MATHEMATICS

The weak closure of the equimeasurable rearrangements of a measurable function

by Peter W. Day*

Emory University, University Computing Center, Atlanta, Georgia 30322, USA

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ABSTRACT

If (X, Λ, μ) is a finite measure space and f is in $L^1(X, \mu)$, then the $\sigma(L^1, L^\infty)$-closure of the set $\Delta(f)$ of all measurable functions equimeasurable with f is shown to be the set to which g belongs if and only if there is a function equimeasurable with f which majorizes g (in the sense of the Hardy-Littlewood-Polya preorder relation) on the non-atomic part of X and which equals g on the union of the atoms of X. If ϱ is a saturated Fatou Banach function norm and $L^\varrho(X, \mu)$ is universally rearrangement invariant such that $L^\infty \subset L^\varrho \subset L^1$, then for all f in L^ϱ the $\sigma(L^\varrho, L^{\varrho'})$-closure of $\Delta(f)$ is shown to be the same as the $\sigma(L^1, L^\infty)$-closure of $\Delta(f)$.

1. INTRODUCTION

In answer to a question of W. A. J. Luxemburg, J. V. Ryff [10; 9, p. 97, Theorem 2] has shown that the weak closure of the equimeasurable rearrangements of $F \in L^1[0, 1]$ is the same as the orbit Ω of F under the semigroup of all doubly stochastic operators on $L^1[0, 1]$. The set Ω has also been characterized [9, Theorem 3] as those functions in L^1 which are majorized by F using a preorder relation introduced by Hardy, Littlewood, and Polya in [5; 6]. This preorder relation and a related one have proved useful in investigations in rearrangement invariant Banach

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function spaces [8; 1], such as in proving interpolation theorems in these spaces [7; 11].

Actually, Luxemburg asked for the weak closure of the rearrangements of an L^1 function on an arbitrary finite measure space, especially one which has atoms. Our answer to this more general question will also involve the preorder relation mentioned above.

2. PRELIMINARIES

In the following, (X, Λ, μ) is a measure space of finite total measure $a=\mu(X)$; $M(X, \mu)$ is the set of all extended real valued measurable functions on X; 1_E denotes the characteristic function of $E \subset X$; $\int \cdot d\mu$ denotes integration over the set X; R denotes the real numbers; m denotes Lebesgue measure on R; and $\bar{R}$ denotes the extended real numbers. If f is a function and E is a set contained in the domain of f, then $f|E$ denotes the restriction of f to E. When necessary, (X_1, Λ_1, μ_1) also denotes a finite measure space with $\mu_1(X_1)=\mu(X)=a$.

Functions $f \in M(X, \mu)$ and $g \in M(X_1, \mu_1)$ are called *equimeasurable* (written $f \sim g$) if $\mu(f^{-1}[a, b])=\mu_1(g^{-1}[a, b])$ for all closed intervals $[a, b]$ of $\bar{R}$ (so $a, b=+\infty$ or $-\infty$ is allowed), in which case $\mu(f^{-1}[B])=\mu_1(g^{-1}[B])$ for all Borel sets B of $\bar{R}$ as well. For each $f \in M(X, \mu)$ there is a unique right-continuous decreasing function δ_f on $[0, a]$ such that $f \sim \delta_f$ [3, p. 28, Theorem 4.2]. The function δ_f is called the *decreasing rearrangement* of f.

Recall that a set $A \in \Lambda$ is called an *atom* of (X, Λ, μ) if $A \supset B \in \Lambda$ implies $\mu(B)=0$ or $\mu(A-B)=0$. (X, Λ, μ) is called *non-atomic* if it has no atoms. A finite (or σ-finite) measure space can have at most countably many atoms, and each member of $M(X, \mu)$ is essentially constant on each atom [3, section 5]. We will denote by A the union of the at most countably many atoms of (X, Λ, μ). The set $X_0=X-A$ is then called the *non-atomic part* of X, because $(X_0, \Lambda \cap X_0, \mu)$ is non-atomic.

A map $\sigma: X \to R$ is called *measure preserving* (*m.p.*) if $\mu(\sigma^{-1}[B])=m(B)$ for all Borel sets B. If (X, Λ, μ) is non-atomic, then for any set $E \in \Lambda$ and any interval J with $m(J)=\mu(E)$, there is a measure preserving map $\sigma: E \to J$ [4, p. 385, Lemma 3.2].

(2.1) LEMMA. *If X is non-atomic, and I is a union of at most a countable number of intervals, and $\mu(X)=m(I)<\infty$, then there is a m.p. $\sigma: X \to I$.*

PROOF. Since $I=\bigcup_{n\geq 1} I_n$, there are pairwise disjoint X_n such that $X=\bigcup_{n\geq 1} X_n$ and $\mu(X_n)=m(I_n)$. For each n there is a m.p. $\sigma_n: X_n \to I_n$. Define $\sigma=\sigma_n$ on X_n. ||

We now define two preorder relations introduced by Hardy, Littlewood and Polya in [6]. If $f \in M(X_1, \mu_1)$ and $g \in M(X, \mu)$, then $g \prec\prec f$ means $f^+ \in L^1(X_1, \mu_1)$, $g^+ \in L^1(X, \mu)$ and $\int_0^t \delta_g \leq \int_0^t \delta_f$ for all $0 \leq t \leq a$, while $g \prec f$ means $g \prec\prec f$ and $\int_0^a \delta_g=\int_0^a \delta_f$. Observe that $g \prec\prec f$ and $f \prec\prec g$ if and only if $g \prec f$ and $f \prec g$ if and only if $g \sim f$. The preorder relation $\prec\prec$ can also be characterized as follows (see [2, p. 1326, Cor. 1.8]).

(2.2) PROPOSITION. *For* $f^+ \in L^1(X_1, \mu_1)$ *and* $g^+ \in L^1(X, \mu)$ *we have* $g \prec\prec f$ *if and only if* $\int (g-t)^+ d\mu \leq \int (f-t)^+ d\mu_1$ *for all* $t \in R$. ||

The following may be proved in a straight-forward manner using (2.2).

(2.3) PROPOSITION. *Let* $f^+ \in L^1(X_1, \mu_1)$ *and* $g^+ \in L^1(X, \mu)$.
(1) *If* $g \prec f$ *and* $f|E_1 \prec g|E$ *where* $E_1 \in \Lambda_1$ *and* $E \in \Lambda$, *then*
(i) $g|X-E \prec f|X_1 - E_1$,
(ii) $g1_{X-E} + h1_E \prec f1_{X_1 - E_1} + h_1 1_{E_1}$ *whenever* $h|E \prec h_1|E_1$.
(2) *Suppose* P, $Q \in \Lambda$ *are disjoint, and* P_1, $Q_1 \in \Lambda_1$ *are disjoint. If* $g|P \prec f|P_1$ *and* $g|Q \prec f|Q_1$, *then*
(i) $g|P \cup Q \prec f|P_1 \cup Q_1$,
(ii) $g1_{P \cup Q} \prec f1_{P_1 \cup Q_1}$.
(3) *Let* $E \in \Lambda$ *and* $E_1 \in \Lambda_1$. *If* $\mu(E) = \mu_1(E_1)$, *then* $g|E \prec f|E_1$ *if and only if* $g1_E \prec f1_{E_1}$.
(4) *If* $f_n, f \in L^1(X_1, \mu_1)$ *and* $g_n, g \in L^1(X, \mu)$ *and* $f_n \to f$, $g_n \to g$ *in* L^1*-norm and* $g_n \prec f_n$ *for each* n, *then* $g \prec f$.
The above results are also true if $\prec$ *is replaced throughout by* $\prec\prec$.

PROOF. The straightforwardness of the proof may be illustrated by proving part (1.i). Hence let f and g satisfy the hypotheses of (1.i). Then

$$(g|X-E)^+ = g^+|X-E \in L^1(X, \mu),$$

and similarly for f, and

$$\begin{aligned}\int_{X-E} (g-t)^+ d\mu &= \int_X (g-t)^+ d\mu - \int_E (g-t)^+ d\mu \\ &\leq \int_{X_1} (f-t)^+ d\mu_1 - \int_{E_1} (f-t)^+ d\mu_1 \\ &= \int_{X_1 - E_1} (f-t)^+ d\mu_1,\end{aligned}$$

because $f|E_1 \prec g|E$ implies

$$\int_{E_1} (f-t)^+ d\mu_1 \leq \int_E (g-t)^+ d\mu.$$

Similarly, $\int_{X_1 - E_1} f \, d\mu_1 = \int_{X-E} g \, d\mu$. ||

(2.4) COROLLARY. *Let* $f, g \in L^1(X, \mu)$. *If* $f|A = g|A$, *then* $g \prec f$ *if and only if* $g|X_0 \prec f|X_0$. ||

3. THE WEAK CLOSURE OF $\Delta(f)$

If $f \in L^1(X, \mu)$, it is known that the set $\Omega(f) = \{g \in M(X, \mu) : g \prec f\}$ is a convex, $\sigma(L^1, L^\infty)$-compact subset of L^1 [8, p. 134, Theorem 15.3] and that the set $\Delta(f)=\{g \in M(X,\mu)\colon g \sim f\}$ is contained in the set of all extreme points of $\Omega(f)$ [3, p. 156. W. A. J. Luxemburg has asked that the weak closure of $\Delta(f)$ be determined [8, p. 142, problem 2], and J. V. Ryff has given the following answer [10]: *If* $F \in L^1(0, 1)$, *then*

$$\Omega(F)=\{G \in M(0,1)\colon G \prec F\}$$

is the weak closure of

$$\Delta(F)=\{G \in M(0,1)\colon G \sim F\}.$$

(3.1) THEOREM. *If* $f \in L^1(X,\mu)$, *then the* $\sigma(L^1, L^\infty)$-*closure of* $\Delta(f)$ *is the set* $Z(f)$ *to which* g *belongs if and only if there is an* $h \sim f$ *such that* $g|X_0 \prec h|X_0$ *and* $g|A=h|A$.

PROOF. The proof consists of showing that $\Delta(f)$ is dense in $Z(f)$, and that $Z(f)$ is closed. For the first part, let $g \in Z(f)$, so there is an $f' \sim f$ such that $g|X_0 \prec f'|X_0$ and $g|A=f'|A$. Let $h \in L^\infty(X,\mu)$. Then

$$\int_{X_0} gh\, d\mu \in \{\int_{X_0} f''h\, d\mu\colon f'' \prec f'|X_0\}=\{\int_{X_0} f''h\, d\mu\colon f'' \sim f'|X_0\}$$

[3, p. 81, Theorem 13.4, and p. 85, Theorem 13.8], so there is an $f'' \in M(X_0,\mu)$ such that $f'' \sim f'$ on X_0 and $\int_{X_0} f''h\, d\mu = \int_{X_0} gh\, d\mu$. Since $f'' \sim f'$ on X_0, if f'' is extended to A by defining $f''|A=f'|A$, then $f'' \sim f' \sim f$, and

$$\int_X f''h\, d\mu = \int_{X_0} f''h\, d\mu + \int_A f''h\, d\mu = \int_{X_0} gh\, d\mu + \int_A gh\, d\mu = \int_X gh\, d\mu.$$

Thus $\Delta(f)$ is dense in $Z(f)$.

For the rest, let $\{g_\alpha\}$ be a net in $Z(f)$ with $g_\alpha \to g$ weakly. Let $h_\alpha \sim f$ such that $g_\alpha|X_0 \prec h_\alpha|X_0$ and $g_\alpha|A=h_\alpha|A$. Let B be an atom of (X, Λ, μ). Then for each index α, h_α is constant on B, and

$$0<\mu(B)\leqslant\mu(h_\alpha^{-1}(h_\alpha|B))=\mu(f^{-1}(h_\alpha|B)).$$

Also, for any two indices α, β, $f^{-1}(h_\alpha|B) \cap f^{-1}(h_\beta|B)=\emptyset$ whenever $h_\alpha|B\neq h_\beta|B$. Since $\mu(X)$ is finite, it follows that $\{h_\alpha|B\}$ is finite. But $h_\alpha|B \to g|B$, so for some index α_0, $\alpha \geqslant \alpha_0$ implies $h_\alpha|B=g|B$. Let $A_1, A_2, A_3, \ldots$ be the at most countably many atoms of (X, Λ, μ), where if there are only finitely many atoms, say N, then $A_i=\emptyset$ for $i>N$. Then there is an increasing sequence $\{\alpha_n\}_{n\geqslant1}$ such that $\alpha \geqslant \alpha_n$ implies $h_\alpha|A_k=h_{\alpha_n}|A_k=g|A_k$, $k=1, \ldots, n$.

Now for each n, $h_{\alpha_n}|A_1 \cup \ldots \cup A_n=g|A_1 \cup \ldots \cup A_n$, and $h_{\alpha_n} \sim f \sim \delta_f$, so for each value t of g on A,

$$\mu((A_1 \cup \ldots \cup A_n) \cap g^{-1}[t])\leqslant m(\delta_f^{-1}[t]).$$

Hence $\mu(A \cap g^{-1}[t])\leqslant m(\delta_f^{-1}[t])$. Thus there are disjoint intervals $\{J_n\}_{n\geqslant1}$

with δ_f constant on each J_n such that $m(J_n)=\mu(A_n)$ and $\delta_f|J_n=g|A_n$. Define $J_0=[0,a[-\bigcup_{n\geq 1}J_n$, whence $m(J_0)=\mu(X_0)$. Now J_0 is a union of at most a countable number of intervals, so there is a m.p. $\sigma: X_0\to J_0$. Define $h|X_0=\delta_f\circ\sigma|X_0$ and $h|A_n=\delta_f|J_n$. Then $h\sim\delta_f\sim f$ and $h|A= =\delta_f|\bigcup_{n\geq 1}J_n=g|A$. But $g\prec f\sim h$ so $g\prec h$. Then $g|X_0\prec h|X_0$ by (2.4), so $g\in Z(f)$. ||

(3.2) REMARK. In view of (2.4), the weak closure of $\Delta(f)$ can also be described as those members of $\Omega(f)$ which equal a rearrangement of f on the union of the atoms.

(3.3) THEOREM. *$\Omega(f)$ is the $\sigma(L^1, L^\infty)$-closure of $\Delta(f)$ for all $f\in L^1(X,\mu)$ if and only if (X, Λ, μ) is non-atomic or X is an atom.*

PROOF. If X is non-atomic, then $Z(f)=\Omega(f)$, so the result follows from (3.1).

If X is an atom, then $\Delta(f)=\{f\}=\Omega(f)$.

Suppose then that X is not an atom, and (X, Λ, μ) is not non-atomic. Then $X=A_1\cup A_2$ where A_1 and A_2 are disjoint sets of positive measure, and X has an atom B. Let $f=(2)1_{A_1}+1_{A_2}$, let $b=(1/a)\int f\,d\mu=1+\mu(A_1)/\mu(X)$ (so $1<b<2$), and let $g=(b)1_X$ so $g\in\Omega(f)$ [3, p. 63, Theorem 10.2.v]. If $h\in\Delta(f)$, then $h=(2)1_{B_1}+1_{B_2}$ where $\mu(B_i)=\mu(A_i)$, $i=1, 2$, and B_1, B_2 are disjoint [3, p. 12, Theorem 2.2], so

$$\int (g-h)1_B\,d\mu=b\mu(B)-2\mu(B_1\cap B)-\mu(B_2\cap B).$$

Since B is an atom, and B_1 and B_2 are disjoint, exactly one of $\mu(B_1\cap B)$ and $\mu(B_2\cap B)$ equals $\mu(B)$ and the other equals zero. Hence

$$\left|\int (g-h)1_B\,d\mu\right|\geq\mu(B)\min(b-1, 2-b)>0,$$

so $\Delta(f)$ is not dense in $\Omega(f)$. ||

The sets $\Omega(f)$ and $\Delta(f)$ play an important role in the theory of rearrangement invariant Banach function spaces, where $\Omega(f)$ is known to be compact and when X is non-atomic to be the closed convex hull of $\Delta(f)$ in a certain associated weak topology. Thus it is natural to determine the closure of $\Delta(f)$ in this weak topology. The reader is referred to [8] and [3, Chapters V and VI] for definitions of concepts and basic results of this theory.

In what follows, ϱ will denote a saturated Fatou function norm on $M(X,\mu)$ such that $L^\infty\subset L^\varrho\subset L^1$ and such that L^ϱ is universally rearrangement invariant (u.r.i.). Then the same is true for $L^{\varrho'}$ and $L^{\varrho''}$ [3, p. 93, and p. 97, Theorem 16.5.i], where as usual, ϱ' and ϱ'' denote the first and second associate function norms on $M(X,\mu)$. The L^ϱ and $L^{\varrho'}$ are known to be a dual pair, where each g in $L^{\varrho'}$ corresponds to the linear functional $F_g: f\mapsto\int fg\,d\mu$ [3, p. 92, Theorem 15.4].

(3.4) PROPOSITION. *For all f in L^ϱ, the $\sigma(L^\varrho, L^{\varrho'})$-closure of any set $S \subset \Omega(f)$ is the same as the $\sigma(L^1, L^\infty)$-closure of S.*

PROOF. Since f is in L^ϱ, $\Omega(f) \subset L^\varrho$ [3, p. 97, Theorem 16.5.iii] and $\Omega(f)$ is ϱ-bounded [3, p. 106, Lemma 17.1]. Hence let $M>0$ be such that $\varrho(g) \leqslant M$ for all g in $\Omega(f)$. Let $\bar{S}=\sigma(L^1, L^\infty)$-closure of S, and ${}^\varrho\bar{S}=\sigma(L^\varrho, L^{\varrho'})$-closure of S. Then $\sigma(L^1, L^\infty)$ on $L^\varrho = \sigma(L^\varrho, L^\infty) \subset \sigma(L^\varrho, L^{\varrho'})$, so $\bar{S}$ is $\sigma(L^\varrho, L^{\varrho'})$-closed, and thus ${}^\varrho\bar{S} \subset \bar{S}$. It remains to show the reverse inclusion $\bar{S} \subset {}^\varrho\bar{S}$.

Let g_0 be in $\bar{S}$. Then there is a net $\{g_\alpha\}$ in S with $g_\alpha \to g_0$ in $\sigma(L^1, L^\infty)$. If h is in $L^{\varrho'}$ and g is in L^ϱ, let $F_h(g) = \int gh\, d\mu$. It suffices to show that $F_h(g_\alpha) \to F_h(g_0)$ for all h in $L^{\varrho'}$. Hence let h be in $L^{\varrho'}$ and let $\varepsilon>0$. If $L^{\varrho'}=L^\infty$, there is nothing to prove, so suppose $L^{\varrho'} \neq L^\infty$. Then the Banach dual of $L^{\varrho'}$ is $L^{\varrho''}$ [4, p. 390, Prop. 5.1], so $\varrho'(f_n) \downarrow 0$ whenever $f_n \downarrow 0$ pointwise everywhere [3, p. 93]. It follows that there is a simple function v such that $\varrho'(h-v)<\varepsilon$. Then for all g in $\Omega(f)$ (so in particular, for all g in $\bar{S}$),

$$|F_h(g)-F_v(g)| = |\int (h-v)g\, d\mu| \leqslant \varrho(g)\varrho'(h-v) < M\varepsilon$$

[3, p. 92, Theorem 15.4]. Now there is an α_0 such that $\alpha \geqslant \alpha_0$ implies $|F_v(g_\alpha)-F_v(g_0)|<\varepsilon$. Hence for $\alpha \geqslant \alpha_0$,

$$|F_h(g_\alpha)-F_h(g_0)| \leqslant |F_h(g_\alpha)-F_v(g_\alpha)| + |F_v(g_\alpha)-F_v(g_0)| + \\ + |F_v(g_0)-F_h(g_0)| < M\varepsilon+\varepsilon+M\varepsilon.$$

Thus $F_h(g_\alpha) \to F_h(g_0)$. ||

(3.5) THEOREM. *If ϱ is a saturated Fatou Banach function norm and L^ϱ is u.r.i., then for all f in L^ϱ the $\sigma(L^\varrho, L^{\varrho'})$-closure of $\Delta(f)$ is the same as the $\sigma(L^1, L^\infty)$-closure of $\Delta(f)$.* ||

REFERENCES

1. Boyd, David – The Hilbert transform on rearrangement invariant spaces, Canadian J. Math. **19**, 599–616 (1967).
2. Chong, K. M. – Some extensions of a theorem of Hardy, Littlewood and Polya and their applications, Canad. J. Math. **26**, 1321–1340 (1974).
3. Chong, K. M. and N. M. Rice – Equimeasurable rearrangements of functions, Queen's Papers in Pure and Applied Mathematics **28**, (1971).
4. Day, Peter W. – Decreasing rearrangements and doubly stochastic operators, Trans. AMS **178**, 383–392 (1973).
5. Hardy, G. H., J. E. Littlewood and G. Polya – Some simple inequalities satisfied by convex functions, Messenger of Mathematics **58**, 145–152 (1929).
6. Hardy, G. H., J. E. Littlewood and G. Polya – Inequalities, Cambridge University Press, Cambridge, 1952.
7. Lorentz, G. G. and T. Shimogaki – Interpolation theorems for operators in function spaces, J. Functional Analysis **2**, 31–51 (1968).
8. Luxemburg, W. A. J. – Rearrangement invariant Banach function spaces, Queen's Papers in Pure and Applied Math. **10**, 83–144 (1967).

9. Ryff, John V. – Orbits of L^1-functions under doubly stochastic transformations, Trans AMS 117, 92–100 (1965).
10. Ryff, John V. – Majorized functions and measures, Proc. Acad. Sci. Amsterdam A71 (Indag 30), 431–437 (1968).
11. Shimogaki, T. – Hardy-Littlewood majorants in function spaces, J. Math. Soc. Japan 17, 365–373 (1965).