneric element is internal.

**Proof.** Let \( \mathcal{B} = \mathcal{P}(I). \) Then we have \( \| \mathcal{P}(\bar{u}) \|_\mathcal{B} = \| \mathcal{P}(u) \|_\gamma \| = 1 \) for all \( u \in V. \) It follows that \( \| V_n(\bar{X})_\mathcal{B} = V_n(X) \|_\gamma \| = 1 \) for all \( n, \) so that we have

\[ \hat{V}(X) = \bigcup_{n \in \mathbb{N}} (V_n(X) \|_\gamma \|^{(B)}). \]

Let \( u \in (V_n(X) \|_\gamma \|^{(B)}). \) Define \( \bar{u} : I \to V_n(X) \) by

\[ \bar{u}(i) = \{ v \in V_n(X) \mid i \in [\bar{v} \in u] \} \]

Let \( \tilde{u} : I \to \hat{V}(X) \) be defined by

\[ \tilde{u}(i) = \{ v \in \hat{V}(X) \mid i \in [\bar{v} \in u] \} \]

Then \( \tilde{u}(I) \subseteq \gamma V(Y) \) and \( \tilde{u} : I \to \gamma V(Y) \) is a bijection, therefore \( \langle V(X), V(Y), *, \tilde{u} \rangle \) is isomorphic with \( \langle V(X), V(Y), *, \bar{u} \rangle \).
for all \( i \in I \). Then the correspondence \( u \rightarrow \bar{u} \) is a bijection between \( (V_\kappa(X)')^B \) and \( V_\kappa(X)' \) such that \( [u = v] = \{ i \in I \mid \bar{u}(i) = \bar{v}(i) \} \) and \( [u \in v] = \{ i \in I \mid \bar{u}(i) \in \bar{v}(i) \} \). Now, the comparison between the construction of a bounded ultrapower and that of a bounded ultrasheaf is straightforward, and the detail is left to the reader.

The following theorem holds for every bounded ultrasheaf.

**Theorem 4.9 (Generic Comprehensiveness Theorem).** Let \( \langle V(X), V(Y), \ast, \gamma \rangle \) be a bounded ultrasheaf. Let \( C \in V(X) \setminus X, \ D \in \hat{V}(X) \setminus W_0(X). If \( f : C \rightarrow \gamma(D), \) then there exists an ingeneric function \( g : \ast C \rightarrow \gamma(D) \) such that \( g(\ast u) = f(u) \) for all \( u \in C \).

**Proof.** Without any loss of generality we may assume \( \|D \neq \emptyset\| = 1 \). We can represent \( D \) as \( D = D^B \times \{ 1 \} \). Let \( f_0 : C \rightarrow D^B \) be such that \( \gamma(f_0(u)) = f(u) \) for all \( u \in C \). By the Boolean Comprehensiveness Theorem, there is a function \( g_0 : \hat{C} \rightarrow D \in V(B) \) such that \( \|g_0(\bar{u}) = f_0(u)\| = 1 \) for all \( u \in C \). Since \( \|g_0 \subseteq (\hat{C} \times D)^B\| = 1 \), we have \( g_0 \in \hat{V}(X) \) by the ZFC Transfer Principle. Let \( g = \gamma(g_0) \). Then

\[
g = \{ \gamma(\langle u, v \rangle_B) \mid [u \in \hat{C}] \in \mathcal{U}, \ [\langle u, v \rangle_B \in g_0] \in \mathcal{U} \}
\]

Thus, if \( \langle \gamma(u), \gamma(v) \rangle \in g \) and \( \langle \gamma(u), \gamma(v') \rangle \in g \), then \( [\langle u, v \rangle_B \in g_0] \in \mathcal{U} \) and hence \( [v = v'] \in \mathcal{U} \) so that \( \gamma(v) = \gamma(v') \). It follows that \( g \) is an ingeneric function. Let \( u \in C \). Then \( g(\ast u) = g(\gamma(\bar{u})) = \gamma(g_0(\bar{u})) = \gamma(f_0(u)) = f(u) \). \( \Box \)

The following theorem is a counterpart of the comprehensiveness theorem for bounded ultrapowers [4, Theorem 4.4.23, p. 284].

**Theorem 4.10.** Let \( C, D \in V(X) \setminus X \). If \( f : C \rightarrow \ast D \), then there exists an ingeneric function \( g : \ast C \rightarrow \ast D \) such that \( g(\ast u) = f(u) \) for all \( u \in C \).

**Proof.** Apply Theorem 4.9 to \( C \in V(X) \setminus X \) and \( D \in \hat{V}(X) \setminus W_0(X) \). \( \Box \)

**Remark.** Partial answers to the question as to when the above ingeneric function \( g \) can be chosen as an internal function is as follows:

1. \( C \) is countable. This follows from the Saturation Principle discussed in the next section.
2. \( B \) is \([C]\)-distributive. This follows from a forcing argument; see [11, p. 158; Lemma 19.6, p. 180].

5. Saturation principle

The following principle is useful in many applications of nonstandard analysis. A pre-nonstandard universe \( \langle V(X), V(Y), \ast \rangle \) is called **countably saturated**, or \( \aleph_1 \)-**saturated** if it satisfies the following condition:

...
(Saturation Principle) Any countable sequence of internal sets $A_n \in V(Y) \setminus Y$ with the finite intersection property (i.e., $\bigcap_{n=0}^{\infty} A_n \neq \emptyset$ for all $m \in \mathbb{N}$) has a nonempty intersection.

A filter $\mathcal{U}$ on $\mathcal{B}$ is said to be countably incomplete if there is a countable set $C \subseteq \mathcal{U}$ such that $\inf C \not\in \mathcal{U}$.

**Theorem 5.1.** Let $X$ be an infinite base set and $\langle V(X), V(Y), \ast, \gamma \rangle$ a bounded ultrasheaf of $V(X)$ modulo an ultrafilter $\mathcal{U}$ on a complete Boolean algebra $\mathcal{B}$. Then the following conditions are equivalent:

1. $\mathcal{U}$ is countably incomplete.
2. For every infinite subset $A$ of $X$, $A$ is a proper subset of $^*A$.
3. $\langle V(X), V(Y), \ast \rangle$ is countably saturated.
4. **(Generic Saturation Principle)** Any countable sequence of ingeneric sets $A_n \in V(Y) \setminus Y$ with the finite intersection property has a nonempty intersection.

**Proof.** (1)$\Rightarrow$(4). Let $\{A_n \mid n \in \mathbb{N}\}$ be a countable sequence of ingeneric sets in $V(Y)$ with the finite intersection property. By replacing $A_n$ by the finite intersections if necessary, we can assume that $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$. Thus all $A_n$ are elements of some $V_k(Y)$. For any $n \in \mathbb{N}$, let $C_n \in V_k(X)^{(B)}$ be such that $\gamma(C_n) = A_n$. Obviously, $[C_{n+1} \subseteq C_n] \in \mathcal{U}$ and $[C_n \neq \emptyset] \in \mathcal{U}$. By the Maximum Principle, there is some $B$-valued set $f(n) \in V_k(X)^{(B)}$ such that $[f(n) \in C_n] = [C_n \neq \emptyset]$ for all $n \in \mathbb{N}$. Then $[f(n) \in C_n] \geq [f(n+1) \in C_{n+1}]$. Since $\mathcal{U}$ is countably incomplete, there is a sequence $b_n \in \mathcal{U}$ such that $l = b_0 \geq b_1 \geq \cdots$ and $\inf_{n \in \mathbb{N}} b_n = 0$. Let $a_0 = b_0$ and $a_n = b_n \land [f(n) \in C_n]$ for all $n > 0$, and let $c_n = a_n \setminus a_{n+1}$ for all $n \in \mathbb{N}$. Then $c_k \land c_l = 0$ for $k \neq l$, $c_n = \sup_{k \geq n} c_k$, and $\sup_{n \in \mathbb{N}} c_n = 1$. Let $g$ be such that $[g = f(k)] = c_k$ for all $k \in \mathbb{N}$. Then $[g \in V_k(X)] = 1$. Now we will show that $\gamma(g) \in A_n$ for arbitrary $n \in \mathbb{N}$. Since $c_k \leq [g = f(k)] \land [f(k) \in C_k] \leq [g \in C_k]$, we have $c_k \leq [g \in C_k] \leq [g \in C_n]$ for any $k \geq n$. Thus $a_n = \sup_{k \geq n} c_k \leq [g \in C_n]$. Since $a_n \in \mathcal{U}$, it is concluded that $[g \in C_n] \in \mathcal{U}$, and hence $\gamma(g) \in A_n$.

(4)$\Rightarrow$(3). Since every internal set is ingeneric, the assertion follows.

(3)$\Rightarrow$(2). Suppose that there is an infinite subset $A$ of $X$ such that $^*A \neq ^*A$. We may assume without any loss of generality that $N$ is a subset of $A$. Then $^*N \subseteq \ast A$ by the Transfer Principle, and hence every element of $^*N$ is standard, so that it is an element of $^*N$ by the Transfer Principle. Thus $^*N = \ast N$. By the Saturation Principle, the sequence of internal sets $A_n = ^*N \setminus \{0, \ldots, ^*n\}$ has a nonempty intersection. Thus there is some $\nu \in ^*N \setminus \ast N$. This is a contradiction.

(2)$\Rightarrow$(1). We may assume $N \subseteq X$. By assumption, there is some $\nu \in ^*N \setminus ^*N$. Then, there is some $\nu_0 \in \tilde{N}^{(B)}$ such that $\gamma(\nu_0) = \nu$. It follows that $\bigvee_{n \in \mathbb{N}} [\nu_0 = \tilde{n}] = 1$. If $\mathcal{U}$ is countably complete, then $[\nu_0 = \tilde{n}] \in \mathcal{U}$ for some $n \in \mathbb{N}$ so that $\nu = ^*n$, and hence $\nu \in ^*N$. Therefore, $\mathcal{U}$ must be countably incomplete. □

Thus we have reached the following.
Theorem 5.2. Let $X$ be an infinite base set, $B$ a complete Boolean algebra, and $U$ a countably incomplete ultrafilter of $B$. Let $(V(X), V(Y), \star, \gamma)$ be the bounded ultrasheaf of $V(X)$ modulo $U$. Then the triple $(V(X), V(Y), \star)$ is an $\aleph_1$-saturated nonstandard universe.

The following theorem generalizes [4, Proposition 4.4.20, p. 282] to ingeneric sets.

Theorem 5.3. Let $(V(X), V(Y), \star, \gamma)$ be a bounded ultrasheaf modulo a countably incomplete ultrafilter. Each infinite ingeneric set in $V(Y) \setminus Y$ has cardinality at least $2^{\aleph_0}$. Every countably infinite set in $V(Y) \setminus Y$ is not ingeneric.

Proof. We may assume $N \subseteq X$. Let $A$ be an infinite ingeneric set in $V(Y) \setminus Y$. We have $A = \gamma(C)$ for some $C \in W(X)$ with $k > 0$. Suppose $\|C\|$ is an infinite set in $V(Y) \setminus Y$. Then by the ZFC Transfer Principle, there is a function $f$ in $V(B)$ such that

$$\|f\| = \text{a function from } C \text{ onto } \mathbb{N} \in U.$$ 

Then we can choose $f$ so that $f \in \hat{V}(X)$, and then $\gamma(f)$ is a function from $A$ onto $\ast N$. Since $\ast N$ is an infinite internal set, we have $|A| \geq |\ast N| \geq 2^{\aleph_0}$ by [4, Proposition 4.4.20, p. 282]. Suppose $\|C\|$ is a finite set in $U$. By the ZFC Transfer Principle, there is a natural number $\nu_0$ in $V(B)$ and a function $f$ in $\hat{V}(X)$ such that

$$\|f\| = \text{a one-to-one function from } \{\ast 0, \ldots, \nu - 1\} \text{ onto } C \in U.$$ 

Then $\gamma(f)$ is a one-to-one function from $\{\ast 0, \ldots, \gamma(\nu) - \ast 1\}$ onto $A$. Thus the cardinality of $A$ is the same as an infinite internal set, and hence it is at least $2^{\aleph_0}$. \(\square\)

6. Boolean ultrapowers

The construction of a Boolean ultrapower of a model, developed in the late 60's and early 70's gives an $\aleph_1$-saturated elementary extension of the model and generalizes the construction of an ultrapower. We can easily generalize the construction of a bounded ultrapower of a superstructure to the construction of a bounded Boolean ultrapower. This construction, however, has not attracted the specialists of nonstandard analysis, because it adds no new features to the bounded ultrapower construction and because no one has proved the existence of a Boolean ultrapower which is not an ultrapower.

The construction of a Boolean ultrasheaf als generalizes the construction of an ultrapower, but realizes the generic extension of a Boolean ultrapower within ZFC without any countable model assumption. Thus, a bounded Boolean ultrasheaf of a superstructure has at least one feature which an ultrapower does not have.

In this section, we investigate the relationship between the bounded Boolean ultrasheaf of a superstructure and a Boolean ultrapower of a model more closely. First, we shall review definitions and elementary properties of Boolean-valued interpretations and Boolean ultrapowers. For the detail we refer to Mansfield [19].

Let $L = \{P, \ldots\}$ be a first-order language with a collection of $m$-placed relation symbols $P$, where $m$ depends on $P$. Let $B$ be a complete Boolean algebra. A $B$-valued
interpretation \( A = (A, E, R, \ldots) \) of \( \mathcal{L} \) consists of a set \( A \), a function \( E : A^2 \to B \), and a function \( R : A^m \to B \) corresponding to each relation symbol \( P \) of \( \mathcal{L} \) which satisfy the following conditions: For all \( a, b, a_1, b_1, \ldots, a_m, b_m \in A \) and all functions \( R \),

\begin{equation}
\begin{aligned}
(\text{E1}) & \quad E(a, a) = 1, \\
(\text{E2}) & \quad E(a, b) = E(b, a), \\
(\text{E3}) & \quad E(a, b) \wedge E(b, c) \leq E(a, c), \\
(\text{E4}) & \quad E(a_1, b_1) \wedge \cdots \wedge E(a_m, b_m) \wedge R(a_1, \ldots, a_m) \leq R(b_1, \ldots, b_m), \\
(\text{E5}) & \quad E(a, b) = 1 \text{ then } a = b.
\end{aligned}
\end{equation}

The set \( A \) is called the universe, the function \( E \) is called the \( B \)-valued equality of \( A \), and each \( R \) is called a \( B \)-valued relation on \( A \). The truth function \( \mathbb{[}\cdot\mathbb{]}_A \) is an assignment of a value in \( B \) to each sentence \( \phi \) of \( \mathcal{L} \cup A \) and is defined as follows:

\begin{enumerate}
\item \( \mathbb{[a = b]}_A = E(a, b) \) for all \( a, b \in A \).
\item \( \mathbb{[P(a_1, \ldots, a_m)]}_A = R(a_1, \ldots, a_m) \) for all \( a_1, \ldots, a_m \in A \).
\item \( \mathbb{[-\phi]}_A = -\mathbb{[\phi]}_A \).
\item \( \mathbb{[\phi \lor \psi]}_A = \mathbb{[\phi]}_A \lor \mathbb{[\psi]}_A \).
\item \( \mathbb{[(\exists x)\phi(x)]}_A = \bigvee_{a \in A} \mathbb{[\phi(a)]}_A \).
\end{enumerate}

For an arbitrary ultrafilter \( \mathcal{U} \) of \( B \) we define a model \( \mathcal{A}/\mathcal{U} = (A/\mathcal{U}, R/\mathcal{U}, \ldots) \) for \( \mathcal{L} \), called the reduced model of \( A \) modulo \( \mathcal{U} \), as follows. Define the binary relation \( \equiv_\mathcal{U} \) on \( A \) by \( a \equiv_\mathcal{U} b \) iff \( E(a, b) \in \mathcal{U} \). The relation \( \equiv_\mathcal{U} \) is an equivalence relation because of properties (E1)-(E3). The universe \( A/\mathcal{U} \) is the quotient set \( A/\equiv_\mathcal{U} \); the equivalence class of \( a \in A \) is denoted by \( a/\mathcal{U} \). For each \( B \)-valued relation \( R \) of \( A \), define the relation \( R/\mathcal{U} \) on \( A/\mathcal{U} \) by

\begin{equation}
\begin{aligned}
\langle u_1/\mathcal{U}, \ldots, u_m/\mathcal{U} \rangle \in R/\mathcal{U} \text{ iff } R(u_1, \ldots, u_m) \in \mathcal{U}.
\end{aligned}
\end{equation}

Then, for any formula \( \phi(x_1, \ldots, x_m) \) of \( \mathcal{L} \) and all \( u_1, \ldots, u_m \in A \),

\( \mathcal{A}/\mathcal{U} \models \phi(u_1/\mathcal{U}, \ldots, u_m/\mathcal{U}) \text{ iff } \mathbb{[\phi(u_1, \ldots, u_m)]}_A \in \mathcal{U} \).

Let \( \mathcal{M} = \langle M, R, \ldots \rangle \) be a model for \( \mathcal{L} \) with universe \( M \) and relations \( R \). The Boolean power \( \prod^B M = (\prod^B M, E, \prod^B R, \ldots) \) of \( \mathcal{M} \) over \( B \) is a \( B \)-valued interpretation defined as follows. The universe \( \prod^B M \) is the set of all functions from \( M \) into \( B \) whose ranges partition \( B \), i.e.,

\begin{equation}
\prod^B M = \left\{ u \in B^M \mid (\forall a, b \in M)[a \neq b \Rightarrow u(a) \wedge u(b) = 0] \wedge \bigvee_{a \in M} u(a) = 1 \right\}.
\end{equation}

If \( R \) is an \( m \)-placed relation on \( M \), we extend it to a \( B \)-valued relation \( \prod^B R \) by

\begin{equation}
\prod^B R(u_1, \ldots, u_m) = \bigwedge_{\langle a_1, \ldots, a_m \rangle \in R} \bigvee_{i=1}^m u_i(a_i).
\end{equation}

The equality relation determines \( E \) by the same rule as \( R \), i.e.,
\[ E(u, v) = \bigvee_{a \land b} u(a) \land v(b) = \bigvee_{a \in M} u(a) \land v(a). \]

For each \( a \in M \), define a function \( \delta_a \in \prod^B M \) by

\[
\delta_a(x) = \begin{cases} 
1 & \text{if } x = a, \\
0 & \text{if } x \neq a.
\end{cases}
\]

Then the map \( a \mapsto \delta_a \) is an elementary embedding of \( M \) into \( \prod^B M \). The Boolean ultrapower \( \prod^B_M A = (\prod^B_M A, \prod^B_M R, \ldots) \) of \( A = (M, R, \ldots) \) modulo \( U \) is the reduced model of the \( B \)-valued interpretation \( \prod^B M = (\prod^B_M M, E, \prod^B_M R, \ldots) \) modulo \( U \).

Two \( B \)-valued interpretations \( (A, E, R, \ldots) \) and \( (A', E', R', \ldots) \) are isomorphic if there is a bijection \( h : A \to A' \), called an isomorphism, such that \( E(a_1, a_2) = E'(h(a_1), h(a_2)) \) and \( R(a_1, \ldots, a_m) = R'(h(a_1), \ldots, h(a_m)) \) for each \( B \)-valued relation \( R \), where \( a_1, \ldots, a_m \in A \).

Now we shall turn to the Boolean-valued universe \( V(B) \). A \( B \)-valued model \( \mathcal{B} = (A, E, R, \ldots) \) for \( L \) consists of a separated and definite \( B \)-valued set \( A \) and a \( B \)-valued sets \( R \) corresponding to each relation symbol \( P \) of \( L \), which is an "\( m \)placed relation on \( A \)" in \( V(B) \), i.e., \( \forall R \subseteq (A^m)^B \) \( \exists ! R \). The \( B \)-valued set \( A \) is called the universe of the \( B \)-valued model \( \mathcal{B} \).

Every \( B \)-valued model \( \mathcal{B} = (A, E, R, \ldots) \) is associated with a \( B \)-valued interpretation \( \dot{\mathcal{B}} = (\dot{A}, E, \dot{R}, \ldots) \) for \( L \) such that

\[
\dot{A} = \text{dom}(A),
\]

\[
E(a_1, a_2) = \llbracket a_1 = a_2 \rrbracket,
\]

\[
\dot{R}(a_1, \ldots, a_m) = \llbracket (a_1, \ldots, a_m) \in R \rrbracket,
\]

for all \( a_1, \ldots, a_m \in \text{dom}(A) \). We call \( \dot{\mathcal{B}} \) the \( B \)-valued interpretation of \( \mathcal{B} \).

We will now show that every \( B \)-valued interpretation arises from a \( B \)-valued model in the above manner.

Let \( V(X) \) be the superstructure with base set \( X \), and \( \dot{V}(X) \) be the \( B \)-valued superstructure with base set \( X \). A \( B \)-valued interpretation \( \mathcal{A} = (A, E, R, \ldots) \) is said to be in \( V(X) \) if \( A \in V(X) \setminus X \), and a \( B \)-valued model \( \mathcal{B} = (A, E, R, \ldots) \) is said to be in \( \dot{V}(X) \) if \( A \in \dot{V}(X) \setminus \dot{X}^B \). In this case, by the \( D \)-Absoluteness Principle, each \( R \) is in \( \dot{V}(X) \setminus \dot{X}^B \).

**Theorem 6.1.** Let \( \mathcal{A} = (A, E, R, \ldots) \) be a \( B \)-valued interpretation in \( V(X) \) for \( L \). Define \( \bar{a} \in \dot{V}(B) \), \( \dot{A} \in \dot{V}(B) \), and \( \dot{R} \in \dot{V}(B) \), for all \( a \in A \) and each \( B \)-valued relation \( R \) as follows:

\[
\bar{a} = \{ \langle \bar{a}, E(a, u) \rangle \mid u \in A \},
\]

\[
\dot{A} = \{ \langle \bar{a}, 1 \rangle \mid a \in A \},
\]

\[
\text{dom}(\dot{R}) = \{ (\bar{a}_1, \ldots, \bar{a}_m)_B \mid a_1, \ldots, a_m \in A \},
\]

\[
\dot{R}(\langle \bar{a}_1, \ldots, \bar{a}_m \rangle)_B = R(a_1, \ldots, a_m).
\]
Then $\mathcal{A} = \langle \bar{A}, \bar{R}, \ldots \rangle$ is a $B$-valued model in $\hat{V}(X)$ for $L$ and the map $a \mapsto \bar{a}$ is an isomorphism from $A$ to the $B$-valued interpretation of $B$. Thus, any $B$-valued interpretation is isomorphic to the $B$-valued interpretation of a $B$-valued model.

**Proof.** Since the assertion concerns only atomic formulas, it suffices to prove the assertion in the case that the language $L$ has only one relation symbol $P$. Corresponding to $A \in V(X) \setminus X, E$, and $R$ of a given $B$-valued interpretation $A = \langle A, E, R \rangle$, consider the $B$-valued sets $\bar{A}, \bar{E},$ and $\bar{R}$. Then by the $\Delta_0$-Absoluteness Principle, $\bar{A} \in \hat{V}(X)$, $\llbracket \bar{E} : (\bar{A}^2) \rightarrow \hat{B} \rrbracket = 1$, and $\llbracket \bar{R} : (\bar{A}^m) \rightarrow \hat{B} \rrbracket = 1$. Let $\hat{B}$ be the “completion of $\hat{B}$” in $V(B)$. Then by the $\Delta_0$-Absoluteness Principle

$$\llbracket \langle \bar{A}, \bar{E}, \bar{R} \rangle_B \rrbracket = 1.$$ 

Put $\mathcal{B} = \langle \bar{A}, \bar{E}, \bar{R} \rangle_B$. Let $\mathcal{G}$ be an “ultrafilter of $\hat{B}$ extending $G$” in $V(B)$, where $G$ is the canonical generic filter (see the next section for the definition). By the ZFC Transfer Principle the construction of the reduced model of $\mathcal{B}$ modulo $\mathcal{G}$ is carried out in $V(B)$.

Thus by the Maximum Principle we have $B$-valued sets $(\bar{A}/U)_B, (\bar{R}/U)_B$ such that

$$\llbracket (\bar{A}/U)_B, (\bar{R}/U)_B \rrbracket_B$$

is the reduced model of $\mathcal{B}$ modulo $\mathcal{G} = 1$,

and that $(\bar{A}/U)_B \in \hat{V}(X)$. Denote by $[u]$ the “equivalence class of $u \in \bar{A}$” in $V(B)$. Then $\llbracket [u] \rrbracket = 1$ and the $B$-valued set $[u]$ is represented by

$$[u] = \{ (\bar{x}, [\bar{x} \sim u]) \mid x \in A \},$$

where $\sim$ is the “equivalence relation on $\bar{A}$ modulo $\mathcal{G}$” in $V(B)$. By the property of the reduced model, for all $a, b \in A$,

$$\llbracket \bar{a} = [\bar{b}] \rrbracket = E(a, b) \in G = E(a, b),$$

and similarly, for all $a_1, \ldots, a_m \in A$,

$$\llbracket ([\bar{a}_1], \ldots, [\bar{a}_m]) \in (\bar{R}/U)_B \rrbracket = R(a_1, \ldots, a_m).$$

Consider the map $h : a \to [\bar{a}]$ from $A$ onto $\{ [\bar{a}] \mid a \in A \}$. If $[\bar{a}] = [\bar{b}]$ then $\llbracket [\bar{a}] \rrbracket = \llbracket [\bar{b}] \rrbracket = 1$, so that $E(a, b) = 1$, and hence $a = b$ by property (E5). Thus $h$ is a bijection. Now it is easy to check the following relations:

$$\bar{a} = [\bar{a}], \quad \text{dom}(\bar{A}) = \{ [\bar{a}] \mid a \in A \},$$

$$\llbracket \bar{A} = (\bar{A}/G)_B \rrbracket = 1, \quad \llbracket \bar{R} = (\bar{R}/G)_B \rrbracket = 1.$$ 

Thus the mapping $h$ is the desired isomorphism. \(\square\)

We call the $B$-valued model $\mathcal{A} = \langle \bar{A}, \bar{R}, \ldots \rangle$ the $B$-valued model of a $B$-valued interpretation of $A = \langle A, R, \ldots \rangle$. The following corollary is a special case of Theorem 6.1.

**Corollary 6.2.** The Boolean power $\prod^B M$ of a model $M = \langle M, R, \ldots \rangle$ with $M \in V(X) \setminus X$ for $L$ is isomorphic to the $B$-valued interpretation of the $B$-valued model $B = \langle M(B) \times \{1\}, \bar{R}, \ldots \rangle$. 

Note that the B-valued interpretation of the B-valued model \( \langle A, \tilde{R}, \ldots \rangle \) is isomorphic to \( M \) rather than \( \prod^B M \).

Let \( A = \langle A, R, \ldots \rangle \) be a B-valued model in \( V(X) \) for \( \mathcal{L} \). Abusing notation, we denote by \( \gamma(A) \) the model for \( \mathcal{L} \) such that \( \gamma(A) = \langle \gamma(A), \gamma(R), \ldots \rangle \). For a model \( A = \langle A, R, \ldots \rangle \), we denote by \( ^*A \) the model for \( \mathcal{L} \) such that \( ^*A = \langle ^*A, ^*R, \ldots \rangle \).

**Theorem 6.3.** Let \( \langle V(X), V(Y), *, \gamma \rangle \) be the bounded ultrafilter of \( V(X) \) modulo an ultrafilter \( \mathcal{U} \) of \( B \). Let \( A = \langle A, E, R, \ldots \rangle \) be a B-valued interpretation in \( V(X) \) for \( \mathcal{L} \). The reduced model \( A/\mathcal{U} = \langle A/\mathcal{U}, R/\mathcal{U}, \ldots \rangle \) of \( A \) modulo \( \mathcal{U} \) is isomorphic to the model \( \gamma(A) = \langle \gamma(A), \gamma(R), \ldots \rangle \) by the correspondence \( u/\mathcal{U} \mapsto \gamma(u) \), where \( A = \langle \tilde{A}, \tilde{R}, \ldots \rangle \) is the B-valued model of \( A \).

**Proof.** Let \( u, v \in A \). Then we have

\[
\gamma(u) = \gamma(v)
\]

iff \( [u = v] \subseteq \mathcal{U} \) by the Łoś–Mostowski Principle

iff \( E(u, v) \in \mathcal{U} \) by Theorem 6.1

iff \( u/\mathcal{U} = v/\mathcal{U} \) by definition,

and hence the relation \( h(u/\mathcal{U}) = \gamma(u) \) for all \( u \in A \) defines a bijection from \( A/\mathcal{U} \) to \( \gamma(A) \). It is similar to verify that

\[
\langle \gamma(u_1), \ldots, \gamma(u_m) \rangle \in \gamma(R) \text{ iff } \langle u_1/\mathcal{U}, \ldots, u_m/\mathcal{U} \rangle \in R/\mathcal{U}
\]

for any \( u_1, \ldots, u_m \in A \). Therefore, \( h \) is the desired isomorphism. \( \Box \)

The following corollary is an immediate consequence.

**Corollary 6.4.** Let \( \langle V(X), V(Y), *, \gamma \rangle \) be the bounded ultrafilter of \( V(X) \) modulo an ultrafilter \( \mathcal{U} \) of \( B \). Let \( M = \langle M, R, \ldots \rangle \) be a model with \( M \in V(X) \setminus X \) for \( \mathcal{L} \). The Boolean ultrapower \( \prod^B M = (\prod^B M, \prod^B R, \ldots) \) of \( M \) modulo \( \mathcal{U} \) is isomorphic to the model \( ^*M = \langle ^*M, ^*R, \ldots \rangle \), and the restriction of \( * \) to \( M \) is an elementary embedding of \( M \) into \( ^*M \).

### 7. Existence of external ingeneric sets

We have introduced a new framework of nonstandard analysis in which we may manipulate some external sets called ingeneric sets. However, in an ordinary bounded ultrapower every ingeneric set is internal. In this section, we will show that external ingeneric sets indeed exist in a certain bounded ultrafilter.

Let \( B \) be a complete Boolean algebra. The **canonical generic filter of** \( B \) is a \( B \)-valued set \( G \in V(B) \) defined by \( G = \{ \langle \tilde{b}, b \rangle \mid b \in B \} \). We have for all \( u \in V(B) \),

\[
[ u \in G ] = \bigvee_{b \in B} ( b \land [ u = \tilde{b} ] ),
\]

(1)
and, for $b \in B$,

$$\llbracket \hat{b} \in G \rrbracket = b. \quad (2)$$

Given a Boolean algebra $B$ and a subset $F \subseteq \mathcal{P}(B)$, we say that $B$ is $F$-complete if for all $A \in F$, $\bigvee A$ and $\bigwedge A$ exist in $B$. Thus our Boolean algebra $B$ is $\mathcal{P}(B)$-complete. Since the predicate "$B$ is an $F$-complete Boolean algebra" is clearly a $\Delta_0$-formula (with parameters $B$ and $F$), it follows from the $\Delta_0$-Absoluteness Principle that

$$\llbracket B \text{ is a } \mathcal{P}(B) \text{-complete Boolean algebra} \rrbracket = 1,$$

and furthermore we have [2, Theorem 4.2.1, p. 96]

$$\llbracket G \text{ is a } \mathcal{P}(B) \text{-complete ultrafilter of } \hat{B} \rrbracket = 1.$$

Let $V(X)$ be the superstructure with an infinite base set $X$. We suppose $B \in V(X) \setminus X$. Let $\mathcal{U}$ be an ultrafilter of $B$ and $(V(X), V(Y), *, \gamma)$ the bounded ultrasheaf of $V(X)$ modulo $\mathcal{U}$.

**Lemma 7.1.** For any $n \in \mathbb{N}$, $\llbracket V(X)^\gamma \cap V_n(\tilde{X})_B = V_n(X)^\gamma \rrbracket = 1$.

**Proof.** It suffices to show the relation

$$\llbracket (\forall u \in V(X)^\gamma)(u \in V_n(\tilde{X})_B \iff u \in V_n(X)^\gamma) \rrbracket = 1.$$

By the $\Delta_0$-Absoluteness Principle, $\llbracket \tilde{x} \in V_n(\tilde{X})_B \rrbracket = \llbracket \tilde{x} \in V_n(X)^\gamma \rrbracket$, for all $x \in V(X)$. Thus the assertion follows from the Bounded Evaluation Principle. $\square$

**Lemma 7.2.** An ingeneric element $\gamma(u) \in V(Y)$ is internal if and only if $\llbracket u \in V(X)^\gamma \rrbracket \in \mathcal{U}$.

**Proof.** Let $u \in \hat{V}(X)$. Then $\llbracket u \in V_n(\tilde{X})_B \rrbracket = 1$ for some $n \in \mathbb{N}$. If $\llbracket u \in V(X)^\gamma \rrbracket \in \mathcal{U}$, then $\llbracket u \in V_n(X)^\gamma \rrbracket \in \mathcal{U}$ by Lemma 7.1, so that $\gamma(u) \in ^*V_n(X)$, and hence $\gamma(u)$ is internal. If $\gamma(u)$ is internal, then $\gamma(u) \in ^*V_m(X)$ for some $m \in \mathbb{N}$, and thus $\llbracket u \in V(X)^\gamma \rrbracket \in \mathcal{U}$ by the relation $\llbracket V_m(X)^\gamma \subseteq V(X)^\gamma \rrbracket = 1$. $\square$

An atom of $B$ is an element $b_0$ such that $b < b_0$ implies $b = 0$. Denote by $\text{atom}(B)$ the set of atoms in $B$ and by $P(B)$ the set of principal filters of $B$. Obviously, $\text{atom}(B) = \text{atom}(B)$.

**Lemma 7.3.** $\llbracket G \in V(X)^\gamma \rrbracket = \bigvee \text{atom}(B)$.

**Proof.** There is a $\Delta_0$-formula $\phi(x, y, z)$ of $\mathcal{L}_e$ such that $(\forall \in) \models \phi(U, F, B)$ if $U$ is an $F$-complete ultrafilter of a Boolean algebra $B$. Let $\mathcal{U} \in V(X)$ and suppose $\llbracket G = \tilde{w} \rrbracket > 0$. Since $\llbracket \phi(G, \mathcal{P}(B)^\gamma, \tilde{B}) \rrbracket = 1$, we have

$$\llbracket \phi(\tilde{w}, \mathcal{P}(B)^\gamma, \tilde{B}) \rrbracket \supseteq \llbracket G = \tilde{w} \rrbracket \land \llbracket \phi(G, \mathcal{P}(B)^\gamma, \tilde{B}) \rrbracket > 0.$$
It follows from the $\Delta_0$-Absoluteness Principle that $\langle \forall e \rangle \models \phi(w, P(B), B)$ so that $w$ is a complete ultrafilter of $B$. Since $B$ is complete, $w$ is a principal ultrafilter of $B$ generated by an atom $b_0$ of $B$. Thus, by the $\Delta_0$-Absoluteness Principle,

$$\ll\bar{w} = \{ x \in B \mid b_0 \subseteq x \} \gg = 1,$$

and hence $\bar{w}$ is a principal ultrafilter of $\hat{B}$ generated by the atom $\hat{b}_0$ of $\hat{B}$ in $V(B)$. Then, by the ZFC Transfer Principle, we have

$$\ll G = \bar{w} \gg = \ll b_0 \in G \gg = b_0,$$

and hence

$$\ll G \in V(X)^{\forall} \gg = \bigvee_{w \in V(X)} \ll G = \bar{w} \gg = \bigvee_{w \in P(B)} \ll G = \bar{w} \gg = \bigvee_{\text{atom}(B)}.$$  \Box

Let $\Omega$ be the space of ultrafilters on $B$ with the open base $\{ B(b) \mid b \in B \}$, where $B(b) = \{ \omega \in \Omega \mid b \in \omega \}$. In $\Omega$, every principal filter $p$ generated by an atom $b_0$ is an isolated point and $\{ p \} = B(b_0)$. The correspondence $b \mapsto B(b)$ is the Stone representation, and hence $B(\sqrt{\text{atom}(B)}) = P(B)^-$, where $-$ stands for the closure operation. Now we have the following characterization.

**Theorem 7.4.** Suppose $B \in V(X)$. Let $\langle V(X), V(Y), \star, \gamma \rangle$ be the bounded ultrasheaf of $V(X)$ modulo an ultrafilter $\mathcal{U}$ of $B$. Then the following conditions are equivalent:

1. There is an external ingeneric set in $V(Y)$.
2. $\gamma(G)$ is an external ingeneric set in $V(Y)$.
3. $\mathcal{U} \in P(B)^-$.

**Proof.** (3) $\implies$ (2). Suppose (3). Then, $\sqrt{\text{atom}(B)} \notin \mathcal{U}$. It follows that $\ll G \in V(X)^{\forall} \gg \notin \mathcal{U}$ by Lemma 7.3 and that $\gamma(G)$ is external by Lemma 7.2.

(2) $\implies$ (1). Trivial.

(1) $\implies$ (3). If $\mathcal{U} \in P(B)^-$, the bounded ultrasheaf of $V(X)$ modulo $\mathcal{U}$ is isomorphic with the bounded ultrapower of $V(X)$ modulo $\mathcal{U}'$, where $\mathcal{U}'$ is the restriction of $\mathcal{U}$ to the complete subalgebra generated by $\text{atom}(B)$ isomorphic to $P(P(B))$. Therefore, there is no external ingeneric sets by Theorem 4.8. \Box

In the rest of this section we shall give some examples of external ingeneric sets. In the nonstandard universe containing $\hat{\mathbb{N}}$, we say that a subset $A$ of $\hat{\mathbb{N}}$ has internal segments if $A \cap \{ 0, 1, \ldots, \nu \}$ is internal for all $\nu \in \hat{\mathbb{N}}$. We shall denote by $\mathbb{N}_B$, $\mathbb{Q}_B$, $\mathbb{R}_B \in V(B)$ the set of natural numbers, the set of rational numbers, and the set of real numbers in $V(B)$, respectively. Then we have [40, pp. 129–130], [35, p. 11]

$$\ll [\mathbb{N}_B = \hat{\mathbb{N}}] \gg = 1 \quad \text{and} \quad \ll [\mathbb{Q}_B = \hat{\mathbb{Q}}] \gg = 1.$$

However, it is not necessarily the case that $\ll [\mathbb{R}_B = \hat{\mathbb{R}}] \gg = 1$. This fact is closely related to the existence of an external ingeneric subset of $\hat{\mathbb{N}}$ having internal segments.
**Theorem 7.5.** There is a bounded ultrasheaf of $V(N)$ which has an external ingeneric subset of $*N$ having internal segments.

**Proof.** Let $B$ be the complete Boolean algebra of Borel sets of the unit interval $[0, 1]$ modulo the ideal $\mathcal{N}$ of Borel sets with Lebesgue measure zero. We denote by $B \rightarrow B/\mathcal{N} \in B$ the canonical map. Let $(V(N), V(*N), *, \gamma)$ be the bounded ultrasheaf modulo a countably incomplete ultrafilter $U$ of $B$. Let $a \in V(B)$ be the real number in $V(B)$ such that $\|a \leq \delta\| = (\langle s, t \rangle \cap [0, 1])/N$ for all $s \in R$. Then $\|a \in R_B\| = 1$ and $\|0 \leq a \leq 1\| = 1$ but $\|a \in \hat{R}\| = 0$ [35, p. 16]. There is a formula $\phi(x_0, x_1, x_2, x_3)$ of $L_\infty$ strictly bounded in $x_0$ such that, for all $u, v, w \in V(N)$, $(\forall \epsilon \exists N, u, v, w)$ iff $u \in R, v \in N$, and $w = f(u)$ where $f : N \rightarrow [0, 1]$ is such that $u = \sum_{n \in N} f(n)2^{-(n+1)}$.

Let $A_0 \in \hat{V}(N)$ be such that $\|A_0 = \{n \in \hat{N} | \phi(\hat{N}, a, n, \hat{1})\}\| = 1$.

Let $A = \gamma(A_0), \nu \in *N$, and let $\nu_0 \in N(B)$ be a name of $\nu$. Then

$$A \cap \{0, 1, \ldots, \nu_0\} = \gamma(A_0 \cap \{0, 1, \ldots, \nu_0\}_B).$$

By the ZFC Transfer Principle there is a rational number $r$ in $V(B)$, i.e., $\|r \in \hat{Q}\| = 1$, such that $\|A_0 \cap \{0, 1, \ldots, \nu_0\}_B = \{n \in \hat{N} | \phi(\hat{N}, r, n, \hat{1})\}\| = 1$.

Thus by the Łoś–Mostowski Principle,

$$A \cap \{0, 1, \ldots, \nu_0\} = \{n \in *N | (V(*N), \in) \models \phi(*N, \gamma(r), n, *1)\},$$

and $\gamma(r) \in *Q$. Hence, by the Internal Definition Principle $A \cap \{0, 1, \ldots, \nu_0\}$ is internal. Suppose that $A$ is internal. Then, $\|A_0 \in V(N)\| \in U$ by Lemma 7.2 so that $\|A_0 \in V_1(N)\| = b$ for some $b \in U$ by Lemma 7.1. Thus, there is a family $\{M_\alpha\}$ of subsets of $N$ and a partition $\{b_\alpha\}$ of unity in $B$ such that $\|a_\alpha \in M_\alpha\| \geq b_\alpha b$ for all $\alpha$. On the other hand, $\|a = \sum_{n \in A_0} 2^{-(n+1)}\| = 1$, and hence $\|a = (\sum_{n \in M_\alpha} 2^{-(n+1)})^N\| \geq b_\alpha b$ for all $\alpha$. It follows that $\|a \in \hat{R}\| = b \in U$, a contradiction. Therefore, $A$ is external. □

**Remark.** By a well-known forcing argument [11, p. 177], we can strengthen the above theorem so that the number of external ingeneric subsets of $*N$ with internal segments is a given any cardinal number.

The existence of an external subset of $*N$ having internal segments has been studied in the structure theory of the hyperreal number field $*R$ in a nonstandard universe. An ordered field $K$ is said to be **Scott complete** if every initial segment $C \subseteq K$ such that $(\forall x > 0)(\exists y \in C)[y + x \notin C]$ has a least upper bound in $K$. It is well known that in a nonstandard universe $*R$ is Scott complete if and only if every subset of $*N$ having internal segments is internal [17, Proposition 1.3]. Zakon [43] proposed to study the structure of the hyperreal numbers and asked whether the hyperreal numbers are Scott complete. Kamo [14,15] showed that there exist nonstandard universes in which $*R$ is...
not Scott complete. One of his arguments was generalized to the following observation [17]: if the nonstandard universe is $\lambda$-saturated and $^*\mathbb{N}$ has cofinality $\lambda$, then $^*\mathbb{R}$ is not Scott complete. This and CH concludes the existence of an ultrapower of the real number field which is not Scott complete. For the opposite direction of the problem, i.e., the construction of a nonstandard universe with the Scott complete hyperreal number field, we refer to the recent paper by Keisler and Schmerl [17].

The following corollary of Theorem 7.5 settles affirmatively the following problem: Can one prove in ZFC that there exists an $\aleph_1$-saturated elementary extension of $\mathbb{R}$ which is not Scott complete?

**Corollary 7.6.** There is an $\aleph_1$-saturated Boolean ultrapower of $\mathbb{R}$ which is not Scott complete.

**Proof.** The hyperreal number field $^*\mathbb{R}$ constructed in the proof of Theorem 7.5 is an $\aleph_1$-saturated elementary extension by the Transfer Principle and the Saturation Principle. It is not Scott complete by [17, Proposition 1.3]. It is isomorphic to a Boolean ultrapower of $\mathbb{R}$ by Corollary 6.4. □

The next example comes from a collapsing Boolean algebra.

**Theorem 7.7.** For any pair of infinite cardinals $\kappa$ and $\lambda$ with $\kappa < \lambda$, there is a bounded ultrasheaf which has an external generic surjective function $f: ^*\kappa \to ^*\lambda$ such that for any cardinal $\alpha < \kappa$ the restrictions $f|^{\alpha}$ of $f$ to $^*\alpha$ is internal.

**Proof.** Let $P$ be the set of all functions $p$ such that $\text{dom}(p) \subseteq \kappa$, $|\text{dom}(p)| < \kappa$, and $\text{ran}(p) \subseteq \lambda$, and let $p \leq q$ iff $q \subseteq p$. Let $\mathcal{B}$ be the complete Boolean algebra of regular open subsets of $P$ with the base $\{U_p \mid p \in P\}$, where $U_p = \{q \in P \mid q \leq p\}$. Then $p \to U_p$ is an order preserving embedding, and we may assume $P \subseteq \mathcal{B}$. Consider $V(\mathcal{B})$, and the canonical generic filter $G$ of $\mathcal{B}$. Let $F \in V(\mathcal{B})$ be such that $[F = \bigcup (G \cap \mathcal{P})] = 1$. Note that, for all $x \in \kappa$ and $y \in \lambda$,

$$[\exists p \in G \cap \mathcal{P} \exists \bar{y} = F(x)] = [\bigvee_{p \in \mathcal{P}} \mathcal{P} \land \bar{y} = \check{p}(x)] = \bigvee_{y \tau(x)} \check{p}.$$

Then it is well known that

$$[F \text{ is a function from } \check{\kappa} \text{ onto } \bar{\lambda}] = 1,$$

and that for any cardinal $\alpha < \kappa$ there is some $p \in \mathcal{P}(\mathcal{B})$ such that

$$[F \cap (\check{\alpha} \times \bar{\alpha}) = p] = 1.$$

By a suitable bijection, we can assume that $\lambda$ is a base set. Let $\mathcal{U}$ be an ultrafilter of $\mathcal{B}$, and $\langle V(\lambda), V(\check{\alpha}), *, \gamma \rangle$ the bounded ultrasheaf of $V(\lambda)$ modulo $\mathcal{U}$. Let $f = \gamma(F)$. Then $f$ is a function from $^*\kappa$ onto $^*\lambda$. Since $f \cap (^*\alpha \times ^*\lambda) = \gamma(p) \in ^*\mathcal{P}$, the restriction
fl is internal. Suppose that f is internal. Then $f \in \star P(\kappa \times \lambda)$, and hence by the Transfer Principle there is a function from $\kappa$ onto $\lambda$. This contradicts the assumption $\kappa < \lambda$. Thus f is external. □

8. Infinitesimal analysis

Robinson instituted a theory of infinitesimal and infinite numbers based on an elementary extension of the real number field. The nonstandard universe is nowadays a standard setting for Robinson’s theory. Moreover, the recent development has revealed important roles of the $\mathcal{N}_1$-saturation principle played in applications to real analysis. Since a bounded ultrasheaf of a superstructure is an $\mathcal{N}_1$-saturated nonstandard universe, this theory can be developed without any restriction as simple corollaries of the transfer principle and the $\mathcal{N}_1$-saturation principle.

Now, we should remark a new framework of the infinitesimal analysis possible for a bounded ultrasheaf. A bounded ultrasheaf has not only an elementary extension of the standard real number field, called the hyperreal number field, but also a generic extension of the hyperreal number field which we will call the ingeneric real-number field. Although the ingeneric real-number field is not an elementary extension, it shares every property of the real numbers which is provable in ZFC, and hence we can develop a theory of infinitesimal and infinite numbers based on the ingeneric real-number field.

In this section we will develop fragments of infinitesimal analysis based on the ingeneric real or complex numbers in a bounded ultrasheaf $(V(X), V(Y), *, \gamma)$ of $V(X)$ modulo a countably incomplete ultrafilter of a complete Boolean algebra $B$.

A set $A \in V(X)$ is called definable in $V(X)$ if there is a $\Delta_0$-formula $\phi(x_0, x_1)$ of $\mathcal{L}_E$ strictly bounded in $x_0$ such that $\langle \forall \in \rangle \models A = \{x_1 \mid \phi(X, x_1)\}$, or equivalently $A = \{u \in V \mid \langle \forall \in \rangle \models \phi(X, u)\}$. We call such formula $\phi$ the definition of $A$. Note that the definition of $\phi$ is essentially unique in the sense that any two definitions are equivalent in ZFC.

Let $A$ be a definable set in $V(X)$ with definition $\phi$. Then by the Maximum Principle there is an essentially unique $B$-valued set, denoted by $A_B$, such that $\llbracket A_B = \{x \mid \phi(X, x)\} \rrbracket = 1$. We call $A_B \in V^{(B)}$ the $B$-interpretation of a definable set $A \in V$. The $B$-hull of the $B$-interpretation of a definable set $A$ in $V(X)$ is denoted by $A^{(B)}$ instead of $(A_B)^{\ast}$. In this case, we simply call $A^{(B)}$ the $B$-hull of the definable set $A \in V(X)$. If a definable set $A \in V(X)$ is nonempty, then $\llbracket A_B \neq 0 \rrbracket = 1$ and hence we can represent $A_B$ by $A_B = A^{(B)} \times \{1\}$, where $A^{(B)}$ is the $B$-hull of $A$. For a definable set $A \in V(X)$, we will write $\gamma A = \gamma (A_B)$.

In this section, we assume that $C \subseteq X$, so that we have the following identifications $^{\ast}N = N$, $^{\ast}Q = Q$, $^{\ast}R = R$, and $^{\ast}C = C$ by the identification $u = ^{\ast}u$ for all $u \in X$ [4, p. 273]. The following sets are definable sets in $V(X)$: $N$, $Q$, $R$, $C$. For them, we have $\llbracket N_B = \hat{N} \rrbracket = 1$, $\llbracket Q_B = \hat{Q} \rrbracket = 1$, $\llbracket R_B \supseteq \hat{R} \rrbracket = 1$, and $\llbracket C_B \supseteq \hat{C} \rrbracket = 1$ for any complete Boolean algebra $B$. Then obviously, we have $N \subseteq ^{\ast}N = \gamma N$, $Q \subseteq ^{\ast}Q = \gamma Q$, $R \subseteq ^{\ast}R \subseteq ^{\ast}\gamma R$, and $C \subseteq ^{\ast}C \subseteq ^{\ast}\gamma C$.

The set $^{\ast}R$ is called the hyperreal number field. An element of $^{\ast}R$ is called a hyperreal number. By the Transfer Principle, $^{\ast}R$ is an ordered field extension of the real number
field $\mathbb{R}$. The set $\mathbb{R}$ is called the *ingenic real-number field*. The elements of $\mathbb{R}$ are called *ingenic real-numbers*. By the ZFC Transfer Principle, $\mathbb{R}$ is an ordered field extension of $\mathbb{R}$. Thus we have the chain $\mathbb{R} \subseteq *\mathbb{R} \subseteq \mathbb{R}$ of ordered field extensions of $\mathbb{R}$. An ingenic real-number $x$ is called *infinite* if $|x| > n$ for any $n \in \mathbb{N}$, *finite* if $|x| < n$ and *infinitesimal* if $|x| < 1/n$ for any $n \in \mathbb{N}$. We will write $x \approx y$ if $|x - y|$ is infinitesimal, $|x| < \infty$ if $x$ is infinite. The *principal galaxy* of $\mathbb{R}$, denoted by $\text{gal} (\mathbb{R})$, is the set of all finite ingenic real-numbers and the *principal monad* of $\mathbb{R}$, denoted by $\text{mon} (\mathbb{R})$, is the set of all infinitesimal ingenic real-numbers. The following theorem holds for any ordered field extensions.

**Theorem 8.1 (Standard Part Theorem).** For any finite ingenic real-number $x$, there is a unique real number $r$ such that $r \approx x$.

This $r$ is called the *standard part* of $x$ and denoted by $\circ x$. The map $x \mapsto \circ x$ is called the *standard part map* and it is an algebraic homomorphism from $\text{gal}(\mathbb{R})$ onto $\mathbb{R}$. The standard part $\circ z$ of $z \in \mathbb{C}$ is defined by $\circ z = \Re(z) + i\Im(z)$, which is the unique complex number such that $|\circ z| \approx 0$.

Let $\mathcal{N}(n)$ be the definable set of normed linear spaces $\mathcal{X}$ over $\mathbb{C}$ such that $\mathcal{X} \in V_\mathcal{N}(X) \setminus X$. A set $\mathcal{X} \in V(Y)$ is called an *internal normed linear space* if $\mathcal{X} \in \bigcup_{n \in \mathbb{N}} *\mathcal{N}(n)$. A set $\mathcal{X} \in V(Y)$ is called an *ingenic normed linear space* if $\mathcal{X} \in \bigcup_{n \in \mathbb{N}} *\mathcal{N}(n)$. We will use the same symbol for a normed linear space $\mathcal{X}$ and the set of vectors of $\mathcal{X}$ as usual.

Let $\mathcal{X}$ be an ingenic normed linear space with $\mathbb{R}$-valued norm $\| \cdot \|$. The *principal galaxy* $\text{gal}(\mathcal{X})$ and the *principal monad* $\text{mon}(\mathcal{X})$ are defined by

$$\text{gal}(\mathcal{X}) = \{ x \in \mathcal{X} \mid \| x \| < \infty \},$$

$$\text{mon}(\mathcal{X}) = \{ x \in \mathcal{X} \mid \| x \| \approx 0 \}.$$

Then both of them are linear spaces over $\mathbb{C}$. The *generic hull* of $\mathcal{X}$ is the quotient linear space $\hat{\mathcal{X}} = \text{gal}(\mathcal{X})/\text{mon}(\mathcal{X})$ equipped with the norm given by

$$\| \circ x \| = \circ \| x \|,$$

for all $x \in \text{gal}(\mathcal{X})$, where $\circ x = x + \text{mon}(\mathcal{X})$.

**Theorem 8.2 (Generic Hull Completeness Theorem).** The *generic hull* $\hat{\mathcal{X}}$ of an ingenic normed linear space $\mathcal{X}$ is a Banach space.

**Proof.** Let $f : \mathbb{N} \to \text{gal}(\mathcal{X})$ be a Cauchy sequence in $\text{gal}(\mathcal{X})$. By the Generic Completeness Theorem there is an ingenic function $g : *\mathbb{N} \to \mathcal{X}$ such that $g(n) = f(n)$ for all $n \in \mathbb{N}$. For any $k \in \mathbb{N}$, let $N(k) \in \mathbb{N}$ be such that $\| f(n) - f(N(k)) \| \leq 1/k$ for all $n \geq N(k)$. Let $A_k$ be an ingenic subset of $*\mathbb{N}$ such that

$$A_k = \{ n \in *\mathbb{N} \mid \| g(n) - f(N(k)) \| \leq 1/k \}.$$

Then the sequence $\{ A_k \mid k \in \mathbb{N} \}$ has the finite intersection property, since $\{ f(n) \mid n \in \mathbb{N} \}$ is a Cauchy sequence. It follows from the Generic Saturation Principle that there is
some \( m \in \mathbb{N} \) such that \( m \in \bigcap_{k \in \mathbb{N}} A_k \). Then it is easy to see that \( g(m) \in \text{gal}(\mathcal{X}) \) and 
\[
\lim_{m \to \infty} \| f(n) - g(m) \| = 0.
\]
Thus \( \mathcal{X} \) is complete with respect to the metric induced by the norm \( \| \cdot \| \). \( \square \)

**Remark.** A generalization of the above theorem to arbitrary metric spaces can be formulated and proved without any difficulties.

Let \( \mathcal{M}(n) \) be the definable set of finite measure spaces \( \mathcal{M} = (\mathcal{M}, F, \mu) \) with a finitely additive measure \( \mu \) on a field of subsets of \( \mathcal{M} \) such that \( \mathcal{M} \in V(X) \setminus X \). A set \( \mathcal{M} \in V(Y) \) is called an internal measure space if \( \mathcal{M} \in \bigcup_{n \in \mathbb{N}} \mathcal{M}(n) \). A set \( \mathcal{M} \in V(Y) \) is called an ingeneric measure space if \( \mathcal{M} \in \bigcup_{n \in \mathbb{N}} \gamma \mathcal{M}(n) \).

**Theorem 8.3** (Loeb Measure Construction). Let \( \mathcal{M} = (\mathcal{M}, F, \mu) \) be an ingeneric measure space. Then there is a measure space \( L(\mathcal{M}) = (\mathcal{M}, \sigma(\mathcal{F}), L(\mu)) \) with a countably additive finite measure \( L(\mu) \) on the \( \sigma \)-field \( \sigma(\mathcal{F}) \) of subsets of \( \mathcal{M} \) generated by \( \mathcal{F} \) such that \( L(\mu)(A) = \circ \mu(A) \) for all \( A \in F \).

**Proof.** By the Generic Transfer Principle, it is easy to see that \( \mathcal{F} \) is a field of subsets of \( \mathcal{M} \) and that \( \mu \) is a \( \mathbb{R} \)-valued finitely additive measure on \( \mathcal{F} \). By the property of the standard part map, the function \( \circ \mu : A \mapsto \circ \mu(A) \) for all \( A \in \mathcal{F} \) is a real-valued finitely additive measure on \( \mathcal{F} \). Let \( \{A_k\} \) be a countable decreasing sequence of sets in \( \mathcal{F} \) such that \( \bigcap_{k=1}^{\infty} A_k = \emptyset \). Since every \( A_k \) is an ingeneric set, there is some \( n \) such that \( \bigcap_{k=1}^{n} A_k = \emptyset \) by the Generic Saturation Principle. Thus by finite additivity, we have
\[
\circ \mu \left( \bigcap_{k=1}^{\infty} A_n \right) = 0,
\]
and hence the function \( \circ \mu \) is countably additive on \( \mathcal{F} \). Therefore, by the Hopf-Caratheodory extension theorem \( \circ \mu \) can be extended to a countably additive measure \( L(\mu) \) on the \( \sigma \)-field \( \sigma(\mathcal{F}) \) of subsets of \( \mathcal{M} \) generated by \( \mathcal{F} \). \( \square \)

The above construction generalizes Loeb's construction of measure spaces from internal measure spaces based on the Transfer Principle and the Saturation Principle [18].

9. **Reduction of the Boolean-valued complex numbers**

Let \( B \) be a complete Boolean algebra and \( \Omega \) the space of ultrafilters of \( B \) with the topology given by the open base \( B_0 = \{B(b) \mid b \in B\} \) defined by
\[
B(b) = \{\omega \in \Omega \mid b \in \omega\}.
\]
The space \( \Omega \) is a Stonean space, namely, a compact Hausdorff space such that the closure of every open set is open, and \( B_0 \) coincides with the field of clopen subsets of \( \Omega \). The mapping \( b \mapsto B(b) \) is an isomorphism of \( B \) onto \( B_0 \), called the Stone representation of
B. Then every regular open set is clopen and the supremum and the infimum in \( B_0 \) is given by

\[
\bigvee_{i \in I} B_i = \left( \bigcup_{i \in I} B_i \right), \quad \bigwedge_{i \in I} B_i = \left( \bigcap_{i \in I} B_i \right)
\]

for any family \( \{B_i \mid i \in I\} \) in \( B_0 \), where \( - \) and \( ^o \) are the closure operation and the interior operation, respectively. We say that a subset \( A \) of \( \Omega \) is congruent with a subset \( B \) of \( \Omega \), in symbols \( A \sim B \), if \( (A - B) \cup (B - A) \) is a meager set, and a function \( f \) on \( \Omega \) is congruent with a function \( g \) on \( \Omega \), in symbols \( f \sim g \), if \( \{\omega \in \Omega \mid f(\omega) = g(\omega)\} \) is a comeager set. By the Baire category theorem, every comeager set is dense. Let \( B(\Omega) \) be the space of all complex-valued Borel functions on \( \Omega \) and \( N(\Omega) \) the space of all functions in \( B(\Omega) \) vanishing outside a meager Borel set, the functions in \( N(\Omega) \) will be called negligible functions. Then \( B(\Omega) \) is a *-algebra and \( N(\Omega) \) is a *-ideal of \( B(\Omega) \) by the pointwise operations, where the *-operation is the complex conjugation. Let \( B(\Omega) \) be the quotient space \( B(\Omega) / N(\Omega) \). The *-subalgebra of \( B(\Omega) \) or \( B(\Omega) \) consisting of the real-valued functions is called the real part. A Borel function \( f \in B(\Omega) \) is called a normal Borel function if it is continuous on an open dense subset \( D(f) \), called the continuous domain, and if \( \lim_{\omega \to \omega_0} |f(\omega)| = \infty \) for any \( \omega \in \Omega \setminus D(f) \). A self-adjoint Borel function on \( \Omega \) is a real-valued normal Borel function on \( \Omega \). If two normal Borel functions \( f \) and \( g \) are congruent, then their continuous domains are the same and they coincide on the continuous domain [13, Lemma 5.6.6, p. 344]. A normal function is the unique continuous extension of a normal Borel function from its continuous domain to \( \Omega \) with values in the Riemann sphere \( \mathbb{C} \cup \{\infty\} \), and a self-adjoint function is the analogous continuous extension with values in the extended real line \( \mathbb{R} \cup \{-\infty, +\infty\} \); for the uniqueness of the continuous extension of a self-adjoint function \( f \), note that \( \{\omega \in D(f) \mid f(\omega) > 1\} \) and \( \{\omega \in D(f) \mid f(\omega) < -1\} \) are disjoint open sets [13, p. 344]. Denote by \( N(\Omega) \) the space of normal functions and by \( S(\Omega) \) the space of the self-adjoint functions.

A short proof of the following folk theorem is presented for the reader's convenience.

**Theorem 9.1.** Every complex-valued Borel function on \( \Omega \) has a congruent normal Borel function.

**Proof.** For any Borel set \( A \), let \( \tilde{A} \) be the unique congruent clopen set [9, p. 58]. Let \( f \) be a bounded Borel function. There is a sequence of simple Borel functions \( f_n = \sum_{i=1}^n a_{n,i} \chi(A_{n,i}) \) uniformly convergent to \( f \), where the characteristic function of a set \( A \) is denoted by \( \chi(A) \). Let \( g_n \) be a sequence of continuous functions defined by \( g_n = \sum_{i=1}^n a_{n,i} \chi(\tilde{A}_{n,i}) \). Then \( g_n \) converges uniformly to a continuous function \( g \) congruent with \( f \). Thus every bounded Borel function has a congruent continuous function. Let \( g \) be a real-valued Borel function. Then the Cayley transform \( u = (g - i1)(g + i1)^{-1} \) is a bounded Borel function with values in the unit circle, where \( I \) stands for \( \chi(\Omega) \), and hence has a congruent continuous function \( u \). Since \( u \) has values in the unit circle on a dense set, the range of \( u \) is contained in the unit circle by the continuity of \( u \). Since \( u(\omega) \neq 1 \) for all \( \omega \in \Omega \), \( D = \{\omega \in \Omega \mid u(\omega) \neq 1\} \) is a dense open set. It
follows that the relation \( f(\omega) = i(1 + v(\omega))(1 - v(\omega))^{-1} \) for all \( \omega \in D \) gives a self-adjoint Borel function \( f \) with continuous domain \( D(f) = D \) which is congruent with \( g = i(1+u)(1-u)^{-1} \). Thus, every real-valued Borel function has a congruent self-adjoint Borel function. Therefore, by arguing the real and imaginary parts separately, we conclude that every complex-valued Borel function has a congruent normal Borel function. \( \square \)

The above theorem implies that the space \( N(\Omega) \) of normal functions is a \(*\)-algebra, with pointwise operations on a dense open subset, \(*\)-isomorphic to \( B(\Omega) \) and the space \( S(\Omega) \) of self-adjoint functions is isomorphic to the real part of \( B(\Omega) \). Thus, the space \( C(\Omega) \) of complex-valued continuous functions, which coincides with the \(*\)-subalgebra of \( N(\Omega) \) consisting of the bounded normal functions, is \(*\)-isomorphic to the space \( B^\infty(\Omega) \) of bounded Borel functions modulo the negligible functions. Similarly, the space \( C_R(\Omega) \) of real-valued continuous functions is isomorphic to the space \( B_R^\infty(\Omega) \) of bounded real-valued Borel functions modulo the negligible functions. Note that the structure of \( C(\Omega) \) is characterized as a general commutative \( AW^* \)-algebra [3].

Consider the universe \( V(B) \) of \( B \)-valued sets. The global section algebra of the complex number field \( C_B \) in \( V(B) \), \( (C(B), +, \times, *, \cdot) \), is defined as follows:

- **addition**: \( u + v = \text{the unique } w \in C(B) \text{ such that } [w = u +_B v] = 1 \),
- **multiplication**: \( u \times v = \text{the unique } w \in C(B) \text{ such that } [w = u \times_B v] = 1 \),
- **involution**: \( u^* = \text{the unique } w \in C(B) \text{ such that } [w = u^*_B] = 1 \),
- **scalar multiplication**: \( \alpha \cdot v = \text{the unique } w \in C(B) \text{ such that } [w = \alpha \cdot_B v] = 1 \),

for all \( u, v \in C(B) \) and \( \alpha \in C \), where \( +_B \), \( \times_B \), and \( *_B \) are addition, multiplication and complex-conjugation in \( V(B) \), respectively. Then the global section algebra \( C(B) \) is a \(*\)-algebra. The bounded global section algebra of \( C_B \) is a \(*\)-subalgebra \( C_B^\infty \) defined by

\[
C_B^\infty = \{ u \in C(B) \mid (\exists n \in N) \|[u] < n\| = 1 \}.
\]

The structures of the global section algebra \( R(B) \) and the bounded global section algebra \( R_B^\infty \) are defined analogously. Then they are \(*\)-subalgebras of \( C(B) \) and \( C_B^\infty \) respectively, and we have naturally the relations \( C(B) = R(B) + iR(B) \) and \( C_B^\infty = R_B^\infty + iR_B^\infty \).

The following characterization of the global section algebra is obtained by Ozawa [25].

**Theorem 9.2.** The global section algebra \( C(B) \) is \(*\)-isomorphic to \( B(\Omega) \) and the bounded global section algebra \( C_B^\infty \) is \(*\)-isomorphic to \( C(\Omega) \).

Note that any \(*\)-isomorphism induces an isomorphism between the respective real parts. By virtue of Theorem 9.1, the above theorem concludes that \( C(B) \) is \(*\)-isomorphic to \( N(\Omega) \); this result is also obtained independently by Jech [12]. The advantage of using the space \( B(\Omega) \) for representing \( C(B) \) is that the above \(*\)-isomorphism holds for the regular open algebra \( B \) of any Baire space \( \Omega \) as proved in [25, Theorem 3.5].

In this section, we will show another way to establish the above \(*\)-isomorphism using infinitesimal analysis based on Boolean ultrasheaves.
Let $X$ be a base set containing the complex number field $\mathbb{C}$, where we assume that $\mathbb{C}$ is replaced by a suitable base set with the same size without any loss of generality. Let $\omega$ be an ultrafilter of $B$, i.e., $\omega \in \Omega$. We denote by $(V(X), V(Y_\omega), \ast_\omega, \gamma_\omega)$ the Boolean ultrasiheaf of $V(X)$ modulo $\omega$.

Let $a \in C(B)$. For any $n \in \mathbb{N}$, let $D_n(a) = \{||a| < n\} \in B$ and $D(a) = \bigcup_{n \in \mathbb{N}} D_n(a)$. Since

$$\bigvee_{n \in \mathbb{N}} ||a| < n\} = 1,$$

$D(a)$ is a dense open subset of $\Omega$, and $D(a) = \Omega$ if and only if $a \in C(B)$. It is easy to see that the ingeneric complex number $\gamma_\omega(a)$ in $V(Y_\omega)$ is finite if and only if $\omega \in D(a)$. Define the function $\hat{a} : \Omega \to \mathbb{C} \cup \{\omega\}$ by

$$\hat{a}(\omega) = \begin{cases} \gamma_\omega(a) & \text{if } \omega \in D(a), \\ \infty & \text{otherwise}, \end{cases}$$

where $\gamma_\omega$ stands for the standard part map on the finite ingeneric complex numbers in $V(Y_\omega)$.

**Theorem 9.3.** The correspondence $a \mapsto \hat{a}$ is a $*$-homomorphism of $C(B)$ into $N(\Omega)$.

**Proof.** Let $a \in C(B)$. Let $\omega \in D(a)$. Suppose $\varepsilon \in \mathbb{R}$ and $\varepsilon > 0$. Then $|\gamma_\omega(a) - \hat{a}(\omega)| < \varepsilon$. Let $N(\omega) = \{||a| - \hat{a}(\omega)| < \varepsilon\}$. It follows from the Łoś-Mostowski Principle that $\omega \in N(\omega)$ and that if $\omega' \in N(\omega)$ then $|\gamma_{\omega'}(a) - \hat{a}(\omega)| < \varepsilon$, and hence by taking the standard part in $V(Y_\omega)$ we have $|\hat{a}(\omega') - \hat{a}(\omega)| \leq \varepsilon$. Thus $\hat{a}$ is continuous on $D(a)$. Let $\omega \in \Omega \setminus D(a)$. For any $n \in \mathbb{N}$, $\omega \in \Omega \setminus D_n(a) = \{||a| \geq n\}$, and hence $|\hat{a}(\omega')| \geq n$ for all $\omega' \in \Omega \setminus D_n$. It follows that $\lim_{\omega' \to \omega} |a(\omega')| = \infty$ for any $\omega \in \Omega \setminus D(a)$. Thus $\hat{a}$ is a normal function on $\Omega$. By the properties of $\gamma_\omega$ and the standard part map, it is easy to see that the correspondence $a \mapsto \hat{a}$ is a $*$-homomorphism of $C(B)$ into $N(\Omega)$. $\Box$

Let $\Theta$ be the “space of ultrafilters of the Boolean algebra $B$” in $V(B)$ with the “Stone representation $p \mapsto U(p)$ for $p \in B$” in $V(B)$. Then $[G \in \Theta] = 1$, where $G$ is the canonical generic filter of $B$, and $[\hat{\Omega} \subseteq \Theta] = 1$ in $V(B)$.

**Lemma 9.4.** For any $b \in B$, $[B(b)^V = U(\hat{b}) \cap \hat{\Omega}] = 1$.

**Proof.** By the Bounded Evaluation Principle, it suffices to prove that $\hat{b} \in U(\hat{b})$ if and only if $\hat{\omega} \in B(b)^V$ in $V(B)$ for any $\omega \in \Omega$. By the definition of the Stone representation, $\hat{\omega} \in U(\hat{b})$ if and only if $\hat{b} \in \hat{\omega}$ in $V(B)$. By the $\Delta_0$-Absoluteness Principle, the latter condition in $V(B)$ is equivalent to $b \in \omega$, that is equivalent to $\omega \in B(b)$, and thus by the $\Delta_0$-Absoluteness again this is equivalent to $\hat{\omega} \in B(b)^V$ in $V(B)$. $\Box$

**Theorem 9.5.** The space $\Theta$ is a “Stone–Čech compactification of $\hat{\Omega}$” in $V(B)$.

**Proof.** It suffices to show that the following statements hold in $V(B)$: (1) $\hat{\Omega}$ is dense in $\Theta$, (2) every real-valued bounded continuous function on $\hat{\Omega}$ has an extension to a
real-valued bounded continuous function on \( \Theta \).

(1) Since the set \( \{ U(p) \mid p \in \tilde{B} \} \) is an open base of \( \Theta \) in \( V(\mathbb{B}) \), it suffices to show that if \( U(p) \neq \emptyset \) then \( U(p) \cap \tilde{\Theta} \neq \emptyset \) for all \( p \in \tilde{B} \) in \( V(\mathbb{B}) \), but this follows from Lemma 9.4.

(2) By Lemma 9.4, \( \{ B(b) \mid b \in \mathbb{B} \} \) is an open base for the relative topology on \( \tilde{\Theta} \) in \( V(\mathbb{B}) \). Thus \( \tilde{\Theta} \) is totally disconnected in \( V(\mathbb{B}) \), and hence every real-valued continuous function is a uniform limit of a sequence of finite linear combinations of characteristic functions of clopen sets in \( V(\mathbb{B}) \). Thus it suffices to show that every clopen set in \( \tilde{\Theta} \) can be extended to a clopen set in \( \Theta \). By Lemma 9.4 and the Bounded Evaluation Principle, we have

\[
[\forall b \in \tilde{\Theta}] [U(b) \cap \tilde{\Theta} = \tilde{B}(b)] = 1,
\]

and hence the desired statement is obtained. \( \square \)

Let \( C_\mathbb{B} \cup \{ \infty \} \) be the Riemann sphere in \( V(\mathbb{B}) \). Then \( C \cup \{ \infty \} \subseteq C_\mathbb{B} \cup \{ \infty \} \) and \( \infty = \infty \) in \( V(\mathbb{B}) \).

**Theorem 9.6.** For any \( f \in \mathcal{N}(\Omega) \),

\[
[f \text{ is a continuous function from } \tilde{\Theta} \text{ to } C \cup \{ \infty \}] = 1.
\]

**Proof.** By the \( \Delta_0 \)-Absoluteness Principle, \( \tilde{f} \) is a function from \( \tilde{\Theta} \) to \( C \cup \{ \infty \} \) in \( V(\mathbb{B}) \). To show the continuity, let \( \varepsilon \in \mathbb{Q} \) and \( \varepsilon > 0 \) and \( \omega \in D(f) \). Then there is some \( b \in \mathbb{B} \) such that \( \omega \in B(b) \) and that \( |f(\omega) - f(\omega')| < \varepsilon \) for any \( \omega' \in B(b) \). It follows from the Bounded Evaluation Principle and the \( \Delta_0 \)-Absoluteness Principle that

\[
\left[ \forall \omega \in \tilde{\Theta} \cap D(f) \right] \left[ \forall \varepsilon \in \tilde{\mathbb{Q}} \right] \left[ \varepsilon > 0 \Rightarrow \left( \exists b \in \tilde{\mathbb{B}} \right) \left[ \omega \in \tilde{B}(b) \land \left( \forall \omega' \in \tilde{B}(b) \right) \left[ |\tilde{f}(\omega) - \tilde{f}(\omega')| < \varepsilon \right] \right] \right] = 1.
\]

Similarly, we have

\[
\left[ \forall \omega \in \tilde{\Theta} \setminus D(f) \right] \left[ \forall n \in \tilde{\mathbb{N}} \right] \left( \exists b \in \tilde{\mathbb{B}} \right) \left[ \omega \in \tilde{B}(b) \land \left( \forall \omega' \in \tilde{B}(b) \right) \left[ |\tilde{f}(\omega) - \tilde{f}(\omega')| > n \right] \right] = 1.
\]

This shows that \( \tilde{f} \) is continuous on \( \tilde{\Theta} \) in \( V(\mathbb{B}) \). \( \square \)

By Theorems 9.5 and 9.6, for any \( f \in \mathcal{N}(\Omega) \), the function \( \tilde{f} : \tilde{\Theta} \to C_\mathbb{B} \cup \{ \infty \} \) has the unique continuous extension \( \tilde{f} : \Theta \to C_\mathbb{B} \cup \{ \infty \} \) in \( V(\mathbb{B}) \). Since \([G \in \Theta] = 1\), we have

\[
\lim_{\omega \to G} \tilde{f}(\omega) = \tilde{f}(G) \in C_\mathbb{B} \cup \{ \infty \}
\]

in \( V(\mathbb{B}) \). In what follows, we will show that \( \tilde{f}(G) \in C_\mathbb{B} \) in \( V(\mathbb{B}) \), and that the correspondence \( f \mapsto \tilde{f}(G) \) is a \( * \)-homomorphism of \( \mathcal{N}(\Omega) \) into \( C(\mathbb{B}) \). Then our goal is to show that this \( * \)-homomorphism \( f \mapsto \tilde{f}(G) \) is the inverse of the \( * \)-homomorphism \( \alpha \mapsto \tilde{\alpha} \) of \( C(\mathbb{B}) \) into \( \mathcal{N}(\Omega) \).
Lemma 9.7. For any $b \in B$, $[U(\hat{b}) = (B(b)^\omega)^0] = 1$.

Proof. Since $\hat{D}$ is dense in $\Theta$ and since $B(b)^\omega$ is regular open in $\hat{D}$ in $V(B)$, we have $(B(b)^\omega)^0 = U(\hat{b})$ in $V(B)$ by Lemma 9.4 and [40, Theorem 22.5] together with the ZFC Transfer Principle. □

Lemma 9.8. For any $b \in B$, $f \in S(\Omega)$, and $r \in R$, we have
$[B(b)^\omega \subseteq \{\omega \in \hat{D} \mid \hat{f}(\omega) \leq \hat{r}\}] = [U(\hat{b}) \subseteq \{\theta \in \Theta \mid \hat{f}(\theta) \leq \hat{r}\}]$.

Proof. The assertion follows from the following manipulation of Boolean truth values.

$[B(b)^\omega \subseteq \{\omega \in \hat{D} \mid \hat{f}(\omega) \leq \hat{r}\}]$

$\leq \{(B(b)^\omega)^0 \subseteq \{\omega \in \hat{D} \mid \hat{f}(\omega) \leq \hat{r}\}^0\}$

$\leq [U(\hat{b}) \subseteq \{\theta \in \Theta \mid \hat{f}(\theta) \leq \hat{r}\}^0]$ by Lemma 9.7

$\leq [U(\hat{b}) \cap \hat{D} \subseteq \{\theta \in \Theta \mid \hat{f}(\theta) \leq \hat{r}\} \cap \hat{D}]$

$\leq [B(b)^\omega \subseteq \{\omega \in \hat{D} \mid \hat{f}(\omega) \leq \hat{r}\}]$ by Lemma 9.4. □

The following theorem shows a remarkable property of the canonical generic filter.

Theorem 9.9. For any $f \in N(\Omega)$, we have $[\hat{f}(G) \in C_B] = 1$.

Proof. Let $f \in N(\Omega)$. For any $n \in N$, let $B_n = \{\omega \in \Omega \mid |f(\omega)| \leq n\}^0$. Then $B_n$ is a clopen set, and hence there is some $b(n) \in B$ such that $B_n = B(b(n))$. Since $f$ is a normal function, $(\bigcup_{n \in N} B_n)^\omega = \Omega$. It follows that $\forall\{b(n) \mid n \in N\} = 1$. By the $\Delta_0$-Absoluteness principle, we have $[\bigcup\{b(n) \mid n \in N\}^\omega = \hat{1}] = 1$. Since $[\{b(n) \mid n \in N\}^\omega \in P(B)^\omega] = 1$ and since $[G \text{ is } P(B)^\omega\text{-complete}] = 1$, there is some $\nu \in N(B)$ such that $[\hat{b}(\nu) \in G] = 1$. It follows that $[G \in U(b(\nu))] = 1$. Let $n \in N$. By the $\Delta_0$-Absoluteness Principle,

$[B(b(n))^\omega \subseteq \{\omega \in \hat{D} \mid |\hat{f}(\omega)| \leq \hat{n}\}] = 1,$

and hence by Lemma 9.8

$[U(b(\nu))^\omega \subseteq \{\theta \in \Theta \mid |\hat{f}(\theta)| \leq \hat{n}\}] = 1.$

Since $n \in N$ is arbitrary, we have

$[\forall n \in \hat{N}] [U(\hat{b}(n)) \subseteq \{\theta \in \Theta \mid |\hat{f}(\theta)| \leq n\}] = 1,$

and hence

$[U(\hat{b}(\nu)) \subseteq \{\theta \in \Theta \mid |\hat{f}(\theta)| \leq \nu\}] = 1.$

Thus we have $[|\hat{f}(G)| \leq \nu] = 1$, and therefore $[\hat{f}(G) \in C_B] = 1$. □

By virtue of the above theorem, we define the $B$-valued set $\hat{f}(G)$ so that $\hat{f}(G) \in C(B)$. Then the following theorem can be verified easily.
Theorem 9.10. The correspondence \( f \mapsto \tilde{f}(G) \) is a *-homomorphism of \( N(\Omega) \) into \( C^*(B) \).

Now we will show that one of each *-homomorphism \( a \mapsto \hat{a} \) or \( f \mapsto \tilde{f}(G) \) is the inverse of the other.

Theorem 9.11. For any \( f \in N(\Omega) \), \( \tilde{f}(G)^\wedge = f \).

Proof. It suffices to prove the assertion for \( f \in S(\Omega) \). Suppose \( f^* = f \) and \( \tilde{f}(G) \in R(B) \). Let \( r \in Q \) and \( \omega \in \Omega \). Suppose \( f(\omega) < r \). Then there is some \( b \in B \) such that

\[
\omega \in B(b) \subseteq \{ \omega' \in \Omega \mid f(\omega') \leq r \}.
\]

Then \( \mathcal{B} \cap U(\tilde{b}) = \{ \tilde{b} \in \mathcal{G} \} = b \in \omega \) and

\[
\mathcal{B}(b)^\wedge \subseteq \{ \omega' \in \Omega \mid f(\omega') \leq r \} = 1.
\]

From Lemma 9.8,

\[
[U(\tilde{b}) \subseteq \{ \theta \in \Theta \mid \tilde{f}(\theta) \leq r \}] = 1.
\]

Thus we have \( \tilde{f}(G) \leq r \) \( \in \omega \) and hence \( \gamma_f(\tilde{f}(G)) \leq r \), so that \( \tilde{f}(G)^\wedge(\omega) \leq r \). It follows that \( \tilde{f}(G)^\wedge(\omega) \leq f(\omega) \). Conversely, suppose \( \tilde{f}(G)^\wedge(\omega) < r \). Then \( \gamma_f(\tilde{f}(G)) < r \) and \( \tilde{f}(G) < r \) \( \in \omega \), so that there is some \( p \in B(B) \), such that

\[
[G \in U(\tilde{p})] \wedge [U(p) \subseteq \{ \theta \in \Theta \mid \tilde{f}(\theta) \leq r \}] = 1.
\]

Put

\[
D = [U(p) \subseteq \{ \theta \in \Theta \mid \tilde{f}(\theta) \leq r \}].
\]

By Lemma 9.8,

\[
D = [B(p) \subseteq \{ \omega' \in \tilde{\Omega} \mid \tilde{f}(\omega') \leq r \}] = \bigvee_{B(b) \subseteq \{ \omega' \in \tilde{\Omega} \mid \tilde{f}(\omega') \leq r \}}\bigwedge_{B(b)}[b \in B(b)]\wedge [B(b)^\wedge \subseteq \{ \omega' \in \tilde{\Omega} \mid \tilde{f}(\omega') \leq r \}][p = \tilde{b}]
\]

and hence

\[
\omega \in B([G \in U(p)] \wedge D) \subseteq (\bigcup_{B(b) \subseteq \{ \omega' \in \tilde{\Omega} \mid \tilde{f}(\omega') \leq r \}}[B(b)])_{B(b) \subseteq \{ \omega' \in \tilde{\Omega} \mid \tilde{f}(\omega') \leq r \}} = \{ \omega' \in \tilde{\Omega} \mid f(\omega') \leq r \}^B \subseteq \{ \omega' \in \tilde{\Omega} \mid f(\omega') \leq r \}.
\]
whence \( f(\omega) \leq r \). It follows that \( f(\omega) \leq \tilde{f}(G) \wedge (\omega) \). Therefore, \( f(\omega) = \tilde{f}(G) \wedge (\omega) \), and hence \( \tilde{f}(G) \wedge = f \). \( \square \)

**Theorem 9.12.** For any \( a \in C(B) \), \( (\tilde{a}) \wedge(G) = a \).

**Proof.** Let \( a \in C(B) \) and \( r \in Q \). There is \( a_1, a_2 \in R(B) \) such that \( a = a_1 + ia_2 \), and hence we can assume without any loss of generality that \( a \in R(B) \). Suppose \( \llbracket a \leq \tilde{r} \rrbracket = 1 \). Then \( \tilde{a}(\omega) \leq r \) for all \( \omega \in \Omega \) and hence \( \Omega = \{ \omega \in \Omega \mid \tilde{a}(\omega) \leq r \} \). By Lemma 9.8, \( \llbracket \{ \theta \in \Theta \mid (\tilde{a}) \wedge(\theta) \leq \tilde{r} \} \rrbracket = 1 \), and hence \( \llbracket (\tilde{a}) \wedge(G) \leq \tilde{r} \rrbracket = 1 \). Therefore, \( \llbracket (\tilde{a}) \wedge(G) \leq a \rrbracket = 1 \). Conversely, suppose \( \llbracket (\tilde{a}) \wedge(G) < \tilde{r} \rrbracket = 1 \). Let \( \omega \in \Omega \). Then \( (\tilde{a}) \wedge(G) \wedge(\omega) < r \). By Theorem 9.11, \( \tilde{a}(\omega) < r \), and hence \( \gamma(\omega)(a) < r \). Thus \( \llbracket a < \tilde{r} \rrbracket \in \omega \). Since \( \omega \) is arbitrary, we have \( \llbracket a \leq \tilde{r} \rrbracket = 1 \). It follows that \( \llbracket a \leq (\tilde{a}) \wedge(G) \rrbracket = 1 \), so that \( \llbracket (\tilde{a}) \wedge(G) = a \rrbracket = 1 \). Since \( (\tilde{a}) \wedge(G) \in C(B) \) and \( a \in C(B) \), we conclude that \( (\tilde{a}) \wedge(G) = a \). \( \square \)

We have just proved the following theorem.

**Theorem 9.13.** The correspondence \( a \mapsto \tilde{a} \) is a \( * \)-isomorphism of \( C(B) \) onto \( N(\Omega) \), whose inverse \( * \)-isomorphism is the correspondence \( f \mapsto \tilde{f}(G) \) from \( N(\Omega) \) to \( C(B) \).

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