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On recursively enumerable structures

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Abstract

We state some general facts on r.e. structures, e.g. we show that the free countable structures in quasivarieties are r.e. and construct acceptable numerations and universal r.e. structures in quasivarieties. The last facts are similar to the existence of acceptable numerations of r.e. sets and creative sets. We state a universality property of the acceptable numerations, classify some index sets and discuss their relation to other decision problems. These results show that the r.e. structures behave in some respects better than the recursive structures.

1. Introduction

Recursive algebra was established by several famous mathematicians, among them are Rabin [10] and Malcev [6]. The key notions of a recursive and of a r.e. (or positive) structure are presented in the cited papers. In the subsequent development of recursive algebra, most attention was paid to recursive structures; this work was summarized in [3]. R.e. structures were less popular (though r.e. models of some theories were considered in detail, e.g. r.e. vector spaces [8]).

In this paper we try to argue that r.e. structures are in some respects more "regular" and have a better theory than the recursive structures (a remote analogy to this is the situation with r.e. and recursive sets). Recursive structures seem to be insufficient for a complete treatment of effectiveness in algebra for the following reasons: many interesting finitely presented structures are r.e. but not recursive (for the case of groups this was shown by P. S. Novikov and W.W. Boone); the least model of a logic program (see [5]) is r.e. but not always recursive; natural classes of recursive structures may have no "acceptable" numerations.

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We will show that the class of r.e. structures in a given recursively axiomatizable quasivariety always has an "acceptable" numeration, and this numeration often has a nice characterization in terms of complete numerations. This enables us to consider natural decision problems on r.e. structures in a way similar to the study of index sets in recursion theory. We classify some of these problems and discuss the relation of this with the decidability of elementary theories.

We consider recursively axiomatizable quasivarieties, not confining ourselves, to the finitely axiomatizable case, as is usual in logic programming. The reason is that some natural quasivarieties are not finitely axiomatizable (e.g. by a well-known result of A.I. Malcev such is the quasivariety of semigroups embeddable into a group).

Now some notation and terminology. Fix a language $L = \{f_i, P_j | i < i_0, j < j_0\}$ with $i_0, j_0 \leqslant \omega$, functional symbols f_i and predicate symbols P_j, the arity of any symbol being effectively computable from its index. The case of arity 0 is possible. For technical reasons we assume that $j_0 > 0$ and P_0 is always a binary symbol. This symbol is usually represented as the equality relation (often understood intensionally). In expressions like $f_i(a_1, \ldots, a_k)$ we always assume that k is the arity of f_i.

By a *L-rule* we mean a formula of the form $\theta_1 \wedge \cdots \wedge \theta_k \to \theta$, where all θ_l, θ are atomic L-formulas. By a *L-identity* (*L-quasiidentity*) we mean the universal closure of an atomic L-formula (resp. of a L-rule). An expression without free variables is called *ground*. Natural examples of quasiidentities are the so-called axioms of equality, i.e. sentences stating that $P_0(x, y)$ is an equivalence relation respected by all other functions and relations. By a *quasivariety* we mean a class of structures axiomatizable by quasiidentities. We will use some well-known facts on quasivarieties which may be found in any standard text on universal algebra, e.g. [7] or [1].

By an *extensional* (resp. *intensional*) L-structure we mean an L-structure interpreting P_0 as the equality relation (resp. as a relation satisfying the axioms of equality). It is well-known that the natural factorization of an intensional L-structure is an extensional L-structure. By a *r.e. L-structure* we mean an extensional L-structure $\mathbf{A} = (A; f_i^A, P_j^A)$ having a numeration (i.e. a mapping α from ω onto A) such that any function f_i^A is represented in α by a recursive function uniformly in i, and any predicate P_j^A is r.e. in the numeration α uniformly in j. The pair $(\mathbf{A}; \alpha)$ is called a *numbered r.e. structure*.

Our main objects are the numbered r.e. structures. For technical reasons it is more convenient to consider them in the form of intensional r.e. L-structures $(\omega; f_i, P_j)$ with f_i uniformly recursive and P_j uniformly r.e. Note that any numbered r.e. structure $(\mathbf{A}; \alpha)$ induces such a structure $(\omega; f_i, P_j)$ by taking f_i to be the recursive function representing f_i^A in α, and P_j to be the index set of P_j^A in the numeration α. Conversely, any intensional r.e. structure $(\omega; f_i, P_j)$ induces the unique numbered r.e. structure, namely the factor-structure by the congruence relation P_0 with the numeration α being the natural epimorphism on the factor-structure. Simplifying notation, we usually identify these two presentations of numbered r.e. structures and denote the numeration of a numbered r.e. structure by a "similar" letter (α for $\mathbf{A}$, β for $\mathbf{B}$, and so on).

We will sometimes use notions from category theory, all of them are broadly known and may be found in any text on category theory. Let $\mathscr{R}$ be the category formed by the numbered r.e. L-structures as objects, and by homomorphisms representable by recursive functions in the respective numerations, as morphisms. By $\simeq_r$ we denote the $\mathscr{R}$-isomorphism and by $\simeq$ the abstract isomorphism of the corresponding structures. Note that any numbered r.e. structure is $\mathscr{R}$-isomorphic to the corresponding intensional r.e. structure.

For any r.e. set Q of L-quasiidentities, let $\mathscr{R}_Q$ be the full subcategory of $\mathscr{R}$ formed by the models of Q. Due to our use of intensional structures, it is convenient to think that Q always contains the equality axioms (otherwise, just add them to Q). For a fixed $\mathscr{R}$-object $\mathbf{A}$, let $\mathscr{R}_Q^{\mathbf{A}}$ be the category with objects $(\phi, \mathbf{B})$, where $\phi : \mathbf{A} \to \mathbf{B}$ is a $\mathscr{R}$-morphism to an object $\mathbf{B}$ of $\mathscr{R}_Q$. By a morphism ψ from $(\phi, \mathbf{B})$ to $(\phi_1, \mathbf{B}_1)$ we mean a $\mathscr{R}$-morphism $\psi : \mathbf{B} \to \mathbf{B}_1$ satisfying $\phi_1 = \psi \circ \phi$.

By $dom(f)$ and $rng(f)$ we denote, respectively, the domain and range of a function f.

2. Numeration

Here we will construct some natural numerations of r.e. structures but first let us state some auxiliary facts on the categories introduced above. By the *image* of an object $\mathbf{A} \in \mathscr{R}$ under a morphism $\phi : \mathbf{A} \to \mathbf{B}$ we mean the substructure $\phi(\mathbf{A})$ of the structure $\mathbf{B}$ with the universe $rng(\phi)$. By a *quotient* of a r.e. structure $(\omega; f_i, P_j)$ we mean any r.e. structure of the form $\tilde{\mathbf{A}} = (\omega; f_i, \tilde{P}_j)$ satisfying $P_j \subseteq \tilde{P}_j$ for $j < j_0$. Let us state some properties of the introduced objects.

2.1. Properties. (i) *Any r.e. generated substructure of a r.e. structure is a r.e. structure.*

(ii) *Any quotient* $\tilde{\mathbf{A}}$ *of a r.e. structure* $\mathbf{A}$ *is* $\mathscr{R}$*-isomorphic to an image of* $\mathbf{A}$.

(iii) *Any image* $\phi(\mathbf{A})$ *of a r.e. structure* $\mathbf{A}$ *under a morphism* $\phi : \mathbf{A} \to \mathbf{B}$ *is* $\mathscr{R}$*-isomorphic to a quotient of* $\mathbf{A}$.

(iv) *The category* $\mathscr{R}_Q$ *is closed under finite nonempty direct products.*

(v) *The category* $\mathscr{R}_Q^{\mathbf{A}}$ *has a terminal and an initial object.*

Proof. (i) Let $\mathbf{B} = (B; f_i, P_j)$ be the substructure of a r.e. structure $\mathbf{A} = (\omega; f_i, P_j)$ generated by a nonempty r.e. set W. The set B is clearly r.e., so $B = rng(g)$ for a recursive function g. The function g induces a numeration β of $\mathbf{B}$ so that $(\mathbf{B}; \beta)$ is r.e.

(ii) The structure $\tilde{\mathbf{A}}$ is the image of $\mathbf{A}$ under the standard natural epimorphism $\varepsilon : \mathbf{A} \to \tilde{\mathbf{A}}$.

(iii) The universe of the structure $\phi(\mathbf{A})$ is generated by the set $rng(g)$, where g is a recursive function representing ϕ in the numerations α and β of the structures $\mathbf{A}$ and $\mathbf{B}$. So $\phi(\mathbf{A})$ is r.e. by (i). Define predicates P_j^* on ω by $P_j^*(x_1, \ldots, x_k) \leftrightarrow P_j(g(x_1), \ldots, g(x_k))$. Then $\mathbf{A}/\phi = (\omega; f_i, P_j^*)$ is a quotient of $\mathbf{A}$. A standard algebraic argument shows that g represents a $\mathscr{R}$-isomorphism $\phi^* : \mathbf{A}/\phi \to \phi(\mathbf{A})$ satisfying $\phi = \varepsilon \circ \phi^*$.

(iv) It is clear that the usual direct product $\mathbf{A} \times \mathbf{B}$ of r.e. structures is a r.e. structure which is a product of $\mathbf{A}$ and $\mathbf{B}$ in the category $\mathscr{R}_Q$.

(v) Let $\mathbf{E}$ be the unique singleton L-structure in which all predicates are true. Then $\mathbf{E}$ is an object of $\mathscr{R}_Q$, and for any r.e. structure $\mathbf{B}$ there is a unique morphism $\varepsilon : \mathbf{B} \to \mathbf{E}$. So $(\varepsilon, \mathbf{E})$ is a terminal object in $\mathscr{R}_Q^A$.

Let us seek the initial object $(\phi_Q, \mathbf{A}_Q)$ such that $\mathbf{A}_Q$ is a quotient $(\omega; f_i, \tilde{P}_j)$ of $\mathbf{A} = (\omega; f_i, P_j)$, and ϕ_Q is the natural epimorphism from $\mathbf{A}$ to $\mathbf{A}_Q$. So we need to specify only the predicates $\tilde{P}_j$ $(j < j_0)$. The structure $\mathbf{A}_Q$ should satisfy all the quasiidentities from Q (including the equality axioms) as well as the conditions

$$\forall x_1 \dots \forall x_k (P_j(x_1, \dots, x_k) \to \tilde{P}_j(x_1, \dots, x_k)). \tag{1}$$

It is clear that there is the least sequence of such predicates $\tilde{P}_j$. Moreover, from the form of quasiidentities and the uniformity of the sequences $\{f_i\}$ and $\{P_j\}$ it follows that the sequence $\{\tilde{P}_j\}_{j<j_0}$ is uniformly r.e.

It remains to check that for any object $(\phi, \mathbf{B})$ of $\mathscr{R}_Q^A$ there is a unique $\mathscr{R}$-morphism $\psi : \mathbf{A}_Q \to \mathbf{B}$ satisfying $\phi = \psi \circ \phi_Q$. Let P_j^* be the predicates from the proof of (iii). The structure $\mathbf{A}/\phi = (\omega; f_i, P_j^*)$ is $\mathscr{R}$-isomorphic to the substructure $\phi(\mathbf{A})$ of $\mathbf{B}$, so it satisfies all the quasiidentities from Q and the conditions (1). By the minimality condition, $\tilde{P}_j \subseteq P_j^*$ for all $j < j_0$. So there is a unique epimorphism $\tilde{\phi} : \mathbf{A}_Q \to \mathbf{A}/\phi$ satisfying $\varepsilon = \tilde{\phi} \circ \phi_Q$. Then the morphism $\psi = \phi^* \circ \tilde{\phi}$ has the desired property, because $\phi = \phi^* \varepsilon = \phi^* \tilde{\phi} \phi_Q = \psi \phi_Q$. The uniqueness of ψ is clear. This completes the proof of the properties. □

2.2. Remarks. (i) We have only sketched the existence of the initial object because the proof is almost the same as that of the existence of the least Herbrand model of a logic program (see [5, Section 1.6]). Indeed it is an effective version of an old fact about quasivarieties, see [7].

(ii) For any homomorphism $\phi : \mathbf{A} \to \mathbf{B}$ to an abstract Q-model $\mathbf{B}$ there is a unique homomorphism $\psi : \mathbf{A}_Q \to \mathbf{B}$ satisfying $\phi = \psi \phi_Q$.

(iii) The sequence $\{\tilde{P}_j\}$ is effectively computable from the sequence $\{P_j\}$.

Let us formulate an example of Property 2.1(v) for the case when Q is the r.e. set of quasiidentities axiomatizing the class of all semigroups embeddable into a group.

2.3. Corollary. *For any r.e. semigroup* $\mathbf{A}$ *there is the least congruence relation* P *on* $\mathbf{A}$ *such that* $\mathbf{A}_Q = \mathbf{A}/P$ *is embeddable into a group. The relation* P *and the semigroup* $\mathbf{A}/P$ *are r.e.*

An interesting corollary of Properties 2.1 is the existence of numerations of r.e. structure properties of which are quite similar to the properties of the acceptable numeration $\{W_n\}$ of r.e. sets. To state the corresponding result, let us first recall some relevant information about numerations, see e.g. [2].

By a *numeration* we mean any function ν with $dom(\nu) = \omega$. So numerations may be written also as sequences $\{\nu_n\}_{n<\omega}$. Let numerations μ, ν be given. Then μ is *reducible* to ν (in symbols $\mu \leqslant \nu$) if $\mu = \nu \circ g$ for a recursive function g; μ is *equivalent* to ν if $\mu \leqslant \nu$ and $\nu \leqslant \mu$; μ is *recursively isomorphic* to ν if $\mu = \nu \circ g$ for some recursive permutation g; ν is a *factorization* of μ if ν is equivalent to $f \circ \mu$ for some $f: rng(\mu) \to rng(\nu)$. Numeration ν is *precomplete* if for any partial recursive function ψ, there is a recursive function g such that $\nu\psi(x) = \nu g(x)$ for $x \in dom(\psi)$. It is well-known that any factorization of a precomplete numeration is precomplete, and if a numeration is equivalent to a precomplete numeration, then these numerations are recursively isomorphic.

We call a numeration $\{(\omega; f_i^n, P_j^n)\}$ of r.e. L-structures *r.e.* if the sequence $\{f_i^n\}$ is uniformly recursive and the sequence $\{P_j^n\}$ is uniformly r.e. in i, j, n. We will consider such numerations of structures always modulo $\mathscr{R}$-isomorphism, not mentioning this explicitly. We will apply to the numerations of structures the notions from the preceding paragraph always under this agreement. For example, a r.e. sequence of structures $\{\mathbf{A}_n\}$ is reducible to a r.e. sequence of structures $\{\mathbf{B}_n\}$ if there is a recursive function g such that $\mathbf{A}_n \simeq_r \mathbf{B}_{g(n)}$ for all $n < \omega$. By an *acceptable numeration of* $\mathscr{R}_Q$ we mean a r.e. numeration $\{\mathbf{A}_n\}$ of Q-models such that any r.e. numeration of Q-models is reducible to $\{\mathbf{A}_n\}$.

We are ready to state the main result of this section.

2.4. Theorem. *For any r.e. set Q of L-quasiidentities there is an acceptable numeration of $\mathscr{R}_Q$, and this numeration is unique up to recursive isomorphism.*

Proof. Let $L' = L \cup \{c_k \mid k < \omega\}$ be the enlargement of L by new constant symbols c_k, and let $\mathbf{F}$ be the free L-structure generated by these new constants (recall that the universe of $\mathbf{F}$ is the set of all ground L'-terms, the functional symbols are interpreted as the corresponding syntactic operations on terms, P_0 is interpreted as the equality relation, and P_j for $j > 0$ is interpreted as the false predicate). We can consider $\mathbf{F}$ as a L'-structure interpreting the new constants in the natural way. Note that $\mathbf{F}$ is a r.e. (even a recursive) structure via a natural Gödel numeration which is not visualized explicitly.

Relate to any number n the set of quasiidentities $Q_n = Q \cup \{\theta_x \mid x \in W_n\}$, where $\{\theta_x\}$ is a fixed Gödel numeration of all ground L'-identities. Now define $\mathbf{A}_n$ as the L-reduct of the L'-structure $\mathbf{F}_{Q_n}$ constructed as in the proof of Property 2.1 (v) (note that $\mathbf{F}_{Q_n}$ is just the structure defined by the identities θ_x $(x \in W_n)$ in the quasivariety with the axioms Q, see [7] or [1]).

By Remark 2.3(iii), $\{\mathbf{A}_n\}$ is a r.e. numeration of Q-models. To check the acceptability of $\{\mathbf{A}_n\}$ it clearly suffices to show that from a given $\mathscr{R}_Q$-object $\mathbf{B} = (\omega; f_i^B, P_j^B)$ (i.e. from indices of $\{f_i^B\}$ and $\{P_j^B\}$) one can effectively find a number n with $\mathbf{B} \simeq_r \mathbf{A}_n$. Let $\mathbf{B}'$ be the L'-enlargement of $\mathbf{B}$ obtained by interpreting c_k as $\beta(k)$. Compute a number n satisfying $W_n = \{x \mid \mathbf{B}' \models \theta_x\}$. Let $\phi: \mathbf{F} \to \mathbf{B}'$ be the epimorphism satisfying $\phi(c_k^F) = c_k^{B'}$ for all $k < \omega$. The structure $\mathbf{B}'$ is a model of Q_n, so there is an epimorphism

$\tilde{\phi}:\mathbf{F}_{Q_n} \to \mathbf{B}'$ induced by ϕ. By the choice of n, $\mathbf{F}_{Q_n} \models \theta_x$ for $\mathbf{B}' \models \theta_x$. So $\tilde{\phi}$ is indeed a $\mathscr{R}$-isomorphism, and a fortiori $\mathbf{A}_n \simeq_r \mathbf{B}$.

It remains to show that any acceptable numeration $\{\mathbf{A}'_n\}$ of $\mathscr{R}_Q$ is recursively isomorphic to $\{\mathbf{A}_n\}$. By acceptability these numerations are equivalent. By construction, $\mathbf{A}_n = \mathbf{A}_m$ for $W_n = W_m$, so $\{\mathbf{A}_n\}$ is a factorization of the precomplete numeration $\{W_n\}$. By remarks before the formulation of the theorem, $\{\mathbf{A}_n\}$ and $\{\mathbf{A}'_n\}$ are recursively isomorphic. This completes the proof. □

The next sequel of Theorem 2.4 relates the acceptable numerations of $\mathscr{R}_Q$ for different Q. A numeration ν is said to be a *retract* of a numeration μ (see [2]) if $\nu \leqslant \mu$ and for some $h:rng(\mu) \to rng(\nu)$ we have $h \circ u \leqslant \nu$ and $h \circ \nu = \nu$.

2.5. Proposition. *For any r.e. sets $Q \subseteq Q'$ of L-quasiidentities the acceptable numeration ν of $\mathscr{R}_{Q'}$ is a retract of the acceptable numeration μ of $\mathscr{R}_Q$.*

Proof. Every Q'-model is a Q-model, so $\nu \leqslant \mu$ by the acceptability of μ. Let h: $rng(\mu) \to rng(\nu)$ be the function induced by the construction $\mathbf{A} \mapsto \mathbf{A}_{Q'}$ from the proof of 2.1(v). By Remark 2.2(iii) $h \circ \mu \leqslant \nu$. If $\mathbf{A} \models Q'$, then by uniqueness of the initial object in $\mathscr{R}^{A}_{Q'}$, we have $\mathbf{A} \simeq_r \mathbf{A}_{Q'}$. This completes the proof. □

2.6. Remarks. (i) For the constructed acceptable numeration $\mathbf{A}_n = (\omega;\, f_i^n, P_j^n)$ we have $f_i^n = f_i^m$ for all m, n, i.e. the functions in all structures are indeed the same.

(ii) Any r.e. numeration $\{\mathbf{B}_n\}$ of Q-models is reducible to $\{\mathbf{A}_n\}$ in the stronger sense that there are a recursive function g and uniform sequences $\{u_n\}$ and $\{v_n\}$ of recursive functions such that for any n the functions u_n, v_n represent an $\mathscr{R}$-isomorphism of $\mathbf{B}_n$ and $\mathbf{A}_{g(n)}$.

(iii) Theorem 2.4 is true for the class of n-generated structures for any fixed $n < \omega$ (to see this just take $\{c_k \mid k < n\}$) in place of $\{c_k \mid k < \omega\}$ above.

We conclude this section by the remark that one can get also a natural "numeration" of all (modulo isomorphism) countable Q-models by elements of the Baire space ${}^{\omega}\omega = \{h \mid h:\omega \to \omega\}$. We call such "numerations" *parameterizations* in order to distinguish them from the "true" numerations. First note that one can in the obvious way relativize notions and results of this section to any given oracle $h \in {}^{\omega}\omega$ getting in particular the notion of a h-r.e. structure and the analog of Theorem 2.4 for such structures. As usual, this relativization is uniform in the oracle.

Now define a parametrization $\{\mathbf{A}_h\}$ of countable Q-models by specifying $\mathbf{A}_h$ as the L-reduct of $\mathbf{F}_{Q_h}$, where $Q_h = Q \cup \{\theta_x \mid x \in W^f_{h(0)}\}$ and $f(n) = h(n+1)$, and relativizing 2.1(v) to f. Then $\{\mathbf{A}_h\}$ is acceptable, i.e. it is r.e. and any r.e. parametrization $\{\mathbf{B}_h\}$ of Q-models is reducible to $\{\mathbf{A}_h\}$. A sequence $\{(\omega;\, f_i^h, P_j^h)\}_{h \in {}^{\omega}\omega}$ is called r.e. if there are recursive functionals $F, G: {}^{\omega}\omega \times \omega \to \omega$ such that $f_i^h = \phi^h_{F(h,i)}$ and $P_j^h = W^h_{G(h,j)}$, where ϕ^h and W^h are the relativized versions of the standard numerations of partial recursive functions and of r.e. sets. The reducibility of $\{\mathbf{B}_h\}$ to $\{\mathbf{A}_h\}$ means the existence of a recursive functional $F: {}^{\omega}\omega \to {}^{\omega}\omega$ such that $\mathbf{B}_h$ is $\mathscr{R}^h$-isomorphic to $\mathbf{A}_{F(h)}$ for all $h \in {}^{\omega}\omega$.

Note also that the coding ability of the Baire space is sufficient to get even a parametrization of all (modulo isomorphism) countable Q-models for all sets Q of quasivarieties in all countable languages.

3. Universal structures

In Section 2 we constructed a numeration of structures similar to the standard numeration of r.e. sets. Here we try to find an analog of the "universal", i.e. creative set. We call a r.e. Q-model $\mathbf{B}$ a *component* of a r.e. Q-model $\mathbf{A}$ if $\mathbf{A} \simeq_r \mathbf{B} \times \mathbf{C}$ for some r.e. Q-model $\mathbf{C}$.

3.1. Remarks. (i) For the case of upper semilattices with 0 any component of $\mathbf{A}$ is isomorphic to an initial segment of $\mathbf{A}$.

(ii) For the case of Boolean algebras $\mathbf{B}$ is a component of $\mathbf{A}$ if and only if $\mathbf{B}$ is isomorphic to an initial segment of $\mathbf{A}$.

We call a r.e. Q-model $\mathbf{U}$ *universal* if any r.e. Q-model is a component of $\mathbf{U}$. Somewhat unexpectedly it turns out that such structures exist in many natural cases.

3.2. Theorem. *For any r.e. set Q of quasiidentities there exists a universal r.e. Q-model, and it is unique up to $\mathscr{R}$-isomorphism.*

Proof. Let $\{\mathbf{A}_n\}$ be the numeration from the proof of Theorem 2.4. By Remark 2.6(i), the functions f_i are the same in all the structures $\mathbf{A}_n = (\omega; f_i, P_j^n)$. Let $\mathbf{U} = (U; f_i^U, P_j^U)$ be the substructure of the direct product $\prod_n \mathbf{A}_n$ with the universe consisting of all almost constant sequences $\{a_n\}$ of natural numbers (i.e. $\forall m \geqslant k(a_m = a_k)$ for some k). Note that U has a recursive bijective coding in which the number k above is effectively computable from the code of $\{a_n\}$; we do not mention the coding explicitly.

The structure $\mathbf{U}$ is a Q-model, because quasivarieties are closed under direct products and substructures. We claim that $\mathbf{U}$ is r.e. The uniform recursiveness of the functions f_i^U is clear, and for the predicates we have

$$P_j^U(\{x_n^1\}, \dots, \{x_n^k\}) \leftrightarrow \forall n \leqslant l P_j^n(x_n^1, \dots, x_n^k), \tag{2}$$

where l is a number satisfying $x_n^i = x_l^i$ for $1 \leqslant i \leqslant k, n \geqslant l$, and $W_l = \emptyset$. The implication $\to$ in (2) is clear, because the left-hand side is by definition $\forall n P_j^n(x_n^1, \dots, x_n^k)$. To check the implication $\leftarrow$, note that any $\mathbf{A}_n$ is the image of $\mathbf{A}_l = \mathbf{F}_Q$ (see the proof of Theorem 2.4) under the morphism induced by the identity function on ω. So $P_j^n(x_n^1, \dots, x_n^k)$ is true for $n \geqslant l$ because $P_j^l(x_l^1, \dots, x_l^k)$ is true and $x_n^i = x_l^i$. But l is effectively computable from $\{x_n^1\}, \dots, \{x_n^k\} \in U$, so the predicates P_j^U are uniformly r.e.

To check that $\mathbf{U}$ is universal it suffices to show that any $\mathbf{A}_m$ is a component of $\mathbf{U}$. Let $\mathbf{A}'_n = \mathbf{A}_n$ for $n < m$ and $\mathbf{A}'_n = \mathbf{A}_{n+1}$ for $n \geqslant m$. Then $\{\mathbf{A}'_n\}$ is an acceptable numeration of r.e. Q-models, so by Theorem 2.4 $\mathbf{A}'_n \simeq_r \mathbf{A}_{p(n)}$ for some recursive permutation p. The same is true for the sequence $\mathbf{A}''_0 = \mathbf{A}_m$, $\mathbf{A}''_{n+1} = \mathbf{A}'_n$. Let $\mathbf{U}'$ and $\mathbf{U}''$ be constructed as $\mathbf{U}$, but for the numerations $\{\mathbf{A}'_n\}$ and $\{\mathbf{A}''_n\}$, respectively. From the recursive isomorphism of the numerations $\{\mathbf{A}_n\}$, $\{\mathbf{A}'_n\}$ and $\{\mathbf{A}''_n\}$ it follows that $\mathbf{U} \simeq_r \mathbf{U}' \simeq_r \mathbf{U}''$. But clearly $\mathbf{U}'' \simeq_r \mathbf{A}_m \times \mathbf{U}'$, so $\mathbf{U} \simeq_r \mathbf{A}_m \times \mathbf{U}$ and $\mathbf{A}_m$ is a component of $\mathbf{U}$.

It remains to show that any universal r.e. Q-model $\mathbf{V}$ is $\mathscr{R}$-isomorphic to $\mathbf{U}$. By universality of $\mathbf{V}$, $\mathbf{U}$ is a component of $\mathbf{V}$, i.e. $\mathbf{V} \simeq_r \mathbf{U} \times \mathbf{A}$ for some r.e. Q-model $\mathbf{A}$. Using the preceding paragraph and the commutativity of direct product, we get $\mathbf{V} \simeq_r \mathbf{A} \times \mathbf{U} \simeq_r \mathbf{U}$. This completes the proof of the theorem. □

3.3. Remarks. (i) A Q-model $\mathbf{A}$ is universal iff $\mathbf{U}$ is a component of $\mathbf{A}$. So e.g. $\mathbf{U} \times \mathbf{U}$ is universal, and a fortiori $\mathbf{U} \times \mathbf{U} \simeq_r \mathbf{U}$.

(ii) The structure $\mathbf{U}$ is a subdirect product of $\{\mathbf{A}_n\}$.

(iii) The proof of Theorem 3.2 generalizes the corresponding proof for the case of Boolean algebras in [14]. A similar fact for the Boolean algebras is Theorem 5.1 in [9], but that proof is specific for Boolean algebras and does not generalize to arbitrary structures.

Property 2.1(v), Theorems 2.4 and 3.2 show that r.e. structures behave in some respects better than the recursive structures. We do not know similar general facts on the recursive structures.

The construction of the universal structure seems to have no analogs in algebra. Let us show that for some particular natural classes of structures the universal structure can be obtained by a standard algebraic construction. This will be also useful in further considerations. Assume that our language L contains a unique constant symbol denoted by 0, and that the quasiidentities from Q imply that the terminal structure $\mathbf{E}$ from the proof of Property 2.1(v) is a substructure of any Q-model. We call such a theory Q *special*. Examples of such theories are: semigroups with 0, lattices with 0, rings, as well as any extension of these theories by quasiidentities.

It is well known that any sequence $\{\mathbf{B}_n\}$ of Q-models for any special Q has *a direct sum* $\coprod_n \mathbf{B}_n$, which is also a Q-model. This is the substructure of $\prod_n \mathbf{B}_n$ formed by the sequences almost all of whose elements are 0. If the numeration $\mathbf{B}_n = (\omega; f_i^n, P_j^n)$ is r.e., then the structure $\coprod_n \mathbf{B}_n$ is clearly also r.e. Let us formulate without proof some evident properties of this construction which will be used later. In these properties Q is special, $\{\mathbf{A}_n\}$ is the acceptable numerations of Q-models, and $\mathbf{B}$ is the direct sum of a r.e. numeration $\{\mathbf{B}_n\}$ of Q-models.

3.4. Properties. (i) *For any recursive permutation p, $\mathbf{B} \simeq_r \coprod_n \mathbf{B}_{p(n)}$.*

(ii) $\mathbf{B} \simeq_r \mathbf{B}_0 \times (\coprod_n \mathbf{B}_{n+1})$.

(iii) *Any $\mathbf{B}_n$ is a component and a substructure of $\mathbf{B}$.*

(iv) *The structure $\coprod_n \mathbf{A}_n$ is universal.*

(v) *If* $\mathbf{B}_n = \mathbf{E}$ *for* $n \geqslant k$, *then* $\mathbf{B} \simeq_r \mathbf{B}_0 \times \cdots \times \mathbf{B}_k$.

(vi) *If* $\mathbf{B}_n$ *is universal for some* n, *then* $\mathbf{B}$ *is universal.*

3.5. Remark. The construction of direct sum is easier than the construction of the universal structure, but it is applicable only to special Q. The construction of $\mathbf{U}$ from $\{\mathbf{A}_n\}$ in the proof of Theorem 3.2 is applicable only to the acceptable numeration $\{\mathbf{A}_n\}$ (if this numeration is r.e. but not acceptable, then one can not guarantee that the resulting structure $\mathbf{U}$ is r.e.).

4. Properties of numerations

Hence we show that in many cases the acceptable numerations of structures have some universality properties. In [12] we have shown that some natural numerations are universal h-2-complete for a suitable oracle $h \in {}^{\omega}\omega$. This notion is relevant also to the numerations of structures, so we start with recalling some definitions from [12]. A numeration v is *h-2-complete* (with respect to given elements $a, b \in rng(v)$), if it has the following properties: for any function ψ partial recursive in h there is a recursive function g such that $vg(x) = v\psi(x)$ for $x \in dom(\psi)$ and $vg(x) = a$ otherwise; for any set S r.e. in h there is a recursive function $f(x, y)$ such that $vf(x, y) = vx$ for $y \notin S$ and $vf(x, y) = b$ for $y \in S$.

Let again $\mathbf{E}$ be the trivial singleton structure and $\mathbf{U}$ be the universal structure from Section 3. The next result shows a special role of these structures.

4.1. Theorem. *For any special set Q of quasiidentities the acceptable numeration $\{\mathbf{A}_n\}$ of Q-models is $\emptyset'$-2-complete with respect to $\mathbf{E}$ and $\mathbf{U}$.*

Proof. Let ψ be a partial recursive in $\emptyset'$ function; we have to find a recursive function g described above. It is easy (for details see [12]) to find a recursive function $p(x, s)$ such that $\lim_s p(x, s) = \psi(x)$ for $x \in dom(\psi)$ and $\{p(x, s)\}_s$ changes infinitely often for $x \notin dom(\psi)$. Let $q(x, s)$ be a recursive function with the following properties:

(a) if $(s = 0 \vee p(x, s) \neq p(x, s-1))$ and $\forall t \geqslant s(p(x, t) = p(x, s))$, then $W_{q(x,s)} = W_{p(x,s)}$;

(b) otherwise, $W_{q(x,s)} = \omega$.

From this and from the properties of p we immediately get the following properties:

(a_1) if $x \in dom(\psi)$, then $W_{q(x,s)} = W_{\psi(x)}$ for the least s satisfying the condition $\forall t \geqslant s(p(x, t) = p(x, s))$, and $W_{q(x,t)} = \omega$ for $t \neq s$;

(b_1) if $x \notin dom(\psi)$, then $W_{q(x,t)} = \omega$ for all t.

Now let $\mathbf{B}_x = \coprod_s \mathbf{A}_{q(x,s)}$. It is clear that $\{\mathbf{B}_x\}$ is a r.e. numeration of Q-models, so $\mathbf{B}_x \simeq_r \mathbf{A}_{g(x)}$ for some recursive function g. Note that $\mathbf{A}_a \simeq_r \mathbf{E}$ for $W_a = \omega$ (because $\mathbf{A}_a$ satisfies all the ground L'-identities, see the proof of Theorem 2.4). From (a_1), (b_1), Theorem 3.4 and the evident property $\mathbf{B} \times \mathbf{E} \simeq_r \mathbf{B}$ we get that $\mathbf{A}_{g(x)} \simeq_r \mathbf{A}_{\psi(x)}$ for $x \in dom(\psi)$ and $\mathbf{A}_{g(x)} \simeq_r \mathbf{E}$ for $x \notin dom(\psi)$, as desired.

Now let S be any set r.e. in $\emptyset'$; we have to find a recursive function $f(x, y)$ such that $\mathbf{A}_{f(x,y)} \simeq_r \mathbf{A}_x$ for $y \notin S$ and $\mathbf{A}_{f(x,y)} \simeq_r \mathbf{U}$ for $y \in S$. Let u be a fixed number satisfying $\mathbf{A}_u \simeq_r \mathbf{U}$, and let ψ be the $\emptyset'$-partial recursive function sending all elements of S to u and undefined outside S. Let g be the function from the preceding paragraph for the specified ψ. Then $\mathbf{A}_{g(y)} \simeq_r \mathbf{E}$ for $y \notin S$ and $\mathbf{A}_{g(y)} \simeq_r \mathbf{U}$ for $y \in S$. Let $f(x, y)$ be a recursive function satisfying $\mathbf{A}_{f(x,y)} \simeq_r \mathbf{A}_x \times \mathbf{A}_{g(y)}$. From the properties of $\mathbf{U}$ in Section 3 it follows that f has the desired property. This completes the proof of the theorem. □

The next result follows immediately from the properties of 2-complete numerations in [12].

4.2. Corollary. *Any nontrivial (i.e. different from $\emptyset$ and ω) index set of the numeration $\{\mathbf{A}_n\}$ for any special Q is either Σ^0_2-hard or Π^0_2-hard.*

4.3. Corollary. *Theorem 4.1 does not generalize to arbitrary r.e. set Q of quasiidentities.*

Proof. Let Q be the set of axioms of Boolean algebras in the language $\{\cup, \cap, ^-, 0, 1\}$. Then $\mathbf{A}_n \simeq_r \mathbf{E}$ iff $P^n_0(0, 1)$ (without loss of generality we assume that 0, 1 are interpreted in any structure by themselves). So $\{n \mid \mathbf{A}_n \simeq_r \mathbf{E}\}$ is a nontrivial r.e. index set. By Corollary 4.2 the numeration $\{\mathbf{A}_n\}$ is not $\emptyset'$-2-complete. Note that the same proof is applicable to some other classes of structures: lattices with 0 and 1, rings with 1, and so on. □

Now we will show that in some natural cases the numeration $\{\mathbf{A}_n\}$ has a universality property among the $\emptyset'$-2-complete numerations. Call a set Q of quasiidentities in a finite language L *good* if it is special and there exists a sequence $\{\mathbf{B}_n\}$ of finite Q-models such that $\mathbf{B}_n$ is not embeddable in $\mathbf{B}_m$ for $m \neq n$ and there is an algorithm computing the diagram D_n of the structure $\mathbf{B}_n$ (recall that D_n is a finite set of formulas). Note that good sets of axioms are not very rare in algebra. For example, all examples of special sets mentioned before Properties 3.4 are good. From [13] it follows that the set of axioms of distributive lattices with 0 is also good.

We say that numerations μ and ν are *f-equivalent*, if any of them is a factorization of the other (note that this is an equivalence relation). A numeration μ is called *universal h-2-complete*, if it is h-2-complete, and any h-2-complete numeration is a factorization of μ. It is clear that any two universal h-2-complete-numerations are f-equivalent.

4.4. Theorem. *For any good set Q of quasiidentities the acceptable numeration $\{\mathbf{A}_n\}$ is universal $\emptyset'$-2-complete.*

Proof. By Theorem 4.1 and by the criterium of h-2-universality from [12] it suffices to find a Σ^0_2-sequence $\{S_n\}_{n<\omega}$ of index sets of the numeration $\{\mathbf{A}_n\}$ and a recursive function g such that $g(n) \in S_n \setminus (\bigcup_{k \neq n} S_k)$ for all n. Let S_n be the set of all x such that $\mathbf{B}_n$ is embeddable in $\mathbf{A}_x$, and let g be a recursive function satisfying $\mathbf{B}_n \simeq_r \mathbf{A}_{g(n)}$. Then g clearly has the desired property, so it remains to show that "$x \in S_n$" is a Σ^0_2-predicate.

Note that D_n has the form $\{\theta_i(c_0, \dots, c_p) \mid i \leqslant m\}$ for some $m < \omega$ and some ground quantifier-free L-formulas θ_i from new constants $c_0, \dots, c_p$ representing the elements of $\mathbf{B}_n$; the diagram D_n describes the structure $\mathbf{B}_n$ up to isomorphism. By definition we can effectively compute all these formulas from n. The condition "$x \in S_n$" means that $\mathbf{B}_n$ is embeddable in $\mathbf{A}_x = (\omega; f_i, P_j^x)$, i.e. that for some $a_0, \dots, a_p \in \omega$ all formulas θ_i $(i \leqslant m)$ are true in $\mathbf{A}_x$ with $c_0, \dots, c_p$ replaced by $a_0, \dots, a_p$. By the Tarski–Kuratowski algorithm we see that this predicate is Σ_2^0, This completes the proof of the theorem. □

From [12] and from the remarks before the formulation of the theorem we get the following two assertions.

4.5. Corollary. *Any numaration from Theorem* 4.4 *is f-equivalent to the factorization* $\{W_n^*\}$ *of the numeration* $\{W_n\}$ *modulo finite sets.*

4.6. Corollary. *Any index set of the numeration from Theorem* 4.4 *is recursively isomorphic to an index set of the numeration* $\{W_n^*\}$, *and vice versa.*

Let $(I_\nu; \leqslant_m)$ be the structure of m-degrees containing index sets of a given numeration ν. This structure for different ν was considered by several authors. By Corollary 4.5 this structure is the same for any numeration from Theorem 4.4 as well as for the numeration $\{W_n^*\}$. The results in [12] imply the following intriguing property of this structure.

4.7. Corollary. *Let* ν *be any numeration from Theorem* 4.4. *For all* $n < \omega$ *and* $a_0, \dots, a_n \in I_\nu$ *there are* $b_0, b_1, b_2, b_3 \in I_\nu$ *such that* $a_i \leqslant_m b_j$ *for all* $i \leqslant n, j \leqslant 3$; *if* $x \in I_\nu$ *and* $a_i \leqslant_m x$ *for all* $i \leqslant n$, *then* $b_j \leqslant_m x$ *for some* $j \leqslant 3$; *if* $x \in I_\nu$ *and* $x \leqslant_m b_j$ *for all* $j \leqslant 3$, *then* $x \leqslant_m a_i$ *for some* $i \leqslant n$.

5. Definable index sets

The existence of the acceptable numerations of structures enables us to consider many natural algorithmic problems on such structures of the following type: given a "natural" property of structures, what is the degree of unsolvability of the (index set of the) class of structures having this property? Note that m-degrees seem to be the most suitable for this problem. From Theorem 2.4 it follows that the degree of a property does not depend on the acceptable numeration: index sets of any property in any two given acceptable numerations are recursively isomorphic.

Questions of this type are similar to the popular topic of the classification of index sets in recursion theory. The aim of this section is to discuss some particular cases of the raised problem trying to show the richness of the subject and its relatedness to some more traditional decision problems.

We start the discussion from the classification of some concrete index sets in the acceptable numeration $\{\mathbf{A}_n\}$ of r.e. distributive lattices with 0 in the language $\{\cup, \cap, 0\}$. By ω we denote the lattice $(\omega; max, min, 0)$ and by $k+1$ for $k<\omega$, its sublattice with the universe $\{0, \ldots, k\}$. By LO (WO) we denote the class of lattices which are linearly ordered (respectively well ordered) by the induced partial ordering. By $\simeq$ we denote the abstract isomorphism of structures. Note that $\mathbf{A} \simeq_r \mathbf{B}$ implies $\mathbf{A} \simeq \mathbf{B}$ but not vice versa (although the inverse implication is true in some particular cases, e.g. if one of $\mathbf{A}$, $\mathbf{B}$ is finitely generated.) Let $\{\Sigma^0_n\}$ and $\{\Sigma^1_n\}$ be the arithmetical and the analytical hierarchies, respectively.

5.1. Theorem. (i) *The sets* $M_k = \{n \mid \exists l \leqslant k(\mathbf{A}_n \simeq l+1)\}$ *for* $k<\omega$ *and* $\{n \mid \mathbf{A}_n \in LO\}$ *are* Π^0_2*-complete.*

(ii) *The set* $M = \{n \mid \mathbf{A}_n$ *is finite*$\}$ *is* Σ^0_3*-complete.*

(iii) *The set* $C = \{n \mid \mathbf{A}_n \simeq \omega\}$ *is* Π^0_4*-complete.*

(iv) *The set* $\{n \mid \mathbf{A}_n \in WO\}$ *is* Π^1_1*-complete.*

Proof. (i) follows from Corollary 4.2 because our index sets are nontrivial and (by the Tarski–Kuratowski algorithm) belong to Π^0_2. Note that the sequence $\{M_k\}$ is increasing (i.e. $M_0 \subseteq M_1 \subseteq \cdots$) and Π^0_2. For the next assertion, let us show that this sequence is complete, i.e. for any increasing Π^0_2-sequence $\{S_k\}$ there is a recursive function g such that $S_k = g^{-1}(M_k)$ for all k. Choose a r.e. sequence $\{C_{x,k}\}$ of r.e. sets such that $x \in S_k$ iff $C_{x,k}$ is infinite, and an effective stepwise enumeration $\{C^s_{x,k}\}$ of this sequence. Define a canonically enumerable sequence $\{D^s_{x,k}\}$ of finite sets by induction on s as follows.

Let $D^0_{x,k} = \{k\}$. If $C^s_{x,k} = C^{s+1}_{x,k}$ for all k, let $D^{s+1}_{x,k} = D^s_{x,k}$. Otherwise choose the least k satisfying $C^s_{x,k} \neq C^{s+1}_{x,k}$ and define $D^{s+1}_{x,l} = D^s_{x,l}$ for $l<k$, $D^{s+1}_{x,k} = D^s_{x,k} \cup D^s_{x,k+1}$ and $D^{s+1}_{x,k+l+1} = D^s_{x,k+l+2}$.

Note that $D^s_{x,k} \neq \emptyset$, $\bigcup_k D^s_{x,k} = \omega$, and $u<v$ for $u \in D^s_{x,k}, v \in D^s_{x,k+1}$. This implies that the relation $P^x_0(u,v) \leftrightarrow \exists k, s(u, v \in B^s_{x,k})$ is a congruence relation on the lattice ω. The sequence $\mathbf{B}_x = (\omega; max, min, 0, P^x_0)$ is r.e., so $\mathbf{B}_x \simeq_r \mathbf{A}_{g(x)}$ for some recursive function g. For $x \in S_0$ we have $C^{s+1}_{x,0} \neq C^s_{x,0}$ for infinitely many s, so by construction $\mathbf{A}_{g(x)} \simeq 1$ and $g(x) \in M_0$. For $x \in S_{k+1} \setminus S_k$ there is s_0 satisfying $\forall l \leqslant k\ \forall s \geqslant s_0$ $(C^{s+1}_{x,l} = C^s_{x,l})$, and we have $C^{s+1}_{x,k+1} \neq C^s_{x,k+1}$ for infinitely many s, so by construction $\mathbf{A}_{g(x)} \simeq k+2$ and $g(x) \in M_{k+1} \setminus M_k$. Finally, for $x \notin \bigcup_k S_k$ all sets $C_{x,k}$ $(k<\omega)$ are finite, so by construction $\mathbf{A}_{g(x)} \simeq \omega$ and $g(x) \notin \bigcup_k M_k$. This states the desired property of g.

(ii) By the Tarski–Kuratowski algorithm we again easily get $M \in \Sigma^0_3$, so it remains to check that M is Σ^0_3-hard. Let $S \in \Sigma^0_3$, then S is clearly representable in the form $S = \bigcup_k S_k$ for an increasing Π^0_2-sequence $\{S_k\}$. For the function g from the proof of (i) we have $\mathbf{A}_{g(x)} \simeq \omega$ for $x \notin S$ and $\exists k(\mathbf{A}_{g(x)} \simeq k+1)$ for $x \in S$. So g reduces S to M.

(iii) The estimation $C \in \Pi^0_4$ is again easy, so we have only to reduce any given set $T \in \Pi^0_4$ to C. Let $\bar{T} = \bigcup_t S_t$, where $\{S_t\}$ is a suitable increasing Π^0_3-sequence. By the proof of (ii) there is a recursive function $g(x,t)$ such that $\mathbf{A}_{g(x,t)} \simeq \omega$ for $x \in S_t$ and

$\exists k < \omega(\mathbf{A}_{g(x,t)} \simeq k + 1)$ for $x \notin S_t$. Let $\mathbf{B}_x$ be the union of the disjoint copies of $\mathbf{A}_{g(x,t)}$ $(t < \omega)$, in which $u < v$ for $u \in \mathbf{A}_{g(x,t)}$ and $v \in \mathbf{A}_{g(x,t+1)}$ ($\mathbf{B}_x$ could be denoted as $\mathbf{A}_{g(x,0)} + \mathbf{A}_{g(x,1)} + \cdots$). The sequence $\{\mathbf{B}_x\}$ is r.e., so $\mathbf{B}_x \simeq_r \mathbf{A}_{f(x)}$ for a recursive function f. Now, for $x \notin T$ we have $\mathbf{A}_{g(x,t)} \simeq \omega$ for almost all t, so $\mathbf{A}_{f(x)}$ is well ordered with the corresponding ordinal $\omega + \omega + \cdots = \omega^2$. For $x \in T$ all $\mathbf{A}_{g(x,t)}$ are finite nonzero ordinals, so $\mathbf{A}_{f(x)} \simeq \omega$. This shows that f reduces T to C.

(iv) follows from the corresponding well-known fact for the recursive linear orderings. This completes the proof of the theorem. □

Theorem 5.1 may be applied to the classification of some index sets in the structure $(\mathbf{R}_m; \leqslant)$ of r.e. m-degrees (for other examples see [14]). The structure $\mathbf{R}_m$ has a natural numeration $v_n = deg(W_n)$. For simplicity, we think that all recursive sets form a single m-degree. We classify index sets of some principal ideals $\hat{\mathbf{a}} = \{\mathbf{x} \mid \mathbf{x} \leqslant \mathbf{a}\}$ of $\mathbf{R}_m$.

Let $\mathscr{L}_0$ be the category of the $\emptyset''$-r.e. distributive lattices with 0 (so $\mathscr{L}_0$ is the relativization of the category considered above to the oracle $\emptyset''$), and let $\mathscr{L}_0^1$ be the category of the $\emptyset''$-r.e. distributive lattices with 0 and 1. For an object $\mathbf{B}$ of $\mathscr{L}_0$, let $\mathbf{B}^*$ be obtained from $\mathbf{B}$ by joining the greatest element (one could write $\mathbf{B}^* = \mathbf{B} + \mathbf{E}$). Note that from a given index of $\mathbf{B}$ in the acceptable numeration of $\mathscr{L}_0$ one can effectively compute an index of $\mathbf{B}^*$ in the acceptable numeration of $\mathscr{L}_0^1$. In [13] it was noted that from the characterization of the principal ideals of $\mathbf{R}_m$ in [4] it follows that from a given (index of) object $\mathbf{A}$ of $\mathscr{L}_0^1$ one can effectively compute a degree $\mathbf{a} \in \mathbf{R}_m$ such that $\hat{\mathbf{a}}$ and $\mathbf{A}$ are-isomorphic as upper semilattices with 0 and 1.

5.2. Corollary. (i) *The sets* $\{n \mid \hat{v}_n \in LO\}$ *and* $\{n \mid \exists l \leqslant k(\hat{v}_n \simeq l + 2)\}$ *are* Π_4^0*-complete.*

(ii) *The set* $\{n \mid \hat{v}_n$ *is finite*$\}$ *is* Σ_5^0*-complete.*

(iii) *The set* $C = \{n \mid \hat{v}_n \simeq \omega + 1\}$ *is* Π_6^0*-complete.*

(iv) *The set* $\{n \mid \hat{v}_n \in WO\}$ *is* Π_1^1*-complete.*

Proof. All results are proved in the same way, so consider only (iii) as a typical example. The estimation $C \in \Pi_6^0$ is again straightforward (Π_6^0 in place of Π_4^0 in Theorem 5.1 is explained by the fact that the structure $\mathbf{R}_m$ is r.e. in $\emptyset''$ in place of r.e. structures in Theorem 5.1). Now let any set $T \in \Pi_6^0$ be given. Relativizing the proof of Theorem 5.1(iii) to $\emptyset''$, we get a $\emptyset''$-r.e. sequence $\{\mathbf{B}_n\}$ of $\mathscr{L}_0$-objects such that $\mathbf{B}_n \simeq \omega$ for $n \in T$ and $\mathbf{B}_n \simeq \omega^2$ for $n \notin T$. Let $\mathbf{B}_n^* = \mathbf{B}_n + \mathbf{E}$, then $\{\mathbf{B}_n^*\}$ is a $\emptyset''$-r.e. sequence of $\mathscr{L}_0^1$-objects. So for a recursive function g we have $\hat{v}_{g(n)} \simeq \mathbf{B}_n^*$ (isomorphism in the language of semilattices). Then $\hat{v}_{g(n)} \simeq \omega + 1$ for $n \in T$ and $\hat{v}_{g(n)} \simeq \omega^2 + 1$ for $n \notin T$. So g reduces T to C completing the proof. □

Theorem 5.1 gives only few examples of algorithmic problems which could be considered within the presented framework. We conclude the paper with a discussion of some more general problems and of what kind of results one could expect along these lines.

For the acceptable numeration $\{\mathbf{A}_n\}$ of r.e. Q-models one could try e.g. to classify index sets $\{n \mid \mathbf{A}_n \models \theta\}$, for any given L-sentence θ. Any such index set is arithmetical, so one could try to use the arithmetical hierarchy as a scale for this classification. It turns out that it is indeed insufficient: one should find a suitable refinement of the arithmetical hierarchy. In some cases the sequence $\{\Sigma_n^{-1,\emptyset^k}\}_{n<\omega}$, formed by the classes of the difference hierarchies relativized to the jumps $\emptyset^k$ $(k<\omega)$, is sufficient; by the *long difference hierarchy* we mean the scale formed by these classes and by their duals. Let us call a set S of m-degrees *almost well ordered* if $(S; \leqslant_m)$ is well-founded, S is closed under the operation $\mathbf{a} \mapsto \check{\mathbf{a}} = \{\bar{A} \mid A \in \mathbf{a}\}$, and for all $\mathbf{a}, \mathbf{b} \in S$ either $\mathbf{a} \leqslant_m \mathbf{b}$ or $\check{\mathbf{b}} \leqslant_m \mathbf{a}$. For example, the class of m-degrees of sets m-complete in some level of the long difference hierarchy is almost well ordered with the corresponding ordinal ω^2. The next result is an example of a positive decision of the raised problem.

5.3. Theorem (Selivanov [14]). *For the acceptable numeration $\{\mathbf{A}_n\}$ of Boolean algebras any set $\{n \mid \mathbf{A}_n \models \theta\}$ is m-complete in one of levels of the long difference hierarchy, and the level is effectively computable from the sentence θ. The structure of m-degrees of such index sets is almost well ordered with the corresponding ordinal ω^2.*

One cannot expect to prove analogs of Theorem 5.3 for broad enough classes of structures. For example, the analog of Theorem 5.3 for the numeration $\{\mathbf{A}_n\}$ of r.e. distributive lattices is false (otherwise one could decide the set of sentences θ for which $\{n \mid \mathbf{A}_n \models \theta\}$ is m-complete in $\Pi_0^{-1} = \{\omega\}$, i.e. decide the elementary theory of r.e. distributive lattices, which is impossible). We see that the discussed problem is closely related to the decidability of elementary theories. The proof of Theorem 5.3 heavily uses the Tarski elementary classification of Boolean algebras. It seems plausible that one can hope to get analogs of Theorem 5.3 for the classes of structures with the effective elementary classification, e.g. for the relatively complemented distributive lattices with 0 or for the abelian groups. Another natural question of this type is to get analogs of Theorem 5.3 for "complex" classes of structures (e.g. distributive lattices) but for "simple enough" classes of sentences.

Interesting algorithmic problems may be raised on index sets in the numerations of the universal r.e. structures $\mathbf{U}$ from Section 3. Relate to any L-formula $\theta(v_0, \ldots, v_k)$ with free variables among $v_0, \ldots, v_k$ the index set

$$I_\theta = \{\langle x_0, \ldots, x_k\rangle \mid \mathbf{U} \models \theta(\nu x_0, \ldots, \nu x_k)\},$$

where ν is the numeration of $\mathbf{U}$; we call such I_θ *definable index sets.* Let $\mathbf{S} = (S; \leqslant_m)$ be the structure of m-degrees of definable index sets for the universal r.e. Boolean algebra $\mathbf{U}$. Let $\mathbf{S}_1$ be the substructure of $\mathbf{S}$ formed by the unary definable index sets (i.e. by the sets I_θ for $\theta = \theta(v_0)$).

5.4. Theorem (Selivanov [14]). *The m-degrees of unary definable index sets coincide with the m-degrees from Theorem 5.3, so the structure $\mathbf{S}_1$ is almost well ordered with the corresponding ordinal ω^2.*

For the structure **S** the problem is much harder and requires first to find a suitable refinement of the long difference hierarchy. Such a refinement was found in [11] and has the form $\{\Sigma_\alpha\}_{\alpha<\varepsilon_0}$, where $\varepsilon_0 = sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$. The proof of the next result is quite similar to the proof of Theorem 2 in [14], so we do not repeat it here.

5.5. Theorem. *Any definable index set in the universal r.e. Boolean algebra is m-complete in one of levels Σ_α, Π_α, $\Delta_{\lambda+1}$ ($\alpha, \lambda < \varepsilon_0, \lambda$ is limit), and all the possibilities are realized. The structure* **S** *is almost well ordered with the corresponding ordinal ε_0.*

Our last result is an application of Theorems 5.4 and 5.5 to a natural question about the universal Boolean algebra **U**. By a *coding* on **U** we mean any bijection from $\mathbf{U} \times \mathbf{U}$ onto **U**. By results of Section 3, there is a coding on **U** which is even a $\mathscr{R}$-isomorphism. Is there a definable coding on **U** (i.e. a coding with the definable graph)?

5.6. Corollary. *There is no definable coding on* **U**.

Proof. Suppose the contrary: $(a_0, a_1) \mapsto [a_0, a_1]$ is such a coding. Let $\gamma(v_0, v_1, v)$ be a formula defining the graph $[a_0, a_1] = a$ in **U**, then γ is a Σ_n^0-formula for some $n < \omega$, $n > 0$. Define $[a_0, \dots, a_k]$ by induction on k as follows: $[a_0] = a_0$, $[a_0, \dots, a_{k+1}] = [[a_0, \dots, a_k], a_{k+1}]$. For any k the function $(a_0, \dots, a_k) \mapsto [a_0, \dots, a_k]$ is a bijection between $\mathbf{U}^{k+1}$ and **U**, and it is definable by a Σ_n^0-formula $\gamma_k(v_0, \dots, v_k, v)$ (e.g. for $k = 2$ we have

$$[a_0, a_1, a_2] = a \leftrightarrow \exists b([a_0, a_1] = b \wedge [b, a_2] = a),$$

so we can take $\gamma_2(v_0, v_1, v_2, v) = \exists u(\gamma(v_0, v_1, u) \wedge \gamma(u, v_2, v)))$.

Relate to any formula $\theta(v_0, \dots, v_k)$ the unary formula

$$\theta^*(v) = \exists v_0 \dots \exists v_k(\gamma_k(v_0, \dots, v_k, v) \wedge \theta(v_0, \dots, v_k)).$$

Then $\mathbf{U} \models \theta(a_0, \dots, a_k)$ iff $\mathbf{U} \models \theta^*([a_0, \dots, a_k])$ for all $a_0, \dots, a_k \in \mathbf{U}$.

Now let $h = \emptyset^{n+1}$, $\leqslant_m^h$ be the relativization of $\leqslant_m$ to h (i.e. $A \leqslant_m^h B$ if $A = g^{-1}(B)$ for some function g recursive in h), and let $\equiv_m^h$ be the corresponding equivalence relation. It is easy to see that $I_\theta \equiv_m^h I_{\theta^*}$ for any $\theta(v_0, \dots, v_k)$, so the factorizations $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{S}}_1$ of **S** and $\mathbf{S}_1$ modulo $\equiv_m^h$ are the same. But from Theorem 5.4 it follows that $\tilde{\mathbf{S}}_1$ is well ordered with some ordinal $\leqslant \omega^2$ (indeed from the proof in [14] it follows that with the ordinal ω^2), and from Theorem 5.5 and the definition of the hierarchy $\{\Sigma_\alpha\}$ in [14] it follows that $\tilde{\mathbf{S}}$ is almost well ordered with the ordinal ε_0. This contradiction completes the proof. □

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