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EXACT BOUNDARY AND DISTRIBUTED CONTROLLABILITY OF RADIAL DAMPED WAVE EQUATION

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ABSTRACT. – We study the zero, closed loop and periodic controllability problems for the distributed parameter systems governed by the radial wave equations containing damping terms both in the equation and in the boundary conditions. These equations are obtained by the separation of variables in the spherical coordinates from the 3-dimensional damped equation with spacially nonhomogeneous spherically symmetric coefficients. We consider two types of controls: a) the distributed controls implemented as forcing terms in the right-hand sides of the equations and b) the boundary controls implemented through the boundary conditions. Applying the spectral decomposition method, we give the necessary and sufficient conditions for the exact controllability of the systems and provide explicit formulas for the controls for all three aforementioned problems and for both types of controls. The proofs are based on our recent results concerning the spectral analysis for the class of nonselfadjoint operators and operator pencils generated by the above equations and the boundary conditions. These operators are the dynamic generators of the systems in the energy spaces of two-component initial data. We do not restrict our analysis to the case when the spectra of the dynamic generators are simple and assume that they may have associated vectors, *i.e.*, the algebraic multiplicities of their eigenvalues are greater than one. © Elsevier, Paris

RÉSUMÉ. – Nous étudions des problèmes de contrôle – retour à l'équilibre, en boucle fermée, périodiques – pour des systèmes gouvernés par des équations d'ondes radiales contenant des termes d'amortissement à la fois dans les équations et dans les conditions aux limites. Ces équations sont obtenues par séparation des variables en coordonnées sphériques non homogènes dans l'espace tridimensionnel. Nous considérons ici deux types de contrôle :
a) Les contrôles distribués qui portent sur les seconds membres de l'équation ;
b) Les contrôles frontière qui portent sur les conditions aux limites.
En utilisant la méthode spectrale de décomposition nous donnons des conditions nécessaires et suffisantes de contrôlabilité exacte et des formules explicites de calcul des contrôles pour les trois problèmes mentionnés ci-dessus et pour les deux types de contrôle. Les preuves utilisent des travaux récents de l'auteur sur l'analyse spectrale des classes d'opérateurs non auto adjoints et les faisceaux d'opérateurs associés aux équations et aux conditions aux limites. Ces opérateurs sont générateurs dynamiques du système dans les espaces d'énergie adaptés aux données initiales des deux composantes. Nous ne restreignons pas notre étude au cas où les spectres des générateurs dynamiques sont simples, nous supposons par exemple que la multiplicité algébrique de valeurs propres puisse être supérieure à un. © Elsevier, Paris

1. Introduction

In the present paper, we study controllability problems for the radial damped wave equations using the spectral decomposition method.These equations occur as a result of separation of variables in the 3-dimensional damped wave equation with spherically symmetric damping, density and elasticity coefficients. The radial equations are defined

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on a finite interval with linear first order nonselfadjoint boundary conditions containing damping terms.

We consider three controllability problems: the zero controllability problem, the periodic controllability problem, and a generalization of the periodic problem, which we call the "closed loop" controllability problem. We investigate each of these problems with two different types of controls. First of them is a distributed control, which enters in the equation through a forcing term. This term is a product of the control function of time and the force profile function of a space variable. The second type control is the boundary control implemented as a function of time in the boundary conditions. In all cases, we give explicit formulas for the desired control functions in terms of the spectral characteristics of the corresponding boundary value problem.

We recall that the general scheme of the spectral decomposition method, as well as the solution of the controllability problem for the undamped wave equation, was given in a series of well known works by D. Russell [1-3]. The main difference between the undamped and damped wave equations is the fact that the latter generates a nonselfadjoint operator for which the spectral theory has been developed only recently. Our solution is based on the recent work by the author [7], in which it was shown that the dynamic generators of radial damped wave equations with linear first order dissipative boundary conditions are nonselfadjoint spectral operators in the sense of N. Dunford [8]. In the proof of this result given in [7], we have used the results of our another recent work [9], which was devoted to a detailed asymptotic analysis of the spectra and the eigenfunctions of these dynamic generators. In turn, in both papers [7] and [9], we have used the method developed in a series of our works [5-9] (see also [4] and [11]). These papers were devoted to the asymptotic and spectral analysis of a spacially nonhomogeneous damped string.

Now, we describe the spherically symmetric damped wave equation and the corresponding radial wave equations, and then formulate the control problems.

Let Ω be an open ball of radius a in $\mathbb{R}^3$ and $\partial\Omega$ be its boundary – the sphere of radius a. In Ω, we consider the following wave equation:

$$(1.1)\quad u_{tt} - \frac{1}{\rho(r)}\operatorname{div}(p(r)\nabla u) + 2d(r)u_t + q(r)u = 0, \quad r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}, \quad t \geq 0.$$

All coefficients: ρ (the density of the medium), p (the elasticity coefficient), d (the viscous damping coefficient), and q (the rigidity of an external harmonic force) are positive spherically symmetric functions. (Precise conditions on these functions are formulated later.)

Together with the equation, we consider the initial conditions:

$$(1.2)\qquad u(x_1 0) = u_0(x), \quad u_t(x,0) = u_1(x)$$

and a one-parameter family of boundary conditions:

$$(1.3)\qquad \left(\frac{\partial u}{\partial n} + hu_t\right)_{x\in\partial\Omega} = 0, \quad h \in \mathbb{C} \cup \{\infty\},$$

with $\frac{\partial u}{\partial n}$ being the normal derivative on the boundary. To $h = \infty$, we formally associate the Dirichlet condition: $u(x,t) = 0$ for $x \in \partial\Omega$.

We look for a solution of problem (1.1)-(1.3) in the form of an expansion with respect to the spherical harmonics [12], i.e.,

$$u(x,t) = \sum_{\ell,m,j} u_{\ell mj}(r,t) Y_{\ell mj}(\theta,\varphi). \tag{1.4}$$

For the initial conditions, we have

$$\begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix} = \sum_{\ell mj} \begin{pmatrix} u_0^{\ell mj}(r) \\ u_1^{\ell mj}(r) \end{pmatrix} Y_{\ell mj}(\theta,\varphi). \tag{1.5}$$

Substituting (1.4), (1.5) into Eq. (1.1) and conditions (1.2), (1.3), and using the orthogonality of the spherical harmonics, we transfer 3-dimensional initial-boundary problem (1.1)-(1.3) to the infinite sequence of the following one dimensional problems:

$$\begin{aligned}(u_{\ell mj}(r,t))_{tt} = \frac{1}{\rho(r)}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(p(r)r^2\frac{\partial}{\partial r}u_{\ell mj}(r,t)\right) - \right.\\ \left.\frac{\ell(\ell+1)}{r^2}p(r)u_{\ell mj}(r,t)\right] - 2d(r)(u_{\ell mj}(r,t))_t - q(r)u_{\ell mj}(r,t),\\ \ell = 0,1,2,\ldots; \quad |m| \le \ell; \quad j = 1 \text{ if } m = 0 \text{ and } j = 1,2 \text{ if } m \ne 0,\end{aligned} \tag{1.6}$$

with the initial conditions:

$$u_{\ell mj}(r,0) = u_0^{\ell mj}(r), \quad (u_{\ell mj})_t(r,0) = u_1^{\ell mj}(r), \tag{1.7}$$

and the boundary conditions: $u_{\ell mj}(r)$ must be bounded at $r = 0$ and satisfy at $r = a$:

$$\left((u_{\ell mj})_r + h(u_{\ell mj})_t\right)(a,t) = 0. \tag{1.8}$$

From now on, we will omit the triple index "ℓmj" and instead of $u_{\ell mj}$ simply write u. From this point, we study the following initial-boundary problem:

$$u_{tt} = L_\ell u - 2d(r)u_t, \qquad \ell = 0,1,2,\ldots \tag{1.9}$$

where L_ℓ is the Sturm-Liouville operator defined by

$$L_\ell \varphi = \frac{1}{\rho(r)}\left[\frac{1}{r^2}\frac{d}{dr}\left(p(r)r^2\frac{d\varphi}{dr}\right) - \frac{\ell(\ell+1)}{r^2}p(r)\varphi\right] - q(r)\varphi, \tag{1.10}$$

on any smooth function $\varphi(r)$. We impose the following boundary conditions:

$$\lim_{r\to 0} ru(r,t) = 0, \quad (u_r + hu_t)(a,t) = 0, \quad h \in \mathbb{C} \cup \{\infty\} \tag{1.11}$$

and standard initial conditions

$$u(r,0) = u_0^0(r), \quad u_t(r,0) = u_1^0(r). \tag{1.12}$$

We recall that for $h = \infty$, 0, we have the Dirichlet and Newmann boundary conditions respectively; for $h = 1$ and $\ell = 0$, we have the Sommerfeld radiation conditions [13].

Now we are in a position to describe our control problems.

Let us consider the nonhomogeneous radial damped wave equation:

$$u_{tt} - L_\ell u + 2d(r)u_t = g(r)f(t) \tag{1.13}$$

with the boundary conditions

$$\lim_{r\to 0} ru(r,t) = 0, \quad (u_r + hu_t)(a,t) = \varphi(t), \quad h \in \mathbb{C} \text{ or } u(a,t) = \varphi(t). \tag{1.14}$$

In Eq. (1.13), L_ℓ is given in (1.10) and u obeys standard initial conditions (1.12); $g(r)$ is called a force profile function; the functions $f(t)$ and $\varphi(t)$ are called distributed and boundary controls respectively. In this work, we study the following two cases: $f(t) \neq 0$ and $\varphi(t) = 0$ or $f(t) = 0$ and $\varphi \neq 0$. (The combination of both controls for the equation of damped string is considered in our works [14, 15] where it is shown that such combination can make the control time two times shorter.)

We consider the following control problems.

A) *Zero controllability problem.* Let initial conditions (1.12) and $T > 0$ be given. Do there exist a distributed control $f \in L^2(0,T)(\varphi = 0)$ or a boundary control $\varphi \in L^2(0,T)(f = 0)$ such that the solution of problem (1.13), (1.14) satisfies also an additional condition at $t = T$:

$$u(r,T) = u_t(r,T) = 0, \quad r \in [0,a]?$$

B) *Closed loop control problem.* Do there exist controls $f \in L^2(0,T)(\varphi = 0)$ or $\varphi \in L^2(0,T)(f = 0)$ such that the terminal state of the system at the moment $t = T$ is a multiple of its initial state (δ is a given number):

$$u(r,T) = \delta u(r,0), \quad u_t(r,T) = \delta u_t(r,0)?$$

C) *Periodic control problem.* Let $T_1 > 0$ be given and let control functions $f \in L^2(0,T)(\varphi = 0)$ or $\varphi \in L^2(0,T)(f = 0)$ be switched on at the moment $t = T_1$. Do there exist f and φ such that the state of the system at $t = T_1 + T$ is equal to the initial state

$$u(r,T_1+T) = u(r,0), \quad u_t(r,T_1+T) = u_t(r,0)?$$

We say that for a given initial state $U_0 = \begin{pmatrix} u_0^0 \\ u_1^0 \end{pmatrix}$, the system is controllable in time T if the desired control exists. If the control is also unique, we say that the system is exactly controllable.

In Sections 3 and 4, we first present the solutions of the above problems under the assumption that the dynamic generators (defined in Section 2) of systems (1.9)-(1.12) have only simple eigenvalues. Then we separately investigate the following question:

D) *Multiple eigenvalues case.* What are the formulas for the control functions $f \in L^2(0,T)(\varphi = 0)$ or $\varphi \in L^2(0,T)(f = 0)$ if we admit several multiple eigenvalues of the dynamic generators? (The number of such eigenvalues is always finite and their geometric multiplicities must be equal to 1. So, "multiple" means the presence of associated vectors - Theorem 1.1. below.) We will consider question D) with a complete proof only in the case of problem A) (zero controllability) with a distributed control. In case A) (zero controllability) with a boundary control, we will only state the results. The extension of the results to all other cases is almost straightforward and we do not consider it here only for the sake of brevity.

The paper is organized as follows. In Section 2, we introduce a matrix differential operator $\mathcal{L}_\ell$ (see formulas (2.7)-(2.8) for its definition) naturally related to the operators L_ℓ defined by (1.10), (1.11). These operators $\mathcal{L}_\ell$ are the dynamic generators of our systems. Then we formulate (without proofs) the main spectral properties of these operators (Theorem 2.1). In the next statement (Theorem 2.2), we describe the geometry of the root vectors (associated and eigenvectors together) of $\mathcal{L}_\ell$. In the conclusion of Section 2, we provide the information on the Riesz basis property of the nonharmonic exponentials in the space $L^2(0, \mathcal{M})$, where $\mathcal{M}$ is a positive constant given in (2.4) below (see Theorem 2.3). The proofs of all these results are given in [7].

In Section 3, we formulate the results on all of the controllability problems stated in Section 1 (Theorems 3.1-3.4).

Section 4 is devoted to the reduction of the boundary-distributed control problem to the moment problem. Based on the results for this moment problem, we give the proofs of all controllability results under the assumption that the spectrum of the operator $\mathcal{L}_\ell$ is simple (*i.e.*, there are no associated vectors - only the eigenvectors).

In Section 5, we remove the assumption about the simplicity of the spectrum of $\mathcal{L}_\ell$ and give the solution of Problem D).

2. Statement of main spectral results

We begin with the properties of the coefficients ρ, d, q and p.

We assume that the density ρ satisfies the conditions:

$$\rho \in H^2[0,a], \quad \rho(r) > 0 \text{ for } r \in [0,a]. \tag{2.1}$$

In the case of real h, we need an additional restriction

$$\rho(a)/p(a) \neq h^2. \tag{2.2}$$

The discussion concerning condition (2.2) can be found in our paper [16].

About the damping, rigidity and elasticity coefficients we assume:

(2.3) $d \in H^1(0,a),\ q \in L^1(0,a),\ d(r),\ q(r) \geq 0$ for $r \in [0,a]$;

(2.4) $p \in C^2[0,a],\ p(r) > 0$ for $r \in [0,a]$ and $p(r) = p_0 + p_1 r^2 + O(r^3)$ when $r \to 0$, (p_0, p_1 are two positive constants).

The condition on the behavior of p at the vicinity of zero has been imposed only to simplify the asymptotic analysis in [9]. However, this restriction can be eliminated if necessary. We introduce the quantities $\mathcal{M}$ and $\mathcal{N}$ which are important in the following:

$$(2.5)\qquad \mathcal{M} = \int_0^a \sqrt{\rho(t)/p(t)}dt,\quad \mathcal{N} = \int_0^a d(t)\sqrt{\rho(t)/p(t)}dt > 0.$$

We mention that, in fact, the asymptotic behavior of the eigenvalues depends on the behavior of the function $\tau = \rho/p$ only at the vicinity of the endpoint $r = a$. So, we can admit singularities and zeros somewhere in the inner points of the interval $[0,a]$, even at the endpoint $r = 0$ (see [4, 5]). The leading term of the asymptotics will be the same. We do not allow this behavior only to simplify the formulation of asymptotic results.

Now we introduce the dynamic generator $\mathfrak{L}_\ell$ of our system.

Eq. (1.9) can be represented in the form of the first order evolution equation for 2-component function $U(r,t) = \begin{pmatrix} u_0(r,t) \\ u_1(r,t) \end{pmatrix} = \begin{pmatrix} u(r,t) \\ u_t(r,t) \end{pmatrix}$:

$$(2.6)\qquad U_t = i\mathfrak{L}U,$$

where $\mathfrak{L}$ is the following matrix differential expression:

$$(2.7)\qquad \mathfrak{L} = -i\begin{pmatrix} 0 & 1 \\ L_\ell & -2d(r) \end{pmatrix}.$$

Eq. (2.6) with boundary conditions (1.11) defines a strongly continuous semigroup of transformations in a complex Hilbert space $\mathfrak{H}_\ell$ of 2-component initial data; $\mathfrak{H}_\ell$ is the closure of smooth 2-component functions $U(r) = \begin{pmatrix} u_0(r) \\ u_1(r) \end{pmatrix}$, such that $u_0(r) = 0$ in a vicinity of $r = 0$, in the following energy norm:

$$(2.8)\qquad \|U\|^2_{\mathfrak{H}_\ell} = \frac{1}{2}\int_0^a \left[p(r)|u_0'|^2 + q(r)\rho(r)|u_0|^2 + \frac{\ell(\ell+1)}{r^2}p(r)|u_0|^2 + \rho(r)|u_1|^2\right] r^2 dr.$$

The generator $\mathfrak{L}_\ell$ of the aforementioned semigroup is defined by (2.7), i.e., $\mathfrak{L}_\ell = \mathfrak{L}$, on the domain:

$$(2.9)\qquad D(\mathfrak{L}_\ell) = \left\{U \in \mathfrak{H}_\ell : \mathfrak{L}_\ell U \in \mathfrak{H}_\ell,\quad \lim_{r\to 0} r u_1(r) = 0,\quad (u_0' + h u_1)(a) = 0\right\}.$$

In addition to the operator $\mathfrak{L}_\ell$, we need the following quadratic operator pencil $\mathcal{P}_\ell(\lambda)$. To describe the pencil $\mathcal{P}_\ell(\lambda)$, let us look for a solution of problem (1.9)-(1.12) in the form

$$(2.10)\qquad u(r,t) = e^{i\lambda t}v(r).$$

For $v(r)$, we have the spectral problem:

$$\mathcal{P}_\ell(\lambda)v = 0, \quad \lim_{r\to 0} rv(r) = 0, \quad (v' + i\lambda hv)(a) = 0, \tag{2.11}$$

where

$$\mathcal{P}_\ell(\lambda)v = L_\ell v + \lambda^2 v - 2i\lambda d(r)v, \tag{2.12}$$

with L_ℓ being given in (1.10). The pencil $\mathcal{P}_\ell(\lambda)$ is defined on the domain

(2.13)
$$D(\mathcal{P}_\ell(\lambda)) = \left\{ v \in H^2(0,a) : L_\ell v \in L^2(0,a), \quad \lim_{r\to 0} rv(r) = 0, \quad (v' + i\lambda hv)(a) = 0. \right\}$$

Now we describe the relationship between the operator $\mathfrak{L}_\ell$ and pencil $\mathcal{P}_\ell(\lambda)$. We recall that $\lambda \in \mathbb{C}$ is an eigenvalue of problem (2.11), (2.12) if this problem has a nontrivial solution. This solution is called an eigenmode or eigenfunction. One can easily verify that if λ_n^ℓ is an eigenvalue of $\mathfrak{L}_\ell$, then the corresponding eigenvector $\mathcal{F}_n^\ell$ can be written in the form:

$$\mathcal{F}_n^\ell = \begin{pmatrix} \frac{1}{i\lambda_n^\ell} & F_n^\ell \\ & F_n^\ell \end{pmatrix}, \quad n \in \mathbb{Z}, \tag{2.14}$$

where F_n^ℓ is an eigenvector of pencil (2.12), (2.13). (According to Theorem 2.1 below, the eigenvalues of $\mathfrak{L}_\ell$ are naturally numbered by $n \in \mathbb{Z}$, and for each eigenvalue there is only one eigenvector.) The latter fact means that both the pencil $\mathcal{P}_\ell(\lambda)$ and operator $\mathfrak{L}_\ell$ have the same spectra and the eigenvectors are related through formula (2.14). Note, that relation (2.14) is valid only for the eigenvectors; associated vectors of the operator $\mathfrak{L}_\ell$ and pencil $\mathcal{P}_\ell(\lambda)$ are related through more complicated formula which is not given since we do not need it in the present paper. We would like to emphasize that neither for the operators $\mathfrak{L}_\ell$ nor for pencils $\mathcal{P}_\ell(\lambda)$ the spectral analysis has been developed before.

Our main theorem obtained in [7] (Theorem 2.2 below) describes the geometry of the set of root vectors of the operator $\mathfrak{L}_\ell$. Before we formulate it, we recall several definitions related to the notion of Riesz basis.

As is well known, the convergence of expansions with respect to any complete orthonormal system $\{\varphi_n\}$ in a Hilbert space H is unconditional, i.e., the corresponding Fourier series converges to the same sum after any permutation of its terms. The latter fact becomes true for any system $\{\psi_n\}$ obtained from an orthonormal basis by means of a bounded and boundedly invertible transformation of H.

DEFINITION 2.1. – *Any complete system $\{\psi_n\}_{n\in\mathbb{Z}}$ in a Hilbert space H is called a Riesz basis (R-basis) if there exist an orthonormal basis $\{\varphi_n\}_{n\in\mathbb{Z}}$ and bounded, boundedly invertible operator A such that $\varphi_n = A\psi_n$. The operator A is called an orthogonalizer of the system $\{\psi_n\}_{n\in\mathbb{Z}}$. (Note that the system $\{\psi_n\}$ is almost normalized, i.e., there exist positive constants C_1 and C_2 such that $0 < C_1 \leq \|\psi_n\| \leq C_2 < \infty$).*

To formulate the main result concerning the spectral properties of $\mathfrak{L}_\ell$, we need the description of the spectrum obtained in our paper [9]. We reproduce the necessary information from [9] in the form of the following theorem.

THEOREM 2.1. – *a) The operator* $\mathfrak{L}_\ell$ *has a countable set of complex eigenvalues, which are located in a strip parallel to the real axis, and has only two points of accumulation:* $+\infty$ *and* $-\infty$. *For this reason, the spectrum can be represented in the form* $\{\lambda_n^\ell,\ n \in \mathbb{Z}\}$, *where* $Re\ \lambda_n^\ell \leq Re\ \lambda_{n+1}^\ell$ *and* $Re\ \lambda_n^\ell \to \pm\infty$ *as* $n \to \pm\infty$. *If we denote*

$$\alpha_\pm = \sqrt{\rho(a)/p(a)} \pm h, \tag{2.15}$$

then for any $h \in \mathbb{C}$, *the following asymptotic formula for the eigenvalues of problem (2.11), holds*

$$\begin{aligned} &\lambda_n^\ell = \Lambda_n^\ell + O(\ell n|n|/|n|),\ as\ |n| \to \infty, \\ &\Lambda_n^\ell = \mathcal{M}^{-1}\left(n + \frac{\ell}{2} + \frac{1}{2} sgn\ n\right)\pi + i\mathcal{M}^{-1}\left(\frac{1}{2}\ell n(\alpha_+\alpha_-^{-1}) + \mathcal{N}\right), \end{aligned} \tag{2.16}$$

where $\mathcal{M}$ *and* $\mathcal{N}$ *are given by (2.5), and under* ℓn *we understand the principal value of the logarithm. In (2.16), the numeration is asymptotical but not absolute. It is possible to pass to the limit* $|h| \to \infty$. *In the case* $|h| = \infty$, *which corresponds to the Dirichlet boundary condition* $u(a) = 0$, *the expression under the sign of logarithm in (2.16) should be replaced with* (-1), *so that* $\ell n(-1) = i\pi$.

b) All eigenvalues of the operator $\mathfrak{L}_\ell$ *have geometric multiplicities equal to one, i.e., for each* λ_n^ℓ *there exists only one linearly independent eigenvector* $\mathcal{F}_n^\ell$. *However, a finite number of eigenvalues* $\{\lambda_n^\ell, n \in R_\ell \subset \mathbb{Z}\}$ *may have finite algebraic multiplicities* m_n^ℓ, *i.e., for such* λ_n^ℓ *there exists a finite chain of associated vectors* $\{\mathcal{F}_{n,j}^\ell\}_{j=1}^{m_n^\ell - 1}$:

$$(\mathfrak{L}_\ell - \lambda_n^\ell I)\mathcal{F}_{n,j}^\ell = \mathcal{F}_{n,j-1}^\ell,\quad \mathcal{F}_{n,0}^\ell = \mathcal{F}_n^\ell,\quad \mathcal{F}_{n,-1}^\ell = 0. \tag{2.17}$$

COROLLARY 2.1. – *The eigenvalues are asymptotically equidistant:*

$$\lim_{|n|\to\infty} \lambda_n^\ell/n = \pi\mathcal{M}^{-1}.$$

Now, we are in a position to formulate the main result on the spectral properties of the nonselfadjoint operators $\mathfrak{L}_\ell$ [7].

THEOREM 2.2. – *Assume that the linearly independent eigenvectors* $\{\mathcal{F}_n^\ell, n \in \mathbb{Z}\}$ *are selected in such a way that they are almost normalized (see Definition 2.1). Then the whole set of the root vectors (eigenvectors and associated vectors together) of* $\mathfrak{L}_\ell$ *forms a Riesz basis in the energy space* $\mathfrak{H}_\ell$.

An important corollary of Theorem 2.2 is the fact that the operators $\mathfrak{L}_\ell$ provide a class of nontrivial examples of spectral operators. While an abstract theory of spectral operators has been developed long ago [8], there is still a problem of finding specific examples of such operators.

Below we give the formulation of the results on nonharmonic exponentials from our paper [9]. These results will be used for the solutions of our controllability problems.

THEOREM 2.3. – *1) Assume that the operator $\mathfrak{L}_\ell$ has a simple spectrum $\{\lambda_n^\ell\}_{n\in\mathbb{Z}}$, i.e., there are no associated vectors. Denote by $\mathcal{E}_h^\ell$ the closed linear span of the set of exponentials $\{e^{i\bar{\lambda}_n^\ell t}\}_{n\in\mathbb{Z}}$ in the space $H = L^2(0, 2\mathcal{M})$. Then the set of nonharmonic exponentials $\{e^{i\bar{\lambda}_n^\ell t}\}_{n\in\mathbb{Z}}$ forms a Riesz basis in $\mathcal{E}_h^\ell$. Moreover,*

$$\dim H(mod\ \mathcal{E}_h^\ell) = \begin{cases} \ell, & \text{if } |h| \neq \infty \text{ and } h \neq 0, \\ \ell + 1, & \text{if } |h| = \infty \text{ or } h = 0. \end{cases} \tag{2.18}$$

2) Let $\{\lambda_k^\ell, k \in S\}$ be the set of multiple eigenvalues of $\mathfrak{L}_\ell$ (S is a finite subset of $\mathbb{Z}$). Let $n_k^\ell + 1$ be the algebraic multiplicity of $\lambda_k^\ell (k \in S), n_k^\ell > 0$. The set of nonharmonic exponentials and exponential-polynomial functions $\bigcup_{k\in S}\{t^m e^{i\bar{\lambda}_k^\ell t}\}_{m=1}^{n_k^\ell} \bigcup \{e^{i\bar{\lambda}_n^\ell t}\}_{n\in\mathbb{Z}}$ forms a Riesz basis in its closed linear span $\hat{\mathcal{E}}_h^\ell$ in $L^2(0, 2\mathcal{M})$; $\dim H(mod\hat{\mathcal{E}}_h^\ell)$ satisfies (2.18) as well.

3. Statement of main results

In this section, we formulate the results on controllability Problems A)-D) given in Section 1. We start with Problems A)-C) and restrict ourselves to the case of simple spectrum of the operator $\mathfrak{L}_\ell$. Even in this case (when there are no associated vectors), the results are quite nontrivial. In Section 5, we extend the proofs to the case of multiple eigenvalues.

To answer questions A)-C), we study evolution problem (1.13), (1.14) in the energy space $\mathfrak{H}_\ell$ with the energy norm (2.8). Let us represent initial boundary value problem (1.13), (1.14) in the form of the following operator equation in $\mathfrak{H}_\ell$ for the function $U = \begin{pmatrix} u \\ u_t \end{pmatrix} \equiv \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$:

$$U_t = i\mathfrak{L}_\ell U + \hat{G}, \quad U|_{t=0} = U_0 \in \mathfrak{H}_\ell, \tag{3.1}$$

where $\mathfrak{L}_\ell$ is given by (2.7), (2.9) and

$$U_0(r) = \begin{pmatrix} u_0^0(r) \\ u_1^0(r) \end{pmatrix}, \quad \hat{G}(r,t) = f(t)G(r), \quad G(r) = \begin{pmatrix} 0 \\ g(r) \end{pmatrix}. \tag{3.2}$$

The crucial fact about the dynamics generator $\mathfrak{L}_\ell$ (Theorem 2.2) is that $\mathfrak{L}_\ell$ is a Riesz spectral operator. We also need Theorem 2.4, (case a)) which says that the set of nonharmonic exponentials $\{e^{i\bar{\lambda}_n^\ell t}\}_{n\in\mathbb{Z}}$ forms an R-basis in $\mathcal{E}_h^\ell \subset L^2(0, 2\mathcal{M})$, where $\mathcal{M}$ is given in (2.5). Having these results at our disposal, we are in a position to formulate the controllability results.

Assume that $U_0, G \in \mathfrak{H}_\ell$ have the following expansions with respect to Riesz basis (2.14):

$$U_0(r) = \sum_{n\in\mathbb{Z}} u_n^0 \mathcal{F}_n^\ell(r), \quad G(r) = \sum_{n\in\mathbb{Z}} g_n \mathcal{F}_n^\ell(r). \tag{3.3}$$

THEOREM 3.1 (Problem A). – *Assume that the spectrum of the operator $\mathfrak{L}_\ell$ is simple (no associated functions):*

(i) If $f = 0$, then the following statements hold.

a) Problem (1.13), (1.14) is exactly controllable on the time interval $[0, 2\mathcal{M}]$ through the boundary control φ for any initial state $U_0 \in \mathfrak{H}_\ell$. Let $\{w_n^\ell(t)\}_{n\in\mathbb{Z}}$ be the Riesz basis in $\mathcal{E}_h^\ell$, biorthogonal [17] to the basis $\{e^{i\bar{\lambda}_n^\ell t}\}_{n\in\mathbb{Z}}$, i.e., $\int_0^{2\mathcal{M}} e^{-i\lambda_m^\ell t} w_n^\ell(t)dt = \delta_{m,n}$. The desired boundary control function, which brings the system to zero state on the time interval $[0, 2\mathcal{M}]$, is uniquely defined by the formula:

$$\varphi(t) = \frac{2}{p(a)} \sum_{n\in\mathbb{Z}} u_n^0 w_n^\ell(t). \tag{3.4}$$

b) If $T < 2\mathcal{M}$, then the system is not controllable in time T for an arbitrary initial condition $U_0 \in \mathfrak{H}_\ell$.

c) If $T > 2\mathcal{M}$, then the system is controllable in time T and our control problem has infinitely many solutions $\varphi \in L^2(0, T)$.

(ii) If $\varphi = 0$, then assume

$$g_n \neq 0 \text{ for all } n \in \mathbb{Z}. \tag{3.5}$$

The following statements hold:

a) For a given $g \in L^2(0, a)$ the system (1.13), (1.14) is controllable on the time interval $[0, 2\mathcal{M}]$ if and only if the initial state $U_0 \in \mathfrak{H}_\ell^g$, where $\mathfrak{H}_\ell^g$ is the dense subspace of $\mathfrak{H}_\ell$ defined by the condition

$$\{\gamma_n = u_n^0/g_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}), \text{ i.e., } \sum_{n\in\mathbb{Z}} |\gamma_n|^2 < \infty. \tag{3.6}$$

The desired control function $f(t)$ is defined by the formula

$$f(t) = -\sum_{n\in\mathbb{Z}} \gamma_n w_n^\ell(t), \tag{3.7}$$

where $\{\gamma_n\}_{n\in\mathbb{Z}}$ is given by (3.6) and $\{w_n^\ell(t)\}_{n\in\mathbb{Z}}$ is the Riesz basis in $\mathcal{E}_h^\ell$ defined in (i).

Claims b) and c) which follow (3.4) are valid in this case as well.

THEOREM 3.2 (Problem B). – *If the terminal state of system (1.13), (1.14), is*

$$U(x, T) = \delta U_0(x), \tag{3.8}$$

then the solution of the closed loop problem is given by formulas similar to (3.4) and (3.7) with some additional factors under the summation sign. Namely,

(i) If $f = 0$ then the boundary control is given by

$$\varphi(t) = \frac{2}{p(a)} \sum_{n\in\mathbb{Z}} (1 - \delta e^{-i\lambda_n^\ell T}) u_n^0 w_n^\ell(t), \quad T = 2\mathcal{M}. \tag{3.9}$$

(ii) If $\varphi = 0$ and (3.6) is satisfied, then the distributed control function is given by

$$f(t) = -\sum_{n\in\mathbb{Z}}(1 - \delta e^{-i\lambda_n^\ell T})\gamma_n w_n^\ell(t), \quad T = 2\mathcal{M}. \tag{3.10}$$

We consider the periodic control problem only for the case $f \neq 0, \varphi = 0$. The case of the boundary control can be done in a similar manner.

THEOREM 3.3 (Problem C). – *If the initial state U_0 satisfies (3.6), then for any $T_1 > 0$ there exists a control f which returns the system to the initial state on the time interval $[0, T_1 + T]$ with $T = 2\mathcal{M}$; f is defined by the formula*

$$f(T_1 + t) = \sum_{n\in\mathbb{Z}} \gamma_n(e^{i\lambda_n^\ell T_1} - e^{-i\lambda_n^\ell T})w_n^\ell(t), \quad t \geq 0. \tag{3.11}$$

(Note that formula (3.11) makes sense for any $T_1 \geq 0$. Moreover, when $T_1 = 0$, we obtain (3.10) with $\delta = 1$).

For the case when both $f \neq 0$ and $\varphi \neq 0$, we refer to our work [15] in which the damped string equation is considered. It turns out that in this case, the control time can be reduced to $T = \mathcal{M}$.

Problem D. As was already mentioned, here we consider only problem A), i.e., the steering to zero of the initial state U_0 by the control $f(t)(\varphi = 0)$ or the boundary control $\varphi(t)(f = 0)$. The proof will be given only for the first case.

Assume that the operator $\mathfrak{L}_\ell$ has one multiple eigenvalue λ_0^ℓ with the algebraic multiplicity $(n + 1)$. (Recall that the geometric multiplicity is always equal to 1.) Let $\mathcal{F}_0^\ell$ and $\{\mathcal{F}_m^0\}_{m=1}^n$ be the eigenvector and the chain of associated vectors corresponding to λ_0^ℓ. By Theorem 2.2, the set of all root vectors of the operator $\mathfrak{L}_\ell$: $\{\mathcal{F}_m^0\}_{m=1}^n \cup \{\mathcal{F}_m^\ell\}_{m\in\mathbb{Z}}$ forms a Riesz basis in $\mathfrak{H}_\ell$. Let $U_0(x)$ and $G(x)$ be expanded as follows:

$$U_0 = u_0^0\mathcal{F}_0^\ell + \sum_{m=1}^n u_m^0\mathcal{F}_m^0 + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \widetilde{u}_m^0\mathcal{F}_m^\ell, \tag{3.12}$$

$$G = g_0\mathcal{F}_0^\ell + \sum_{m=1}^n g_m^0\mathcal{F}_m^0 + \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} g_m\mathcal{F}_m^\ell. \tag{3.13}$$

THEOREM 3.4. – *1. (Zero controllability, distributed control.) Assume that $g_m \neq 0$ for $m \neq 0$ and*

$$\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \left|\frac{\widetilde{u}_m^0}{g_m}\right|^2 < \infty. \tag{3.14}$$

Then there exists a unique control function f which steers U_0 to zero on the time interval $[0, 2\mathcal{M}]$. This function is given by the following sequence of formulas.

Let $\mathbb{G}$ *and* $\mathbb{U}$ *be the lower triangular matrices and* $\vec{T}$ *be the vector in* $\mathbb{R}^{n+1}$ *defined by*

$$\mathbb{G} = \begin{pmatrix} g_n^0 & 0 & 0 & \cdots & 0 \\ g_{n-1}^0 & g_n^0 & 0 & \cdots & 0 \\ g_{n-2}^0 & g_{n-1}^0 & g_n^0 & \cdots & 0 \\ -- & -- & -- & -- & -- \\ -- & -- & -- & -- & -- \\ g_1^0 & g_2^0 & g_3^0 & \cdots & 0 \\ g_0 & g_1^0 & g_2^0 & \cdots & g_n^0 \end{pmatrix}, \quad \mathbb{U} = \begin{pmatrix} u_n^0 & 0 & 0 & \cdots & 0 \\ u_{n-1}^0 & u_n^0 & 0 & \cdots & 0 \\ u_{n-2}^0 & u_{n-1}^0 & u_n^0 & \cdots & 0 \\ -- & -- & -- & -- & -- \\ -- & -- & -- & -- & -- \\ u_1^0 & u_2^0 & u_3^0 & \cdots & 0 \\ u_0^0 & u_1^0 & u_2^0 & \cdots & u_n^0 \end{pmatrix}$$

(3.15) $$\vec{T} = (1, T, T^2/2, T^3/3, \dots T^n/n)^*,$$

where "" means transposition and* $T = 2\mathcal{M}$ *(see (2.5)). Assume that* $g_n^0 \neq 0$, *so that* $\mathbb{G}$ *is invertible. Introduce the vector* $\vec{B} \in \mathbb{R}^{n+1}$*:*

(3.16) $$\vec{B} = \mathbb{G}^{-1}\mathbb{U}\vec{T} = \{b_i, \quad i = 0, 1, \dots n\}.$$

The desired control function is

(3.17) $$f(t) = -\sum_{m=0}^{n} b_m m! \theta_m(t) - \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \frac{\widetilde{u}_m^0}{g_m} \omega_m^\ell(t),$$

where the set

(3.18) $$\{\theta_m(t)\}_{m=0}^{n} \cup \{\omega_m^\ell(t)\}_{m \in \mathbb{Z}, m \neq 0}$$

is the Riesz basis in the subspace $\hat{\mathcal{E}}_h^\ell \in L^2(0, 2\mathcal{M})$, *biorthogonal to the Riesz basis of the form*

(3.19) $$\{t^m e^{i\lambda_0^\ell t}\}_{m=0}^{n} \cup \{e^{i\lambda_m^\ell t}\}_{m \in \mathbb{Z}, m \neq 0}.$$

(The fact that (3.19) is a Riesz basis is guaranteed by Theorem 2.3)

2. *(Zero controllability, boundary control.) Define the vector* $\vec{C} \in \mathbb{R}^{n+1}$*:*

(3.20) $$\vec{C} = \mathbb{U}\vec{T} = \{c_i, i = 0, 1, \dots, n\},$$

where $\mathbb{U}$ *and* $\vec{T}$ *are given in (3.15). Then the unique boundary control function* φ, *which steers the initial state* U_0 *(it can be any element of* $\mathfrak{H}_\ell$*) to zero on the time interval* $[0, T]$, *is given by the formula:*

(3.21) $$\varphi(t) = \frac{2}{p(a)} \left[\sum_{m=0}^{n} c_m m! \theta_m(t) + \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \widetilde{u}_m^0 \omega_m^\ell(t) \right].$$

Remark 3.2. – Theorem 3.4 is stated for the case of one multiple eigenvalue only to make the result more observable. The generalization to the case of several multiple eigenvalues is straightforward. Namely, let $\{\lambda_k^\ell, k \in S\}$ be the eigenvalues with algebraic multiplicities $n_k^\ell + 1$ (S is a finite subset of $\mathbb{Z}$). Let $\mathcal{F}_k$ and $\{\mathcal{F}_m^k\}_{m=1}^{n_k}$ be the eigenvector and the chain of associated vectors corresponding to λ_k^l. Expansion (3.12) now takes the form:

$$U_0 = \sum_{k\in S} u_k^0 \mathcal{F}_k^\ell + \sum_{k\in S}\sum_{m=1}^{n_k^\ell} u_{k,m}^0 \mathcal{F}_m^k + \sum_{m\in\mathbb{Z}\setminus S} \widetilde{u}_m^0 \mathcal{F}_m^\ell, \tag{3.22}$$

and a similar formula holds for G. For each $k \in S$ we can define the matricies $\mathbb{G}_k$, $\mathbb{U}_k$ and the vector $\vec{B}_k = \{b_{k,i}, i = 1, \ldots, n_k\}$ by the formulas similar to (3.15) and (3.16). Condition (3.14) takes the form

$$\sum_{m\in\mathbb{Z}\setminus S} \left|\frac{\widetilde{u}_m^0}{g_m}\right|^2 < \infty. \tag{3.23}$$

The desired control function is given by the formula

$$f(t) = -\sum_{k\in S}\sum_{m=0}^{n_k^\ell} b_{k,m} m! \theta_{k,m}(t) - \sum_{m\in\mathbb{Z}\setminus S} \frac{\widetilde{u}_m^0}{g_m}\omega_m^\ell(t), \tag{3.24}$$

where $\cup_{k\in S}\{\theta_{k,m}(t)\}_{m=0}^{n_k^\ell} \cup \{\omega_m^\ell(t)\}_{m\in\mathbb{Z}\setminus S}$ is the Riesz basis, biorthogonal to the Riesz basis $\cup_{k\in S}\{t^m e^{i\bar{\lambda}_k^\ell t}\}_{m=0}^{n_k^\ell} \cup \{e^{i\bar{\lambda}_m^\ell t}\}_{m\in\mathbb{Z}\setminus S}$. Formula (3.22) for the boundary control is generalized to the case of several multiple eigenvalues in a similar way.

4. Proofs of controllability results for problems A), B), C)

This section is devoted to the proofs of Theorems 3.1-3.3. These proofs begin with one common step, which consists of the reduction of the control problem to the moment problem. We stress that this reduction in the presence of a boundary control cannot be done by the method used in [16]. The method appropriate for the problems involving boundary controls for undamped wave equation with $\ell = 0$ is developed in [1, 2]. We begin by carrying out this step for the problem A) in the most general case, when both the distributed and boundary controls are taken into account. The reduction of the control problems B) and C) to the corresponding moment problems will be done by introducing appropriate changes to the derivation obtained for problem A).

We remark that, according to the well known theory of hyperbolic equations [e.g., 18, 19], problem (3.3)-(3.4) has a unique global in time solution U, which is an element of the space $C(0,T;\mathfrak{H}_\ell)$ for any $T > 0$. Depending on the properties of the initial state, forcing term, and the boundary control, this solution should be treated either as the strong or weak solution. We refer to [16] for a discussion of such questions.

1) In addition to problem (3.3)-(3.4), we consider the following initial value problem:

$$V_t = i\mathfrak{L}_\ell^* V, \quad V|_{t=0} = V_0 \in \mathfrak{H}_\ell, \tag{4.1}$$

where $\mathfrak{L}_\ell^*$ is the adjoint operator defined by differential expression (2.7), in which $(-2d(r))$ has been replaced with $2d(r)$, on the domain:

$$D(\mathfrak{L}_\ell^*) = \{U \in \mathfrak{H}_\ell : \mathfrak{L}_\ell^* U \in \mathfrak{H}_\ell, \quad \lim_{r\to 0} ru_1(r) = 0, \quad (u_0' - \bar{h}u_1)(a) = 0\}.$$

We note that problem (4.1) does not have a forcing term. Assume that the vectors $U = \begin{pmatrix} u_0(x,t) \\ u_1(x,t) \end{pmatrix}$ and $V = \begin{pmatrix} v_0(x,t) \\ v_1(x,t) \end{pmatrix}$ are the solutions of problems (3.1), (3.2) and (4.1) respectively. Then we have ($T > 0$ is not yet specified):

$$\int_0^T [(U, V_t - i\mathfrak{L}_\ell^* V)_{\mathfrak{H}_\ell} + (U_t - i\mathfrak{L}_\ell U, V)_{\mathfrak{H}_\ell}]dt = \int_0^T f(t)(G, V)_{\mathfrak{H}_\ell} dt. \tag{4.2}$$

Let us rewrite explicitly both scalar products from (4.2). Using the definition of $\mathfrak{L}_\ell^*$, we obtain:

$$(U, V_t - i\mathfrak{L}_\ell^* V)_{\mathfrak{H}_\ell} = \left(\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_{0t} - v_1 \\ v_{1t} + L_\ell v_0 - 2d(x)v_1 \end{pmatrix} \right)_{\mathfrak{H}_\ell} = \tag{4.3}$$
$$\frac{1}{2}\int_0^a \Big[p(r)u_0'\bar{v}_{0t}' - p(r)u_0'\bar{v}_1' + q(r)u_0\bar{v}_{0t} - q(r)u_0\bar{v}_1 + \ell(\ell+1)r^{-2}p(r)u_0\bar{v}_{0t} -$$
$$\ell(\ell+1)r^{-2}u_0\bar{v}_1 + \rho(r)u_1\bar{v}_{1t} + \rho(r)L_\ell\bar{v}_0 u_1 - 2d(r)\rho(r)u_1\bar{v}_1 \Big] r^2 dr,$$

where L_ℓ is given in (1.10).

For the second scalar product from (4.2), we have:

$$(U_t - i\mathfrak{L}_\ell U, V)_{\mathfrak{H}_\ell} = \left(\begin{pmatrix} u_{0t} - u_1 \\ u_{1t} + L_\ell u_0 + 2d(x)u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right)_{\mathfrak{H}_\ell} = \tag{4.4}$$
$$\frac{1}{2}\int_0^a \Big[p(r)u_{0t}'\bar{v}_0' - p(r)u_1'\bar{v}_0' + q(r)u_{0t}\bar{v}_0 - q(r)u_1\bar{v}_0 + \ell(\ell+1)r^{-2}p(r)u_{0t}\bar{v}_0 -$$
$$\ell(\ell+1)r^{-2}u_1\bar{v}_0 + \rho(r)u_{1t}\bar{v}_1 + \rho(r)L_\ell u_0\bar{v}_1 + 2\rho(r)d(r)u_1\bar{v}_1 \Big] r^2 dr.$$

Collecting together (4.3) and (4.4), we obtain:

$$(U, V_t - i\mathfrak{L}_\ell^* V)_{\mathfrak{H}_\ell} + (U_t - i\mathfrak{L}_\ell U, V)_{\mathfrak{H}_\ell} = \tag{4.5}$$
$$\frac{1}{2}\int_0^a \Big[p(r)(u_0'\bar{v}_0')_t + q(r)\rho(r)(u_0\bar{v}_0)_t + \rho(r)(u_1\bar{v}_1)_t + \ell(\ell+1)r^{-2}p(r)(u_0\bar{v}_0)_t \Big] r^2 dr$$
$$-\frac{1}{2}\int_0^a \Big[p(r)(u_0'\bar{v}_1' + u_1'\bar{v}_0') + q(r)\rho(r)(u_0\bar{v}_1 + u_1\bar{v}_0)$$
$$+ \ell(\ell+1)r^{-2}p(r)(u_0\bar{v}_1 + u_1\bar{v}_0)$$
$$- \rho(x)(L_\ell\bar{v}_0 \cdot u_1 + L_\ell u_0 \cdot \bar{v}_1) \Big] r^2 dr \equiv J_1 + J_2.$$

Taking into account an explicit formula for L_ℓ, we evaluate J_2 and have:

$$2J_2 = -\left[p(r)r^2(\bar{v}_0' u_1 + u_0'\bar{v}_1)\right]_{r=0}^{r=a}. \tag{4.6}$$

Due to the facts that $U \in D(\mathfrak{L}_\ell)$, $V \in D(\mathfrak{L}_\ell^*)$, and because of the boundary conditions at $x = a$, we have:

$$\lim_{r\to 0} ru_1(r) = \lim_{r\to 0} rv_1(r) = 0, \quad u_0'(a) = -hu_1(a) + \varphi(t), \quad \bar{v}_0'(a) = h\bar{v}_1(a). \tag{4.7}$$

Substituting (4.7) into (4.6), we obtain

$$2J_2 = -a^2 p(a)\bar{v}_1(a)\varphi(t). \tag{4.8}$$

Integrating J_1 over $[0, T]$, we have:

$$2\int_0^T J_1 dt = \int_0^a dx\big[p(r)u_0'\bar{v}_0' + q(r)\rho(r)u_0\bar{v}_0 + \rho(r)u_1\bar{v}_1 + \ell(\ell+1)r^{-2}u_0\bar{v}_0\big]_{t=0}^{t=T}. \tag{4.9}$$

Recall that according to the statement of problem A), we have $u_0(r,T) = u_1(r,T) = u_0'(r,T) = 0$. Therefore, (4.9) can be reduced to

$$\int_0^T J_1 dt = -\frac{1}{2}\int_0^a r^2 dr\big[pu_0'\bar{v}_0' + q\rho u_0\bar{v}_0 + \ell(\ell+1)r^{-2}u_0\bar{v}_0 + \rho u_1\bar{v}_1\big](x,0) = - \left(\begin{pmatrix} u_0(\cdot,0) \\ u_1(\cdot,0)\end{pmatrix}, \begin{pmatrix} v_0(\cdot,0) \\ v_1(\cdot,0)\end{pmatrix}\right)_{\mathfrak{H}_\ell}. \tag{4.10}$$

Substituting (4.8) and (4.10) into (4.5) and combining the result with (4.2), we obtain the following integral identity:

$$\left(\begin{pmatrix} u_0(\cdot,0) \\ u_1(\cdot,0)\end{pmatrix}, \begin{pmatrix} v_0(\cdot,0) \\ v_1(\cdot,0)\end{pmatrix}\right)_{\mathfrak{H}_\ell} = \frac{a^2 p(a)}{2}\int_0^T \bar{v}_1(a,t)\varphi(t)dt - \int_0^T (G,V)_{\mathfrak{H}_\ell} f(t)dt. \tag{4.11}$$

Now, let us return to problem (4.1) and consider a sequence of solutions corresponding to the following sequence of initial states: $V_n^0(r) = e^{-i\bar{\lambda}_n^\ell T}\Phi_n^*(r)$ where $\Phi_n^*(r)$ is an eigenfunction of the operator $\mathfrak{L}_\ell^*$ corresponding to the eigenvalue $\bar{\lambda}_n^\ell$. These solutions have the forms

$$V_n(r,t) = e^{-i\bar{\lambda}_n^\ell (T-t)}\Phi_n^*(r), \quad n \in \mathbb{Z}. \tag{4.12}$$

At this moment, we need a description of $\Phi_n^*(r)$ (the eigenfunction of the operator $\mathfrak{L}_\ell^*$ corresponding to the eigenvalue $\bar{\lambda}_n^\ell$). It is known (see [4, 17, 20]), that $\Phi_n^*(r)$ can be represented in the form

$$\Phi_n^*(r) = \begin{pmatrix} \frac{1}{i\bar{\lambda}_n^\ell}\varphi_n^*(r) \\ \varphi_n^*(r)\end{pmatrix}, \tag{4.13}$$

where $\varphi_n^*(r)$ is an eigenfunction of the nonselfadjoint quadratic operator pencil $\mathcal{P}_\ell^*(\lambda)$ adjoint to the pencil $\mathcal{P}_\ell(\lambda)$, given by (2.12) and (2.13). $\mathcal{P}_\ell^*(\lambda)$ is given by formula (2.12) in which $(-2i\lambda d(r))$ has been replaced with $2i\lambda d(r)$, and defined on the domain

$$D(\mathcal{P}_\ell^*(\lambda) = \Big\{rv \in H^2(0,a), \quad \lim_{r\to 0} rv(r) = 0, \quad (v' - i\lambda\bar{h}v)(a) = 0\Big\}.$$

Recall [4], that the set of eigenfunctions of $\mathcal{P}_\ell^*(\lambda)$, specified by the condition $\varphi_n^*(a) = 1$, is almost normalized. Now, let us substitute $V = V_n$ into (4.11). Using formula (4.13) for $\Phi_n^*(r)$ and the normalization condition for $\varphi_n^*(a)$, we obtain the following integral identities

$$(4.14) \qquad u_n^0 = \frac{a^2 p(a)}{2} \int_0^T e^{-i\lambda_n^\ell t} \varphi(t) dt - g_n \int_0^T e^{-i\lambda_n^\ell t} f(t) dt, \quad n \in \mathbb{Z},$$

where $g_n = (G, \Phi_n^*)_{\mathfrak{H}_\ell}$ and $u_n^0 = (U_0, \Phi_n^*)_{\mathfrak{H}_\ell}$ are the coefficients from (3.3). In (4.14), we have taken into account that the Riesz bases $\{\mathcal{F}_n^\ell\}_{n\in\mathbb{Z}}$ and $\{\Phi_n^*\}_{n\in\mathbb{Z}}$ are biorthogonal.

System (4.14) is now our main object of interest. The reconstruction of the functions f and φ from (4.14) is known as a moment problem [3, 21].

2) Now we are in a position to complete the proof of Theorem 3.1 (Problem A)).

(i) $f = 0$, $\varphi \neq 0$ (boundary control). In this case, moment problem (4.14) has the form

$$(4.15) \qquad u_n^0 = \frac{p(a)}{2} \int_0^T e^{-i\lambda_n^\ell t} \varphi(t) dt, \quad n \in \mathbb{Z}.$$

Integrals in the right of Eq. (4.15) can be treated as the generalized Fourier coefficients of the function φ with respect to the set of nonharmonic exponentials $\{e^{i\bar{\lambda}_n^\ell t}\}_{n\in\mathbb{Z}}$, which is a Riesz basis in $\mathcal{E}_h^\ell$ (Theorem 2.3, Statement 2)). Due to the fact that $\{u_n^0\}_{n\in\mathbb{Z}}$ are also the generalized Fourier coefficients of U_0 with respect to the Riesz basis $\{\mathcal{F}_n^\ell\}_{n\in\mathbb{Z}}$ in $\mathfrak{H}_\ell$, we have

$$(4.16) \qquad \{u_n^0\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}).$$

As follows from the moment problem theory [3, 21] in the case $T = 2\mathcal{M}$, the control φ can be uniquely reconstructed from Eqs. (4.15) and is given by (3.4).

In the case $T < 2\mathcal{M}$, the system of the aforementioned exponentials is overloaded in $L^2(0, T)$, i.e., it is not minimal. (Recall that minimality is equivalent to the linear independence of infinite set of vectors.) Therefore, the system of equations (4.15) is overdetermined and, in general, does not have a solution. (For particular choices of U_0, this system may have a solution. However, for a given $T < 2\mathcal{M}$ the characterization of all U_0, for which system (4.15) is consistent, is a nontrivial problem. This problem is related to subtle properties of bases of nonharmonic exponentials, and we do not consider it here.)

In the case $T > 2\mathcal{M}$, our system of exponentials forms a Riesz basis in its closed linear span $\mathcal{E}_h^\ell$, but it is not complete in $L^2(0, T)$. Moreover, dim $L^2(0, T)(mod\ \mathcal{E}_h^\ell) = \infty$. Therefore, in this case, the solution of Eq. (4.15) exists, but it is defined up to an addition of an arbitrary function $\tilde{\varphi} \in \mathcal{E}_h^{\ell\perp}$, which means that the control problem has infinitely many solutions.

(ii) $\varphi = 0$, $f \neq 0$ (distributed control). Formula (3.7) can be obtained exactly as it was done in our paper [16], where we have used the method of the spectral decomposition.

In the present paper, we suggest an alternative to [16] method for the derivation of the moment problem (4.14). For the case of a distributed control either method can be used; for the case of a boundary control only one method works – the one discussed in the present paper.

Proof of Theorem 3.2 (Problem B). – To solve the closed loop control problem, i.e., $U(x,T) = \delta U_0(x)$, we have to derive the corresponding moment problem. Let us return to formula (4.9). Now we have the following conditions at the moment T:

$$u_0(x,T) = \delta u_0^0(x), \quad u_1(x,T) = \delta u_1^0(x), \quad u_0'(x,T) = \delta (u_0^0)'(x). \tag{4.17}$$

Substituting (4.17) into (4.9), we obtain

$$\int_0^T J_1 dt = \delta \left(\begin{pmatrix} u_0(\cdot,0) \\ u_1(\cdot,0) \end{pmatrix}, \begin{pmatrix} v_0(\cdot,T) \\ v_1(\cdot,T) \end{pmatrix} \right)_{\mathfrak{H}_\ell} - \left(\begin{pmatrix} u_0(\cdot,0) \\ u_1(\cdot,0) \end{pmatrix}, \begin{pmatrix} v_0(\cdot,0) \\ v_1(\cdot,0) \end{pmatrix} \right)_{\mathfrak{H}_\ell}. \tag{4.18}$$

Let us substitute $V = V_n$ (V_n from (4.12)) into (4.18) and obtain the following relation:

$$\int_0^T J_1 dt = (\delta - e^{i\lambda_n^\ell T}) u_n^0, \quad \text{where } u_n^0 = (U_0, \Phi_n^*)_{\mathfrak{H}_\ell}. \tag{4.19}$$

Collecting together (4.8), (4.19) and (4.5), we obtain from (4.2) a new moment problem:

$$(1 - \delta e^{-i\lambda_n^\ell T}) u_n^0 = -\frac{p(a)}{2} \int_0^T e^{-i\lambda_n^\ell t} \varphi(t)\, dt + g_n \int_0^T e^{-i\lambda_n^\ell t} f(t)\, dt, \quad n \in \mathbb{Z}. \tag{4.20}$$

Formulas (3.9)-(3.10) can be derived from (4.20) in precisely the same way as (3.4) and (3.7) have been obtained from (4.14).

Theorem is shown.

Remark 4.1. – The controllability result for Problem B) can be strengthened for a special value of the constant δ. Let us explain this briefly. Consider case (ii) of a distributed control: $\varphi = 0$, $f \neq 0$. Assume that restriction (3.6) on the initial state U_0 is not satisfied, i.e., $\{\gamma_n = u_n^0 / g_n\}_{n \in \mathbb{Z}} \notin \ell^2(\mathbb{Z})$, which means $U_0 \notin \mathfrak{H}_\ell^g$. Assume instead that a weaker condition holds:

$$\{u_n^0 / (n g_n)\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}). \tag{4.21}$$

We claim that, in this case, there exists a special value $\delta = \tilde{\delta}$ such that

$$\{(\tilde{\delta} e^{-i\lambda_n^\ell T} - 1) u_n^0 / g_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \tag{4.22}$$

and, therefore, as is clear from moment equations (4.20), the closed loop problem with this particular value of δ has a solution.

Note that (4.21) defines a subspace $\tilde{\mathfrak{H}}_\ell^g \subset \mathfrak{H}_\ell$ such that $\mathfrak{H}_\ell^g \subset \tilde{\mathfrak{H}}_\ell^g$. Note, also, that $\mathfrak{H}_\ell^g$ is always dense in $\mathfrak{H}_\ell$ but can never be equal to $\mathfrak{H}_\ell$ if $g \in L^2(0,a)$. However, if g is such that $|g_n| \geq C n^{-1}$ for some $C > 0$ (this is possible for $g \in L^2(0,a)$), then $\tilde{\mathfrak{H}}_\ell^g = \mathfrak{H}_\ell$. So, the system is controllable for any initial state $U_0 \in \mathfrak{H}_\ell$.

Let us show how to find this $\tilde{\delta}$.

Based on the fact that $T = 2\mathcal{M}$ and using (2.16), we have the following representation:

$$e^{-i\lambda_n^\ell T} = -e^{\mathcal{N}}\alpha_+\alpha_-^{-1} + O(|n|^{-1}\ell n|n|). \tag{4.23}$$

If we take

$$\tilde{\delta} = -\alpha_-\alpha_+^{-1}e^{-\mathcal{N}}, \tag{4.24}$$

then (4.22) is obviously satisfied.

In case (i) of the boundary control, we can obtain a similar result. Namely, if $\delta = \tilde{\delta}$, then the initial state U_0 can be taken from the class defined by the condition:

$$\sum_{n\in\mathbb{Z}} |u_n^0|^2/(1+|n|)^2 < C. \tag{4.25}$$

This class is larger than the energy space $\mathfrak{H}_\ell$ and contains distributions.

Proof of Theorem 3.3 (Problem C). – We discuss only the case of distributed control. We are looking for the function $f(t)$ which is not equal to zero only for $t \geq T_1 > 0$. From the requirement that $U(x, T_1 + T) = U_0(x)$ we obtain, in precisely the same way as above, the following moment problem:

$$(1 - e^{i\lambda_n^\ell (T_1+T)})u_n^0 = g_n \int_{T_1}^{T_1+T} e^{i\lambda_n^\ell (T_1+T-\tau)} f(\tau)d\tau. \tag{4.26}$$

Solution of (4.26) is given by (3.11). The proof is complete.

5. Multiple eigenvalues

We give the proof of Theorem 3.4 only for the case of the distributed control. All steps for the boundary control can be done in a similar and even easier way. We also note that the proof below is completed for one multiple eigenvalue since the generalization for several multiple eigenvalues can be done without any difficulties.

Proof of Theorem 3.4. – Let us consider control problem A) given by Eq. (3.1) with $\varphi(t) = 0$.To solve this problem, we use the method of spectral decomposition. Let us look for a solution of problem (3.1) in the form of an expansion with respect to the Riesz basis of the root vectors of the operator $\mathfrak{L}_\ell$. We have:

$$U(x,t) = \sum_{m=1}^{n} a_m^0(t)\mathcal{F}_m^0(x) + \sum_{m\in\mathbb{Z}} a_m(t)\mathcal{F}_m^\ell(x). \tag{5.1}$$

Let $\mathcal{H}_{\lambda_0}^\ell$ be the linear span of the root vectors corresponding to the eigenvalue λ_0^ℓ; $\dim \mathcal{H}_{\lambda_0}^\ell = n + 1$. It is clear that $a_0(t)\mathcal{F}_0^\ell + \sum_{m=1}^n a_m^0(t)\mathcal{F}_m^0 \in \mathcal{H}_{\lambda_0}^\ell$. We recall that the root vectors satisfy the following system of equations

$$(\mathfrak{L}_\ell - \lambda_0^\ell I)\mathcal{F}_0^\ell = 0, \ (\mathfrak{L}_\ell - \lambda_0^\ell I)\mathcal{F}_j^0 = \mathcal{F}_{j-1}^0, \ j = 1, 2, \ldots, n \text{ and } \mathcal{F}_0^0 \equiv \mathcal{F}_0. \tag{5.2}$$

Substituting (5.1) into Eq. (3.1) and taking all terms which belong to the subspace $\mathcal{H}^{\ell}_{\lambda_0}$ to the left, we have

$$\begin{aligned}(5.3)\qquad & \{-\dot{a}_0(t)\mathcal{F}^{\ell}_0 + i\lambda_0 a_0(t)\mathcal{F}^{\ell}_0 + f(t)g_0\mathcal{F}^{\ell}_0\}+\\ & \sum_{m=1}^{n}(-\dot{a}_m(t)\mathcal{F}^0_m + ia^0_m(t)\mathfrak{L}_\ell\mathcal{F}^0_m + f(t)g^0_m\mathcal{F}^0_m)\\ & = \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}(\dot{a}_m(t) - i\lambda^l_m a_m(t) - f(t)g_m)\mathcal{F}^{\ell}_m.\end{aligned}$$

Here, by the dot over $a_m(t)$, we denote the differentiation with respect to t.

In Eq. (5.3), the left-hand side belongs to $\mathcal{H}^{\ell}_{\lambda_0}$, while the right-hand side belongs to the closed span of the eigenvectors $\{\mathcal{F}^{\ell}_m, m \in \mathbb{Z}, m \neq 0\}$. Since these two subspaces complement each other in $\mathfrak{H}_\ell$, the problem can be naturally separated into two independent problems. From now on, we concentrate on the problem in $\mathcal{H}^{\ell}_{\lambda_0}$. Due to the fact that all associated vectors are linearly independent, we obtain the following system of coupled initial value problems in $\mathcal{H}^{\ell}_{\lambda_0}$:

$$\begin{aligned}& \dot{a}_0(t) = i\lambda^{\ell}_0 a_0(t) + ia^0_1(t) + g_0 f(t),\ \ a_0(0) = u^0_0,\\ (5.4)\qquad & \dot{a}^0_j(t) = i\lambda^{\ell}_0 a^0_j(t) + ia^0_{j+1}(t) + g^0_j f(t),\ \ a^0_j(0) = u^0_j,\ \ 1 \le j \le n-1,\\ & \dot{a}^0_n(t) = i\lambda^{\ell}_0 a^0_n(t) + g^0_n f(t),\ \ a^0_n(0) = u^0_n.\end{aligned}$$

The last problem in (5.4) is independent, and its solution can be given in the form

$$(5.5)\qquad a^0_n(t) = u^0_n e^{i\lambda^{\ell}_0 t} + g^0_n\int_0^t e^{i\lambda^{\ell}_0(t-\tau)}f(\tau)dt.$$

The following recurrence formula is valid for the solutions of the rest n equations $(1 \le p \le n)$:

$$(5.6)\quad a^0_{n-p}(t) = u^0_{n-p}e^{i\lambda^{\ell}_0 t} + g^0_{n-p}\int_0^t e^{i\lambda^{\ell}_0(t-\tau)}f(\tau)d\tau + i\int_0^t e^{i\lambda^{\ell}_0(t-\tau)}a^0_{n-p+1}(\tau)d\tau.$$

Now we introduce the following two-parameter family of linear operators:

$$(5.7)\qquad (\mathcal{T}(\alpha,\beta)f)(t) = \alpha + \beta\int_0^t e^{-i\lambda^{\ell}_0\tau}f(\tau)d\tau.$$

We claim that if we identify $a_0(t)$ with $a^0_0(t)$, then the following formula holds $(1 \le p \le n)$:

$$\begin{aligned}(5.8)\qquad & a^0_{n-p}(t) = e^{i\lambda^{\ell}_0 t}\Bigg[(\mathcal{T}(u^0_{n-p}, g^0_{n-p})f)(t)+\\ & \sum_{k=0}^{p-1} i^{k+1}\int_0^t \frac{(t-\tau)^k}{k!}\big(\mathcal{T}(u^0_{n-p+k+1}, g^0_{n-p+k+1})f\big)(\tau)d\tau\Bigg].\end{aligned}$$

Let us prove (5.8) using the induction argument. We easily verify the above formula for $p = 1$ by a straitforward computation using (5.6) and (5.7). Now, assuming that (5.8) is valid for some p, one has to prove that the same formula holds if p is replaced with $p + 1$.

From (5.6), it follows that for $p + 1$ we have

$$a_{n-p-1}^0(t) = u_{n-p-1}^0 e^{i\lambda_0^\ell t} + g_{n-p-1}^0 \int_0^t e^{i\lambda_0^\ell (t-\tau)} f(\tau)d\tau + i \int_0^t e^{i\lambda_0^\ell (t-\tau)} a_{n-p}^0(\tau)d\tau.$$

Denote by J the right-hand side of the latter equation. Using (5.7) and the induction assumption (5.8) about a_{n-p}, we rewrite J in the form

$$J = e^{i\lambda_0^\ell t}(\mathcal{T}(u_{n-p-1}^0, g_{n-p-1}^0)f)(t) + ie^{i\lambda_0^\ell t}\left\{ \int_0^t (\mathcal{T}(u_{n-p}^0, g_{n-p}^0)f)(\tau)d\tau \right.$$
$$\left. + \sum_{k=0}^{p-1} i^{k+1} \int_0^t d\tau \int_0^\tau \frac{(\tau-\eta)^k}{k!} (\mathcal{T}(u_{n-p+k+1}^0, g_{n-p+k+1}^0)f)(\eta)d\eta \right\}. \tag{5.9}$$

By changing the order of integration in the double integral in (5.9) and then intergrating by parts, we obtain

$$\int_0^t d\tau \int_0^\tau \frac{(\tau-\eta)^k}{k!} (\mathcal{T}(u_{n-p+k+1}^0, g_{n-p+k+1}^0)f)(\eta)d\eta =$$
$$\int_0^t d\eta(\mathcal{T}(u_{n-p+k+1}^0, g_{n-p+k+1}^0)f(\eta)) \int_\eta^t \frac{(\tau-\eta)^k}{k!} d\tau =$$
$$\int_0^t \frac{(t-\eta)^{k+1}}{(k+1)!} (\mathcal{T}(u_{n-p+k+1}^0, g_{n-p+k+1}^0)f)(\eta)d\eta. \tag{5.10}$$

Taking into account Eq. (5.10), one can see that:

$$\sum_{k=0}^{p-1} i^{k+2} \int_0^t \frac{(t-\eta)^{k+1}}{(k+1)!} (\mathcal{T}(u_{n-p+k+1}^0, g_{n-p+k+1}^0)f)(\eta)d\eta$$
$$= \sum_{m=1}^{p} i^{m+1} \int_0^t \frac{(t-\tau)^m}{m!} (\mathcal{T}(u_{n-p+m}^0, g_{n-p+m}^0)f)(\eta)d\eta. \tag{5.11}$$

Substituting (5.11) into (5.9), we arrive at (5.8) with p replaced with $p + 1$. So, formula (5.8) is shown.

In terms of the operators $\mathcal{T}(\alpha, \beta)$, the moment problem corresponding to (5.4) can be written in the form $(1 \le p \le n)$:

$$(\mathcal{T}(u_n^0, g_n^0)f)(T) = 0,$$
$$(\mathcal{T}(u_{n-p}^0, g_{n-p}^0)f)(T)$$
$$+ \sum_{k=0}^{p-1} i^{k+1} \int_0^T \frac{(T-\tau)^k}{k!} (\mathcal{T}(u_{n-p+k+1}^0, g_{n-p+k+1}^0)f)(\tau)d\tau = 0. \tag{5.12}$$

Using definition (5.7) and setting

$$\widehat{f}(t) = \int_0^t e^{-i\lambda_0^\ell \tau} f(\tau)d\tau, \tag{5.13}$$

we obtain

$$\begin{aligned} &\int_0^T (T-\tau)^k (\mathcal{T}(u^0_{n-p+k+1}, g^0_{n-p+k+1})f)(\tau)d\tau = \\ &u^0_{n-p+k+1}\int_0^T (T-\tau)^k d\tau + g^0_{n-p+k+1}\int_0^T (T-\tau)^k \widehat{f}(\tau)d\tau = \\ &\frac{T^{k+1}}{k+1}u^0_{n-p+k+1} + g^0_{n-p+k+1}\int_0^T (T-\tau)^k \widehat{f}(\tau)d\tau. \end{aligned} \tag{5.14}$$

Substituting (5.14) into (5.12), we arrive at the moment problem ($1 \le p \le n$):

$$\begin{aligned} &u^0_n + g^0_n \widehat{f}(T) = 0, \\ &u^0_{n-p} + g^0_{n-p}\widehat{f}(T) + \sum_{k=0}^{p-1} \frac{(iT)^{k+1}}{(k+1)!} u^0_{n-p+k+1} \\ &\quad + \sum_{k=0}^{p-1} g^0_{n-p+k+1} \int_0^T \frac{(T-\tau)^k}{k!} \widehat{f}(\tau)d\tau = 0. \end{aligned} \tag{5.15}$$

Let $\vec{F} \in \mathbb{R}^{n+1}$ be the following vector:

$$\begin{aligned} \vec{F} = \Big(\widehat{f}(T),\ &\int_0^T \widehat{f}(\tau)d\tau,\ \int_0^T (T-\tau)\widehat{f}(\tau)d\tau, \ldots \\ &\int_0^T \frac{(T-\tau)^k}{k!}\widehat{f}(\tau)d\tau, \ldots \int_0^T \frac{(T-\tau)^{n-1}}{(n-1)!}\widehat{f}(\tau)d\tau\Big)^*. \end{aligned} \tag{5.16}$$

Recall that the superscript "*" means the transposition. Using the matrices $\mathbb{G}$ and $\mathbb{U}$ and the vector $\vec{T}$ defined in (3.15), we arrive at the following form for the moment problem (5.15):

$$\mathbb{U}\vec{T} + \mathbb{G}\vec{F} = 0. \tag{5.17}$$

Since $\mathbb{G}$ is invertible, we immediately obtain

$$\vec{F} = -\mathbb{G}^{-1}\mathbb{U}\vec{T} = -\{b_i\}_{i=0}^n, \tag{5.18}$$

where b_i are defined in (3.16). From (5.16) and (5.18), we obtain another moment problem:

$$\widehat{f}(T) = -b_0, \qquad \int_0^T \frac{(T-\tau)^k}{k!}\widehat{f}(\tau)d\tau = -b_{k+1}, \qquad 0 \le k \le n-1. \tag{5.19}$$

Now we use the definition (5.13) of $\widehat{f}$ and have:

$$(5.20)\qquad \int_0^T (T-\tau)^k \widehat{f}(\tau)d\tau = \int_0^T (T-\tau)^k d\tau \int_0^\tau e^{-i\lambda_0^\ell \eta} f(\eta)d\eta = \\ -\left[\frac{(T-\tau)^{k+1}}{k+1}\int_0^\tau e^{-i\lambda_0^\ell \eta} f(\eta)d\eta\right]_0^T + \int_0^T \frac{(T-\tau)^{k+1}}{k+1} e^{-i\lambda_0^\ell \tau} f(\tau)d\tau.$$

Using the change of variables $\tau \mapsto T-\tau$ and taking into account (5.20), we transform system (5.19) to the following moment problem

$$(5.21)\qquad \int_0^T \tau^k e^{i\lambda_0^\ell \tau} f(T-\tau)d\tau = -k! b_k e^{i\lambda_0^\ell T}, \quad 0 \le k \le n.$$

Now let us return to the moment problem corresponding to the original problem (5.3) in $\mathfrak{H}_\ell$, i.e., in addition to the system (5.21), we have the following sequence of equations:

$$(5.22)\qquad \int_0^T e^{-i\lambda_m^\ell \tau} f(T-\tau)d\tau = -\frac{u_m^0}{g_m} e^{i\lambda_m^\ell T}.$$

System (5.21), (5.22) has the solution:

$$(5.23)\qquad f(T-\tau) = -\sum_{m=0}^{n} m! b_m e^{i\lambda_0^\ell T}\theta_m(t) - \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{u_m^0}{g_m}\omega_m^\ell(t) e^{i\lambda_m^\ell T}.$$

where $\{\theta_m(t)\}_{m=0}^n \cup \{\omega_m^\ell(t)\}_{m\in\mathbb{Z}, m\neq 0}$ is the Riesz basis described in (3.18).

The final step consists of the following. If $\omega_m^\ell(t)$ is the function orthogonal to the closure of the linear span of exponentials $\{\chi_k = e^{i\bar{\lambda}_k^\ell t}\}_{k\in\mathbb{Z}, k\neq m}$ and such that $(\chi_m, \omega_m^\ell)_{L^2(0,T)} = 1$, then the function $e^{i\lambda_m^\ell T}\omega_m^\ell(t)$ is biorthogonal to the set $\{\chi_k(T-t) = e^{i\bar{\lambda}_k^\ell T - i\bar{\lambda}_k^\ell t}, k \in \mathbb{Z}\}$. The same is valid for the function $\theta_m(t)$. Therefore, we have:

$$\omega_m^\ell(T-t) = \omega_m^\ell(t) e^{i\lambda_m^\ell T}, \quad m \in \mathbb{Z}, \quad m \neq 0; \quad \theta_m(T-t) = \theta_m(t) e^{i\lambda_0^\ell T}, \quad 0 \le m \le n.$$

Taking into account the latter observation, we arrive at (3.17). Theorem 3.4 is completely shown.

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