Discrete Mathematics 109 (1992) 289–296
North-Holland

Some conditions for 1-transitivity

Mark E. Watkins

Mathematics Department, Syracuse University, Syracuse, NY 13244-1150, USA

Received August 1990
Revised 6 March 1991

Dedicated to Gert Sabidussi on his 60th birthday.

Abstract

Watkins, M.E., Some conditions for 1-transitivity, Discrete Mathematics 109 (1992) 289–296.

Every 1-transitive finite or infinite graph is clearly both vertex-transitive and edge-transitive. The converse being false, this paper considers various sufficient conditions for a vertex-transitive, edge-transitive graph to be 1-transitive. Following a survey of known results, a new sufficient condition is presented for infinite, 3-connected, planar graphs, and a conjecture is proposed for infinite planar graphs of connectivity 2. Finally, a new infinite family of infinite graphs is constructed that are vertex-transitive and edge-transitive but not 1-transitive.

1. Notation and history

In this paper, Γ will denote an undirected simple graph; Γ may be finite or infinite. The symbols $V(\Gamma)$, $E(\Gamma)$, $G(\Gamma)$, $\kappa(\Gamma)$, $\rho(\Gamma)$, and $\varepsilon(\Gamma)$ will denote, respectively, the vertex set, the edge set, the group of automorphisms, the connectivity, the valence (when constant on $V(\Gamma)$), and the number of ends of Γ. The graphical argument Γ will be suppressed when there is no risk of ambiguity.

We say that Γ is 1-*transitive* if given $\{x_i, y_i\} \in E$ $(i = 1, 2)$, there exists $\alpha \in G$ such that $\alpha(x_1) = x_2$ and $\alpha(y_1) = y_2$. Clearly every connected 1-transitive graph is both vertex-transitive and edge-transitive. Here is what I know about the converse.

In [15, p. 59, item 7.53], Tutte proved that every finite, vertex-transitive, edge-transitive graph Γ of odd valence is 1-transitive. He inquired whether his result still held when $\rho(\Gamma)$ is even. Bouwer responded negatively [2] by constructing an infinite family of $(2n+2)$-valent counterexamples having $6 \cdot 9^n$ vertices for each $n \geq 1$. The smallest of these clearly has 54 vertices. A vertex- and edge-transitive graph that is not 1-transitive on only 27 vertices is due to Holt [7]. Holt's graph is 4-valent and has girth 5, whereas Bouwer's graphs are all bipartite, having girth 6.

Finite, planar, vertex- and edge-transitive graphs have been classified (see [3]), and all of them are 1-transitive.

We now turn our attention to infinite graphs, in particular to *locally finite* graphs, that is, graphs wherein all vertices have finite valence. Letting $d(k)$ denote the number of vertices at distance k from some fixed vertex, we say that the growth of Γ is *subexponential* if for every real number $a > 1$ one has

$$\liminf_{k\to\infty} (d(k)/a^k) = 0.$$

It was shown in [14] that Tutte's theorem is extendable to infinite, locally finite connected graphs provided that their growth is subexponential. The authors then provided a variety of counterexamples to the converse. These are vertex-transitive, edge-transitive graphs that have even valence and/or exponential growth and, of course, fail to be 1-transitive. With one exception noted below in Section 2, none of these counterexamples happens to be planar.

In the case of an infinite, locally finite graph Γ, the cardinality $\varepsilon(\Gamma)$ of its set of ends has the following simple characterization: it is the supremum of the number of infinite components of $\Gamma - S$ as S ranges over all finite subsets of $V(\Gamma)$. By combining [6, Corollary, 15] and [10, Theorem 1], one has that $\varepsilon(\Gamma) = 1$, 2 or $2^{\aleph_0}$ whenever Γ is infinite, locally finite, connected, and vertex-transitive. When $\varepsilon(\Gamma) = 1$ the growth of Γ may be either exponential or subexponential. When $\varepsilon(\Gamma) = 2$, its growth is subexponential; in fact, it is linear (see [11, Lemma 5.4]). When $\varepsilon(\Gamma) = 2^{\aleph_0}$, the growth of Γ is exponential. All three possibilities for $\varepsilon(\Gamma)$ are represented among the counterexamples in [14].

When Γ is connected, by $\kappa_\infty(\Gamma)$ we mean the cardinality of a smallest subset $T \subseteq V(\Gamma)$ such that $\Gamma - T$ has at least two infinite components. Thus $\kappa_\infty(\Gamma) < \infty$ if and only if $\varepsilon(\Gamma) > 1$. Clearly $\kappa_\infty(\Gamma) \geq \kappa(\Gamma)$ for any connected graph Γ.

In [16] it is shown that if Γ is edge-transitive and $\varepsilon(\Gamma) = 2$, then all vertices have even valence. The planar, 2-ended, edge-transitive graphs are then completely characterized, and they are all 1-transitive.

Let us further restrict our attention to those infinite, locally finite graphs that are planar and both edge- and vertex-transitive. Grünbaum and Shephard proved that if such a graph is 3-connected and 1-ended, then it must also be 1-transitive [5, Theorem 2]. From this we infer that the hypothesis of subexponential growth provides no clue about 1-transitivity in the case of planar graphs. In Section 2 the following will be proved.

Theorem. *Let Γ be infinite, locally finite, planar, vertex-transitive, and edge-transitive such that some planar embedding has a finite face. If $\rho(\Gamma)$ is odd and $\kappa(\Gamma) \geq 3$, then Γ is 1-transitive and all faces contain the same finite number of edges in their boundaries.*

We will show that the theorem fails when $\kappa = 1$ and conjecture that it is vacuous when $\kappa_\infty = 2$.

In the third and final section a new family of infinite, vertex-transitive, edge-transitive, non-1-transitive graphs will be constructed. Among them is a graph of valence 9, the least odd valence known for a biconnected graph satisfying the given transitivity conditions.

2. Infinite planar graphs

Proof of the theorem. We assume that Γ is infinite, locally finite, planar, vertex-transitive, edge-transitive, and 3-connected. Suppose, moreover, that $\rho(\Gamma)$ is odd.

Let $\{x_0, y_0\}$ be a fixed edge of Γ, and let us suppose that Γ is not 1-transitive. It is not hard to show, in the light of the other hypotheses, that for each edge $\{x, y\} \in E$, exactly one of the following holds:

(1) there exists $\alpha \in G$ such that $\alpha(x_0) = x$ and $\alpha(y_0) = y$, or

(2) there exists $\alpha \in G$ such that $\alpha(x_0) = y$ and $\alpha(y_0) = x$.

Thus, corresponding to Γ there exists a directed graph $\hat{\Gamma}$ such that $V(\hat{\Gamma}) = V(\Gamma)$ and

$$E(\hat{\Gamma}) = \{(x, y) \in V \times V\colon \alpha(x_0) = x \text{ and } \alpha(y_0) = y \text{ for some } \alpha \in G(\Gamma)\}.$$

Clearly $G(\Gamma) = G(\hat{\Gamma})$.

Since $\kappa(\Gamma) \geq 3$, the embedding of Γ in the plane is essentially unique. Not only may one speak of the 'regions' of Γ and of its planar dual, but for each vertex $x \in V$, there corresponds a unique cyclic ordering of the set x^* of edges incident with x. (See [8] or [13] for an extension to infinite graphs of Whitney's classical result [17] concerning the uniqueness of planar embeddings of finite 3-connected graphs.) Let us assume that for some given vertex x, the edges in x^* have been indexed in such a way that as one proceeds in counterclockwise fashion around x, one encounters the edges in x^* in cyclic order: $e_1, e_2, \ldots, e_r$, where $r = \rho(\Gamma)$. The stabilizer G_x of x in $G(\Gamma)$ induces a group H_x of permutations on the set x^* that is a subgroup of the full dihedral group on the cyclic list of symbols $e_1, e_2, \ldots, e_r$.

Clearly H_x must respect the orientation of $\hat{\Gamma}$. Thus, let $e_i = \{x, y_i\}$ $(i = 1, \ldots, r)$, $F_x^+ = \{e_i\colon (x, y_i) \in E(\hat{\Gamma})\}$, and $F_x^- = \{e_i\colon (y_i, x) \in E(\hat{\Gamma})\}$. Each of the sets F_x^+ and F_x^- is an edge-orbit of H_x. If z is any other vertex, one analogously defines F_z^+ and F_z^-, Any automorphism of Γ mapping x onto z also maps F_x^+ onto F_z^+ and F_x^- onto F_z^-. It follows that neither F_x^+ nor F_x^- is empty.

Without loss of generality, we suppose $e_r \in F_x^-$ and $e_1 \in F_x^+$. If $e_2 \in F_x^+$, then any element of H_x mapping e_1 to e_2 must map e_r onto e_3. We conclude that no three consecutive edges in the cyclic ordering $e_1, e_2, \ldots, e_r$ may belong to the same set F_x^+ (respectively, F_x^-). It follows that the only possible distributions of the elements of x^* into F_x^+ and F_x^- are those with the following four cyclic representations:

$$\cdots + - + - \cdots$$

$$\cdots + + - - + + - - \cdots$$

$$\cdots + + - + + - \cdots$$

$$\cdots + - - + - - \cdots$$

Since $\rho(x)$ is odd, we must eliminate the first two possibilities. We may assume, then, that it is the third possibility that holds, for were it the fourth one, we could have reversed x_0 and y_0 at the outset. Moreover, the same pattern of pairs of 'outgoing' edges flanked on each side by a single 'incoming' edge occurs at every vertex of $\hat{\Gamma}$. For definiteness, let us agree that $e_1, e_2 \in F_x^+$.

At this point we invoke the hypothesis that some face is finite. By the *covalence* of a face we mean the number of edges (or vertices) in its boundary. Since some (and hence every) edge is incident with a face of some finite covalence s, at least half of the faces incident with x have covalence s. But if some face were to have covalence a different cardinal t, then by the same argument, at least half of the faces incident with x would be t-covalent. Since $\rho(x) = s + t$ is odd, this is impossible. Hence all faces have the same finite covalence.

Let Φ denote that unique face incident with both e_1 and e_2. The boundary of Φ is a circuit whose vertices may be listed in cyclic order; $z_1(=x)$, $z_2(=y_1)$, $z_3, \ldots, z_s(=y_2)$. Since $\{z_1, z_2\} \in F_{z_1}^+$, we have $\{z_1, z_2\} \in F_{z_2}^-$. Hence $\{z_2, z_3\} \in F_{z_2}^+$, since no vertex is incident with two consecutive incoming edges. One may then argue inductively that $\{z_{i-1}, z_i\} \in F_{z_{i-1}}^+ \cap F_{z_i}^-$ for all i (where subscripts of the symbol z are read modulo s). However, $e_2 = \{z_s, z_1\} \in F_{z_s}^-$, providing a contradiction. It follows that Γ is 1-transitive. □

Remark. The hypothesis in the theorem that a finite face exists may well be superfluous, as I know of no 3-connected, planar, vertex- or edge-transitive graph *all* of whose faces are infinite. There do exist, however, such vertex-transitive graphs with $\kappa = 2$, namely the cartesian product of K_2 with the k-valent tree for $k = 3$ and $k = 4$. There also exist planar, vertex-transitive graphs with $\kappa = 3$ and $\kappa = 4$ that have countably many infinite faces and countably many finite faces in their planar embeddings. These latter examples will be published elsewhere.

Example. We cite a particular example from a family of graphs mentioned in [14]. Let $\Gamma_{2,n}$ be the connected graph of which every maximal biconnected subgraph (*lobe*) is an isomorph of $K_{2,n}$ for some $n \geq 3$ and such that each vertex belongs to exactly two lobes, being on the '2-side' of one lobe and on the 'n-side' of one lobe. Thus $\rho(\Gamma_{2,n}) = n + 2$. When n is odd, the graph $\Gamma_{2,n}$ satisfies all of the hypotheses of the theorem except that $\kappa(\Gamma_{2,n}) = 1$. To see that $\Gamma_{2,n}$ is not 1-transitive, refer to the labeling in Fig. 1 (showing the case for $n = 3$). Suppose

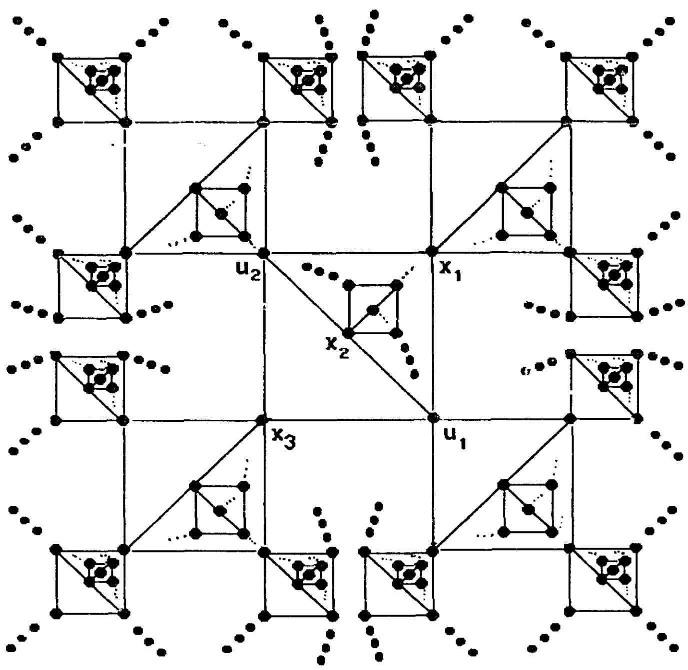

Fig. 1. A planar, vertex-, edge-, non-1-transitive graph with $\kappa = 1$.

$\alpha \in G(\Gamma_{2,n})$ and $\alpha(u_1) = x_1$ and $\alpha(x_1) = u_1$. Hence α maps onto itself the lobe that contains the edge $\{u_1, x_1\}$. However, while u_1 has n neighbors in this lobe, $\alpha(u_1)$ has only two neighbors in this lobe. Hence $\Gamma_{2,n}$ is not 1-transitive for $n \geqslant 3$.

It is interesting to note that one can apply a theorem of Sabidussi to show that $\Gamma_{2,n}$ is not a Cayley graph for $n \geqslant 3$. Otherwise $G(\Gamma_{2,n})$ would contain a subgroup G_0 that acts regularly on $V(\Gamma_{2,n})$. (See [12, Lemma 4].) For $i = 1, 2$, let ϕ_i be the unique element of G_0 such that $\phi_i(x_3) = x_i$. It is easy to see that ϕ_i must fix the lobe that contains both x_i and x_3. Since ϕ_i has no fixed points, $\phi_i(u_1) = u_2$ and $\phi_i(u_2) = u_1$. Since $\phi_2\phi_1^{-1}(u_1) = u_1$ and $\phi_2\phi_1^{-1} \in G_0$, we must have $\phi_2\phi_1^{-1} = \iota$. But $\phi_1 \neq \phi_2$, giving a contradiction. This argument is generalizable to $\Gamma_{m,n}$ whenever $m \neq n$.

The question remains open as to whether a graph is still 1-transitive if it satisfies all of the hypotheses of the theorem except that $\kappa_\infty = 2$. I believe that the question is, in fact, vacuous and offer the following.

Conjecture. Let Γ be infinite, locally finite, planar, and edge-transitive. If $\kappa_\infty(\Gamma) = 2$, then Γ is bipartite and every vertex has even valence. If, moreover, Γ is vertex-transitive, then it is 1-transitive.

As mentioned above, the conjecture holds for 2-ended graphs. (See [16, Fig. 2] for a 2-ended, 4-valent graph satisfying the conditions of the conjecture.) An

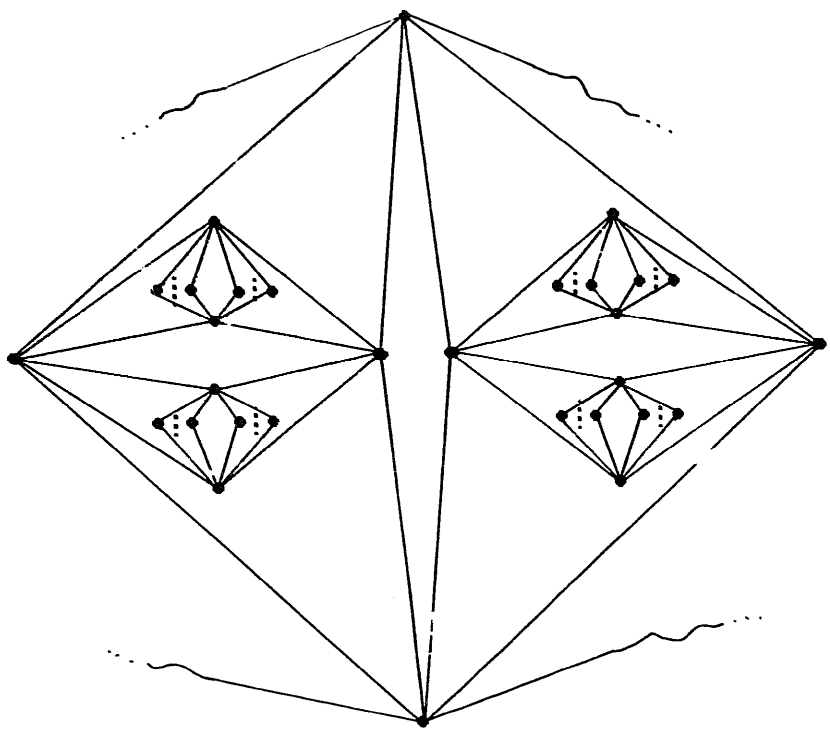

Fig. 2. A planar, 1-transitive graph with $\kappa_\infty = 2$.

infinitely-ended, 6-valent graph satisfying all of the conditions mentioned in the conjecture is shown here in Fig. 2.

3. Some more infinite non-1-transitive graphs

In this section a method will be presented for constructing a new family of infinite, vertex-transitive, edge-transitive, non-1-transitive graphs. The basic building blocks are the graphs whose existence was proved by Folkman [4], who for brevity used the term 'admissible graph'.

Let us call a finite, connected graph *admissible* if it is edge-transitive and ρ-valent for some ρ but not vertex-transitive. An admissible graph Λ is therefore bipartite, admitting a bipartition $\{X, Y\}$ of $V(\Lambda)$ such that $|X| = |Y|$. The sets X and Y are the two orbits of $G(\Lambda)$.

Folkman gave many sufficient conditions and some necessary conditions for an admissible graph Λ to exist. One necessary condition is that $|V(\Lambda)|$ be an even number not less than 20. One sufficient condition is that $|V(\Lambda)| \geqslant 20$ and that 4 divide $|V(\Lambda)|$. In fact, the two smallest admissible graphs are 4-valent with 20 and 24 vertices, respectively. Folkman inquired whether there exists an admissible graph of prime valence. In response, Bouwer [1] constructed a 3-valent admissible graph on 54 vertices. (For an update on the existence of admissible graphs with various parameters, see Ivanov [9].)

We proceed with our construction. Since there are infinitely many admissible graphs, we will have constructed an infinite family, too. Let $\{X, Y\}$ be the bipartition of an admissible graph Λ. Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_m\}$. (Thus $m \geqslant 10$.) Let T be the directed 3-valent tree such that every vertex has in-valence 1 and out-valence 2. Our undirected graph Γ will satisfy

$$V(\Gamma) = V(T) \times \{1, \ldots, m\}.$$

The 2-set $\{(s, i), (t, j)\}$ will be an edge of Γ if and only if (s, t) is an arc of T and $\{x_i, y_j\} \in E(\Lambda)$. Thus $E(\Gamma)$ admits a partition into countably many cells, each of which induces a copy of Λ. It is immediate that Γ is both vertex- and edge-transitive. Clearly the m-sets $\{(t, i): i = 1, \ldots, m\}$ are blocks of imprimitivity of $G(\Gamma)$. If an automorphism α of Γ were to interchange the endvertices (s, i) and (t, j) of an edge of Γ, then the restriction of α to $\{s, t\} \times \{1, \ldots, m\}$ would induce an automorphism of Λ that would interchange the sets X and Y. Hence Γ is not 1-transitive.

We comment that the graphs Γ constructed here satisfy

$$\varepsilon(\Gamma) = 2^{\aleph_0}, \qquad \kappa_\infty(\Gamma) = m, \quad \text{and} \quad \kappa(\Gamma) = \min\{3\rho(\Lambda), m\}.$$

If the admissible graph Λ is the 3-valent Bouwer graph mentioned above, then the graph Γ constructed from it is 9-valent. This is the lowest *odd* valence I know for a biconnected, vertex-transitive, edge-transitive, non-1-transitive graph. (The graph shown in Fig. 1 satisfies $\rho = 5$, but $\kappa = 1$.)

References

[1] I.Z. Bouwer, An edge- but not vertex-transitive cubic graph, Canad. Math. Bull. 11 (1968) 533–535.
[2] I.Z. Bouwer, Vertex and edge transtive but not 1-transitive graphs, Canad. Math. Bull. 13 (1970) 231–237.
[3] H. Fleischner and W. Imrich, Transitive planar graphs, Math. Slovaca 29 (1979) 97–106.
[4] J. Folkman, Regular line-symmetric graphs, J. Combin. Theory 3 (1967) 215–232.
[5] B. Grünbaum and G.C. Shephard, Edge-transitive planar graphs, J. Graph Theory 11 (1987) 141–155.
[6] R. Halin, Automorphisms and endomorphisms of infinite locally finite graphs, Abh. Math. Sem. Univ. Hamburg 39 (1973) 251–283.
[7] D.F. Holt, A graph which is edge-transitive but not arc transitive, J. Graph Theory 5 (1981) 201–204.
[8] W. Imrich, On Whitney's Theorem on the unique embeddability of 3-connected planar graphs, in: M. Fiedler, ed., Recent Advances in Graph Theory, Proceedings of the Symposium held in Prague, June 1974 (Academia Praha, Prague, 1975) 303–306.
[9] A.V. Ivanov, On edge but not vertex transitive regular graphs, Ann. Discrete Math., Vol. 34 (North-Holland, Amsterdam, 1987) 273–286.
[10] H.A. Jung, A note on fragments of infinite graphs, Combinatorica 1 (1981) 285–288.

[11] H.A. Jung and M.E. Watkins, Fragments and automorphisms of infinite graphs, European J. Combin. 5 (1984) 149–162.
[12] G. Sabidussi, On a class of fixed-point-free graphs, Proc. Amer. Math. Soc. 9 (1958) 800–804.
[13] C. Thomassen, Infinite graphs, in: L.W. Beineke and R.J. Wilson, eds., Selected Topics in Graph Theory 2 (Academic Press, London, 1983) 129–160.
[14] C. Thomassen and M.E. Watkins, Infinite vertex-transitive, edge-transitive, non-1-transitive graphs, Proc. Amer. Math. Soc. 105 (1989) 258–262.
[15] W.T. Tutte, Connectivity in Graphs (Univ. of Toronto Press, Toronto, Canada, 1966).
[16] M.E. Watkins, Edge-transitive strips, Discrete Math. 95 (1991) 359–372.
[17] H. Whitney, Non-separable and planar graphs, Trans. Amer. Math. Soc. 34 (1932) 339–362.