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# Identifying and locating-dominating codes on chains and cycles

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**Abstract**

Consider a connected undirected graph $G = (V, E)$, a subset of vertices $C \subseteq V$, and an integer $r \geq 1$; for any vertex $v \in V$, let $B_r(v)$ denote the ball of radius $r$ centered at $v$, i.e., the set of all vertices within distance $r$ from $v$. If for all vertices $v \in V$ (respectively, $v \in V \setminus C$), the sets $B_r(v) \cap C$ are all nonempty and different, then we call $C$ an $r$-identifying code (respectively, an $r$-locating-dominating code). We study the smallest cardinalities or densities of these codes in chains (finite or infinite) and cycles.


## 1. Introduction

Given a connected undirected graph $G = (V, E)$ and an integer $r \geq 1$, we define $B_r(v)$, the *ball* of radius $r$ centered at $v \in V$, by

$$B_r(v) = \{x \in V : d(x, v) \leq r\},$$

where $d(x, v)$ denotes the number of edges in any shortest path between $v$ and $x$. Whenever $d(x, v) \leq r$, we say that $x$ and $v$ *r-cover* each other (or simply *cover* if there is no

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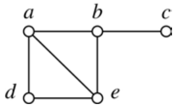


Fig. 1. A graph $G$ admitting no 1-identifying code.

ambiguity). A set $X \subseteq V$ covers a set $Y \subseteq V$ if every vertex in $Y$ is covered by at least one vertex in $X$.

A *code* $C$ is a nonempty set of vertices, and its elements are called *codewords*. For each vertex $v \in V$, we denote by

$$K_{C,r}(v) = C \cap B_r(v)$$

the set of codewords which $r$-cover $v$. Two vertices $v_1$ and $v_2$ with $K_{C,r}(v_1) \neq K_{C,r}(v_2)$ are said to be *r-separated*, or *separated*, by code $C$.

A code $C$ is called *r-identifying*, or *identifying*, if the sets $K_{C,r}(v), v \in V$, are all nonempty and distinct [9]. It is called *r-locating-dominating*, or *locating-dominating*, if the same is true for all $v \in V \backslash C$ [5]. In other words, in the first case all vertices must be covered and pairwise separated by $C$, in the latter case only the noncodewords need to be covered and separated.

**Remark 1.** For given graph $G = (V, E)$ and integer $r$, there exists an $r$-identifying code $C \subseteq V$ if and only if

$$\forall v_1, v_2 \in V (v_1 \neq v_2), B_r(v_1) \neq B_r(v_2).$$

Indeed, if for all $v_1, v_2 \in V$, $B_r(v_1)$ and $B_r(v_2)$ are different, then $C = V$ is $r$-identifying. Conversely, if for some $v_1, v_2 \in V$, $B_r(v_1) = B_r(v_2)$, then for any code $C \subseteq V$, we have $K_{C,r}(v_1) = K_{C,r}(v_2)$. For instance, there is no $r$-identifying code in a complete graph. See also Example 1 below.

**Remark 2.** For given graph $G = (V, E)$ and integer $r$, an $r$-locating-dominating code always exists (simply take $C = V$), and any $r$-identifying code is $r$-locating-dominating.

**Example 1.** Consider the graph $G$ in Fig. 1. We see that $B_1(a) = \{a, b, d, e\}$, $B_1(b) = \{a, b, c, e\}$, $B_1(c) = \{b, c\}$, $B_1(d) = \{a, d, e\}$, $B_1(e) = \{a, b, d, e\}$; consequently, because $B_1(a) = B_1(e)$, there is no 1-identifying code in $G$ (cf. Remark 1 above). On the other hand, $C = \{a, b\}$ is 1-locating-dominating, since the sets $K_{C,1}(c) = \{b\}$, $K_{C,1}(d) = \{a\}$, and $K_{C,1}(e) = \{a, b\}$, are all nonempty and different.

The motivations come, for instance, from fault diagnosis in multiprocessor systems. Such a system can be modeled as a graph where vertices are processors and edges are links between processors. Assume that at most one of the processors is malfunctioning and we wish to test the system and locate the faulty processor. For this purpose, some processors (constituting the code) will be selected and assigned the task of testing their neighborhoods (i.e., the vertices at distance at most $r$). Whenever a selected processor (=a codeword) detects a fault, it sends an alarm signal, saying that one element in its neighborhood is malfunctioning. We require that we can uniquely tell the location of the malfunctioning processor based only on the information which ones of the codewords gave the alarm, and in this case an identifying code is what we need.

If the selected codewords are assumed to work without failure, or if their only task is to test their neighborhoods (i.e., they are not considered as processors anymore) and we assume that they perform this simple task without failure, then we shall search for locating-dominating codes. These codes can also be considered for modeling the protection of a building, the rooms of which are the vertices of a graph.

Locating-dominating codes were introduced in [5], identifying codes in [9], and they constitute now a topic of their own: both were studied in a large number of various papers, investigating particular graphs or families of graphs (such as planar graphs, certain infinite regular grids, or the $n$-cube), dealing with complexity issues, or using heuristics such as the noising methods for the construction of small codes. See, e.g., [3, 4], and references therein, or [12]. For instance, the best possible densities for a 1-locating-dominating code in the infinite grid [11, Theorem 11] or for an $r$-identifying code ($r \geq 1$) in the infinite king grid [2] are exactly known.

Here, we shall study identifying and locating-dominating codes in chains (Sections 2 and 3) and cycles (Section 4).

More specifically:

- In Section 2, we determine the exact value of the smallest possible density of an $r$-identifying code in the *infinite* chain, for all $r \geq 1$. Then we do the same for $r$-locating-dominating codes, for all $r > 1$ (the case $r = 1$ is stated in [11, after Theorem 11]; see Theorem 2 here).
- In Section 3, we investigate *finite* chains. We give the exact value of the smallest possible cardinality of a 1-identifying code; for $r > 1$, we give a lower bound and, for an infinite set of values of the length of the chain (for given $r$), an upper bound which coincides with the lower bound. The smallest cardinality of a 1-locating-dominating code in a finite chain is known [10, Theorem 9] (cf. Theorem 6 here); for $r > 1$, we give a lower and an upper bound, which differ by $cr$, where $c$ is close to one third for an infinite set of values of the chain length.
- In Section 4, we study cycles. For all $r \geq 1$, we give a lower bound on the smallest size of an $r$-identifying code in a cycle, and for cycles of even length we provide a construction meeting this bound. The smallest cardinality of a 1-locating-dominating code in a cycle is known [10, Theorem 9] (cf. Theorem 12 here); we give, for $r > 1$, a lower bound and, for an infinite set of values of the cycle length, we exhibit a construction meeting this bound.
- In Section 5, we deepen the study of infinite chains by changing slightly the notion of ball—this is part of a more general problem, involving patterns other than balls.

Chains can be seen as 1-ary complete trees, and codes in trees are studied in [1], in particular 1-identifying codes in the complete $q$-ary trees, $q \geq 2$.

## 2. The infinite chain

The infinite chain $G = \mathcal{CH}_\infty$ has vertex set $V = Z$ and edge set $E = \{\{i, i+1\} : i \in Z\}$. We denote by $d_r^I(\mathcal{CH}_\infty)$ and $d_r^{LD}(\mathcal{CH}_\infty)$ the smallest density of an $r$-identifying and of an

$r$-locating-dominating code, respectively, in $\mathcal{CH}_\infty$. In the next two subsections, we provide the exact values of $d_r^I(\mathcal{CH}_\infty)$ and $d_r^{LD}(\mathcal{CH}_\infty)$, for all $r \geq 1$.

Before that, we give a definition and an easy but useful lemma. We say that, given a code $C$ in $\mathcal{CH}_\infty$, two vertices $x$ and $y$ *not in* $C$ are $C$*-consecutive* if all vertices between $x$ and $y$ belong to $C$.

**Lemma 1.** *Let $r \geq 1$ be an integer and $C$ be a code in $\mathcal{CH}_\infty$.*

(i) *If all vertices are $r$-covered by $C$ and all pairs of consecutive vertices are $r$-separated by $C$, then $C$ is $r$-identifying.*
(ii) *If all vertices are $r$-covered by $C$ and all pairs of $C$-consecutive vertices are $r$-separated by $C$, then $C$ is $r$-locating-dominating.*
(iii) *A codeword can $r$-separate at most two pairs of consecutive vertices.*
(iv) *A codeword can $r$-separate at most two pairs of $C$-consecutive vertices.*

**Proof.** (i) and (ii): Can be seen from the fact that a ball consists of consecutive integers.

(iii): A codeword $x \in C$ $r$-separates the two pairs of consecutive integers $(x-r-1, x-r)$ and $(x+r, x+r+1)$.

(iv): Let $(i_k)$ be the sequence of vertices not in $C$, sequence which can be finite or infinite; let $\ell$ and $\ell'$ be integers such that $0 < \ell \leq r$ and $\ell' > r$. A codeword $x$ can at most $r$-separate the following two types of $C$-consecutive noncodewords:

- $i_k = x \pm \ell \in B_r(x)\backslash\{x\}$ and $i_{k+1} = x + \ell' \notin B_r(x)$;
- $i_k = x - \ell' \notin B_r(x)$ and $i_{k+1} = x \pm \ell \in B_r(x)\backslash\{x\}$. □

### 2.1. *Identifying codes in the infinite chain*

The best density of an identifying code in the infinite chain does not depend on $r$.

**Theorem 1.** *For all $r \geq 1$,*

$$d_r^I(\mathcal{CH}_\infty) = 1/2.$$

**Proof.** First, we prove the lower bound. Let $C$ be an $r$-identifying code which is not equal to $Z$ (clearly, the infinite chain always admits one). Let $i \in Z$ be a vertex which is not a codeword. The symmetric difference $B_r(i+r)\Delta B_r(i+r+1)$ is equal to $\{i, i+2r+1\}$; consequently, if $i$ is not a codeword, then $i+2r+1$ must be, in order to separate $i+r$ and $i+r+1$. So any noncodeword $i$ induces a codeword $i+2r+1$. This proves that $d_r^I(\mathcal{CH}_\infty) \geq 1/2$.

Now for the upper bound, we exhibit the construction of a code $C$ which is $r$-identifying and has density one half. Actually, we give two different constructions, one depending on $r$; see Fig. 2. The second code, periodic, is obtained by repeating the pattern given between brackets (containing $4r+2$ vertices). Both constructions are easy to check: first, all vertices are covered by $C$; second, moving from left to right, one sees that each vertex $i$ is separated by $C$ from its predecessor $i-1$, because $i$ either "gains" a new codeword to its right or "loses" a codeword to its left. Using Lemma 1(i), this shows that $C$ is identifying. □

### 2.2. *Locating-dominating codes in the infinite chain*

We distinguish two cases, $r = 1, r \geq 2$.

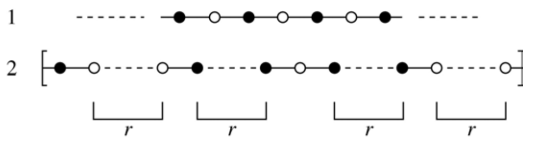


Fig. 2. Two periodic $r$-identifying codes with density 1/2 in $\mathcal{CH}_\infty$. Codewords are in black.

**Theorem 2.**

$$d_1^{LD}(\mathcal{CH}_\infty) = 2/5;$$
$$\textit{for all } r \geq 2, \qquad d_r^{LD}(\mathcal{CH}_\infty) = 1/3.$$

**Proof.** (a) The case $r = 1$ (stated without proof in [11]): we consider five consecutive integers, say 1, 2, 3, 4, 5 and prove that at least two of them must be codewords in a 1-locating-dominating code $C$.

- if there are two or three codewords among 2, 3, and 4, we are done;
- if there is no codeword among 2, 3, and 4, then 3 cannot be covered by $C$;
- if there is one codeword among 2, 3, and 4, without loss of generality we can assume that this codeword is 2 or 3. If it is 2, then 4 must be covered by $C$ and 5 is a codeword; if it is 3, then 1 or 5 is a codeword, because 2 and 4 must be separated by $C$.

This proves the lower bound. Fig. 3(a) gives a pattern of five vertices yielding a periodic 1-locating-dominating code, which is easy to check.

(b) The case $r \geq 2$: again, we first prove the lower bound. Let $C$ be an $r$-locating-dominating code, $Q_n = \{-n, \ldots, n\}$ and $p_n = |C \cap Q_n|$, where $n$ is a positive integer.

In $Q_n$, there are $2n + 1 - p_n$ noncodewords, and there are $\max\{0, 2n - p_n\}$ pairs of $C$-consecutive vertices. Because no codeword outside $Q_{n+r}$ can act on $Q_n$, and because Lemma 1(iv) still works here (see Remark 3 below), we obtain:

$$2p_{n+r} \geq 2n - p_n;$$

consequently, since obviously $p_{n+r} \leq p_n + 2r$,

$$3p_n + 4r \geq |Q_n| - 1,$$

which leads to

$$\frac{p_n}{|Q_n|} \geq \frac{1}{3}\left(1 - \frac{1 + 4r}{|Q_n|}\right)$$

for all $n$. This proves that the density of an $r$-locating-dominating code in $\mathcal{CH}_\infty$ is at least 1/3.

The construction of a periodic $r$-locating-dominating code $C$ with density 1/3 is given in Fig. 3(b), where a pattern containing $3r + 3$ (respectively, $3r$) vertices is given for $r$ odd (respectively, $r$ even).

Again, it is tedious but straightforward to see that all noncodewords are covered and separated by $C$, by comparing the sets $K_{C,r}(i)$ and $K_{C,r}(j)$, where $i$ and $j$

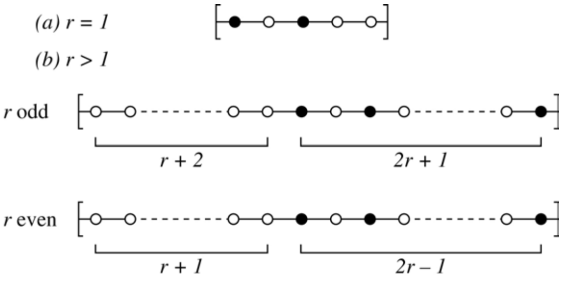


Fig. 3. Periodic $r$-locating-dominating codes with density $2/5$ ($r = 1$) and $1/3$ ($r \geq 2$) in $\mathcal{CH}_\infty$. Codewords are in black.

are $C$-consecutive noncodewords and $i$ or $j$ belong to the patterns in Fig. 3(b). By Lemma 1(ii), this is sufficient. $\square$

## 3. Finite chains

The finite chain with $n$ vertices, $G = \mathcal{CH}_n$, has vertex set $V_n = \{1, \ldots, n\}$ and edge set $E_n = \{\{i, i+1\} : 1 \leq i \leq n-1\}$. We denote by $M_r^I(\mathcal{CH}_n)$ and $M_r^{LD}(\mathcal{CH}_n)$ the smallest cardinality of an $r$-identifying and of an $r$-locating-dominating code, respectively, in $\mathcal{CH}_n$.

In the next subsection, we establish the exact value of $M_1^I(\mathcal{CH}_n)$ for all $n \geq 1$; for $r > 1$, we give a lower bound on $M_r^I(\mathcal{CH}_n)$ and, for an infinite set of values of $n$, an upper bound which coincides with the lower bound.

Then, in Section 3.2, we give the exact value of $M_1^{LD}(\mathcal{CH}_n)$, and for $r > 1$, we determine a lower and an upper bound on $M_r^{LD}(\mathcal{CH}_n)$, which differ by $cr$, where $c$ is close to $1/3$ for an infinite set of values of $n$.

**Remark 3.** The four statements of Lemma 1 still entirely apply if we replace $\mathcal{CH}_\infty$ by $\mathcal{CH}_n$.

### 3.1. *Identifying codes in finite chains*

We first consider the case $r = 1$. Note that there is no identifying code in $\mathcal{CH}_2$, and no $r$-identifying code in $\mathcal{CH}_n$ if $n \leq 2r$, because in this case there exist two distinct vertices $v_1, v_2$ verifying $B_r(v_1) = B_r(v_2)$ (cf. Remark 1); for instance, if $n = 2r$, then $B_r(r) = B_r(r+1) = V_n$.

**Theorem 3.**

$$M_1^I(\mathcal{CH}_n) = \begin{cases} \frac{n+1}{2} & \text{if } n \geq 1 \text{ is odd,} \\ \frac{n}{2} + 1 & \text{if } n \geq 4 \text{ is even.} \end{cases}$$

**Proof.** The case $n = 1$ is trivial. Now consider a 1-identifying code $C$ in $\mathcal{CH}_n$, where $n \geq 3$. At most $|C|$ vertices $i$ can have $|K_{C,1}(i)| = 1$ (otherwise, at least two of them could not be separated) and the other vertices must be covered by at least two codewords, therefore we have the following inequality:

$$1 \cdot |C| + 2 \cdot (n - |C|) \leq \sum_{c \in C} |B_1(c)| \leq 3|C|,$$

which leads to

$$|C| \geq \frac{n}{2}. \tag{1}$$

(a) $n$ odd, $n \geq 3$: by (1), we obtain $|C| \geq (n+1)/2$. On the other hand, $C = \{1, 3, \ldots, n\}$ contains $(n+1)/2$ codewords and is 1-identifying: all codewords are covered by themselves and only by themselves, any noncodeword $i$ is covered by $i - 1 \in C$ and $i + 1 \in C$.

(b) $n$ even, $n \geq 4$: (1) gives $|C| \geq n/2$. Let us try to pick every other vertex as a codeword: $C = \{1, 3, \ldots, n-1\}$; this fails because $K_{C,1}(n-1) = K_{C,1}(n) = \{n-1\}$. Therefore, if $|C| = n/2$, there must be, at least once, at least two consecutive codewords, say $i$ and $i+1$. Necessarily, $i-1$ or $i+2$ is also a codeword (otherwise, $K_{C,1}(i) = K_{C,1}(i+1) = \{i, i+1\}$), which means that at least once, we have at least three consecutive codewords.

Consider now any four consecutive vertices, $j, j+1, j+2, j+3$. Among them, at least two must be codewords, whether or not $j-1$ and $j+4$ are codewords, because $j+1$ and $j+2$ must be covered and separated by $C$.

Finally, let us consider an occurrence of three consecutive codewords $i-1, i, i+1, 2 \leq i \leq n-1$. Every group of four vertices to the right of $i+1$, $\{i+2, i+3, i+4, i+5\}, \ldots, \{i+4k-2, i+4k-1, i+4k, i+4k+1\}$, $k \geq 0$, $n-3 \leq i+4k+1 \leq n$, contains at least two codewords. The same is true for the groups to the left of $i-1$, $\{i-2, i-3, i-4, i-5\}, \ldots, \{i-4k'+2, i-4k'+1, i-4k', i-4k'-1\}$, $k' \geq 0$, $1 \leq i-4k'-1 \leq 4$.

Apart from the codewords $i-1, i, i+1$, and these groups of four vertices, which all in all amount to $3 + 4k + 4k'$ vertices and contain at least $3 + 2k + 2k'$ codewords, we can have additional vertices at both ends of the chain $\mathcal{CH}_n$. At each end, we can have:

- no vertex;
- or one vertex, which is not necessarily a codeword;
- or two vertices, one of them necessarily being a codeword (to cover the end of the chain);
- or three vertices, two of them necessarily being codewords (to cover and separate the end of the chain and its neighbor).

So, at both ends, we can have $a = 1, 3$, or 5 additional vertices—remember that $n$ is even—containing, in the best case, $b = 0, 1$, or 3 codewords, respectively. This shows that the cardinality of a 1-identifying code is at least

$$\begin{aligned} 3 + 2k + 2k' + b &= \frac{(5 + 4k + 4k' + a) + (2b - a + 1)}{2} \\ &= \frac{(n+2) + (2b - a + 1)}{2} \\ &\geq \frac{n+2}{2}. \end{aligned}$$

On the other hand, $C = \{1, 3, \ldots, n-7, n-5, n-3, n-2, n-1\}$ contains $\frac{n}{2}+1$ codewords and is 1-identifying: the first $\frac{n}{2} - 2$ codewords are covered by themselves and only by themselves, $K_{C,1}(n-3) = \{n-3, n-2\}$, $K_{C,1}(n-2) = \{n-3, n-2, n-1\}$, $K_{C,1}(n-1) = \{n-2, n-1\}$, and for the noncodewords, any noncodeword $i$ is covered by $i-1 \in C$ and $i+1 \in C$, except $n$ which is covered by $n-1 \in C$. □

When $r > 1$, we first give a lower bound.

**Theorem 4.** *For $r \geq 2$, $n \geq 2r+1$,*

$$M_r^I(\mathcal{CH}_n) \geq \left\lceil \frac{n+1}{2} \right\rceil.$$

**Proof.** Let $C$ be an $r$-identifying code in $\mathcal{CH}_n$. Assume that $i$ is a noncodeword, with $1 \leq i \leq n-(2r+1)$; then $i+2r+1$ is a codeword, because $i+r$ and $i+r+1$ must be separated by $C$. Therefore:

$$|(V_n \backslash C) \cap [1, n-2r-1]| \leq |C \cap [2r+2, n]|.$$

On the other hand,

$$n = |C \cap [1, 2r+1]| + |C \cap [2r+2, n]| \\ + |(V_n \backslash C) \cap [1, n-2r-1]| + |(V_n \backslash C) \cap [n-2r, n]|,$$

and therefore

$$n \leq 2|C| - |C \cap [1, 2r+1]| + |(V_n \backslash C) \cap [n-2r, n]|.$$

Now the $r$ pairs of vertices $(i, i+1)$, $1 \leq i \leq r$, can be separated only by $i+1+r$, so

$$C \cap [1, 2r+1] \supseteq \{r+2, r+3, \ldots, 2r+1\},$$

but 1 is not covered by $C$ yet; therefore $|C \cap [1, 2r+1]| \geq r+1$.

Also, in the same way, $|C \cap [n-2r, n]| \geq r+1$, and $|(V_n \backslash C) \cap [n-2r, n]| \leq r$. Finally:

$$n \leq 2|C| - (r+1) + r = 2|C| - 1,$$

as was claimed. □

With given $r$, for infinitely many values of $n$, it is possible to construct codes with $(n+1)/2$ elements, which meets the lower bound of Theorem 4.

**Theorem 5.** *Let $k$ be a nonnegative integer. For any fixed $r \geq 2$, and for $n = (4r+2)k+1$,*

$$M_r^I(\mathcal{CH}_n) \leq \frac{n+1}{2}.$$

**Proof.** The case $k = 0$ is trivial, so we assume that $k \geq 1$. Consider the pattern with $4r+2$ vertices and $2r+1$ codewords, already seen in Fig. 2. Repeat this pattern $k-1$ times to the left and append a codeword to the right, to obtain a code $C \subseteq V_n$ which has $k(2r+1)+1 = (n+1)/2$ elements. The same argument as in the proof of Theorem 1 shows that $C$ is indeed $r$-identifying. □

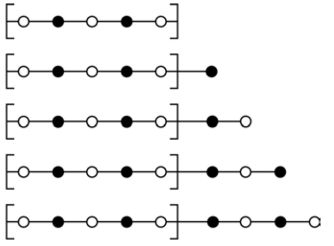


Fig. 4. 1-Locating-dominating codes in $\mathcal{CH}_n$. Codewords are in black.

### 3.2. Locating-dominating codes in finite chains

First, we consider the case $r = 1$.

**Theorem 6** ([10]). *For all $n \geq 1$,*

$$M_1^{LD}(\mathcal{CH}_n) = \left\lceil \frac{2n}{5} \right\rceil.$$

**Proof.** For the lower bound, we give an alternative proof, using a technique quite different from the proof in [10]: consider a 1-locating-dominating code $C$ in $\mathcal{CH}_n$. Now $n - |C|$ noncodewords must be covered by $C$, at most $|C|$ of these vertices are covered by one codeword, and the remaining vertices are covered by at least two codewords; therefore we have the following inequality:

$$1 \cdot |C| + 2 \cdot (n - 2|C|) \leq \sum_{c \in C} |B_1(c) \backslash \{c\}| \leq 2|C|,$$

which leads to

$$|C| \geq \left\lceil \frac{2n}{5} \right\rceil.$$

This lower bound can be met with equality, as shown in Fig. 4, where the left pattern, containing five vertices, is repeated $k-1$ times to the left, with $k = \lfloor \frac{n}{5} \rfloor$. It is easy to check that these codes are 1-locating-dominating. Their sizes are $2k$, $2k + 1$, $2k + 1$, $2k + 2$, and $2k + 2$, respectively, which in all cases is equal to $\lceil \frac{2n}{5} \rceil$. □

When $r > 1$, we first give a lower bound.

**Theorem 7.** *For $r \geq 2, n \geq 1$,*

$$M_r^{LD}(\mathcal{CH}_n) \geq \left\lceil \frac{n+1}{3} \right\rceil.$$

**Proof.** Let $C$ be an $r$-locating-dominating code in $\mathcal{CH}_n$. By Lemma 1(iv) combined with Remark 3, a codeword can separate at most two pairs of $C$-consecutive noncodewords. Therefore, $2 \cdot |C| \geq n - |C| - 1$. But one can go further: because 1 and $n$ must either belong to $C$ or be covered by $C$,

$$\text{if } p = |C \cap [1, r+1]| \text{ and } q = |C \cap [n-r, n]|, \text{ then } p \geq 1 \text{ and } q \geq 1.$$

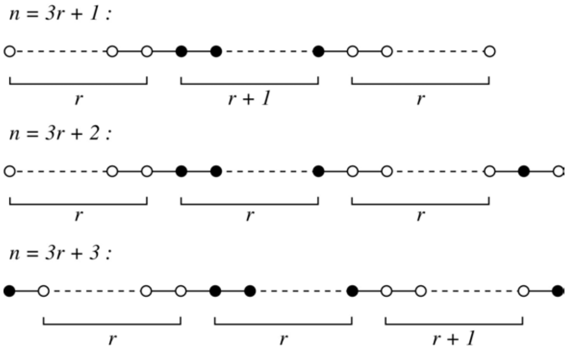


Fig. 5. $r$-Locating-dominating codes in $\mathcal{CH}_n$. Codewords are in black.

Now a codeword in $[1, r+1]$ (respectively, in $[n-r, n]$) cannot separate a pair of vertices to its left (respectively, to its right); therefore,

$$2 \cdot (|C| - p - q) + 1 \cdot (p+q) \geq n - |C| - 1,$$
$$\text{i.e., } 3|C| \geq n - 1 + p + q \geq n + 1,$$

hence the claim. □

The lower bound of the previous theorem can be met with equality for some values of $n$, as can be easily seen in Fig. 5. However, we did not succeed in reducing the gap between the lower bound of Theorem 7 and the following upper bounds to less than $cr$, where $c$ is close to 1/3 for infinitely many values of the chain length $n$.

Still, we believe that the lower bound of Theorem 7 can be met with equality for infinitely many values of $n$ (see Conjecture 1 below).

**Theorem 8.** *Let $k$ be a nonnegative integer and $r \geq 2$;*

(i) *if $r$ is even and $n = 3kr + 2r + 1$, then*

$$M_r^{LD}(\mathcal{CH}_n) \leq \frac{n+1}{3} + \frac{r+1}{3};$$

(ii) *if $r$ is odd and $n = k(3r+3) + 2r + 1$, then*

$$M_r^{LD}(\mathcal{CH}_n) \leq \frac{n+1}{3} + \frac{r+1}{3};$$

(iii) *for any $n \geq 2r + 1$,*

$$M_r^{LD}(\mathcal{CH}_n) \leq \left\lceil \frac{n+1}{3} + \frac{7r+5}{3} \right\rceil.$$

**Proof.** (i) Consider the case $r$ even in Fig. 6 (cf. Fig. 3). If $k = 1$, simply take the construction as it is. If $k = 0$, take only the left part, with $2r+1$ vertices. If $k \geq 2$, then repeat $k - 1$ times to the right the pattern given between brackets.

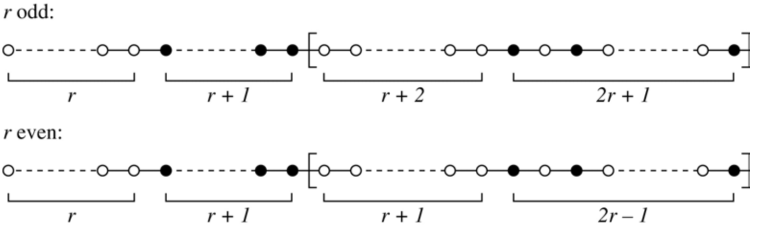


Fig. 6. Patterns for $r$-locating-dominating codes in $\mathcal{CH}_n$. Codewords are in black.

The code $C$ thus constructed has $kr + r + 1$ elements, in a chain of length $n = 3kr + 2r + 1$. Therefore $|C| = (n + r + 2)/3$ and it is straightforward to check that $C$ is $r$-locating-dominating.

(ii) Now we construct $C^*$ considering the case $r$ odd in Fig. 6; code $C^*$ has $k(r + 1) + r + 1$ elements, in a chain of length $n = k(3r+3)+2r+1$. Therefore $|C^*| = (n + r + 2)/3$ and it is straightforward to check that $C^*$ is $r$-locating-dominating.

(iii) If $r$ is even, add $s$ codewords to the left of the previous construction of $C$, where $1 \leq s \leq 3r - 1$. The code $C_s$ thus constructed has $kr + r + 1 + s$ elements in a chain of length $n_s = 3kr + 2r + 1 + s$, running between $3kr + 2r + 2$ and $3kr + 2r + 1 + 3r - 1 = 3r(k + 1) + 2r$: all lengths $n \geq 2r + 1$ have been considered and $C_s$, which is obviously $r$-locating-dominating, has size $(n_s + r + 2 + 2s)/3 \leq (n_s + 7r)/3$.

If $r$ is odd, add $s$ codewords to the left of the previous construction of $C^*$, where $1 \leq s \leq 3r + 2$. The code $C_s^*$ thus constructed has $k(r + 1) + r + 1 + s$ elements in a chain of length $n_s^* = k(3r + 3) + 2r + 1 + s$, running between $k(3r + 3) + 2r + 2$ and $k(3r + 3) + 2r + 1 + 3r + 2 = (k + 1)(3r + 3) + 2r$: all lengths $n \geq 2r + 1$ have been considered and $C_s^*$, which is obviously $r$-locating-dominating, has size $(n_s^* + r + 2s + 2)/3 \leq (n_s^* + 7r + 6)/3$. $\square$

With further investigation, the results of Theorem 8(iii) could be slightly improved, since clearly not all the $s$ vertices added to the left of the construction need to be codewords.

Constructions for $r = 2$ and $r = 3$ give us grounds for stating the following conjecture (see also note added in proof).

**Conjecture 1.** *For any fixed $r \geq 2$, there exist infinitely many values of $n$ for which*

$$M_r^{LD}(\mathcal{CH}_n) \leq \left\lceil \frac{n+1}{3} \right\rceil.$$

## 4. Cycles

The cycle with $n$ vertices, $G = \mathcal{CY}_n$, has vertex set $V_n = \{1, \ldots, n\}$ and edge set $E_n = \{\{i, i + 1\} : 1 \leq i \leq n - 1\} \cup \{\{n, 1\}\}$. We denote by $M_r^I(\mathcal{CY}_n)$ and $M_r^{LD}(\mathcal{CY}_n)$ the smallest cardinality of an $r$-identifying and of an $r$-locating-dominating code, respectively, in $\mathcal{CY}_n$.

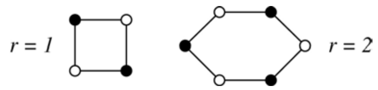


Fig. 7. Codes which are not $r$-identifying. Codewords are in black.

**Remark 4.** Unlike the case of chains, in order to prove that a code $C$ is identifying in a cycle, it is not sufficient to check that any two consecutive vertices are separated by $C$, as shown in Fig. 7, where the codes are not identifying, although all pairs of consecutive vertices are separated by $C$. The same is true for locating-dominating codes and $C$-consecutive vertices. This means that the statements (i) and (ii) of Lemma 1 do not apply to cycles.

However, it is easy to see that the statements (iii) and (iv) of Lemma 1 are still valid if we replace $\mathcal{CH}_\infty$ by $\mathcal{CY}_n$.

In the next subsection, for all $r$ we give a lower bound on the smallest size of an $r$-identifying code in $\mathcal{CY}_n$, and for even $n$ we provide a construction which meets this bound.

Then in Section 4.2, we give the exact value of $M_1^{LD}(\mathcal{CY}_n)$, and determine, for all $r \geq 2$, a lower bound on $M_r^{LD}(\mathcal{CY}_n)$, and infinitely many particular values of the cycle length $n$ for which the lower and upper bounds coincide.

### 4.1. *Identifying codes in cycles*

We first give a lower bound which is valid for all $r$ and $n \geq 2r + 2$—note that if $n \leq 2r + 1$, then no $r$-identifying code can exist in $\mathcal{CY}_n$, because for all vertices $i$, we have $B_r(i) = V_n$ (cf. Remark 1).

**Theorem 9.** *For $r \geq 1, n \geq 2r + 2$,*

$$M_r^I(\mathcal{CY}_n) \geq \left\lceil \frac{n}{2} \right\rceil.$$

**Proof.** The proof is similar to the first part of the proof of Theorem 1: let $C$ be an $r$-identifying code; if $i$ is not a codeword, then $i + 2r + 1 \pmod n$ (which cannot be equal to $i$) must be, so $n - |C| \leq |C|$, i.e., $|C| \geq n/2$. $\square$

This lower bound is met with equality for all even $n \geq 2r + 4$ (see also note added in proof).

**Theorem 10.** *For all $r \geq 1$, and for all even $n$, $n \geq 2r + 4$,*

$$M_r^I(\mathcal{CY}_n) \leq \frac{n}{2}.$$

**Proof.** We claim that the code $C = \{1, 3, \ldots, n - 1\}$, consisting of every other vertex, is $r$-identifying. First, all vertices are covered by $C$.

Next, since all codewords play the same role, and all noncodewords play the same role, without loss of generality we can consider a particular codeword, say 1, and the noncodeword 2, and it will be sufficient to show that no other vertex $j$ can be such that $K_{C,r}(j) = K_{C,r}(1)$ or $K_{C,r}(j) = K_{C,r}(2)$.

Assume first that $r$ is even. We see that

$$K_{C,r}(1) = \{n+1-r, n+3-r, \ldots, n-1, 1, 3, \ldots, r-1, r+1\},$$
$$K_{C,r}(2) = \{n+3-r, n+5-r, \ldots, 1, 3, \ldots, r-1, r+1\}.$$

If $j$ is such that $K_{C,r}(j) = K_{C,r}(1)$, then in particular

- $j$ is covered by 1: this means that

$$n+1-r \leq j \leq r+1;$$

- $j$ is not covered by $r+3$ (which is smaller than $n+1-r$ and cannot belong to $K_{C,r}(1)$): this means that

$$j \notin \{3, 4, 5, \ldots, 2r+2, 2r+3\};$$

- $j$ is not covered by $n-1-r$ (which is greater than $r+1$ and cannot belong to $K_{C,r}(1)$): this means that

$$j \notin \{n-1-2r, n-2r, \ldots, n-2, n-1\}.$$

This leaves only $n$, 1 or 2 as possible values for $j$; but $n$ cannot be covered by $r+1$, and 2 cannot be covered by $n+1-r$. So these two vertices are separated by $C$ from 1 and necessarily $j = 1$.

If $j$ is such that $K_{C,r}(j) = K_{C,r}(2)$, then similarly:

$$n+1-r \leq j \leq r+1;$$
$$j \notin \{3, 4, 5, \ldots, 2r+2, 2r+3\}; \quad j \notin \{n+1-2r, n+2-2r, \ldots, n, 1\}.$$

So necessarily $j = 2$.

If $r$ is odd, then

$$K_{C,r}(1) = \{n+2-r, n+4-r, \ldots, n-1, 1, 3, \ldots, r-2, r\},$$
$$K_{C,r}(2) = \{n+2-r, n+4-r, \ldots, 1, 3, \ldots, r, r+2\}.$$

If $j$ is such that $K_{C,r}(j) = K_{C,r}(1)$, then:

$$n+1-r \leq j \leq r+1;$$
$$j \notin \{2, 3, 4, \ldots, 2r+1, 2r+2\}; \quad j \notin \{n-2r, n-2r+1, \ldots, n-1, n\}.$$

So necessarily $j = 1$.

If $j$ is such that $K_{C,r}(j) = K_{C,r}(2)$, then:

$$n+1-r \leq j \leq r+1;$$
$$j \notin \{4, 5, 6, \ldots, 2r+3, 2r+4\}; \quad j \notin \{n-2r, n-2r+1, \ldots, n-1, n\}.$$

This leaves only 1, 2, or 3 as possible values for $j$; but 1 cannot be covered by $r+2$, and 3 cannot be covered by $n+2-r$. So these two vertices are separated by $C$ from 2 and necessarily $j = 2$. $\square$

The condition $n \geq 2r+4$ is crucial for Theorem 10. If $n = 2r+2$, there is a serious degradation in the quality of the codes, compared to Theorem 10, since the next theorem states that all vertices but one must be taken!

**Theorem 11.** *For all* $r \geq 1$,

$$M_r^I(\mathcal{CY}_{2r+2}) = 2r+1.$$

**Proof.** That $n-1 = 2r+1$ codewords are sufficient is easy to check.

We prove now that if $C$ is an $r$-identifying code in $\mathcal{CY}_{2r+2}$, then at most one vertex is not in $C$.

For any $i \in V_n, B_r(i) = V_n \setminus \{i+r+1 \pmod n\}$, and for any $i, j \in V_n, i \neq j, B_r(i) \Delta B_r(j) = \{i+r+1 \pmod n, j+r+1 \pmod n\}$. Now, we know that in $B_r(i)\Delta B_r(j)$, there must be at least one codeword, which separates $i$ and $j$.

Assume, without loss of generality, that $1 \notin C$. Then for all $j \neq r+2, B_r(r+2)\Delta B_r(j) = \{1, j+r+1 \pmod n\}$ and

$$\emptyset \neq (B_r(r+2)\Delta B_r(j)) \cap C \subseteq \{j+r+1 \pmod n\}.$$

So for all values of $j$ but one, the $n-1$ distinct vertices $j+r+1 \pmod n$ are necessarily codewords. $\square$

### 4.2. Locating-dominating codes in cycles

The case $r=1$ in a cycle is similar to the case of the finite chain.

**Theorem 12** ([10]). *For all* $n \geq 1$,

$$M_1^{LD}(\mathcal{CY}_n) = \left\lceil \frac{2n}{5} \right\rceil.$$

**Proof.** For the lower bound, see the proof of Theorem 6. The only, unimportant, difference is that in a cycle, $|B_1(c)\setminus\{c\}|$ is always equal to 2 for $n \geq 3$.

This lower bound can be met with equality, using Fig. 4: the left pattern, containing five vertices, is repeated $k-1$ times to the left, and to obtain a cycle of length $n$, one links the leftmost and rightmost vertices. It is easy to check that these codes are 1-locating-dominating in $\mathcal{CY}_n$. $\square$

When $r > 1$, we first give a lower bound.

**Theorem 13.** *For* $r \geq 2, n \geq 1$,

$$M_r^{LD}(\mathcal{CY}_n) \geq \left\lceil \frac{n}{3} \right\rceil.$$

**Proof.** Let $C$ be an $r$-locating-dominating code in $\mathcal{CY}_n$. By Lemma 1(iv) together with Remark 4, and since there are $n$ pairs of consecutive vertices, we have: $2 \cdot |C| \geq n - |C|$. $\square$

With given $r$, for infinitely many values of $n$, the previous lower bound can be met with equality.

**Theorem 14.** *Let $r = 2$ and $k \geq 2$, or $r > 2$ and $k \geq 1$; if $r$ is odd and $n = k(3r + 3)$, or if $r$ is even and $n = 3kr$, then*

$$M_r^{LD}(\mathcal{CY}_n) \leq \frac{n}{3}.$$

**Proof.** According to the parity of $r$, use $k$ times the appropriate pattern of Fig. 3(b), and link together the leftmost and rightmost vertices. □

The case when $r = 2$ and $k = 1 (n = 6)$ would leave two noncodewords covered by the same two codewords. Actually, it is easy to see that $M_2^{LD}(\mathcal{CY}_6) = 3$; the lower bound comes from the fact that with two codewords, only three nonempty sets of codewords are available, for four noncodewords.

## 5. New balls

Note that in the infinite chain $\mathcal{CH}_\infty$, the ball $B_r(i)$ is the set of $2r + 1$ consecutive integers $\{i - r, \ldots, i, \ldots, i + r\}$, i.e., a segment of odd length. One can instead consider segments of even length $s$ and the problem is how to best place these segments in $\mathcal{CH}_\infty$, in such a way that every integer belongs to at least one segment, and no two integers belong to the same set of segments (identifying codes) or no two noncodewords belong to the same set of segments (locating-dominating codes). More generally, in [8] this problem is considered in $Z^2$, for identifying codes, and with various patterns.

Now, in an even segment, the center is not in $Z$. As we shall see below, this does not affect the nature of the issue for identifying codes, but leads to add a new parameter in the case of locating-dominating codes.

### *5.1. Identifying codes*

Here, all ways of associating a segment with a codeword are equivalent, since codewords and noncodewords have the same status with respect to the properties that an identifying code must satisfy: every vertex must belong to at least one segment, and no two vertices can belong to the same set of segments—of course this remark is true for even as well as for odd segments.

**Theorem 15.** *Let $s \in N^*$, $s$ even. The best density of an identifying code $C$ using segments of length $s$ in $\mathcal{CH}_\infty$ is $2/3$ if $s = 2$, and $1/2$ if $s > 2$.*

We recall that for all odd values of $s \geq 3$, the best density is $1/2$ (Theorem 1).

**Proof.** In this proof, in order to specify the segments and use easily their covering and separating properties, we choose to associate a segment with, for instance, its smallest element, which will be the codeword representing this segment, and we change accordingly the notion of covering: a codeword $i$ will $s$-cover the $s$ integers $i, i + 1, \ldots, i + s - 1$. See Fig. 8, where the leftmost codewords are represented with their segments.

We first show the lower bound for $s = 2$: we consider three consecutive integers, say 1, 2, 3, and prove that at least two of them must be codewords of $C$. Suppose that only 3 is

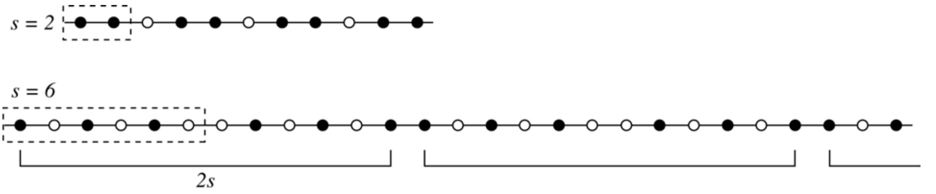


Fig. 8. Optimal identifying codes in $\mathcal{CH}_\infty$. Codewords are in black.

in $C$; then 2 is not covered by $C$. Suppose next that only 2 is a codeword; then 2 and 3 are not separated by $C$. Finally, if only 1 is a codeword, then 3 is not covered by $C$.

To prove that the density is at least one half for all even $s > 2$, it suffices to prove that every pair $\{j - s, j\}$ contains at least one codeword, and this holds, because if $j$ is not a codeword, then the only codeword that can separate $j - 1$ and $j$ is $j - s$.

The upper bound for $s = 2$ comes from the construction in Fig. 8, which is easy to check.

Finally, we provide a construction for $s > 2$, $s$ even. Let

$$C_0 = \{0, 2, \dots, s-4, s-2, s+1, s+3, \dots, 2s-1\} \text{ and } C = \bigcup_{k \in Z} (C_0 + k2s)$$

(see Fig. 8 for $s = 6$). We claim that $C$ is identifying. Consider a pair of consecutive points $i, i+1$; we show that necessarily there is a codeword which covers exactly one of them.

If $i + 1$ is a codeword, we are done, since $i + 1$ does not cover $i$. So we are left with two cases:

(i) $i \in C, i + 1 \notin C$; we can assume, without loss of generality, that $i$ belongs to $C_0 \backslash \{2s - 1\}$. Now $i - s + 1$ covers $i$ and not $i + 1$, and $i - s + 1$ is a codeword: indeed, if $i$ is of the form $s + 1 + 2j\,(0 \le 2j \le s - 4)$, then $i - s + 1 = 2j + 2$ is in $C_0$, and if $i$ is of the form $2j\,(0 \le 2j \le s - 2)$, then $i - s + 1 = (2j + s + 1) - 2s$ is in $(C_0 - 2s) \subset C$.

(ii) $i, i + 1 \notin C$; we can assume, without loss of generality, that $i = s - 1$, and we see that 0 is a codeword covering $i$ and not $i + 1$. $\square$

### 5.2. *Locating-dominating codes*

For odd as well as for even segment lengths, the best density of a locating-dominating code depends on how we choose to associate a segment with a codeword, as shows the next example, for length four.

**Example 2.** (a) Consider segments of length $s = 4$, in which the smallest element is the codeword representing the segment—see Fig. 9(a). We claim that the best density that a locating-dominating code using these segments can achieve is $2/5$.

First, a periodic locating-dominating code $C$ is obtained by repeating the pattern of Fig. 3(a). For the lower bound, consider five consecutive vertices, say 6, 7, 8, 9, and 10. Obviously, at least one of them must be a codeword; now assume that only one of them

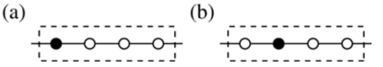


Fig. 9. Two ways of choosing a codeword in a segment of length four. Codewords are in black.

belongs to $C$. If it is 6 that belongs to $C$, then 10 cannot be covered by $C$; if it is 7 or 8, then 9 and 10 cannot be separated by $C$; if it is 10, then 9 cannot be covered by $C$. This leaves $C \cap \{6, 7, 8, 9, 10\} = \{9\}$ as the only possibility. But then, in order to separate 6 and 7, 7 and 8, and to cover 8, necessarily 3, 4, and 5 are codewords.

So we have proved that

either $|C \cap \{6, 7, 8, 9, 10\}| \geq 2$

or $|C \cap \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}| \geq 1 + 3 = 4$,

which shows that the density of the code is at least $2/5$.

(b) On the other hand, if we consider segments of length four in which now it is the second smallest element which represents the segment—see Fig. 9(b)—then the density can reach $1/3$, since it is easy to check that the code $C = \{3p : p \in Z\}$ is locating-dominating (see also the proof of Theorem 17).

Therefore, for each value of $s$, it would be necessary to consider all possible positions of the codeword in the segment and try to derive the corresponding best densities. Since we do not wish to go into such a detailed investigation, we content ourselves with the following results and conjecture.

**Theorem 16.** *Let $s \in N^*$; the best density of a locating-dominating code $C$ using segments of length $s$ in $\mathcal{CH}_\infty$ is at least* $1/3$.

**Proof.** Apply *mutatis mutandis* Lemma 1(iv) and adapt part (b) of the proof of Theorem 2. The asymptotic result of the counting arguments does not depend on the position of the codeword in the segment. $\square$

**Theorem 17.** *Let $s \in N^*$. If $s$ is odd ($s \geq 5$), or $s = 6k+2$ ($k \geq 1$), or $s = 6k + 4$ ($k \geq 0$), then, choosing appropriately the position of the codeword in the segment, we can achieve* $1/3$ *for the best density of a locating-dominating code using segments of length $s$ in $\mathcal{CH}_\infty$.*

**Proof.** If $s \geq 5$ is odd, see Theorem 2, where the codeword is the center of the segment.

When $s = 6k + 2$, $s \geq 8$, if we choose to represent the segment by its smallest element, then the code $C = \{3p : p \in Z\}$ is locating-dominating: all we have to check is that two consecutive noncodewords $i = 3p + 1$, $i + 1 = 3p + 2$, are separated by $C$. This is so, because $3p + 1 - (s - 1) = 3p - 6k$ is a codeword covering $i$, not $i + 1$.

When $s = 6k + 4$, $s \geq 4$, if we choose to represent the segment by its second smallest element, then the code $C = \{3p : p \in Z\}$ is locating-dominating; since now a codeword $a$ covers the vertices $a - 1, a, a + 1, \ldots, a + s - 2$, all we have to check is that two $C$-consecutive noncodewords $i = 3p + 2$, $i + 2 = 3p + 4$, are separated by $C$. This is so, because $3p + 2 - (s - 2) = 3p - 6k$ is a codeword covering $i$, not $i + 2$. $\square$

Things get more complicated when the length of the segment is a multiple of six, but a small study led us to the following conjecture.

**Conjecture 2.** *Let* $s \in N^*$. *If* $s = 6k$ $(k \geq 1)$, *then, choosing appropriately the position of the codeword in the segment, we can achieve* $1/3$ *for the best density of a locating-dominating code using segments of length* $s$ *in* $\mathcal{CH}_\infty$.

## 6. Conclusion

Below, we summarize some results on identifying and locating-dominating codes in cycles and chains.

- For infinite chains:
  - For $r \geq 1$, $d_r^I(\mathcal{CH}_\infty) = 1/2$.
  - $d_1^{LD}(\mathcal{CH}_\infty) = 2/5$ [11]; for $r \geq 2$, $d_r^{LD}(\mathcal{CH}_\infty) = 1/3$.
- For finite chains:
  - $M_1^I(\mathcal{CH}_n) = \begin{cases} \frac{n+1}{2} & \text{if } n \geq 1 \text{ is odd,} \\ \frac{n}{2}+1 & \text{if } n \geq 4 \text{ is even;} \end{cases}$
    for $r \geq 2, n \geq 2r+1$, $M_r^I(\mathcal{CH}_n) \geq \lceil \frac{n+1}{2} \rceil$, with equality for infinitely many $n$.
  - $M_1^{LD}(\mathcal{CH}_n) = \lceil \frac{2n}{5} \rceil$ [10]; for $r \geq 2, n \geq 1$, $M_r^{LD}(\mathcal{CH}_n) \geq \lceil \frac{n+1}{3} \rceil$ and $M_r^{LD}(\mathcal{CH}_n) \leq \frac{n+1}{3} + \frac{r+1}{3}$ for infinitely many $n$.
- For cycles:
  - For $r \geq 1, n \geq 2r+2$, $M_r^I(\mathcal{CY}_n) \geq \lceil \frac{n}{2} \rceil$, with equality for even $n \geq 2r+4$.
  - $M_1^{LD}(\mathcal{CY}_n) = \lceil \frac{2n}{5} \rceil$ [10]; for $r \geq 2, n \geq 1$, $M_r^{LD}(\mathcal{CY}_n) \geq \lceil \frac{n}{3} \rceil$, with equality for infinitely many $n$.

### Note added in proof

The case of 1-identifying codes in cycles (cf. Theorems 9 and 10) is completely solved, since it is shown in [6] that when $n$ is odd, $n \geq 7$,

$$M_1^I(\mathcal{CY}_n) = \frac{n+3}{2}.$$

Using variations on the pattern given in Fig. 3(b) with $r = 2$, i.e., $C = \{4, 6\} \subset \{1, 2, 3, 4, 5, 6\}$, Honkala [7] proves that

(1) if $r \equiv 1, 2, 3,$ or $4 \pmod 6$, $r \neq 1$, then for all $n \geq 2r+1$,

$$M_r^{LD}(\mathcal{CH}_n) \leq \left\lceil \frac{n+2r+3}{3} \right\rceil;$$

(2) for all $n \geq 1$,

$$M_2^{LD}(\mathcal{CH}_n) \leq \left\lceil \frac{n+1}{3} \right\rceil,$$

cf. Theorem 8. Inequality (2), together with Theorem 7, shows that $M_2^{LD}(\mathcal{CH}_n) = \lceil \frac{n+1}{3} \rceil$, and Conjecture 1 “strongly” holds for $r = 2$.

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