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Optimization problems for general simple population with n-impulsive harvest

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Abstract

In this paper, n-impulsive harvest problems of general simple population are discussed by models with Dirac function. The optimal impulsive harvest policies to protect the renewable resource better are obtained under conditions of fixed quantity per impulsive harvest. Then, a concept of the sequence for $\mathcal{F}$-optimal harvest moments for general simple population is presented which is beneficial to protect resource better and sustainable development. Finally, we apply the conclusions to some special models.

Keywords: General simple population; Impulsive differential equation; n-Impulsive harvest; Sustainable development; Sequence for $\mathcal{F}$-optimal harvest moments

1. Introduction

The optimal management of renewable resources, which has a direct relationship to sustainable development, has been studied extensively by many authors [3,7–9]. As we know, however, most of them studied the optimal problems with management objective of the maximum sustainable yield. In the real world, we sometimes need to keep the population more and more

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in particular time under a fixed harvest. For example, we exploit a bare hill or other open resources or contract some government-owned property in a period. In the exploitation of population resources, both the economic benefits and the environment effects should be considered. Sometimes, concluding some contracts, we cannot return them unimproved in terminal time of the period but keep the resource's quantity more and more or a fixed value at last. Namely, we should not only exploit the resource but also protect it.

On the other hand, impulsive differential systems are suitable mathematical models to simulate the evolution of large classes of real processes. Specially, in many situations the impulsive harvest is more convenient to be operated than the continuous harvest. Considering this fact, some authors have studied the problems of impulsive harvest. In [10], Nenov studied the optimization problems with general n-impulsive control. Besides, Zhang et al. [11] and Bai et al. [4] also considered the optimization problem with impulsive periodic solutions, respectively.

In 1990, Bainov and Dishliev considered impulsive harvest and resource's reasonable exploitation simultaneously for Logistic model [5]. The moments of impulsive harvest and the magnitudes of every impulse are determined so that the biomass of population at terminal time of exploitation is maximum provided the fixed quantity of total impulsive harvest. In 2000, Angelova and Dishliev studied impulsive harvest for general population model for similar optimal aim, however, they focused on one-impulsive harvest [2]. In 1998, Angelova and Dishliev considered the optimal problem for general n-impulsive population model [1]. In their paper, they assumed the quantity of total harvest is fixed and determine the optimal number of impulsive harvest and quantities of every harvest so that the biomass in the moment T will be maximum.

In this paper, we study optimization problems for general population model, namely, to keep much more population's quantity in terminal time of exploitation period for protecting resource provided fixed number of impulsive harvest and quantities of every impulsive harvest. Moreover, we consider the maximum number of impulsive harvest with same quantity of impulsive harvest during the exploitation period.

Consider a general simple population, X, satisfies the following system

$$\begin{cases} \dfrac{dx}{dt} = f(x), \\ x(0) = x_0. \end{cases} \tag{1.1}$$

Here, assume that $f(x)$ satisfies some conditions as follow:

(H1) For some fixed $K > 0$, $f : [0, K] \to R$ is continuous and locally Lipschitzian with respect to x, namely, f can affirm that there exists a unique solution $x(t, t_0, x_0)$ of (1.1);
(H2) $f(0) = f(K) = 0$, $K > 0$;
(H3) if $0 < x < K$, then $f(x) > 0$;
(H4) there exists $0 < p < K$ such that $f'(p) = 0$; and if $0 < x < p$, then $f'(x) > 0$, if $p < x < K$, then $f'(x) < 0$.

It is clear that Logistic equation keeps all above properties of $f(x)$ and the population of (1.1) ought to have the same properties as Logistic equation. In fact, in biological point of view, K just expresses the maximum capacity of environment for the population X, and p stands for the population level with maximum growth rate. Therefore, we can list some conclusions of system (1.1) similar to Logistic models.

Lemma 1.1. *Assume* (H1)–(H3) *hold, then the unique positive equilibrium of* (1.1) *is globally asymptotically stable.*

For the proof, we refer to [6]. The following lemma firstly appeared in [1, Lemma 1].

Lemma 1.2. *If* (H2)–(H4) *hold and* $0 < h < K$*, then there exists unique* $a^* \in (\max\{h, p\}, K)$ *such that* $f(a^*) = f(a^* - h)$*.*

Lemma 1.3. *If* (H1) *is valid and* $0 < x_1 < x_2 < K$*, then population* X *should spend time* $\Delta t = \int_{x_1}^{x_2} \frac{1}{f(x)}\,dx$ *to change the biomass from* x_1 *to* x_2*.*

Proof. By system (1.1), we have $\frac{dx}{f(x)} = dt$. Integrate both sides of the equality from t_1 to t_2 simultaneously, where t_1, t_2 obey $x(t_1) = x_1$, $x(t_2) = x_2$, respectively. We can reach $\int_{t_1}^{t_2} \frac{dx(t)}{f(x(t))} = t_2 - t_1$. Thus, $\Delta t := t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{f(x)}$. □

Similarly to [11], we establish our mathematical model using Dirac function for the general simple population with n-impulsive harvest of per harvest's quantity being constant E on time interval $[0, T]$,

$$\begin{cases} \dfrac{dN(t)}{dt} = f(N) - \delta\big(s(t, \tau_1, \ldots, \tau_n)\big)E, \\ N(0) = N_0. \end{cases} \tag{1.2}$$

Here, $N(t)$ is the density of the population X at time t and denote the initial population level with N_0; the quantity of population harvested at any time τ_i $(i = 1, 2, \ldots)$ is always E, a positive fixed constant; δ is the Dirac impulsive function, which satisfies $\delta(0) = \infty$, $\delta(s) = 0$ for $s \neq 0$ and $\int_{-\infty}^{\infty} \delta(s)\,ds = 1$; s is defined as follows:

$$s(t, \tau_1, \ldots, \tau_n) = \begin{cases} 0, & t = \tau_i,\ i = 1, 2, \ldots, n, \\ 1, & t \in [0, T],\ t \neq \tau_i,\ i = 1, 2, \ldots, n. \end{cases} \tag{1.3}$$

Without loss of generality, suppose that $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant \tau_n \leqslant T$. Denote the solution of system (1.2) with $N(t)[\tau_1, \tau_2, \ldots, \tau_n]$. It is obvious that the solution $N(t)[\tau_1, \tau_2, \ldots, \tau_n]$ depends on $\tau_1, \tau_2, \ldots, \tau_n$, and is right continuous for t. In this paper, our optimization problem is to find the impulse moments $\tau_1, \tau_2, \ldots, \tau_n$ so that $N(T)[\tau_1, \tau_2, \ldots, \tau_n]$ achieves maximum value under the fixed quantity of every impulsive harvest during the fixed exploitation period. In a word, how should we take to protect the resource under a fixed harvest? At last, it is need to point out that we always assume $f(x)$ satisfies assumptions (H1)–(H4) later in this paper.

2. Optimization problem for long enough managing period *T*

In this section, we only discuss the case of large enough time interval $[0, T]$. The other case will be studied in the next section. As we know, if T is large enough, it means that people own long enough time to manage the resource and adopt some plausible approaches for the protection of resource. Unfortunately, we could not give the clean interpretation of large enough T now and people also do not know whether T is large enough or not. Luckily, we will give the quantitative interpretation later and now let us lie down under the case temporarily.

First, we study the case of $N_0 < K$, then, by Lemma 1.1, the fact $N(T)[\tau_1, \tau_2, \ldots, \tau_n] < K$ for any $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant \tau_n \leqslant T$ holds. Denote that $a_1 = N(\tau_1)[\tau_1, \tau_2, \ldots, \tau_n] + E$, $a_2 =$

$N(\tau_2)[\tau_1, \tau_2, \dots, \tau_n] + E, \ \dots, \ a_n = N(\tau_n)[\tau_1, \tau_2, \dots, \tau_n] + E$, respectively. Thus, we declare the following proposition is valid.

Proposition 2.1. *If $K > N_0$, then the following two optimization problems for system* (1.2) *are equivalent*:

Problem 1 *Select appropriate impulsive moments $\tau_1, \tau_2, \dots, \tau_n$ and harvest constant quantity E per moment for population X such that the population level in terminal time keeps maximum.*

Problem 2 *Look for the applicable vector $(a_1^*, a_2^*, \dots, a_n^*)$ such that*

$$\mathcal{T}\left(a_1^*, a_2^*, \dots, a_n^*\right) \leqslant \mathcal{T}(a_1, a_2, \dots, a_n),$$

where

$$\mathcal{T}(a_1, a_2, \dots, a_n) = \int_{N_0}^{a_1} \frac{1}{f(x)}\,dx + \int_{a_1 - E}^{a_2} \frac{1}{f(x)}\,dx + \dots + \int_{a_{n-1} - E}^{a_n} \frac{1}{f(x)}\,dx + \int_{a_n - E}^{b} \frac{1}{f(x)}\,dx, \tag{2.1}$$

$K - \epsilon < b < K$, $0 < \epsilon \ll 1$, and $(a_1, a_2, \dots, a_n)$ satisfies that any integral term of (2.1) *is greater than or equal to* 0 *and $a_i > E$, for $i = 1, 2, \dots, n$.*

Proof. If vector $(a_1, a_2, \dots, a_n)$, satisfying $a_i > 0$, $i = 1, 2, \dots, n$, and any integral term is not less than 0, sets $\mathcal{T}$ get to its minimum value, then let

$$\tau_1 = \int_{N_0}^{a_1} \frac{1}{f(x)}\,dx, \quad \tau_2 = \int_{a_1 - E}^{a_2} \frac{1}{f(x)}\,dx, \quad \dots, \quad \tau_n = \int_{a_{n-1} - E}^{a_n} \frac{1}{f(x)}\,dx.$$

Namely, we obtain a policy of impulsive harvest for population X, $(\tau_1, \tau_2, \dots, \tau_n)$, and denote the level of population X at time T with N_T under this harvest policy. Then, we select any other policy of impulsive harvest, $(t_1, t_2, \dots, t_n)$, and set $a_{11} = N(t_1)[t_1, t_2, \dots, t_n] + E$, $a_{22} = N(t_2)[t_1, t_2, \dots, t_n] + E, \ \dots, \ a_{nn} = N(t_n)[t_1, t_2, \dots, t_n] + E$, respectively. Besides, denote the level of population X at time T with N_{TT} for the harvest policy $(t_1, t_2, \dots, t_n)$. Since $(a_1, a_2, \dots, a_n)$ assure $\mathcal{T}$ minimal, then we assert

$$\tau_1 + \tau_2 + \dots + \tau_n + \int_{a_n - E}^{b} \frac{1}{f(x)}\,dx \leqslant t_1 + t_2 + \dots + t_n + \int_{a_{nn} - E}^{b} \frac{1}{f(x)}\,dx.$$

On the other hand,

$$T = \tau_1 + \tau_2 + \dots + \tau_n + \int_{a_n - E}^{N_T} \frac{1}{f(x)}\,dx,$$

$$T = t_1 + t_2 + \dots + t_n + \int_{a_{nn} - E}^{N_{TT}} \frac{1}{f(x)}\,dx.$$

Therefore,

$$\int_b^{N_T} \frac{1}{f(x)}\,dx \geqslant \int_b^{N_{TT}} \frac{1}{f(x)}\,dx.$$

Further, $N_T \geqslant N_{TT}$. Note that all processes above are reversible, which completes our proof. □

Theorem 2.1. *Let the following conditions are valid*:

(a) $0 < N_0 < a^*$, *where* a^* *is determined by Lemma* 1.2, *and* h *is replaced by* E, $0 < E < K$.
(b) $T \geqslant \tau_1^* + (n-1)\tau^{\Delta}$, *where* $\tau_1^* = \int_{N_0}^{a^*} \frac{1}{f(x)}\,dx$, $\tau^{\Delta} = \tau_i^* - \tau_{i-1}^* = \int_{a^*-E}^{a^*} \frac{1}{f(x)}\,dx$, $i = 2, 3, \ldots, n$.

Then for every choice of impulsive moments $0 \leqslant \tau_1 \leqslant \cdots \leqslant \tau_n \leqslant T$, *the following inequality is fulfilled*:

$$N(T)\bigl[\tau_1^*, \tau_2^*, \ldots, \tau_n^*\bigr] \geqslant N(T)[\tau_1, \tau_2, \ldots, \tau_n].$$

Proof.

$$\frac{\partial \mathcal{T}}{\partial a_i} = \frac{1}{f(a_i)} - \frac{1}{f(a_i - E)} = \frac{f(a_i - E) - f(a_i)}{f(a_i) f(a_i - E)}, \quad i = 1, 2, \ldots, n.$$

Let $\frac{\partial \mathcal{T}}{\partial a_i} = 0$, from Lemma 1.2, $a_i = a^*$. By the ecological meaning of problem, $(a^*, a^*, \ldots, a^*)$ sets $\mathcal{T}$ minimal, thus, at the same time, we may obtain the optimal impulsive harvest moments. □

Theorem 2.1 firstly appeared in [1, Corollary 3]. In the following, we will study the case that the conditions of Theorem 2.1 are invalid and the case $K \leqslant N_0$. For the discussion for $N_0 \geqslant a^*$, we begin with $n = 1$, namely, there is only one harvest at time interval $[0, T]$.

Lemma 2.1. *Suppose* $N_0 \geqslant a^*$, $0 < E < K$, *and* $n = 1$. *Then for any* $0 \leqslant t \leqslant T$, *the following inequality is fulfilled*:

$$N(T)[0] \geqslant N(T)[t].$$

Proof. Since $n = 1$, $\mathcal{T} = \int_{N_0}^{a_1} \frac{1}{f(x)}\,dx + \int_{a_1-E}^{b} \frac{1}{f(x)}\,dx$. It is clear

$$\frac{d\mathcal{T}}{da_1} = 0 \quad \Longrightarrow \quad a_1 = a^*,$$

however,

$$\int_{N_0}^{a^*} \frac{1}{f(x)}\,dx \leqslant 0.$$

Besides, if $a_1 > a^*$, then

$$\frac{d\mathcal{T}}{da_1} = \frac{f(a_1 - E) - f(a_1)}{f(a_1) f(a_1 - E)} > 0.$$

Thus, from $a_1 \geqslant N_0$, we may conclude $a_1^* = N_0$, which means we complete our proof. □

Theorem 2.2. *Let the following conditions are valid*:

(a) $N_0 \geqslant a^*$, $0 < E < K$.
(b) $T \geqslant \tau_{i+1}^{**} + (n-i-1)\tau_\Delta$, *where positive integer* i *satisfies* $N_0 - iE < a^* \leqslant N_0 - (i-1)E$, *and*

$$\tau_{i+1}^{**} = \int_{N_0-iE}^{a^*} \frac{1}{f(x)}\,dx, \quad \tau_\Delta = \tau_j^{**} - \tau_{j-1}^{**} = \int_{a^*-E}^{a^*} \frac{1}{f(x)}\,dx, \quad j = i+2, i+3, \ldots, n.$$

Then for any choice $(a_1, a_2, \ldots, a_n)$ *satisfying conditions in Proposition* 2.1*, the following inequality is fulfilled*:

$$\mathcal{T}\big(N_0, N_0 - E, \ldots, N_0 - (i-1)E, a^*, \ldots, a^*\big) \leqslant \mathcal{T}(a_1, a_2, \ldots, a_n).$$

In other words, for any choice $(\tau_1, \tau_2, \ldots, \tau_n)$*, the following inequality is fulfilled*:

$$N(T)\big[\overbrace{0, 0, \ldots, 0}^{i}, \tau_{i+1}^{**}, \tau_{i+2}^{**}, \ldots, \tau_n^{**}\big] \geqslant N(T)[\tau_1, \tau_2, \ldots, \tau_n].$$

Proof. For any given impulsive harvest moments $(t_1, t_2, \ldots, t_n)$, we apply Theorem 2.1 and Lemma 2.1 again and again, then

$$\begin{aligned}
&N(T)\big[0, 0, \ldots, 0, \tau_{i+1}^{**}, \tau_{i+2}^{**}, \ldots, \tau_n^{**}\big] \\
&\quad\geqslant N(T)[0, 0, \ldots, 0, t_{i+1}, t_{i+2}, \ldots, t_n] \\
&\quad\geqslant N(T)[0, 0, \ldots, 0, t_i, t_{i+1}, t_{i+2}, \ldots, t_n] \\
&\quad\geqslant N(T)[0, 0, \ldots, 0, t_{i-1}, t_i, t_{i+1}, t_{i+2}, \ldots, t_n] \\
&\qquad\vdots \\
&\quad\geqslant N(T)[0, t_2, \ldots, t_i, t_{i+1}, t_{i+2}, \ldots, t_n] \\
&\quad\geqslant N(T)[t_1, t_2, \ldots, t_i, t_{i+1}, t_{i+2}, \ldots, t_n].
\end{aligned}$$

Thus we complete the proof of Theorem 2.2. □

Remark. Now, we give our quantitative interpretation of large enough T. From Theorem 2.1, it is easy to say that $T \geqslant \tau_n^* = \tau_1^* + (n-1)\tau_\Delta$ means T is large enough as all conditions of Theorem 2.1 are valid; on the other side, for all cases in Theorem 2.2, large enough T means $T \geqslant \tau_n^{**} = \tau_{i+1}^{**} + (n-i-1)\tau_\Delta$.

3. Optimization problem without limit of long enough *T*

It is clear that our discussions in the last section are not entire and not in reality, we have to meet with the development with a short period. Thus, for the fixed harvest, it is inevitable that the level of population developed will be brought down, even the population will extinguish. Unfortunately, sometime, even so, we still could not obtain our expectative harvest, then how to do? This is just our question in this section.

To answer this question, we first study another optimization problem: at the time interval $[0, T]$, how many impulsive harvests with constant E can system (1.1) afford, namely, what is the maximal positive integer which times we can harvest with constant E per impulsive harvest such that $N(T) \geqslant 0$? If there exists the maximal integer m for the above question, then we call m the maximal impulsive harvest times with E. We start this section with the lemma as follows.

Lemma 3.1. *Suppose* $n = 1$, $0 < N_0 < a^*$, $T < \int_{N_0}^{a^*} \frac{1}{f(x)}\,dx$, *then* $N(T)[T] \geqslant N(T)[\hat{t}\,]$ *for* $\forall \hat{t} \in [0, T)$.

Proof. For any given $0 < \hat{t} \leqslant T$, denote $c = N(\hat{t}, 0, N_0)$, $b = N(T)[\hat{t}\,]$, from which

$$T = \int_{N_0}^{c} \frac{1}{f(x)}\,dx + \int_{c-E}^{b} \frac{1}{f(x)}\,dx.$$

We derivate with c both sides of the above expression then yield to

$$\frac{1}{f(c)} + \frac{1}{f(b)}\frac{db}{dc} - \frac{1}{f(c-E)} = 0,$$

namely,

$$\frac{db}{dc} = \left(\frac{1}{f(c-E)} - \frac{1}{f(c)}\right) f(b).$$

Since $0 < N_0 < a^*$, $T < \int_{N_0}^{a^*} \frac{1}{f(x)}\,dx$, $N_0 < c \leqslant N(T, 0, N_0) < a^*$, therefore, $\frac{1}{f(c-E)} - \frac{1}{f(c)} > 0$. Besides, from assumption of $f(x)$ and Lemma 1.1, we can obtain $f(b) > 0$, thus, $\frac{db}{dc} > 0$, which completes this proof. □

Set $\tau_\Delta = \int_{a^*-E}^{a^*} \frac{1}{f(x)}\,dx$, $\hat{\tau} = \int_{N_0-uE}^{a^*} \frac{1}{f(x)}\,dx$, where

$$u = \min_{i \geqslant 0}\bigl\{i \in Z\colon\ N_0 - iE < a^*\bigr\}. \tag{3.1}$$

Obviously, $a^* - E \leqslant N_0 - uE < a^*$, $\hat{\tau} \leqslant \tau_\Delta$. Denote

$$v = \begin{cases} 0, & T < \hat{\tau}, \\ 1 + \min_{i \geqslant 0}\{i \in Z\colon\ T - \hat{\tau} - i\tau_\Delta < \tau_\Delta\}, & T \geqslant \hat{\tau}. \end{cases} \tag{3.2}$$

Therefore, $0 \leqslant T < \hat{\tau}$ or $0 \leqslant T - \hat{\tau} - (v-1)\tau_\Delta < \tau_\Delta$. If $T \geqslant \hat{\tau}$, define $h(x) = \int_{a^*-E}^{x} \frac{1}{f(s)}\,ds$. It is easy to know that $h(x)$ is strictly monotone increasing function at the interval $(a^* - E, a^*)$, then there exists a function h^{-1}, which is the inverse function of $h(x)$ in $(a^* - E, a^*)$. In the case of $T < \hat{\tau}$, define $\bar{h}(x) = \int_{N_0-uE}^{x} \frac{1}{f(s)}\,ds$. Similarly, denote $\bar{h}^{-1}$ with the inverse function of $\bar{h}(x)$ in $(N_0 - uE, a^*)$. Thus, we can denote

$$N_{T^-} = \begin{cases} \lim_{t \to T_*^-} h^{-1}(t), & T \geqslant \hat{\tau}, \text{ where } T_* = T - \hat{\tau} - (v-1)\tau_\Delta, \\ \lim_{t \to T^-} \bar{h}^{-1}(t), & T < \hat{\tau}, \end{cases}$$

and

$$w = \max_{i \geqslant 0}\{i \in Z\colon\ N_{T^-} - iE \geqslant 0\}. \tag{3.3}$$

Obviously, $0 \leqslant N_{T^-} - wE < E$. Then we come to the following conclusion.

Theorem 3.1. *The maximal number of impulsive harvest times with E for system* (1.2) *in* $[0, T]$ *is* $M := u + v + w$, *where* u, v, w *depend, respectively, on* (3.1), (3.2), (3.3).

Proof. For the proof of this theorem, we only show for given any $m > M$ times impulsive harvest policy (moments) $(t_1, t_2, \dots, t_m)$, $N(T)[t_1, t_2, \dots, t_m] < 0$ holds, and there exists an M times impulsive harvest policy $(\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_M)$ such that $N(T)[\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_M] \geqslant 0$. For the latter, we only need to set

$$(\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_M) = \big(\overbrace{0, 0, \dots, 0}^{u}, \hat{\tau}, \hat{\tau} + \tau_\Delta, \dots, \hat{\tau} + (v-1)\tau_\Delta, \overbrace{T, T, \dots, T}^{w}\big).$$

Obviously, it just is our request. On the other hand, for given any $(t_1, t_2, \dots, t_m)$, $m > M$, we assert that there exists $t^* \in [0, T]$ such that $N(t^*)[t_1, t_2, \dots, t_m] < 0$, which means our conclusion is correct; otherwise, by Theorems 2.1, 2.2 and Lemma 3.1, we can conclude

$$\begin{aligned}
0 &> N(T)\big[\overbrace{0, 0, \dots, 0}^{u}, \hat{\tau}, \hat{\tau} + \tau_\Delta, \dots, \hat{\tau} + (v-1)\tau_\Delta, \overbrace{T, T, \dots, T}^{w}, \overbrace{T, T, \dots, T}^{m-M}\big] \\
&\geqslant N(T)[t_1, t_2, \dots, t_{u+v}, \overbrace{T, T, \dots, T}^{m-M+w}] \\
&\geqslant N(T)[t_1, t_2, \dots, t_{u+v}, t_{u+v+1}, \overbrace{T, T, \dots, T}^{m-M+w-1}] \\
&\ \ \vdots \\
&\geqslant N(T)[t_1, t_2, \dots, t_{u+v}, t_{u+v+1}, \dots, t_{m-1}, T] \\
&\geqslant N(T)[t_1, t_2, \dots, t_m],
\end{aligned}$$

which is a contradiction. So the maximal impulsive harvest times with E for system (1.2) is M in $[0, T]$. □

From above, we obtain the maximal impulsive harvest times in fixed time interval as the initial population level and harvest's quantity per impulsive harvest are given. Thereinafter, we return our main optimization problem, namely, study the impulsive harvest policy for resource's protecting better in particular period.

Theorem 3.2. *Assumed that there exists M-times impulsive harvest with constant per harvest at the interval* $[0, T]$ *for system* (1.2), *there exists impulsive harvest moments*

$$\big(t_1^*, t_2^*, \dots, t_M^*\big) = \big(\overbrace{0, 0, \dots, 0}^{u}, \hat{\tau}, \hat{\tau} + \tau_\Delta, \dots, \hat{\tau} + (v-1)\tau_\Delta, \overbrace{T, T, \dots, T}^{w}\big)$$

such that the population level at time T reaches its maximum, that is to say, for given any other impulsive policy $(t_1, t_2, \dots, t_M)$ the followed relation

$$N(T)\big[t_1^*, t_2^*, \dots, t_M^*\big] \geqslant N(T)[t_1, t_2, \dots, t_M]$$

holds, furthermore, the impulsive harvest moments are unique.

Proof. By Lemma 3.1, we can now easily get

$$\begin{aligned}
&N(T)\big[\overbrace{0,0,\ldots,0}^{u},\hat{\tau},\hat{\tau}+\tau_\Delta,\ldots,\hat{\tau}+(v-1)\tau_\Delta,\overbrace{T,T,\ldots,T}^{w}\big]\\
&\quad\geqslant N(T)[t_1,t_2,\ldots,t_{u+v},\overbrace{T,T,\ldots,T}^{M-u-v}]\\
&\quad\geqslant N(T)[t_1,t_2,\ldots,t_{u+v},t_{u+v+1},\overbrace{T,T,\ldots,T}^{M-u-v-1}]\\
&\quad\quad\vdots\\
&\quad\geqslant N(T)[t_1,t_2,\ldots,t_{u+v},t_{u+v+1},\ldots,t_{M-1},T]\\
&\quad\geqslant N(T)[t_1,t_2,\ldots,t_M]. \qquad (3.4)
\end{aligned}$$

That is to say, the impulsive harvest policy given above, $(t_1^*,t_2^*,\ldots,t_M^*)$, just is the optimal exploitation policy for the protection of resource under M times harvest with impulse. We further discuss the equal condition for (3.4), then we can yield the uniqueness of the optimal policy. □

Theorem 3.3. *Assume there exists* $0<n<M$ *times impulsive harvest with constant E per harvest for system* (1.2). *Then, the optimal impulsive harvest policy for resource's protection is given by* $(t_1^*,t_2^*,\ldots,t_n^*)$, *namely, harvesting the resource n times with constant quantity, E, at every moment, which is the one of the first n moments by Theorem* 3.2. *Furthermore, the optimal policy is unique.*

Proof. From $0<n\leqslant u+v$, yield to $T\geqslant\tau_{u+1}^{**}+(n-u-1)\tau_\Delta$, by Theorem 2.2, the conclusion thus hold obviously. If $u+v+1<n<M$, then $t_{u+v+1}^*=\cdots=t_n^*=T$. Further, for given any policy $(t_1,t_2,\ldots,t_n)$, we have

$$\begin{aligned}
&N(T)\big[t_1^*,t_2^*,\ldots,t_{n-1}^*,t_n^*\big]\\
&\quad\geqslant N(T)[t_1,t_2,\ldots,t_{u+v},\overbrace{T,\ldots,T}^{n-u-v}]\\
&\quad\geqslant N(T)[t_1,t_2,\ldots,t_{u+v},t_{u+v+1},\overbrace{T,\ldots,T}^{n-u-v-1}]\\
&\quad\quad\vdots\\
&\quad\geqslant N(T)[t_1,t_2,\ldots,t_{u+v},t_{u+v+1},\ldots,t_{n-1},T]\\
&\quad\geqslant N(T)[t_1,t_2,\ldots,t_{u+v+1},t_{u+v+2},\ldots,t_n].
\end{aligned}$$

Therefore, we complete the proof of Theorem 3.3. □

So far, in fact, we have given our answer for the optimization problem completely. In the following, we will give a unitary description for Theorems 3.2 and 3.3. Assume $N_0>0$, $0<E<K$, $T>0$. Further we ignore the case of no optimal policies, that is, only discuss the case that there exists at least one kind of harvest policy with impulse such that the population level of X at terminal time T is greater than 0. For convenience, we call the latter case by general condition for population X.

Define a sequence of time

$$\bar{t}_s = \begin{cases} 0, & 0 < s \leqslant u, \\ \hat{\tau} + (s - u - 1)\tau_\Delta, & s > u. \end{cases}$$

Note that $\tau_\Delta > 0$, $\hat{\tau} > 0$, then $\lim_{s\to\infty} \bar{t}_s = \infty$. Thus, for $\forall T > 0$, there exists s_1, $1 + s_1$ such that $\bar{t}_{s_1} \leqslant T < \bar{t}_{1+s_1}$. Obviously, $1 + s_1 > u$. From above Theorems 3.2 and 3.3, we obtain easily under-mentioned theorem.

Theorem 3.4. *Assume system* (1.2) *satisfies the general condition for population X. If $s_1 \geqslant n$, then the optimal harvest policy with n times impulse for resource's protection just is* $(\bar{t}_1, \bar{t}_2, \dots, \bar{t}_n)$; *if $s_1 < n$, then the corresponding optimal policy is given by*

$$(\bar{t}_1, \bar{t}_2, \dots, \bar{t}_{s_1}, \overbrace{T, T, \dots, T}^{n-s_1}).$$

Here, set $u = 0$, $s_1 \geqslant n$, then we can conclude Theorem 2.1; set $u > 0$, $s_1 \geqslant n$, we thus obtain Theorem 2.2; if $s_1 < n$, then Theorem 3.4 gives the corresponding optimal impulsive harvest policy as T is not large enough. Thus far, we complete our all discussion of optimal optimization problems with n times impulse.

4. Sustainable development and applications

As we know, biological resources are renewable resources. How to exploit biological resources without heavy exploitation is relevant to not only developer's economic benefits but also sustainable development. So it is significant to study the problem for both biology and economic highly appreciated by some administrations and many authors. At last in this paper, we give our viewpoint about it.

First, the impulsive harvest in the finite interval is considered.

Definition 4.1. Assume $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant \tau_n \leqslant T_0 < \infty$. If there exists once impulsive harvest at every τ_i by quantity E, then the harvest policy is called n-impulsive harvest with E in $[0, T_0]$, denoted $[\tau_1, \tau_2, \dots, \tau_n]$. The solution of system (1.2) is denoted $N(t)[\tau_1, \tau_2, \dots, \tau_n]$.

Then we consider the sequence of impulsive harvest moments in infinite interval $\{\tau_n\}_{n=1}^{\infty}$, where $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant \tau_n \leqslant \cdots$, and $\tau_n \to +\infty$, as $n \to +\infty$.

Definition 4.2. If the sequence of impulsive harvest moments in infinite interval $\{\tau_n\}_{n=1}^{\infty}$ satisfies:

(1) the quantity of every impulsive harvest at τ_k is E;
(2) if $m = \max\{n \in N\colon\ \tau_n \leqslant T\}$, where T is given any constant, $N(T)[s_1, s_2, \dots, s_m] \leqslant N(T)[\tau_1, \tau_2, \dots, \tau_m]$. Here, $[s_1, s_2, \dots, s_m]$ is an arbitrary m-impulsive harvest with E in $[0, T]$;

then the sequence of impulsive harvest moments $\{\tau_n\}_{n=1}^{\infty}$ is called the sequence of $\mathcal{F}$-optimal harvest moments.

By our above series of conclusions, it is easy to conclude the following theorem:

Theorem 4.1. *Assume $E < K$, then there exists a unique sequence of $\mathcal{F}$-optimal harvest moments for system* (1.2).

Further, we can obtain their explicit expressions. If $E < K$, then the sequence of $\mathcal{F}$-optimal harvest moments $\{\tau_n\}_{n=1}^{\infty}$ is formed

$$\tau_1 = \tau_2 = \cdots = \tau_u = 0,$$
$$\tau_{u+1} = \hat{\tau},$$
$$\tau_{u+n} = \hat{\tau} + (n-1)\tau_\Delta, \quad n \geqslant 2,$$

where $u = \min_{i \geqslant 0}\{i \in Z\colon N_0 - iE < a^*\}$, $\tau_\Delta = \int_{a^*-E}^{a^*} \frac{1}{f(x)}\,dx$, $\hat{\tau} = \int_{N_0-uE}^{a^*} \frac{1}{f(x)}\,dx$.

From the above conclusions, we can find that the sequence of $\mathcal{F}$-optimal harvest moments is an essential criterion of developed resource, which only depends on the internal properties of population and extrinsic environment. Once these factors are fixed, then the sequence also is certain. Besides, the sequence of $\mathcal{F}$-optimal harvest moments is the expansion of the sequence in Theorem 3.4. Therefore, we can easily yield the optimal impulsive harvest policy at any finite time interval for resource's protection better by the sequence of $\mathcal{F}$-optimal harvest moments. In the following, we will give some practical applications.

Example 1 *(Logistic model).* Set $f(x) = rx(1 - \frac{x}{K})$ for system (1.1), and it is obvious $f(x)$ satisfies (H1)–(H4). By Lemma 1.2, $a^* = \frac{K+E}{2}$ is the unique positive solution of $f(a^*) = f(a^* - E)$, $a^* - E > 0$. Then for system (1.2) we have

$$\tau_\Delta = \int_{\frac{K-E}{2}}^{\frac{K+E}{2}} \frac{1}{rx(1-\frac{x}{K})}\,dx = \frac{2}{r}\ln\frac{K+E}{K-E},$$

$$\hat{\tau} = \int_{N_0-uE}^{\frac{K+E}{2}} \frac{1}{rx(1-\frac{x}{K})}\,dx = \frac{1}{r}\ln\left[\frac{K-(N_0-uE)}{N_0-uE}\frac{K+E}{K-E}\right],$$

where $u = \min_{i \geqslant 0}\{i \in Z\colon N_0 - iE < \frac{K+E}{2}\}$. Therefore the sequence of $\mathcal{F}$-optimal harvest moments for Logistic model, $\{\tau_n\}_{n=1}^{\infty}$, satisfies

$$\tau_1 = \tau_2 = \cdots = \tau_u = 0,$$
$$\tau_{u+1} = \hat{\tau},$$
$$\tau_{u+n} = \hat{\tau} + (n-1)\tau_\Delta, \quad n \geqslant 2.$$

Now, let $r = 0.03$, $K = 100$, $E = 20$, thus, $a^* = 60$, $\tau_\Delta = 27.0310$.

(1) As $N_0 = 15$, then $u = 0$, $\hat{\tau} = 71.3355$. So the sequence of $\mathcal{F}$-optimal harvest moments is

$$\{71.3355, 98.3665, 125.3975, 152.4285, \ldots, 71.3355 + 27.0310j, \ldots\}.$$

For given $T = 150$, we have $v = 3$, $w = 2$, that is to say, from Theorem 3.1, the maximal impulsive harvest times is 5. Without protection for resource, our maximum harvest in $[0, T]$ from the resource is $5 \times 20 = 100$, and corresponding harvest moments with impulse are 71.3355, 98.3665, 125.3975, 150, 150.

Assume that we plan to take 3 times harvest with impulse in $[0, T]$. Then the optimal harvest moments for resource's protection better are 71.3355, 98.3665, 125.3975. Assumed that there exists 4 times impulsive harvest at the time interval $[0, T]$, the corresponding optimal policy is given by 71.3355, 98.3665, 125.3975, 150. For 5 times impulsive harvest, the corresponding optimal harvest moments are 71.3355, 98.3665, 125.3975, 150, 150.

(2) If $N_0 = 90$ holds, then $u = 2$, $\hat{\tau} = 13.5155$. The sequence of $\mathcal{F}$-optimal harvest moments is given by

$$\{0, 0, 13.5155, 40.5465, 67.5775, 94.6085, 121.6395, 148.6705, 175.7015, \ldots, \\ 13.5155 + 27.0310j, \ldots\}.$$

For given $T = 150$, we yield $v = 6$, $w = 2$, and further the maximal impulsive harvest times is 10 by Theorem 3.1. Similar to discussion in (1), we easily obtain the maximum harvest is 200 without protection for resource. If there exists 8 times impulsive harvest in $[0, T]$, then the optimal harvest moments are 0, 0, 13.5155, 40.5465, 67.5775, 94.6085, 121.6395, 148.6705. Furthermore, for 10 times impulsive harvest in $[0, T]$, the optimal harvest moments are given by 0, 0, 13.5155, 40.5465, 67.5775, 94.6085, 121.6395, 148.6705, 150, 150.

Example 2 *(General Logistic model).* In system (1.1), let $f(x) = rx(1 - \frac{x}{K})\frac{1}{1+\beta x}$, $\beta > 0$, and $f(x)$ satisfies (H1)–(H4) obviously. From Lemma 1.2,

$$a^* = \frac{1}{2}\frac{\beta E - 2 + \sqrt{\beta^2 E^2 + 4 + 4\beta K}}{\beta}$$

is the unique positive solution of $f(a^*) = f(a^* - E)$, $a^* - E > 0$. Then for corresponding system (1.2), we have

$$\tau_\Delta = \int_{a^*-E}^{a^*} \frac{1}{rx\left(1 - \frac{x}{K}\right)\frac{1}{1+\beta x}}\,dx = \frac{1}{r}\left[\ln\frac{\beta E - 2 + A}{-\beta E - 2 + A} + (1 + \beta K)\ln\frac{2\beta K + \beta E + 2 - A}{2\beta K - \beta E + 2 - A}\right],$$

$$\hat{\tau} = \int_{N_0-uE}^{a^*} \frac{1}{rx\left(1 - \frac{x}{K}\right)\frac{1}{1+\beta x}}\,dx = \frac{1}{r}\left[\ln\frac{K - (N_0 - uE)}{N_0 - uE} + \ln\frac{\beta E - 2 + A}{\beta} - (\beta K + 1)\ln\frac{-2\beta K + \beta E - 2 + A}{\beta} + K\beta\ln\bigl(2(K - N_0 + uE)\bigr)\right],$$

where $u = \min_{i \geqslant 0}\{i \in Z\colon N_0 - iE < a^*\}$, $A = \sqrt{\beta^2 E^2 + 4 + 4\beta K}$. Therefore, the sequence of $\mathcal{F}$-optimal harvest moments for general Logistic model ($\beta > 0$), $\{\tau_n\}_{n=1}^{\infty}$ is formed as follows:

$$\tau_1 = \tau_2 = \cdots = \tau_u = 0,$$
$$\tau_{u+1} = \hat{\tau},$$
$$\tau_{u+n} = \hat{\tau} + (n-1)\tau_\Delta, \quad n \geqslant 2.$$

For given any parameters, we also have similar discussions to Example 1.

Example 3 *(Depensation model).* Let $f(x) = rx^2(1 - \frac{x}{K})$ for system (1.1), which satisfies (H1)–(H4). So

$$a_t^* = \frac{K}{3} + \frac{E}{2} + \frac{1}{6}\sqrt{4K^2 + 3E^2}$$

is the unique positive solution of $f(a_t^*) = f(a_t^* - E)$, $a_t^* - E > 0$. Then

$$\tau_{\Delta t} = \int_{a_t^* - E}^{a_t^*} \frac{1}{rx^2(1 - \frac{x}{K})}\, dx,$$

$$\hat{\tau}_t = \int_{N_0 - uE}^{a_t^*} \frac{1}{rx^2(1 - \frac{x}{K})}\, dx,$$

where $u = \min_{i \geqslant 0}\{i \in Z\colon N_0 - iE < a_t^*\}$. The sequence of $\mathcal{F}$-optimal harvest moments for the model, $\{\tau_n\}_{n=1}^{\infty}$, is formed as follows:

$$\tau_1 = \tau_2 = \cdots = \tau_u = 0,$$
$$\tau_{u+1} = \hat{\tau}_t,$$
$$\tau_{u+n} = \hat{\tau}_t + (n-1)\tau_{\Delta t}, \quad n \geqslant 2.$$

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