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P_5-free augmenting graphs and the maximum stable set problem

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Abstract

The complexity status of the maximum stable set problem in the class of P_5-free graphs is unknown. In this paper, we first propose a characterization of all connected P_5-free augmenting graphs. We then use this characterization to detect families of subclasses of P_5-free graphs where the maximum stable set problem has a polynomial time solution. These families extend several previously studied classes.

Keywords: Stable set; Augmenting graphs; Polynomial algorithm

1. Introduction

A *stable* set S in a graph G is a set of pairwise non-adjacent vertices. A stable set S is *maximum* if its cardinality $|S|$ is maximum, while it is *maximal* if it is not strictly contained in another stable set of G. The maximum cardinality of a stable set in G is denoted $\alpha(G)$ and is called the *stability number* of G. The problem of finding a maximum stable set in a graph is called the *maximum stable set problem* (MSP). It is well known that the MSP is NP-hard, even when restricted, for example, to triangle-free graphs [19] or cubic planar graphs [8]. The class of P_5-free

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graphs (where a P_5 is a chordless chain on five vertices) is of special interest since it is the only minimal class defined by a single connected forbidden-induced subgraph where the complexity status of the MSP is unknown. Polynomial algorithms have been developed for several subclasses of P_5-free graphs [5,6,11,13,16]. We use in this paper the so-called *augmenting graph technique* which has proven to be a useful approach to solve the MSP in various classes of graphs [2,9,10,13,15–17,20]. Our developments are based on a characterization of all connected bipartite P_5-free graphs. This characterization allows us to detect new families of subclasses of P_5-free graphs where the MSP has a polynomial time solution. These new families extend several previously studied classes.

As usual, $K_{r,s}$ denotes a complete bipartite graph whose parts have, respectively, r and s vertices, and P_k denotes a chordless chain on k vertices. All graphs considered are undirected, without loops and multiple edges. The vertex set and the edge set of a graph G are, respectively, denoted $V(G)$ and $E(G)$. For a vertex $x \in V(G)$, we denote by $N(x)$ the neighbourhood of x, i.e., the set of vertices adjacent to x. For $A \subseteq V(G)$, we denote $G[A]$ the subgraph of G induced by the vertex set A, and $N_A(x) = N(x) \cap A$ the neighbourhood of x in $G[A]$. For two subsets A and B of vertices, we use the notation $N_A(B) = \bigcup_{b \in B} N_A(b)$ for the set of vertices in B which have a neighbour in A, and we denote $A - B$ the set of vertices which are in A but not in B. If a graph G contains a graph H as an induced subgraph, we simply say that G contains H. Many classes of graphs, studied in the literature, are defined by a set $\{H_1, \ldots, H_k\}$ of forbidden induced subgraphs. A graph in such a class is said $(H_1, \ldots, H_k)$-free (or simply H_1-free when $k = 1$).

In the next section, we describe the augmenting graph technique and give a characterization of all connected P_5-free augmenting graphs. We then use this characterization in Sections 3 and 4 to determine subclasses of P_5-free graphs where the MSP can be solved in polynomial time.

2. P_5-free augmenting graphs

A bipartite graph $H = (V_1, V_2, E)$ with parts V_1 and V_2 is called *augmenting* for a stable set S in a graph G if $|V_2| > |V_1|$, $V_1 \subseteq S$, $V_2 \subseteq V(G) - S$ and $(N(v) \cap S) \subseteq V_1$ for all v in V_2. We call V_1 the S-part and V_2 the $\bar{S}$-part of H. The *increment* of H is defined as $\Delta(H) = |V_2| - |V_1|$. An augmenting graph is said *minimal* if it does not contain an induced subgraph which is also augmenting with the same increment.

Clearly, if $H = (V_1, V_2, E)$ is an augmenting graph for a stable set S in G, then S is not of maximum cardinality since $S' = (S - V_1) \cup V_2$ is a stable set of size $|S'| > |S|$ in G. Now, assume S is not a maximum stable set, and let S' be a stable set such that $|S'| > |S|$. Then, the subgraph of G induced by set $(S - S') \cup (S' - S)$ is augmenting for S. Hence, we have the following theorem.

Theorem of augmenting graphs. *A stable set S in a graph G is maximum if and only if there are no augmenting graphs for S.*

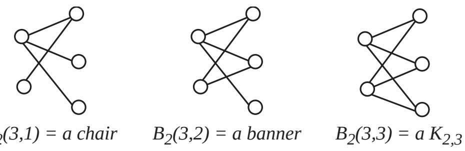

Fig. 1. The three non-isomorphic connected P_5-free augmenting graphs with 2 vertices in the S-part and 3 in the other part.

Notice that every connected $K_{1,3}$-free bipartite graph is either a chain or an even cycle. Since the increment of a even cycle is zero, it follows that every connected $K_{1,3}$-free augmenting graph is a chain. Minty [15] has designed a polynomial algorithm for detecting such augmenting chains. This has lead to his famous polynomial algorithm for the MSP in the class of $K_{1,3}$-free graphs. This technique has recently been extended to other classes of graphs [2,10,13,16,17]. We use it for the class of P_5-free graphs.

A bipartite graph H is said to be *chain bipartite* [23] if either $N(x) \subseteq N(y)$ or $N(y) \subset N(x)$ for any choice of two vertices x and y in the same part of H. It follows from this definition that chain bipartite graphs are P_5-free. It is easy to prove (see, for example, [16]) that every connected bipartite P_5-free graph is chain bipartite. We can therefore state the following property.

Property 1. *A connected augmenting graph is P_5-free if and only if it is chain bipartite*

The following notation will be used in Sections 3 and 4. To every integer vector $(d_1,\dots,d_n)$ such that $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$, we associate the chain bipartite graph denoted $B_n(d_1,\dots,d_n)$ with parts $V_1=\{a_1,\dots,a_n\}$ and $V_2=\{b_1,\dots,b_{d_1}\}$, and in which there is an edge linking a vertex $a_i \in V_1$ to a vertex $b_j \in V_2$ if and only if $j \leqslant d_i$. Notice that a_1 is adjacent to all b_j $(j=1,\dots,d_1)$, and b_1 is adjacent to all a_i $(i=1,\dots,n)$. We say that the pair (a_1,b_1) is a *dominating pair* in $B_n(d_1,\dots,d_n)$. As a particular case, $B_n(d,\dots,d)$ is a complete bipartite $K_{n,d}$. Property 1 can now be reformulated as follows.

Property 1′. *A connected augmenting graph is P_5-free if and only if it is isomorphic to a $B_n(d_1,\dots,d_n)$ with $n < d_1$ and $d_n > 0$.*

As an illustration, the above property states that there are only three non-isomorphic connected P_5-free augmenting graphs $H=(V_1,V_2,E)$ with $|V_1|=2$ and $|V_2|=3$: $B_2(3,1)$ (also called a *chair*), $B_2(3,2)$ (also called a *banner*) and $B_2(3,3)$ (the complete bipartite graph $K_{2,3}$) (see Fig. 1).

The following two lemmas provide additional useful information on connected augmenting graphs (see also [3] for Lemma 1).

Lemma 1. *Let H be a minimal connected augmenting graph for a stable set S, with S-part V_1 and $\bar{S}$-part V_2. Then each vertex in V_1 has at least two neighbours in V_2.*

Proof. Notice first that each vertex in V_1 has at least one neighbour, else H is not connected. Assume now that V_1 contains a vertex x with a unique neighbour y in V_2. Then the graph H' obtained from H by removing vertices x and y is also augmenting with $\Delta(H') = \Delta(H)$, which contradicts the minimality of H. □

Lemma 2. *Let S be a stable set in a P_5-free graph G, and let $B_n(d_1,\ldots,d_n)$ be an augmenting graph for S. If G does not contain any augmenting $K_{1,2}$, then $n > 1$ and $d_2 \geqslant d_1 - 1$.*

Proof. Let $V_1 = \{a_1,\ldots,a_n\}$ and $V_2 = \{b_1,\ldots,b_{d_1}\}$ be the two parts of $B_n(d_1,\ldots,d_n)$. If $n = 1$, then vertices a_1, b_1 and b_2 induce an augmenting $K_{1,2}$ for S in G, a contradiction. Similarly, if $d_2 < d_1 - 1$, then a_1, b_{d_1} and b_{d_1-1} induce an augmenting $K_{1,2}$ for S in G, a contradiction. □

3. Stable sets in (P_5, $K_{3,3} - e$)-free graphs

Let $K_{3,3} - e$ denote the graph obtained by deleting an edge in the complete bipartite graph $K_{3,3}$. The next theorem characterizes connected $(P_5, K_{3,3} - e)$-free augmenting graphs.

Theorem 1. *Let S be a maximal stable set in a $(P_5, K_{3,3} - e)$-free graph G, and assume that G does not contain any augmenting $K_{1,2}$ for S. Then each connected minimal augmenting graph H for S is either a $B_n(d,\ldots,d)$ or a $B_n(d, d-1,\ldots,d-1)$ with $1 < n < d$.*

Proof. Consider any connected minimal augmenting graph H for S in G. By Property $1'$ and Lemma 1, we know that H is isomorphic to a $B_n(d_1,\ldots,d_n)$ with $d_n > 1$. If there exists an index $i > 2$ such that $2 \leqslant d_i < d_2$, then vertices a_1, a_2, a_i, b_1, b_2 and b_{d_i+1} induce a $K_{3,3} - e$ in G, a contradiction. Hence, $d_i = d_2$ for each index $i > 2$ such that $d_i > 1$. It follows from Lemma 2 that $n > 1$ and $d_1 - 1 \leqslant d_2 = \cdots = d_n$. Hence, H is either a $B_n(d,\ldots,d)$ or a $B_n(d, d-1,\ldots,d-1)$ with $1 < n < d$. □

Notice that $B_n(d,\ldots,d)$ is a $K_{n,d}$ while $B_n(d, d-1,\ldots,d-1)$ is the graph obtained by adding a pending edge to one vertex of degree $d-1$ in a $K_{n,d-1}$. The latter graph is denoted $K^+_{n,d-1}$. The following result is a direct corollary of Theorem 1.

Corollary 1. *Let S be a maximal but non-maximum stable set in a $(P_5, K_{3,3} - e)$-free graph G, and assume that G does not contain any augmenting $K_{1,2}$ for S. Then there exists an augmenting graph H for S such that*:

- $\Delta(H) = \alpha(G) - |S|$, *and*
- *each connected component of H is either a $K_{n,d}$ or a $K^+_{n,d-1}$ with $1 < n < d$.*

In order to solve the MSP in polynomial time in $(P_5, K_{3,3} - e)$-free graphs, it is sufficient to design a polynomial algorithm that finds augmenting $K_{n,d}$ and $K^+_{n,d-1}$ in

$(P_5, K_{3,3}-e)$-free graphs. Such an algorithm is not yet available. Brandstädt and Lozin [6] have proposed a polynomial algorithm that solves the MSP in $(P_5, K_{3,3}-e, TH)$-free graphs, where *TH* (also called *twin-house*) is a particular graph with 6 vertices. We show in this section that the MSP has a polynomial time solution in the class of $(P_5, K_{3,3}-e, K^+_{m,m})$-free graphs, with fixed m. Such a result is already known for $m=1$ and 2. Indeed, $K^+_{1,1}$ is a $K_{1,2}$ and $K^+_{2,2}$ is a *banner*, and the stability number of a $K_{1,2}$-free graph G is its number of connected components, while Lozin [13] has designed a polynomial algorithm that solves the MSP in $(P_5, banner)$-free graphs.

Let S be a maximal stable set in a $(P_5, K_{3,3}-e, K^+_{m,m})$-free graph G, with fixed m. Assume there is no augmenting $K^+_{r,r}$ for S with $r<m$. Then there is no augmenting $K^+_{r,s-1}$ for S with $1<r<s$ and $r<m$ since by removing $s-r-1$ vertices in the $\bar{S}$-part one would get an augmenting $K^+_{r,r}$ with $r<m$. Moreover, there is no augmenting $K^+_{r,s-1}$ for S with $1<r<s$ and $r \geqslant m$ since G is $K^+_{m,m}$-free. Hence, it follows from Corollary 1 that if S is not maximum, then there exists an augmenting graph H for S such that $\Delta(H)=\alpha(G)-|S|$, and each connected component of H is an augmenting complete bipartite graph.

Let S be a stable set in G and let x and y be two vertices outside S. Vertices x and y are said *similar* if $N_S(x)=N_S(y)$. Clearly, the similarity is an equivalence relation, and we denote $Q_1,\ldots,Q_k$ the similarity classes. It follows from the definitions that if $K_{r,s}$ $(1<r<s)$ is an augmenting graph for a stable set S, then its S-part is a $N_S(Q_i)$ for some similarity class Q_i with $|N_S(Q_i)|>1$, while its $\bar{S}$-part is a stable set in $G[Q_i]$. A similarity class Q_i is said *interesting* if $|N_S(Q_i)|>1$ and $\alpha(G[Q_i])>|N_S(Q_i)|$. A vertex $q_i \in Q_i$ is said to be *non-dominating* in Q_i if there exists a vertex $q_j \neq q_i$ in Q_i which is no adjacent to q_i in G. Notice that every interesting similarity class contains at least $\alpha(G[Q_i])>1$ non-dominating vertices.

Lemma 3. *Let S be a stable set in a $(P_5, K_{3,3}-e)$-free graph G, and let Q_i and Q_j be two interesting similarity classes such that G contains at least one edge linking a non-dominating vertex in Q_i to a non-dominating vertex in Q_j. Then either $N_S(Q_i) \subseteq N_S(Q_j)$ or $N_S(Q_j) \subset N_S(Q_i)$.*

Proof. Assume G contains an edge between a non-dominating vertex $q_i \in Q_i$ and a non-dominating vertex $q_j \in Q_j$. If neither $N_S(Q_i) \subseteq N_S(Q_j)$ nor $N_S(Q_j) \subset N_S(Q_i)$, then there exists a vertex $x_i \in N_S(Q_i)$ and a vertex $x_j \in N_S(Q_j)$ such that x_i is not linked to q_j and x_j is not linked to q_i is G. Consider any vertex $y_i \in Q_i$ which is not adjacent to q_i, and any vertex $y_j \in Q_j$ which is not adjacent to q_j. Vertex q_i is adjacent to y_j else vertices x_i, q_i, q_j, x_j and y_j induce a P_5 in G, a contradiction. Similarly, q_j is adjacent to y_i. Hence, y_i is adjacent to y_j else vertices x_i, y_i, q_j, x_j and y_j induce a P_5 in G, a contradiction. But now, vertices x_i, y_i, q_i, x_j, y_j and q_j induce a $K_{3,3}-e$ in G, a contradiction. □

Corollary 2. *Let S be a stable set in a $(P_5, K_{3,3}-e)$-free graph G. Let Q_i and Q_j be two interesting similarity classes such that $N_S(Q_i) \cap N_S(Q_j)=\emptyset$, and let S_i and S_j be two maximum stable sets in $G[Q_i]$ and $G[Q_j]$, respectively. Then $S_i \cup S_j$ is a stable set in G.*

Proof. Notice first that $|S_i| > 1$ and $|S_j| > 1$ since Q_i and Q_j are interesting similarity classes. Hence, all vertices in S_i are non-dominating in Q_i and all vertices in S_j are non-dominating in Q_j. Since $N_S(Q_i) \cap N_S(Q_j) = \emptyset$, we know by Lemma 3, that there is no edge linking a vertex in S_i to a vertex in S_j. □

Lemma 4. *Let S be a stable set in a $(P_5, K_{3,3} - e)$-free graph G, and let Q_i and Q_j be two interesting similarity classes such that $N_S(Q_i) \cap N_S(Q_j) \neq \emptyset$. Then either $N_S(Q_i) \subseteq N_S(Q_j)$ or $N_S(Q_j) \subset N_S(Q_i)$.*

Proof. Consider any non-dominating vertices $q_i \in Q_i$ and $q_j \in Q_j$, and let x be any vertex in $N_S(Q_i) \cap N_S(Q_j)$. If neither $N_S(Q_i) \subseteq N_S(Q_j)$ nor $N_S(Q_j) \subset N_S(Q_i)$, then S contains two vertices y_i and y_j such that y_i is adjacent to q_i but not to q_j, and y_j is adjacent to q_j but not to q_i in G. Moreover, it follows from Lemma 3 that q_i is not adjacent to q_j. Hence, vertices y_i, q_i, x, q_j and y_j induce a P_5 in G, a contradiction. □

In summary, we have proved that if S is a stable set in a $(P_5, K_{3,3}-e, K^+_{m,m})$-free graph G with fixed m, and if there is no augmenting $K^+_{r,r}$ for S with $r < m$, then determining an augmenting graph H for S in G with maximum increment $\Delta(H) = \alpha(G) - |S|$ reduces to determining a subset $\mathcal{Q}$ of interesting similarity classes such that $N_S(Q_i) \cap N_S(Q_j) = \emptyset$ for each pair (Q_i, Q_j) of elements in $\mathcal{Q}$ and with $\sum_{Q_i \in \mathcal{Q}} \alpha(G[Q_i]) - |N_S(Q_i)| = \alpha(G) - |S|$. This is done as in [13]. More precisely, let $\mathcal{I}$ denote the set of interesting similarity classes. We define a graph, denoted $F(S)$, with vertex set $\mathcal{I}$ and in which two vertices Q_i and Q_j are linked by an edge if and only if $N_S(Q_i) \cap N_S(Q_j) \neq \emptyset$. With each vertex Q_i in $F(S)$ we associate a weight equal to $\alpha(G[Q_i]) - |N_S(Q_i)|$. The weight of a subset of vertices is the sum of weights of its elements. It is now sufficient to determine a stable set $\mathcal{S}$ with maximum weight in $F(S)$. We then associate a connected augmenting graph H_i for S with each vertex $Q_i \in \mathcal{S}$, the S-part of H_i being equal to $N_S(Q_i)$ while its $\bar{S}$-part is any stable set of maximum size in $G[Q_i]$. The disjoint union of all these augmenting graphs H_i is an augmenting graph H for S with maximum increment. The proposed algorithm for the solution of the MSP in the class of $(P_5, K_{3,3} - e, K^+_{m,m})$-free graphs, with fixed m, is summarized below.

Procedure ALPHA(G)
Input: a $(P_5, K_{3,3} - e, K^+_{m,m})$-free graph G with fixed m.
Output: a maximum stable set S in G.

1. Find an arbitrary maximal stable set S in G.
2. If G contains an augmenting $H = K^+_{r,r}$ for S with $r < m$, then replace the S-part of H in S by its $\bar{S}$-part, and repeat Step 2.
3. Partition the vertices of $V(G) - S$ into similarity classes $Q_1, \ldots, Q_k$, and remove the classes Q_i with $|N_S(Q_i)| < 2$.
4. For each remaining class Q_i, determine a maximum stable set S_i in $G[Q_i]$ by calling **ALPHA**$(G[Q_i])$.

5. Remove all similarity classes Q_i with $|S_i| \leqslant |N_S(Q_i)|$.
6. Construct graph $F(S)$ and find a stable set $\mathscr{S}$ of maximum weight in it.
7. Exchange $N_S(Q_i)$ with S_i for each Q_i in $\mathscr{S}$.
8. Return S and stop.

In order to find a stable set of maximum weight in $F(S)$, it is sufficient to observe (as was done in [3]) that $F(S)$ is (P_4, C_4)-free (where a P_4 is a chordless chain on 4 vertices and a C_4 is a chordless cycle on 4 vertices).

Lemma 5 (Alekseev and Lozin [3]). *Graph $F(S)$ is (P_4, C_4)-free.*

Proof. Assume $F(S)$ is not (P_4, C_4)-free. Consider four vertices Q_1, Q_2, Q_3, Q_4 in $F(S)$ such that Q_2 is adjacent to Q_1 and Q_3 but not to Q_4, and Q_3 is adjacent to Q_2 and Q_4 but not to Q_1 in $F(S)$. Hence, vertices Q_1, Q_2, Q_3 and Q_4 induce a P_4 (if Q_1 is not adjacent to Q_4) or a C_4 in $F(S)$. Since $N_S(Q_2) \cap N_S(Q_3) \neq \emptyset$, we may assume by Lemma 4 that $N_S(Q_2) \subseteq N_S(Q_3)$ in G. Hence, $N_S(Q_1) \cap N_S(Q_3) = \emptyset$ implies $N_S(Q_1) \cap N_S(Q_2) = \emptyset$ which contradicts the fact that there exists an edge between Q_1 and Q_2 in $F(S)$. □

The graphs containing no P_4 and no C_4 as induced subgraphs have been extensively studied in the literature under different names, like trivially perfect graphs [12] and quasi-threshold graphs [22]. The problem of finding a stable set of maximum weight can be solved in that class in linear time using modular decomposition [14].

Theorem 2. *The stability number of a $(P_5, K_{3,3,} - e, K^+_{m,m})$-free graph with n vertices and fixed $m > 1$ can be determined in $O(n^{m+2})$.*

Proof. Correctness of algorithm **ALPHA** follows from the theorems proved in this section. To estimate the time complexity, we note that steps 1, 3, 5, 6, 7 and 8 take in the worst case $O(n^3)$ time. An augmenting $K^+_{r,r}$ for S with $r < m$ can be found in $O(n^m)$ time. Since step 2 is repeated at most n times, the total time complexity of this step is $O(n^{m+1})$. The graph G' obtained by making the disjoint union of all $G[Q_i]$ with $|N_S(Q_i)| > 1$ has strictly less vertices than G since graphs $G[Q_1], \ldots, G[Q_k]$ are vertex disjoint while G' does not contain any vertex from S. But Step 4 reduces to finding a maximum stable set in G'. Hence, the recursion in step 4 results in the total time $O(n^{m+2})$. □

Lozin [13] and Mosca [16] have proposed polynomial algorithms for the solution of the MSP in $(P_5, banner)$-free and $(P_5, K_{2,3})$-free graphs, respectively. The above theorem extends both results since $K_{3,3} - e$ and $K^+_{3,3}$ contain an induced *banner* and an induced $K_{2,3}$. Notice also that if p and q are two fixed integers, then the MSP has a polynomial solution in the class of $(P_5, K_{3,3,} - e, K^+_{p,q})$-free graphs since these graphs do not contain any induced $K^+_{m,m}$ with $m \geqslant \max\{p, q\}$.

4. An infinite family of subclasses of P_5-free graphs

In this section, we illustrate the use of the characterization of all connected P_5-free augmenting graphs by identifying an infinite family of subclasses of P_5-free graphs for which the MSP has a polynomial time solution. Given a graph H and an integer $t \geqslant 0$, we denote $A(t,H)$ the graph obtained by adding a clique $K = \{k_1, \ldots, k_t\}$ and a stable set $L = \{l_1, \ldots, l_t\}$ to H, by linking each vertex of K to each vertex of H, and by linking a vertex k_i to a vertex l_j if and only if $i \geqslant j$. As an illustration, graphs $A(t,H)$ are depicted in Fig. 2 for various graphs H and for various values of t. We prove in this section that if the MSP can be solved in polynomial time in the class of (P_5,H)-free graphs, then the MSP can also be solved in polynomial time in the class of $(P_5,A(t,H))$-free graphs, for any fixed t.

Theorem 3. *Let H be any graph. If one can solve the MSP in a (P_5,H)-free graph G in time $O(|V(G)|^p)$, then one can solve the MSP in a $(P_5,A(1,H))$-free graphs G in time $O(|V(G)|^{p+1} \cdot |E(G)|)$.*

Proof. Let G be a $(P_5,A(1,H))$-free graph. Consider any stable set S in G as well as two adjacent vertices $x \in S$ and $y \notin S$. Let R denote the subset of vertices z in $V(G) - (S \cup \{y\})$ which are adjacent to x but not to y, and such that $N_S(z) \subseteq N_S(y)$. There exists an augmenting $B_n(d_1, \ldots, d_n)$ for S with dominating pair (x, y) if and only if R contains a stable set with $d_1 - 1$ vertices. Hence, to determine whether (x, y) is a dominating pair in an augmenting graph for S, it is sufficient to determine a maximum stable set S' in $G[R]$: $|S'| \geqslant |N_S(y)|$ if and only if $N_S(y) \cup (S' \cup \{y\})$ induces an augmenting $B_n(d_1, \ldots, d_n)$ with $n = |N_S(y)|$, $d_1 = |S'| + 1$, and with dominating pair (x, y). But $G[R]$ is H-free, else $G[R \cup \{x, y\}]$ contains an $A(1,H)$. Hence $\alpha(G[R])$ can be determined in polynomial time.

Now, one can determine whether G contains an augmenting graph for S by considering all pairs (x, y) of adjacent vertices with $x \in S$ and $y \notin S$, and by checking whether (x, y) is a dominating pair in an augmenting graph for S. Since a maximum stable set in G is necessarily reached after at most $|V(G)|$ augmentations, one can solve the MSP in G by running $O(|V(G)| \cdot |E(G)|)$ times the polynomial algorithm which solves the MSP in the class of (P_5,H)-free graphs. □

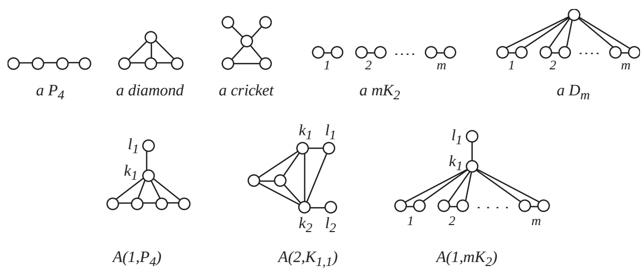

Fig. 2. Special graphs and illustration of the construction of $A(t,H)$ graphs.

The following stronger result was proved independently by Mosca [18]. Let WMSP denote the problem of finding a stable set of maximum weight in a graph, and let H be any graph. If one can solve the WMSP in a (P_5,H)-free graph G in time $O(|V(G)|^p)$, then one can solve the WMSP in a $(P_5,A(1,H))$-free graph G in time $O(|V(G)|^{p+2})$.

Since $A(t,H)=A(1,A(t-1,H))$, we can state the following corollary.

Corollary 3. *Let H be any graph. If the MSP has a polynomial time solution in the class of (P_5,H)-free graphs, then it also has a polynomial time solution in the class of $(P_5,A(t,H))$-free graphs G, for any positive integer t.*

As a first illustration of the above result, consider the graph $H=K_{1,1}$ (i.e., H contains only two vertices linked by an edge). The MSP is particularly easy to solve in the class of $K_{1,1}$-free graphs since the stability number of such a graph $G=(V,E)$ is equal to $|V|$. As a consequence, for any fixed integer t, the MSP has an $O(|E|^t\cdot|V|^{t+1})$ time solution in the class of $(P_5,A(t,K_{1,1}))$-free graphs. But $A(t,K_{1,1})$ contains an induced clique with $t+2$ vertices. Hence, if the size of the largest clique in a P_5-free graph $G=(V,E)$ is bounded by some fixed number m, then the stability number of G can be determined in $O(|E|^{m-1}\cdot|V|^m)$ time. Notice also that $A(2,K_{1,1})$ contains a *diamond* and a *cricket* (see Fig. 2). It is proved in [4,16], respectively, that the MSP has a polynomial time solution in the classes of $(P_5,\mathit{diamond})$-free and $(P_5,\mathit{cricket})$-free graphs. Corollary 3 therefore generalizes these two results.

As a second illustration, consider $H=P_4$. Obviously, a graph is (P_5,P_4)-free if and only if it is P_4-free. Moreover, it is well known that the MSP has a linear time solution in the class of P_4-free graphs [7,14]. Hence, Theorem 3 and Corollary 3 show that the MSP can be solved in $O(|E|^{t+1}\cdot|V|^t+|E|^t\cdot|V|^{t+1})$ time in the class of $(P_5,A(t,P_4))$-free graphs, for any fixed t. Notice that $A(1,P_4)$ contains a *diamond* and a *cricket* (see Fig. 2). We therefore get a second generalization of the results contained in [4,16].

As a third illustration, consider the class of $(P_5,K_{1,m})$-free graphs with fixed $m>1$. Mosca [16] has shown that the MSP has an $O(|V(G)|^{m+1})$ time solution in this class of graphs. This result is in fact a simple corollary of Theorem 3. Indeed, define H as the graph made of $m-1$ isolated vertices. The MSP can obviously be solved in H-free graphs in $O(|V(G)|^{m-2})$ time. Since $A(1,H)$ is a $K_{1,m}$, Theorem 3 shows that the MSP has an $O(|E(G)|\cdot|V(G)|^{m-1})$ time solution in $(P_5,K_{1,m})$-free graphs.

Finally, let mK_2 denote the graph made of m disjoint edges. Alekseev [1] has proved that the number of maximal stable sets in mK_2-free graphs is bounded by a polynomial for any fixed m. In combination with the algorithm of Tsukiyama et al. [21] that generates all maximal stable sets, this leads to a polynomial algorithm for the MSP in mK_2-free graphs with a fixed m. It follows from Theorem 3 that the MSP has a polynomial time solution in the class of $(P_5,A(1,mK_2))$-free graphs. But $A(1,mK_2)$ contains a cricket for $m\geqslant 2$. Hence, Theorem 3 provides a third generalization of Mosca's result on $(P_5,\mathit{cricket})$-free graphs. Now let D_m denote the graph obtained from mK_2 by adding a vertex linked to all vertices in mK_2 (see Fig. 2). Notice that D_{m+1} contains $A(1,mK_2)$ which contains D_m. Gerber and Lozin [10] have proved recently that the MSP has a polynomial solution in the class of (P_5,D_m)-free graphs, for any fixed m. Theorem 3 provides another simple proof of this result.

5. Conclusion

In this paper, we have first characterized all connected P_5-free augmenting graphs. Such a characterization is very helpful when using the augmenting graph technique for the solution of the MSP in P_5-free graphs. Unfortunately, we are not yet able to determine in polynomial time whether an augmenting graph exists in a general P_5-free graph. However, we have used the above characterization to develop polynomial algorithms for the MSP in families of subclasses of P_5-free graphs. All families of graphs studied in this paper extend previous results.

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