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Fractal Interpolation Surfaces derived from Fractal Interpolation Functions ☆

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Abstract

Based on the construction of Fractal Interpolation Functions, a new construction of Fractal Interpolation Surfaces on arbitrary data is presented and some interesting properties of them are proved. Finally, a lower bound of their box counting dimension is provided.

Keywords: Fractal Interpolation Functions; Fractal Interpolation Surfaces; Box counting dimension; Fractals; Smooth fractal surfaces

1. Introduction

Fractal Interpolation Surfaces (FIS) are usually constructed as graphs of continuous functions with the help of Iterated Function Systems (IFS) or Recurrent Iterated Function Systems (RIFS). However, their construction encounters some difficulties that have not yet been overcome. Several constructions have been introduced that confront these problems. Most of them take the interpolation points on a triangular grid and use affine mappings that define an IFS (see [11,13,15]), thus the emerging surface is self-affine. In addition, they constrain the interpolation points (to be coplanar at the boundary of the triangular region) or the contraction factors of the

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affine maps (all contraction factors should be equal). Hence, these constructions lack the flexibility needed to model complex natural surfaces. In [9] and [12] bivariate functions were used to address the problem on a rectangular grid. This approach was generalized and extensively studied in [5] and [7] where RIFS were used. In the same paper the box-counting dimension of the FIS was explicitly computed. The latter construction, however flexible may be, still constrains the interpolation points and the contraction factors.

In this paper we construct fractal interpolation surfaces as graphs of continuous functions on arbitrary data points (placed on rectangular grids) using fractal interpolation functions. This construction enables the control of the box dimension of the fractal surface, giving a lower bound of it, independently of the interpolation points. One may produce a fractal interpolation surface as rough as he wants it to be. The mathematical background on IFS, RIFS and FIF together with a lemma concerning the stability of FIF is given in Section 2. In Section 3 we describe the new construction in detail and prove some interesting properties. Finally, in Section 4 we give a lower bound for the box-counting dimension of the FIS produced by the aforementioned construction.

2. Fractal Interpolation Functions

2.1. IFS–RIFS

A *hyperbolic Iterated Function System*, or IFS for short, is defined as a pair consisted of a complete metric space (X, ρ) together with a finite set of continuous contractive mappings $w_i : X \to X$, with respective contraction factors s_i for $i = 1, 2, \ldots, N$ $(N \geqslant 2)$. The attractor of a hyperbolic IFS is the unique set E for which $E = \lim_{k\to\infty} W^k(A_0)$ for every starting compact set A_0, where

$$W(A) = \bigcup_{i=1}^{N} w_i(A) \quad \text{for all } A \in \mathcal{H}(X),$$

and $(\mathcal{H}(X), h)$ is the metric space of all nonempty compact subsets of X with respect to the Hausdorff metric h. Iterated Function Systems are able to produce very complicated attractors using only a handful of mappings.

A more general concept, that allows the construction of even more complicated sets, is that of the *Recurrent Iterated Function System*, or *RIFS* for short, which consists of the IFS $\{X; w_i,\ i = 1, 2, \ldots, N\}$ (or more briefly $\{X; w_{1-N}\}$) together with an irreducible row-stochastic matrix $(p_{n,m} \in [0, 1]\colon\ n, m = 1, \ldots, N)$, such that

$$\sum_{m=1}^{N} p_{n,m} = 1, \quad n = 1, \ldots, N. \tag{1}$$

The recurrent structure is given by the (irreducible) *connection matrix* $C = (C_{nm})^N$ which is defined by

$$C_{n,m} = \begin{cases} 1, & \text{if } p_{m,n} > 0, \\ 0, & \text{if } p_{m,n} = 0, \end{cases}$$

where $n, m = 1, 2, \ldots, N$. The transition probability for a certain discrete time Markov process is $p_{n,m}$, which gives the probability of transfer into state m given that the process is in state n. Condition (1) says that whichever state the system is in (say n), a set of probabilities is available that sum to one and describe the possible states to which the system transits at the next step.

We define mappings

$$W_{i,j} : \mathcal{H}(X) \to \mathcal{H}(X), \quad \text{with } W_{i,j}(A) = \begin{cases} w_i(A), & p_{j,i} > 0, \\ \emptyset, & p_{j,i} = 0, \end{cases} \tag{2}$$

for all $A \in \mathcal{H}(X)$ and the metric space

$$\tilde{\mathcal{H}}(X) = \mathcal{H}(X)^N = \mathcal{H}(X) \times \mathcal{H}(X) \times \cdots \times \mathcal{H}(X)$$

equipped with the metric

$$\tilde{h}\left(\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix}, \begin{pmatrix} B_1 \\ B2 \\ \vdots \\ B_N \end{pmatrix}\right) = \max\bigl\{h(A_i, B_i);\ i = 1, 2, \ldots, N\bigr\}.$$

Easily we can prove that $\langle \hat{H}, \hat{h} \rangle$ is a complete metric space. Now, we define the map

$$\boldsymbol{W} : \tilde{\mathcal{H}}(X) \to \tilde{\mathcal{H}}(X): \quad \boldsymbol{W}\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \ldots & W_{1N} \\ W_{21} & W_{22} & \ldots & W_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ W_{N1} & W_{N2} & \ldots & W_{NN} \end{pmatrix} \cdot \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix}$$

$$= \begin{pmatrix} \bigcup_{j \in I(1)} w_1(A_j) \\ \bigcup_{j \in I(2)} w_2(A_j) \\ \vdots \\ \bigcup_{j \in I(N)} w_N(A_j) \end{pmatrix},$$

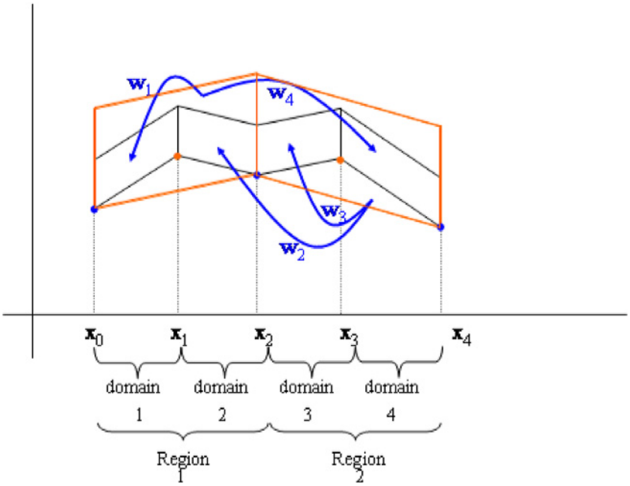
where $I(i) = \{j\colon\ p_{j,i} > 0\}$, for $i = 1, 2, \ldots, N$. If all w_i are contractions, then $\boldsymbol{W}$ is a contraction and there is an $\boldsymbol{E} = (E_1, E_2, \ldots, E_N)^t \in \tilde{\mathcal{H}}(X)$ such that $\boldsymbol{W}(\boldsymbol{E}) = \boldsymbol{E}$ and $E_i = \bigcup_{j \in I(i)} w_i(E_j)$, for $i = 1, 2, \ldots, N$.

Let $A \in \mathcal{H}(X)$. We define sequences $\{\boldsymbol{A}_n\}_{n \in \mathbb{N}}$ in $\tilde{\mathcal{H}}(X)$ and $\{A_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}(X)$ as follows: $\boldsymbol{A}_0 = (A, A, \ldots, A)^t$, $\boldsymbol{A}_n = \boldsymbol{W}(\boldsymbol{A}_{n-1})$ and $A_n = \bigcup_{i=1}^N (\boldsymbol{A}_n)_i$, for $n \in \mathbb{N}$ where $\boldsymbol{A}_n = ((\boldsymbol{A}_n)_1, (\boldsymbol{A}_n)_2, \ldots, (\boldsymbol{A}_n)_N)$. Then, the set $G = \bigcup_{i=1}^N E_i$ is called the attractor of the RIFS $\{X, w_{1-N}, P\}$. Evidently

$$G = \lim_n A_n.$$

2.2. Fractal Interpolation Functions and their stability

Barnsley in [3] was the first to introduce *Fractal Interpolation Functions* (*FIFs*) that are derived as attractors of IFSs or RIFSs and interpolate given data points. Here we briefly describe this construction based on RIFSs as we will use it in our method (for further details see [1–3]). Let $X = [0, 1] \times \mathbb{R}$ and $\Delta = \{(x_i, y_i)\colon\ i = 0, 1, \ldots, N\}$ be an interpolation set with $N + 1$ interpolation points such that $0 = x_0 < x_1 < \cdots < x_N = 1$. The interpolation points divide $[0, 1]$ into N intervals $I_i = [x_{i-1}, x_i]$, $i = 1, \ldots, N$, which we call *domains*. In addition, let $Q = \{(\hat{x}_j, \hat{y}_j)\colon\ j = 0, 1, \ldots, M\}$ be a subset of Δ, such that $0 = \hat{x}_0 < \hat{x}_1 < \cdots < \hat{x}_M = 1$. We, also, assume that for every $j = 0, 1, \ldots, M - 1$ there is at least one i such that $\hat{x}_j < x_i < \hat{x}_{j+1}$. Thus, the points of Q divide $[0, 1]$ into M intervals $J_j = [\hat{x}_{j-1}, \hat{x}_j]$, $j = 1, \ldots, M$, which we call *regions*. Finally, let $\mathbb{J}$ be the labelling map such that $\mathbb{J} : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, M\}$ with



$$(p_{nm}) = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}, \qquad (c_{nm}) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \qquad V = (1, 2, 2, 1)$$

Fig. 1. In the above figure, the set Δ consists of five interpolation points, while the set Q consists of three points. The stochastic matrix, the connection matrix and the connection vector are shown below the figure.

$\mathbb{J}(i) = j$. Let $x_i - x_{i-1} = \delta_i$, $i = 1, 2, \ldots, N$, and $\hat{x}_j - \hat{x}_{j-1} = \psi_j$, $j = 1, 2, \ldots, M$. It is evident that each region contains an integer number of domains. In the special case where the interpolation points are equidistant (that is $x_i - x_{i-1} = \delta$, $i = 1, 2, \ldots, N$, and $\hat{x}_j - \hat{x}_{j-1} = \psi$, $j = 1, 2, \ldots, M$), each region contains exactly $\alpha = \psi/\delta \in \mathbb{N}$ domains.

We define N mappings of the form:

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_i(x) \\ F_i(x, y) \end{pmatrix}, \quad \text{for } i = 1, 2, \ldots, N, \tag{3}$$

where $L_i(x) = a_i x + b_i$ and $F_i(x, y) = s_i y + q_i(x)$ where $q_i(x)$ is a polynomial. Each map w_i is constrained to map the endpoints of the region $J_{\mathbb{J}(i)}$ to the endpoints of the domain I_i (see Fig. 1). That is,

$$w_i \begin{pmatrix} \hat{x}_{j-1} \\ \hat{y}_{j-1} \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix}, \quad w_i \begin{pmatrix} \hat{x}_j \\ \hat{y}_j \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad \text{for } i = 1, 2, \ldots, N. \tag{4}$$

Vertical segments are mapped to vertical segments scaled by the factor s_i. The parameter s_i is called the *contraction factor* of the map w_i.

It is easy to show that if $|s_i| < 1$, then there is a metric d equivalent to the Euclidean metric, such that w_i is a contraction (i.e., there is $\hat{s}_i$: $0 \leqslant \hat{s}_i < 1$ such that $d(w_i(\vec{x}), w_i(\vec{y})) \leqslant \hat{s}_i d(\vec{x}, \vec{y})$, see [4]).

The $N \times N$ stochastic matrix $(p_{nm})^N$ is defined by

$$p_{nm} = \begin{cases} \frac{1}{\gamma_n}, & \text{if } I_n \subseteq J_{\mathbb{J}(m)}, \\ 0, & \text{otherwise,} \end{cases}$$

where γ_n is the number of positive entries of the line n, $n = 1, 2, \ldots, N$. This means that $p_{n,m}$ is positive, iff there is a transformation L_m, which maps the region containing the nth domain (i.e. I_n) to the mth domain (i.e. I_m). Let us take a point in $I_n \times \mathbb{R}$, $i = 1, \dots, N$. We say that we are in state n. The number p_{nm} shows the probability of applying the map w_m to that point, so that the system transits to state m. Sometimes, it is more efficient to describe the matrix $(p_{nm})^N$ through the connection matrix $C = (c_{nm})^N$ or the connection vector V, which is defined as follows:

$$c_{nm} = \begin{cases} 1, & p_{mn} > 0, \\ 0, & \text{otherwise}, \end{cases}$$
$$V = \big(\mathbb{J}(1), \mathbb{J}(2), \dots, \mathbb{J}(N)\big).$$

Next, we consider $\langle C([x_0, x_N]), \|\cdot\|_\infty \rangle$, where $\|\phi\|_\infty = \max\{|\phi(x)|,\ x \in [x_0, x_N]\}$ and the complete metric subspace $\mathcal{F}_\Delta = \{g \in C([x_0, x_N]):\ g(x_i) = y_i,\ \text{for } i = 0, 1, \dots, N\}$. The Read–Bajraktarevic operator $T_{\Delta,Q} : \mathcal{F}_\Delta \to \mathcal{F}_\Delta$ is defined as follows

$$(T_{\Delta,Q}\, g)(x) = F_i\big(L_i^{-1}(x), g\big(L_i^{-1}(x)\big)\big), \quad \text{for } x \in [x_{i-1}, x_i],\ i = 1, 2, \dots, N.$$

It is easy to verify that $T_{\Delta,Q}\, g$ is well defined and that $T_{\Delta,Q}$ is a contraction with respect to the $\rho := \|\cdot\|_\infty$ metric. According to the Banach fixed-point theorem, there exists a unique $f \in \mathcal{F}_\Delta$ such that $T_{\Delta,Q} f = f$. If f_0 is any interpolation function and $f_n = T^n_{\Delta,Q} f_0$, where $T^n_{\Delta,Q} = T_{\Delta,Q} \circ T_{\Delta,Q} \circ \cdots \circ T_{\Delta,Q}$, then $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f. The graph of the function f is the attractor of the RIFS $\{X, w_{1-N}, (p_{ij})^N\}$ associated with the interpolation points (see [4]). Note that f interpolates the points of Δ for any selection of the parameters of the polynomials p_i that satisfies (4). We will refer to a function of this nature as Fractal Interpolation Function (FIF). In [14] it is shown that FIFs (based on IFS) generalize the Hermite-type interpolation functions.

Let us consider the case where the w_i are affine:

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_i(x) \\ F_i(x, y) \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ c_i & s_i \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}, \quad \text{for } i = 1, 2, \dots, N. \tag{5}$$

Here, $p_i(x) = c_i x + f_i$. The FIF that corresponds to the above RIFS is called *affine* FIF.

From Eq. (4) four linear equations arise, which can always be solved for a_i, c_i, e_i, f_i in terms of the coordinates of the interpolation points and the vertical scaling factor s_i. Thus, once the contraction factor s_i for each map has been chosen, the remaining parameters may be easily computed (see [4]). Figures 2 and 3 show some examples of affine FIF.

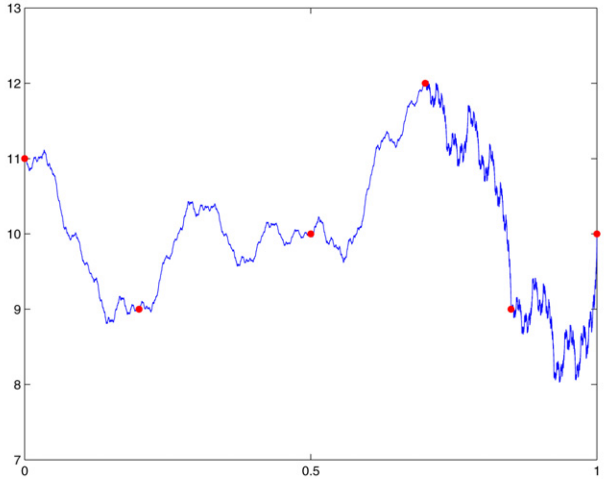
We will prove that if the interpolation points of two distinct FIFs are “almost equal,” then the values of the corresponding FIFs will, also, be “almost equal.”

Lemma 1. *Consider $X = [0, 1] \times \mathbb{R}$, the sets $\Delta_1 = \{(x_i, y_i),\ i = 0, 1, \dots, N\}$, $\Delta_2 = \{(x_i, \tilde{y}_i),\ i = 0, 1, \dots, N\}$ and the corresponding sets $Q_1 = \{(\hat{x}_j, \hat{y}_j),\ j = 0, 1, \dots, M\}$, $Q_2 = \{(\hat{x}_j, \hat{\tilde{y}}_j),\ j = 0, 1, \dots, M\}$. In addition, let f, g be the attractors of the RIFSs associated with the points Δ_1, Q_1 and Δ_2, Q_2, respectively, with the same choice of contraction factors $|s_1|, |s_2|, \dots, |s_N| < 1$ and stochastic matrix $(p_{nm})^N$. If $|\tilde{y}_i - y_i| < \epsilon$, $i = 0, 1, \dots, N$, for some $\epsilon > 0$, then*

$$\|g - f\|_\infty \leqslant \frac{\epsilon(1 + s_{\max})}{(1 - s_{\max})},$$

where $s_{\max} = \max\{|s_i|,\ i = 1, 2, \dots, N\}$.

Proof. For $x \in I_i$ we have that



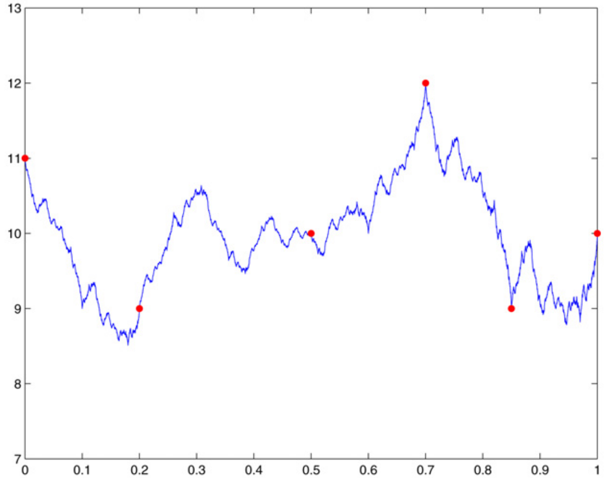
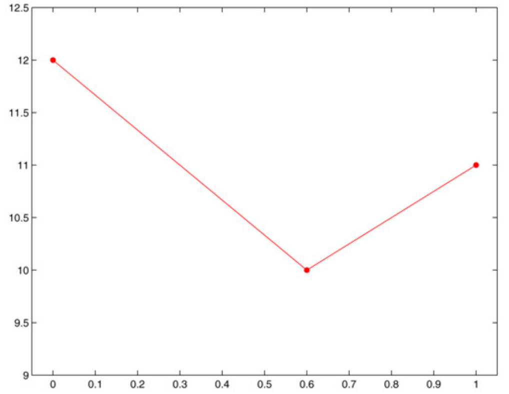


Fig. 2. The two FIFs shown above interpolate the points of the same set Δ (consisting of six points). The difference is due to the selection of two distinct stochastic matrices.

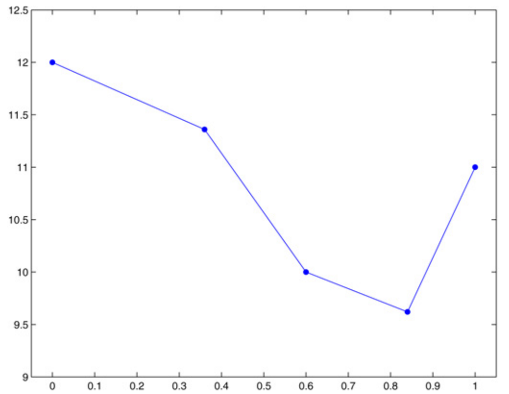
$$\begin{aligned}\left|g(x)-f(x)\right| &= \left|(T_{\Delta_2,Q_2}g)(x)-(T_{\Delta_1,Q_1}f)(x)\right| \\ &= \left|s_i g\left(L_i^{-1}(x)\right)+\tilde{p}_i\circ L_i^{-1}(x)-s_i f\left(L_i^{-1}(x)\right)-p_i\circ L_i^{-1}(x)\right| \\ &\leqslant |s_i|\left|g\left(L_i^{-1}(x)\right)-f\left(L_i^{-1}(x)\right)\right|+\left|\tilde{p}_i\circ L_i^{-1}(x)-p_i\circ L_i^{-1}(x)\right|.\end{aligned}$$

The functions $p_i\circ L_i^{-1}$, $\tilde{p}_i\circ L_i^{-1}$ are polynomials of degree one defined on I_i, where

$$\begin{aligned}&p_i(\hat{x}_{j-1})=y_{i-1}-s_i\hat{y}_{j-1},\\ &p_i(\hat{x}_j)=y_i-s_i\hat{y}_j,\\ &\tilde{p}_i(\hat{x}_{j-1})=\tilde{y}_{i-1}-s_i\hat{\tilde{y}}_{j-1},\\ &\tilde{p}_i(\hat{x}_j)=\tilde{y}_i-s_i\hat{\tilde{y}}_j.\end{aligned}$$



(a)



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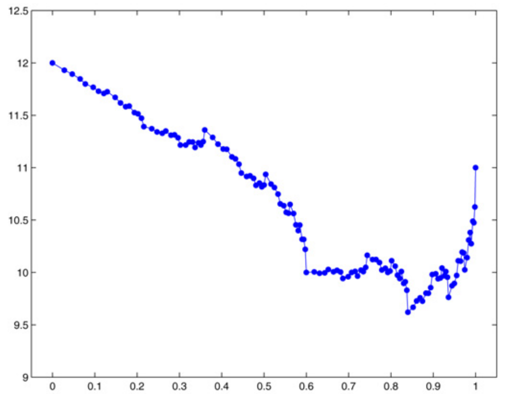


(c)

Fig. 3. The sequence f_n for the RIFS associated with the interpolation points $\Delta = \{(0, 12), (0.6, 10), (1, 11)\}$, $Q = \{(0, 12), (1, 11)\}$ and the contraction factors $s_1 = -0.4$, $s_2 = 0.7$. (a) f_0, (b) f_1, (c) f_6.

Therefore, we may easily deduce that

$$\left|\tilde{p}_i \circ L_i^{-1}(x) - p_i \circ L_i^{-1}(x)\right| \leqslant \left(1 + |s_i|\right)\epsilon,$$

for $x \in I_i$. Hence,

$$\left|g(x) - f(x)\right| \leqslant |s_i|\left|g\left(L_i^{-1}(x)\right) - f\left(L_i^{-1}(x)\right)\right| + \epsilon\left(1 + |s_i|\right),$$

for $x \in I_i$. From this relation we deduce the result. □

3. Fractal Interpolation Surfaces derived from Fractal Interpolation Functions

3.1. The construction

In this section we give a new construction that uses FIF to construct FIS on a rectangular grid of arbitrary interpolation points. We prove that the constructed surface is the graph of a continuous function.

Consider the interpolation points $\Delta = \{(x_i, y_j, z_{ij})\colon\ i = 0, 1, \ldots, N;\ j = 0, 1, \ldots, M\} \subseteq [0, 1] \times [0, p] \times \mathbb{R}$ with $0 = x_0 < x_1 < \cdots < x_N = 1$, $0 = y_0 < y_1 < \cdots < y_M = p$ and $x_i - x_{i-1} = \delta_i$, $i = 0, 1, \ldots, N-1$, $y_j - y_{j-1} = \tilde{\delta}_j$, $j = 0, 1, \ldots, M-1$. Let $S = \{s_1, s_2, \ldots, s_N\}$, $\tilde{S} = \{\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_M\}$ be two sets of contraction factors and let $P = (p_{nm})^N$, $\tilde{P} = (\tilde{p}_{nm})^M$ be two stochastic matrices with dimensions $N \times N$ and $M \times M$, respectively. Also, let $Q = \{(\hat{x}_k, \hat{y}_l, \hat{z}_{kl})\colon k = 0, 1, \ldots, K; l = 0, 1, \ldots, L\}$ be a subset of Δ such that $\hat{x}_0 = 0$, $\hat{x}_K = 1$, $\hat{y}_0 = 0$, $\hat{y}_L = p$ and $\hat{x}_k - \hat{x}_{k-1} = \psi_k$, $\hat{y}_l - \hat{y}_{l-1} = \tilde{\psi}_l$, $k = 0, 1, \ldots, K$, $l = 0, 1, \ldots, L$. Let $\mathbb{J}$ and $\tilde{\mathbb{J}}$ be defined as in Section 2.2 associated with the matrices P and $\tilde{P}$, respectively, with $\mathbb{J}(i) = k$, $\tilde{\mathbb{J}}(j) = l$. The points $\{x_0, x_1, \ldots, x_N\}$ divide $[0, 1]$ into N domains $I_1, I_2, \ldots, I_N$, while the points $\{y_0, y_1, \ldots, y_M\}$ divide $[0, p]$ into M domains $\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_M$. Consequently, the points $\{\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K\}$ divide $[0, 1]$ into K regions $J_1, J_2, \ldots, J_K$, while the points $\{\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_L\}$ divide $[0, p]$ into L regions $\tilde{J}_1, \tilde{J}_2, \ldots, \tilde{J}_L$. In addition, we define the mappings

$$\mathbb{I}\colon \{0, 1, \ldots, K\} \to \{0, 1, \ldots, N\},$$
$$\tilde{\mathbb{I}}\colon \{0, 1, \ldots, L\} \to \{0, 1, \ldots, M\}$$

such that $\hat{x}_k = x_{\mathbb{I}(k)}$ and $\hat{y}_l = y_{\tilde{\mathbb{I}}(l)}$.

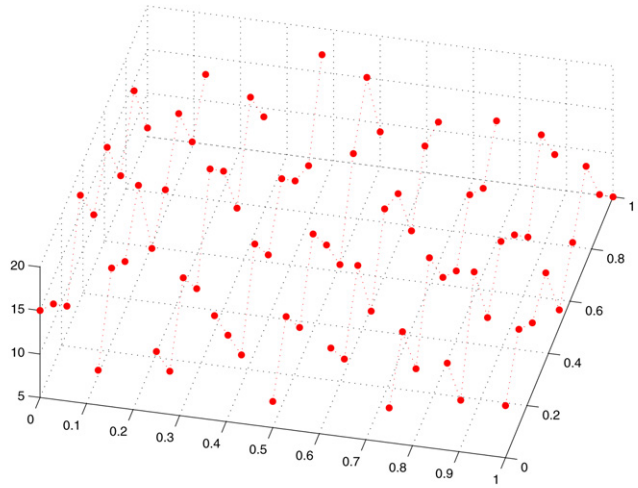
We consider arbitrary continuous functions u_i, that interpolate the sets $\tilde{\Delta}_{x_i} = \{(x_i, y_j, z_{ij})\colon j = 0, 1, \ldots, M\}$, for $i = 0, 1, \ldots, N$ (see Fig. 4). Then, for $y \in [0, p]$, we construct a RIFS associated with the interpolation points $\Delta_y = \{(x_i, y, u_i(y))\colon\ i = 0, 1, \ldots, N\}$, $Q_y = \{(\hat{x}_k, y, u_{\mathbb{I}(k)}(y)),\ k = 0, 1, \ldots, K\}$, the set of contraction factors S together with the matrix P, which produce a FIF $f_y : [0, 1] \to \mathbb{R}$ (see Fig. 4). We define the function

$$F : [0, 1] \times [0, p] \to \mathbb{R} \quad \text{such that } F(x, y) = f_y(x).$$

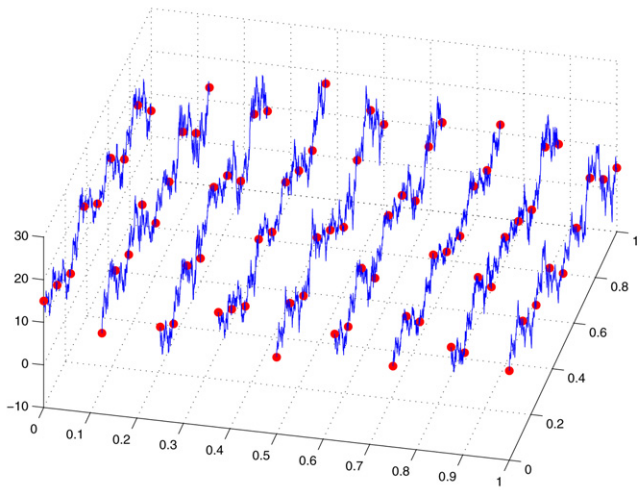
Similarly, we consider arbitrary continuous functions v_j, that interpolate the sets $\Delta_{y_j} = \{(x_i, y_j, z_{ij})\colon\ i = 0, 1, \ldots, N\}$ for $j = 0, 1, \ldots, M$. As before, for $x \in [0, 1]$ we construct a RIFS associated with the interpolation points $\tilde{\Delta}_x = \{(x, y_j, v_j(x))\colon\ j = 0, 1, \ldots, M\}$, $\tilde{Q}_x = \{(x, \hat{y}_l, v_{\tilde{\mathbb{I}}(l)}(x)),\ l = 0, 1, \ldots, L\}$, the set of contraction factors $\tilde{S}$ together with the matrix $\tilde{P}$, which produce a FIF $\tilde{f}_x : [0, p] \to \mathbb{R}$. Thus, we define the function

$$\tilde{F} : [0, 1] \times [0, p] \to \mathbb{R} \quad \text{such that } \tilde{F}(x, y) = \tilde{f}_x(y).$$

The functions F, $\tilde{F}$ interpolate the data Δ. We will prove that F, $\tilde{F}$ are continuous functions.



(a)

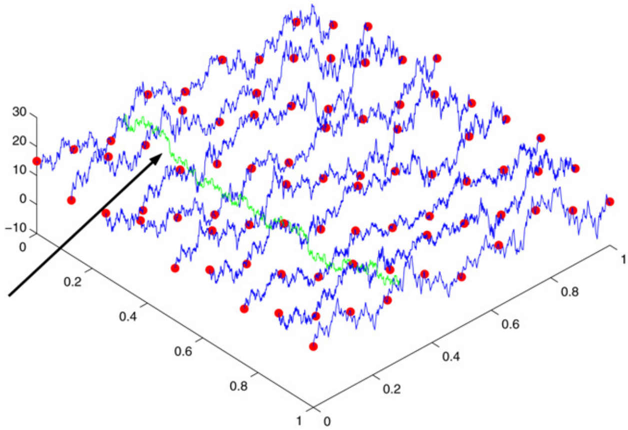


(b)

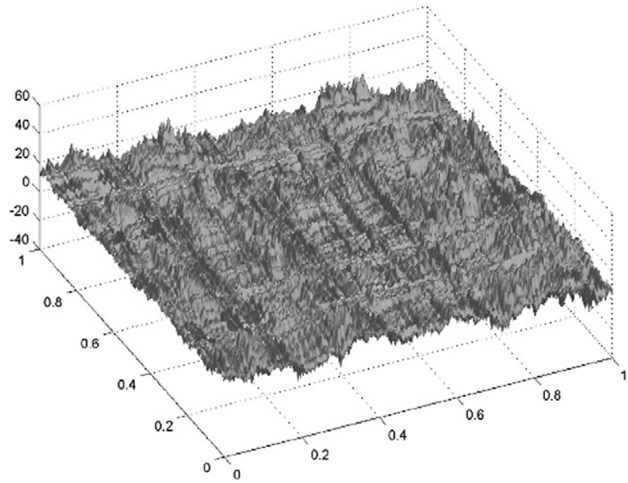
Fig. 4. An example of the construction of the function F is shown (see Table 1). (a) The points of Δ, where $N = M = 8$, $p = 1$. (b) The nine interpolation functions $u_0, u_1, \ldots, u_8$. (c) One of the FIFs f_y (shown by the arrow). (d) The graph of the function F.

Proposition 1. *The functions F, $\tilde{F}$ are continuous.*

Proof. We will prove that F is continuous. We claim that the set $\mathcal{F} = \{f_y\colon\ y \in [0, p]\} \subset C([0, 1])$ (as in our construction) is compact in $\langle C([0, 1]), \|\cdot\|_\infty\rangle$. To prove this claim we consider $(f_{y_n})_{n\in\mathbb{N}}$ to be a sequence in $\mathcal{F}$. As $(y_n)_{n\in\mathbb{N}}$ is a sequence of $[0, p]$ there exists a subsequence with $\lim_{n\to\infty} y_{k_n} = y_0 \in [0, p]$. Then $\lim_{n\to\infty} u_i(y_{k_n}) = u_i(y_0)$, $i = 0, 1, \ldots, N$, and by Lemma 1 it holds that $\lim_{n\to\infty} f_{y_{k_n}} = f_{y_0}$ with respect to the $\|\cdot\|_\infty$ metric. Hence $\mathcal{F}$ is sequentially compact in $C([0, 1])$ and therefore $\mathcal{F}$ is compact.



(c)



(d)

Fig. 4. (*continued*)

Thus the family $\mathcal{F}$ is equicontinuous in $C([0,1])$ at each point $x_0 \in [0,1]$ (i.e. for $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f_y(x) - f_y(x_0)| < \epsilon$ for any $y \in [0,1]$). (For the proof see [8, p. 164].) Let $(x^*, y^*) \in [0,1] \times [0,p]$ and $\epsilon > 0$. The function $F(x^*, \cdot)$ is continuous at y^* as

$$\left|F\left(x^*, y_n\right) - F\left(x^*, y^*\right)\right| = \left|f_{y_n}\left(x^*\right) - f_{y^*}\left(x^*\right)\right| \leqslant \|f_{y_n} - f_{y^*}\|_\infty$$

and $\lim_{n\to 0} f_{y_n} = f_{y^*}$, for $\lim_{n\to 0} y_n = y^*$. Let $\delta_1 > 0$ be such that $|F(x^*, y) - F(x^*, y^*)| < \epsilon/2$, if $|y - y^*| < \delta_1$. As $\mathcal{F}$ is equicontinuous at x^*, there exists $\delta_2 > 0$ such that if $|x - x^*| < \delta_2$, then $|F(x, y) - F(x^*, y)| < \epsilon/2$, for any $y \in [0, p]$. Hence for $(x, y) \in [0,1] \times [0,p]$ with $|x - x^*| + |y - y^*| < \min\{\delta_1, \delta_2\}$ we obtain

$$\left|F(x,y) - F\left(x^*, y^*\right)\right| \leqslant \left|F(x,y) - F\left(x^*, y\right)\right| + \left|F\left(x^*, y\right) - F\left(x^*, y^*\right)\right| < \epsilon.$$

Hence F is continuous at (x^*, y^*). $\square$

Table 1
The interpolation points, the contraction factors and the connection vectors used for the surface of Fig. 4

Δ										
		x								
		0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	0	**15**	12	08	17	**11**	15	08	14	**06**
	0.125	09	17	14	19	08	11	16	09	13
	0.25	12	06	13	08	18	14	06	15	09
	0.375	17	11	05	14	09	14	10	08	17
	0.5	**08**	14	09	16	**11**	05	14	19	**09**
	0.625	15	10	17	08	16	14	06	12	11
	0.75	09	14	06	15	09	06	11	08	12
	0.875	15	07	18	09	14	11	07	15	09
	1	**11**	16	13	15	**07**	11	16	09	**05**

The points of Q are shown with bold letters.
$S_x = \{0.6, -0.6, 0.7, -0.5, 0.5, -0.7, 0.6, -0.8\}$, $S_y = \{-0.7, 0.6, -0.6, -0.6, -0.6, 0.7, -0.5, 0.7\}$,
$V_x = (1, 1, 2, 1, 1, 2, 2, 1)$, $V_y = (1, 2, 1, 2, 1, 2, 2, 1)$, $\psi = 4\delta$.

Figure 5 shows some examples of the above construction using arbitrary data points.

3.2. A notable result

In the general case, where u_i, v_j are arbitrary continuous functions interpolating the sets $\tilde{\Delta}_{x_i}$ and Δ_{y_j}, $i = 0, 1, \ldots, N$, $j = 0, 1, \ldots, M$, respectively, the functions F and $\tilde{F}$ are distinct.

We draw special attention to the case where u_i, v_j are affine fractal interpolation functions constructed as we describe below. The RIFS associated with the interpolation points $\tilde{\Delta}_{x_i} = \{(x_i, y_j, z_{ij})\colon\ j = 0, 1, \ldots, M\}$, $\tilde{Q}_{x_i} = \{(x_i, \hat{y}_l, z_{i,\tilde{\mathbb{I}}(l)})\colon\ l = 0, 1, \ldots, L\}$, the set of contraction factors $\tilde{S}$ together with the stochastic matrix $\tilde{P}$ produces a FIF $u_i : [0, p] \to \mathbb{R}$ (see Fig. 4), for all $i = 0, 1, \ldots, N$. Similarly, the RIFS associated with the interpolation points $\Delta_{y_j} = \{(x_i, y_j, z_{ij})\colon\ i = 0, 1, \ldots, N\}$, $Q_{y_j} = \{(\hat{x}_k, y_j, z_{\mathbb{I}(k),j})\colon\ k = 0, 1, \ldots, K\}$, the set of contraction factors S together with the stochastic matrix P produces a FIF $v_j : [0, 1] \to \mathbb{R}$, $j = 0, 1, \ldots, M$. Evidently, $\tilde{f}_{x_i} = u_i$, $i = 0, 1, \ldots, N$ and $f_{y_j} = v_j$, $j = 0, 1, \ldots, M$. In this case the two functions F and $\tilde{F}$ are coincide, as stated in the following proposition.

Proposition 2. *If in the construction of F, $\tilde{F}$ described as in Section* 3.1, *u_i are the affine FIFs associated with $\tilde{\Delta}_{x_i}$, $\tilde{Q}_{x_i}$, $\tilde{S}$, $\tilde{P}$, $i = 0, 1, \ldots, N$, and v_j are the affine FIFs associated with Δ_{y_j}, Q_{y_j}, S, P, $j = 0, 1, \ldots, M$, then*

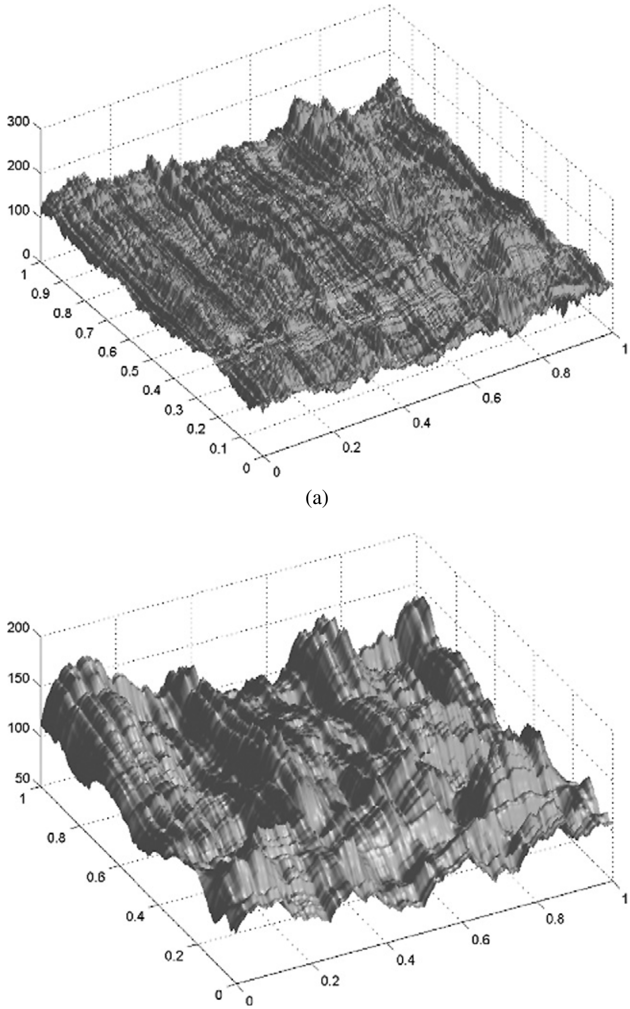
$$F = \tilde{F}.$$

Proof. Let $\{[0, 1], w_{y,1-N}, P\}$ be the RIFS whose attractor is the graph of the affine FIF f_y, where

$$w_{y,i}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} L_i(x) \\ F_{y,i}(x, z) \end{pmatrix} = \begin{pmatrix} L_i(x) \\ s_i z + q_{y,i}(x) \end{pmatrix}$$

and $\{[0, p], \tilde{w}_{x,1-M}, \tilde{P}\}$ be the RIFS whose attractor is the graph of the affine FIF $\tilde{f}_x$, where

$$\tilde{w}_{x,j}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \tilde{L}_j(y) \\ \tilde{F}_{x,j}(y, z) \end{pmatrix} = \begin{pmatrix} \tilde{L}_j(y) \\ \tilde{s}_j z + \tilde{q}_{x,j}(y) \end{pmatrix},$$



(a)

(b)

Fig. 5. Two more continuous surfaces that interpolate a set of 9×9 interpolation points.

for $x \in [0,1]$, $y \in [0,p]$, $z \in \mathbb{R}$, $i = 1,2,\ldots,N$, $j = 1,2,\ldots,M$. Then the following functional equations hold

$$f_y(x) = F_{y,i}\bigl(L_i^{-1}(x), f_y\bigl(L_i^{-1}(x)\bigr)\bigr), \tag{6}$$

$$\tilde{f}_x(y) = \tilde{F}_{x,j}\bigl(\tilde{L}_j^{-1}(y), \tilde{f}_x\bigl(\tilde{L}_j^{-1}(y)\bigr)\bigr), \tag{7}$$

for $x \in I_i$, $y \in \tilde{I}_j$, $i = 1,2,\ldots,N$, $j = 1,2,\ldots,M$.

Consider the RIFS $\{[0,1], L_{1-N}, P\}$, $\boldsymbol{L} = (L_1, L_2, \ldots, L_N)$ the map defined on $\mathcal{H}([0,1])^N$ (as $\boldsymbol{W}$ in Section 2), the set $A_0 = \{x_0, x_1, \ldots, x_N\} \subset \mathcal{H}([0,1])$ and the sequence $\{A_n\}_{n\in\mathbb{N}}$ defined as in Section 2. The attractor of the RIFS is the set $[0,1] = \lim_n A_n$. Similarly, we define the RIFS $\{[0,1], \tilde{L}_{1-M}, P\}$, $\tilde{\boldsymbol{L}} = (\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_M)$ the map defined on $\mathcal{H}([0,p])^N$, the set $\tilde{A}_0 = \{y_0, y_1, \ldots, y_M\} \subset \mathcal{H}([0,p])$ and the sequence $\{\tilde{A}_n\}_{n\in\mathbb{N}}$. The attractor of the latter RIFS is the set $[0,p]=\lim_n \tilde{A}_n$. We may easily deduce that

$$A_n=\bigcup_{i=1}^{N} L_i(A_{n-1}\cap J_{\mathbb{J}(i)}) \quad \text{and} \quad \tilde{A}_n=\bigcup_{j=1}^{M} \tilde{L}_j(\tilde{A}_{n-1}\cap \tilde{J}_{\tilde{\mathbb{J}}(j)}).$$

Evidently for $n=0$,

$$f_x(y)=\tilde{f}_y(x)=F(x,y), \tag{8}$$

for any $(x,y)\in(A_0\times[0,p])\cup([0,1]\times\tilde{A}_0)$.

We assume that (8) holds for any $(x,y)\in(A_n\times[0,p])\cup([0,1]\times\tilde{A}_n)$. Then, we will show that (8) holds for $(x,y)\in(A_{n+1}\times[0,p])\cup([0,1]\times\tilde{A}_{n+1})$.

To prove the latter we use induction. For $m=0$, Eq. (8) holds for $(x,y)\in(A_{n+1}\times\tilde{A}_0)\cup(A_0\times\tilde{A}_{n+1})\subseteq([0,1]\times\tilde{A}_0)\cup(A_0\times[0,p])$. Assuming that (8) holds for $(x,y)\in(A_{n+1}\times\tilde{A}_m)\cup(A_m\times\tilde{A}_{n+1})$, we will prove that it holds for $(x,y)\in(A_{n+1}\times\tilde{A}_{m+1})\cup(A_{m+1}\times\tilde{A}_{n+1})$.

Let $x\in I_i$, $y\in\tilde{I}_j$. Then, $x^*=L_i^{-1}(x)\in J_k$ and $y^*=\tilde{L}_j^{-1}(y)\in\tilde{J}_l$. As $(x^*,y)\in(A_n\times\tilde{A}_{m+1})\cup(A_m\times\tilde{A}_{n+1})\subseteq(A_n\times[0,p])\cup(A_m\times\tilde{A}_{n+1})$ and $(x^*,y^*)\in(A_n\times A_m)\cup(A_m\times\tilde{A}_n)\subseteq(A_{n+1}\times\tilde{A}_m)\cup(A_m\times\tilde{A}_{n+1})$, we have that

$$f_{x^*}(y)=\tilde{f}_y(x^*)=F(x^*,y) \quad \text{and} \quad f_{x^*}(y^*)=\tilde{f}_{y^*}(x^*)=F(x^*,y^*).$$

Therefore we obtain:

$$\begin{aligned}
f_y(x)&=F_{y,i}\big(L_i^{-1}(x),f_y\big(L_i^{-1}(x)\big)\big)\\
&=F_{y,i}\big(x^*,f_y(x^*)\big)\\
&=s_i f_y(x^*)+q_{y,i}(x^*)\\
&=s_i F(x^*,y)+q_{y,i}(x^*)\\
&=s_i\tilde{f}_{x^*}(y)+q_{y,i}(x^*)\\
&=s_i\tilde{F}_{x^*,j}\big(\tilde{L}^{-1}(y),\tilde{f}_{x^*}\big(\tilde{L}^{-1}(y)\big)\big)+q_{y,i}(x^*)\\
&=s_i\tilde{F}_{x^*,j}\big(y^*,\tilde{f}_{x^*}(y^*)\big)+q_{y,i}(x^*)\\
&=s_i\tilde{s}_j\tilde{f}_{x^*}(y^*)+s_i\tilde{q}_{x^*,j}(y^*)+q_{y,i}(x^*)\\
&=s_i\tilde{s}_j F(x^*,y^*)+s_i\tilde{q}_{x^*,j}(y^*)+q_{y,i}(x^*).
\end{aligned}$$

We note that $q_{t,i}$, $t\in[0,1]$ and $\tilde{q}_{t,j}$, $t\in[0,p]$ are polynomials of degree one, where

$$\begin{aligned}
&q_{t,i}(\hat{x}_{k-1})=f_t(x_{i-1})-s_i f_t(\hat{x}_{k-1}), \quad q_{t,i}(\hat{x}_k)=f_t(x_i)-s_i f_t(\hat{x}_k), \quad \text{for } t\in[0,p],\\
&\tilde{q}_{t,j}(\hat{y}_{l-1})=\tilde{f}_t(y_{j-1})-\tilde{s}_j\tilde{f}_t(\hat{y}_{l-1}), \quad \tilde{q}_{t,j}(\hat{y}_l)=\tilde{f}_t(y_j)-\tilde{s}_j\tilde{f}_t(\hat{y}_l), \quad \text{for } t\in[0,1],
\end{aligned}$$

$i=1,2,\dots,N$, $j=1,2,\dots,M$.

Therefore, considering that $f_t(x_i)=u_i(t)=\tilde{f}_{x_i}(t)$ $\forall t\in[0,p]$, we get:

$$\begin{aligned}
q_{t,i}(x^*)&=q_{t,i}(\hat{x}_{k-1})+\frac{x^*-\hat{x}_{k-1}}{\psi_x}\big(q_{t,i}(\hat{x}_k)-q_{t,i}(\hat{x}_{k-1})\big)\\
&=f_t(x_{i-1})-s_i f_t(\hat{x}_{k-1})+\frac{\omega_x}{\psi_x}\big(f_t(x_i)-s_i f_t(\hat{x}_k)-f_t(x_{i-1})+s_i f_t(\hat{x}_{k-1})\big)\\
&=F(x_{i-1},t)-s_i F(\hat{x}_{k-1},t)\\
&\quad+\frac{\omega_x}{\psi_x}\big(F(x_i,t)-s_i F(\hat{x}_k,t)-F(x_{i-1},t)+s_i F(\hat{x}_{k-1},t)\big)
\end{aligned}$$

$$= \left(1 - \frac{\omega_x}{\psi_x}\right)\tilde{f}_{x_{i-1}}(t) - s_i\left(1 - \frac{\omega_x}{\psi_x}\right)\tilde{f}_{\hat{x}_{k-1}}(t) + \frac{\omega_x}{\psi_x}\tilde{f}_{x_i}(t) - s_i\frac{\omega_x}{\psi_x}\tilde{f}_{\hat{x}_k}(t) \tag{9}$$

and similarly

$$\begin{aligned}\tilde{q}_{t,j}\big(y^*\big) = {}& \left(1 - \frac{\omega_y}{\psi_y}\right)F(t, y_{j-1}) - \tilde{s}_j\left(1 - \frac{\omega_y}{\psi_y}\right)F(t, \hat{y}_{l-1}) + \frac{\omega_y}{\psi_y}F(t, y_j)\\ & - \tilde{s}_j\frac{\omega_y}{\psi_y}F(t, \hat{y}_l),\end{aligned} \tag{10}$$

for $t \in [0,1]$, where $\omega_x = x^* - \hat{x}_{k-1}$, $\omega_y = y^* - \hat{y}_{l-1}$, $\psi_x = \psi_k$, $\psi_y = \psi_l$.

In addition, with the help of Eq. (10), we obtain

$$\begin{aligned}\tilde{f}_{x_i}(y) &= \tilde{F}_{x_i,j}\big(y^*, \tilde{f}_{x_i}\big(y^*\big)\big) = \tilde{s}_j F\big(x_i, y^*\big) + \tilde{q}_{x_i,j}\big(y^*\big)\\ &= \tilde{s}_j F\big(x_i, y^*\big) + \left(1 - \frac{\omega_y}{\psi_y}\right)F(x_i, y_{j-1}) - \tilde{s}_j\left(1 - \frac{\omega_y}{\psi_y}\right)F(x_i, \hat{y}_{l-1})\\ &\quad + \frac{\omega_y}{\psi_y}F(x_i, y_j) - \tilde{s}_j\frac{\omega_y}{\psi_y}F(x_i, \hat{y}_l).\end{aligned}$$

We obtain similar relations for $\tilde{f}_{x_{i-1}}(y)$, $\tilde{f}_{\hat{x}_{k-1}}(y)$, $\tilde{f}_{\hat{x}_k}(y)$.

Thus, the value of $f_y(x)$ is

$$\begin{aligned}f_y(x) = {}& s_i\tilde{s}_j F\big(x^*, y^*\big) + \frac{\omega_y}{\psi_y}s_i F\big(x^*, y_j\big) - \frac{\omega_y}{\psi_y}s_i\tilde{s}_j F\big(x^*, \hat{y}_l\big) + \left(1 - \frac{\omega_y}{\psi_y}\right)s_i F\big(x^*, y_{j-1}\big)\\ & - \left(1 - \frac{\omega_y}{\psi_y}\right)s_i\tilde{s}_j F\big(x^*, \hat{y}_{l-1}\big) + \left(1 - \frac{\omega_x}{\psi_x}\right)\tilde{s}_j F\big(x_{i-1}, y^*\big)\\ & + \left(1 - \frac{\omega_x}{\psi_x}\right)\left(1 - \frac{\omega_y}{\psi_y}\right)F(x_{i-1}, y_{j-1}) - \left(1 - \frac{\omega_y}{\psi_y}\right)\left(1 - \frac{\omega_x}{\psi_x}\right)\tilde{s}_j F(x_{i-1}, \hat{y}_{l-1})\\ & + \left(1 - \frac{\omega_x}{\psi_x}\right)\frac{\omega_y}{\psi_y}F(x_{i-1}, y_j) - \left(1 - \frac{\omega_x}{\psi_x}\right)\frac{\omega_y}{\psi_y}\tilde{s}_j F(x_{i-1}, \hat{y}_l)\\ & - \left(1 - \frac{\omega_x}{\psi_x}\right)s_i\tilde{s}_j F\big(\hat{x}_{k-1}, y^*\big) - \left(1 - \frac{\omega_x}{\psi_x}\right)\left(1 - \frac{\omega_y}{\psi_y}\right)s_i F(\hat{x}_{k-1}, y_{j-1})\\ & + \left(1 - \frac{\omega_x}{\psi_x}\right)\left(1 - \frac{\omega_y}{\psi_y}\right)s_i\tilde{s}_j F(\hat{x}_{k-1}, \hat{y}_{l-1}) - \left(1 - \frac{\omega_x}{\psi_x}\right)\frac{\omega_y}{\psi_y}s_i F(\hat{x}_{k-1}, y_j)\\ & + \left(1 - \frac{\omega_x}{\psi_x}\right)\frac{\omega_y}{\psi_y}s_i\tilde{s}_j F(\hat{x}_{k-1}, \hat{y}_l) + \frac{\omega_x}{\psi_x}\tilde{s}_j F\big(x_i, y^*\big) + \frac{\omega_x}{\psi_x}\left(1 - \frac{\omega_y}{\psi_y}\right)F(x_i, y_{j-1})\\ & - \frac{\omega_x}{\psi_x}\left(1 - \frac{\omega_y}{\psi_y}\right)\tilde{s}_j F(x_i, \hat{y}_{l-1}) + \frac{\omega_x}{\psi_x}\frac{\omega_y}{\psi_y}F(x_i, y_j) - \frac{\omega_x}{\psi_x}\frac{\omega_y}{\psi_y}\tilde{s}_j F(x_i, \hat{y}_l)\\ & - \frac{\omega_x}{\psi_x}s_i\tilde{s}_j F\big(\hat{x}_k, y^*\big) - \frac{\omega_x}{\psi_x}\left(1 - \frac{\omega_y}{\psi_y}\right)s_i F(\hat{x}_k, y_{j-1})\\ & + \frac{\omega_x}{\psi_x}\left(1 - \frac{\omega_y}{\psi_y}\right)s_i\tilde{s}_j F(\hat{x}_k, \hat{y}_{l-1}) - \frac{\omega_x}{\psi_x}\frac{\omega_y}{\psi_y}s_i F(\hat{x}_k, y_j) + \frac{\omega_x}{\psi_x}\frac{\omega_y}{\psi_y}s_i\tilde{s}_j F(\hat{x}_k, \hat{y}_l).\end{aligned}$$

Working similarly we obtain the same relation for $\tilde{f}_x(y)$.

Thus, relation (8) holds for $(x,y) \in (A_{n+1} \times \tilde{A}_m) \cup (A_m \times \tilde{A}_{n+1})$, $m \in \mathbb{N}$. Since

$$\lim_m A_m = \overline{\bigcup_{m=1}^{\infty} A_m} = [0, 1], \qquad \lim_m \tilde{A}_m = \overline{\bigcup_{m=1}^{\infty} \tilde{A}_m} = [0, p],$$

and F, $\tilde{F}$ are continuous, we may deduce that (8) holds for $(x, y) \in (A_{n+1} \times [0, p]) \cup ([0, 1] \times \tilde{A}_{n+1})$. Thus, by induction, (8) holds for $(x, y) \in (A_n \times [0, p]) \cup ([0, 1] \times \tilde{A}_n)$ and any $n \in \mathbb{N}$. Therefore, it holds for $(x, y) \in [0, 1] \times [0, p]$. $\square$

4. Lower bound of the dimension of the Constructed Fractal Surfaces

We will prove a general result that gives a lower bound for the box-counting dimension of the graph of a continuous function, if a lower bound of the box-counting dimension of its plane sections is known.

If E is a bounded set in $\mathbb{R}^n$, then the δ-parallel body of E is the set of all points at a distance less than δ from E, i.e.,

$$E(\delta) = E + \delta B_n = \big\{x \in \mathbb{R}^n \colon \exists y \in E \text{ with } \|x - y\| \leqslant \delta\big\},$$

where $\delta \geqslant 0$ and $B_n = B(0, 1)$ the closed unitary sphere of $\mathbb{R}^n$ with center at 0. Denoting the volume by V_n, we get the lower and upper box-counting (Minkowski–Bouligand) dimension, respectively,

$$\underline{\dim}_B(E) = n - \limsup_{\delta \to 0^+} \frac{\log V_n(E(\delta))}{\log \delta},$$
$$\overline{\dim}_B(E) = n - \liminf_{\delta \to 0^+} \frac{\log V_n(E(\delta))}{\log \delta}$$

(see [10]), and if $\underline{\dim}_B(E) = \overline{\dim}_B(E)$ we write $\dim_B(E)$.

Proposition 3. *Let $F : [0, 1] \times [0, p] \to \mathbb{R}$ be a continuous function, F_y its restriction on $[0, 1] \times \{y\}$ and G_F, G_{F_y} their graphs for $y \in [0, p]$. If $\underline{\dim}_B(G_{F_y}) \geqslant s$ for almost all $y \in [0, p]$, then $\underline{\dim}_B(G_F) \geqslant s + 1$.*

Proof. We restrict the Lebesgue measure V_3 on $[0, 1] \times [0, p] \times \mathbb{R}$ and the V_2 on $[0, 1] \times \mathbb{R}$. The continuity of the function F ensures the measurability of the function $h(\delta, y) = V_2(G_{F_y} + \delta B_2)$, $y \in [0, p]$, $\delta \geqslant 0$. From Fubini's theorem we have that

$$V_3(G_F + \delta B_3) = \int_0^1 V_2(G_{F_y} + \delta B_2)\, dy, \tag{11}$$

for $\delta \geqslant 0$. In addition, using Jensen's inequality we obtain

$$\log \int_0^1 V_2(G_{F_y} + \delta B_2)\, dy \geqslant \int_0^1 \log V_2(G_{F_y} + \delta B_2)\, dy, \tag{12}$$

for $\delta \geqslant 0$. Let $\delta_n > 0$ be a sequence with $\lim_n \delta_n = 0$. For all $y \in [0, p]$ we have

$$\frac{\log V_2(G_{F_y} + \delta_n B_2)}{\log \delta_n} < 2,$$

for $n \geqslant n_0(y)$.

In view of Fatou's lemma and the relations (11), (12) we have

$$\begin{aligned}
\limsup_n \frac{\log V_3(G_F + \delta_n B_3)\,dy}{\log \delta_n} &= \limsup_n \frac{\log \int_0^1 V_2(G_{F_y} + \delta_n B_2)\,dy}{\log \delta_n} \\
&\leqslant \int_0^1 \limsup_n \left(\frac{\log V_2(G_{F_y} + \delta_n B_2)}{\log \delta_n} \right) dy \\
&\leqslant 2 - \int_0^1 \underline{\dim}_B(G_{F_y})\,dy \\
&\leqslant 2 - s \quad \text{(from the hypothesis).}
\end{aligned}$$

Therefore,

$$\underline{\dim}_B(G_F) = 3 - \limsup_{\delta \to 0^+} \frac{\log V_3(G_f + \delta B_3)}{\log \delta} \geqslant 3 - (2 - s) = 1 + s. \qquad \square$$

Remark 1. One may prove that the function $h(\delta, y) = V_2(G_{F_y} + \delta B_2)$ used above is actually continuous.

We will use the above result to derive a lower bound of the box-counting dimension of FIS constructed as in Section 3.1.

Let f be an affine recurrent FIF given by $\{[0,1], w_{1-N}, P\}$ with irreducible connection matrix C and graph G_f. Let

$$S(d) = \operatorname{diag}\left\{|s_1| a_1^{D-1}, |s_2| a_2^{D-1}, \ldots, |s_N| a_N^{D-1}\right\}$$

(diagonal matrix) and D be the unique value so that $\rho(S(D) \cdot C) = 1$ ($\rho(\cdot)$ is the spectral radius of the matrix). If $\rho(S(1) \cdot C) > 1$ and the interpolation points contained in $J_k \times \mathbb{R}$ are not colinear for all $k = 0, 1, \ldots, K$, then the box-counting dimension of the graph G_f is

$$\dim_B(G_f) = D,$$

otherwise $\dim_B(G_f) = 1$ (see [2]). In the special case, where the points of Δ and the points of Q are equidistant (i.e. $x_i - x_{i-1} = \delta$, $\hat{x}_k - \hat{x}_{k-1} = \psi$), the box counting dimension is given by

$$D = 1 + \log_\alpha \big(\rho \big(S(1) \cdot C \big) \big),$$

where $\alpha = \psi/\delta$.

Consider the construction presented in Section 3.1. If there is an index $j_0 \in \{0, 1, \ldots, M\}$ such that $\dim_B(G_{f_{y_{j_0}}}) = D_1 > 1$, then there is a subinterval of $[0, p]$ of positive length, such that $\dim_B(G_{f_y}) = D_1$, for any y in the interval. Therefore by Proposition 3

$$\underline{\dim}_B(G_F) \geqslant 1 + D_1 > 2.$$

Similarly if there is $i_0 \in \{0, 1, \ldots, N\}$ such that $\dim_B(G_{\tilde{f}_{x_{i_0}}}) = D_2 > 1$, then there is a subinterval of $[0, 1]$ of positive length, such that $\dim_B(G_{\tilde{f}_x}) = D_2$, for any x in the interval. Therefore

$$\underline{\dim}_B(G_{\tilde{F}}) \geqslant 1 + D_2 > 2.$$

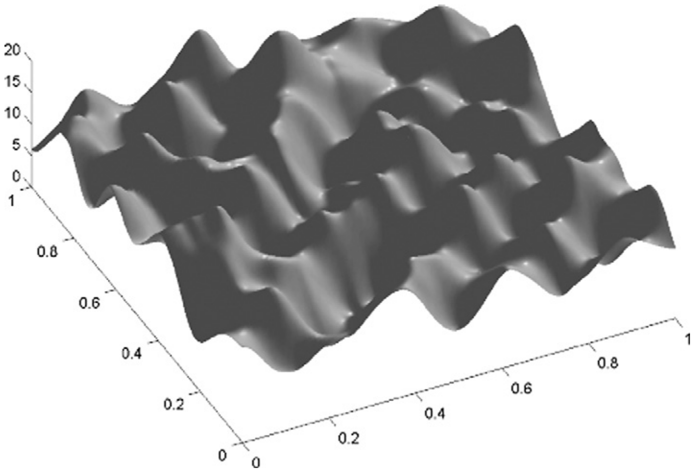


Fig. 6. A C^1 fractal interpolation surface.

In the case where u_i, v_j $(i = 1, 2, \dots, N,\ j = 1, 2, \dots, M)$ are affine FIFs with box-counting dimensions D_1 and D_2, respectively, we have

$$\underline{\dim}_B(G_F) = \underline{\dim}_B(G_{\tilde{F}}) \geqslant \max\{1 + D_1, 1 + D_2\}.$$

5. Conclusions

The construction we describe in the above sections may be applied to arbitrary interpolation points. The emerging surface is the graph of a continuous function that interpolates the data. In Table 1 the data for the construction of the surface shown in Fig. 4(d) are given. Figure 5 shows two more continuous surfaces. One may observe the roughness of the produced surfaces shown in the figures.

We should note that this construction may be generalized to construct fractal interpolation functions defined on $[0, 1]^n$, $n \in \mathbb{N}$, that interpolate arbitrary data (placed in rectangular grids). It would be interesting to see if the result presented in Section 3.2 holds in that case also. In addition, if we choose u_i, $i = 0, 1, \dots, N$, to be C^1 functions (e.g. splines) and construct f_y using Hermite-type polynomials (as in [14]), we may construct smooth FIS that generalizes spline surfaces (see [6] and Fig. 6).

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