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Completing some Partial Latin Squares

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We show that any partial $3r \times 3r$ Latin square whose filled cells lie in two disjoint $r \times r$ sub-squares can be completed. We do this by proving the more general result that any partial $3r$ by $3r$ Latin square, with filled cells in the top left $2r \times 2r$ square, for which there is a pairing of the columns so that in each row there is a filled cell in at most one of each matched pair of columns, can be completed if and only if there is some way to fill the cells of the top left $2r \times 2r$ square.

1. INTRODUCTION

Imagine a square array of n^2 cells containing n different *symbols*. If the symbols are arranged in the array so that no symbol appears more than once in any row or column, then the array is called an $n \times n$ Latin square. To construct examples of Latin squares is a simple task, but what if some of the entries have already been filled and your task is to complete the square, if possible? The impetus for the work on completing partial Latin squares is due to a paper of Evans [3]. In his article, Evans outlined the intrinsic interest of completing a variety of partial structures, including Latin squares of course, but also groups and projective planes. He also made a number of conjectures, among them that every $n \times n$ partial Latin square which has at most $n - 1$ filled entries can be completed. This conjecture remained unsettled for some 20 years until it was finally proved to be true by Häggkvist for $n \geq 1111$ [5] and in its entirety by Andersen and Hilton [1] and Smetaniuk [7]. Evans' conjecture is, in some sense, best possible, since there are obvious configurations of n symbols in an $n \times n$ partial Latin square which cannot be completed. However, if we insist on some additional structure for the filled cells and symbols, a variety of configurations can be shown to always have completions. One of these possibilities gave rise to a conjecture of Häggkvist:

CONJECTURE (1980). *Any partial $nr \times nr$ Latin square whose filled cells lie in $(n - 1)$ disjoint $r \times r$ squares can be completed.*

This paper provides a proof of Häggkvist's conjecture when $n = 3$.

2. THE RESULTS

For the proof we require two existing results from opposite ends of the history of completing partial Latin squares. The first is probably the first result about completing partial Latin squares, due to Ryser.

THEOREM A (RYSER'S THEOREM [6]). *An $r \times s$ Latin rectangle with entries from n symbols can be extended to an $n \times n$ Latin square on those symbols if and only if each symbol occurs in the rectangle at least $r + s - n$ times.*

The configuration of Häggkvist's conjecture when $n = 3$ is actually only a special case of a much broader class of partial Latin squares which all have completions. The key to this is the following more general theorem.

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THEOREM 1. *Let $\mathcal{L}$ be a partial $3r$ by $3r$ Latin square with filled squares in the top left $2r$ by $2r$ square $\mathcal{T}$. Suppose further that in each row of $\mathcal{T}$ there is a filled cell in at most one of columns i and $i+r$ for each $i = 1, \dots, r$. Then $\mathcal{L}$ can be completed if and only if there is some way to fill the cells of $\mathcal{T}$.*

PROOF. The necessity of the condition is trivial of course. Suppose then that all the cells of $\mathcal{T}$ have been filled. Our objective will be to redistribute the symbols in the fillable cells of $\mathcal{T}$ in such a way that every symbol appears at least $2r + 2r - 3r = r$ times; that $\mathcal{L}$ can be completed then follows from Theorem A.

To hopefully avoid confusion, when we wish to refer to the filled cells of the original partial Latin square we shall speak of $\mathcal{L}$, and when we wish to speak of the new filled cells we shall speak of $\mathcal{T}$. We define a bipartite graph H which has bipartition (R, S), where R is the set of rows of $\mathcal{T}$ and S is the set of $3r$ symbols. In H we join a row ρ to a symbol s with an edge $f_{\rho s}$ whenever the symbol s does not occur in row ρ in the original partial Latin square $\mathcal{L}$. Now, to introduce the way in which the fillable cells in $\mathcal{T}$ were filled, we colour the edge $f_{\rho s}$ with colour c if the symbol s now appears in cell (ρ, c) of $\mathcal{T}$. We shall refer to those edges which receive no colour in this way as the *uncoloured* edges. Observe that when $\mathcal{T}$ is filled there are $2r$ used symbols in each row, and so r unused symbols, each vertex $\rho \in R$ is incident with r uncoloured edges. Finally, we provide each vertex $\rho \in R$ with a list $F(\rho)$ of columns in which row ρ was filled in $\mathcal{L}$.

Recall that our aim is to fill the fillable cells in $\mathcal{T}$ in such a way that every symbol appears at least r times. If this has not already been achieved, then there must be a symbol s_0 which is used less than r times in $\mathcal{T}$. Given this symbol, there must be an $1 \le i \le r$ so that symbol s_0 never appears in columns i and $i+r$. Our aim will be to redistribute the symbols so that after the redistribution s_0 will appear in one of these columns, and overall will appear one more time than it did before—creeping closer to the magic count of r.

Let B_0 be the subset of symbols which appear at most r times in $\mathcal{T}$ and also appear in at most one of columns i and $i+r$. Note that since the $4r^2$ cells of $\mathcal{T}$ are filled with only $3r$ symbols, there must be at least one symbol used more than r times which, therefore, is not a member of B_0. Let H_0 be the subgraph of H induced by the uncoloured edges incident with the vertices of B_0. Then for each vertex $x \in B_0$, $d_{H_0}(x) \ge r$ and certainly for each vertex $\rho \in R$, $d_{H_0}(\rho) \le r$. Thus H_0 has a matching M of B_0 into R, which we shall regard as edges in H. We shall use these matching edges to construct a path P which begins at s_0 and ends at one of the vertices outside B_0. We will use this path to redistribute the symbols.

Let $f_{\rho_0 s_0}$ be the edge of M incident with s_0. By assumption, either column i or column $i+r$ is not a member of $F(\rho_0)$, i.e., there is a column $c_0 \in \{i, i+r\} \setminus F(\rho_0)$. Then ρ_0 is incident with an edge $f_{\rho_0 s_1}$ which is coloured c_0 in H. If this new symbol s_1 is a member of B_0, then we continue constructing the path, otherwise we stop. If we continue, since $s_1 \in B_0$ it is incident with an edge of M joining it to a row ρ_1, and as before there must be some column $c_1 \in \{i, i+r\} \setminus F(\rho_1)$ and an edge coloured c_1 joining ρ_1 to s_2. We construct the path by repeating these steps until we meet a symbol outside B_0. P must be a path, since there exists at most one edge coloured i or $i+r$ which is incident with each symbol other than the last (each of these symbols is a member of B_0 and that is one of the conditions for membership), and the edges joining the symbols to the rows form a matching in M. Since B_0 is a finite set this path must eventually meet a vertex outside B_0 and there it stops, by construction, at a symbol s_n.

We recolour $f_{\rho_i s_i}$ with colour c_i, and uncolour $f_{\rho_i s_{i+1}}$, for $i = 0, \dots, n-1$ (or if you wish, make the analogous changes to $\mathcal{T}$). The crux of the argument relies on what has happened to the symbols in this redistribution. Observe that after these changes the symbol s_0 will

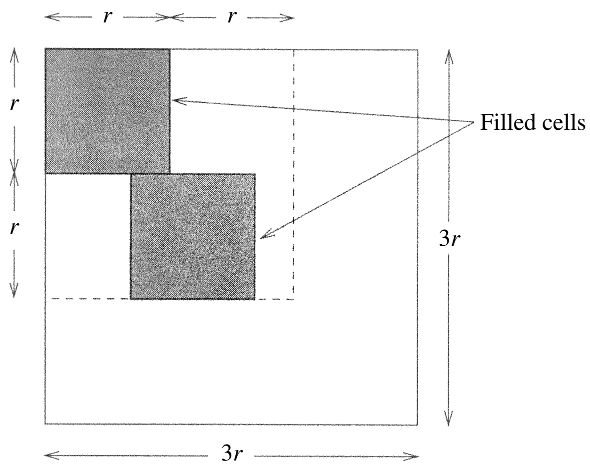

FIGURE 1. A configuration of filled squares in Theorem 2.

appear, in addition to all its previous positions, in precisely one of columns i and $i + r$. The final symbol s_n will appear one less time than before, but by definition it was *greedy* and so initially it appeared more than r times or in both columns i and $i + r$. After the redistribution s_n will appear either at least r times in $\mathcal{T}$ or in one of columns i and $i + r$. Finally, the only changes which were made during the redistribution took place amongst the columns i and $i + r$, and consisted of switching a symbol from column i to column $i + r$ or vice versa. Thus every other symbol still appears amongst these columns as often in $\mathcal{T}$ as it did before, and precisely as often overall in $\mathcal{T}$ as it did before.

As we repeat this process let us keep count of the total number of times a symbol which appears at most r times appears in neither column i nor column $i + r$, for $i = 1, \dots, r$. Observe that after each step of the process the number of such occurrences decreases by one, since one such occurrence for symbol s_0 is eliminated, no such occurrence for symbol s_n is created, and no others are created for the other vertices in the path. Thus by repeating this process eventually the number such occurrences is reduced to zero. Thus every symbol will either appear greater than r times, or at most r times, and at least once in each pair of columns i and $i + r$—thus precisely r times. Hence every symbol appears at least r times and the result follows. □

Using Theorem 1, our main theorem now follows easily, employing the recent result of Borodin, Kostochka and Woodall, which extends Galvins' proof of the Dinitz conjecture [4].

THEOREM B (BORODIN, KOSTOCHKA, WOODALL [2]). *Let G be a bipartite graph and $L_e(uv)$ be a set (list) of at least $\max\{d_G(u), d_G(v)\}$ positive integers for each edge $uv \in E(G)$. Then G has an L_e-list colouring (a proper edge colouring in which each edge f receives a colour from its list $L_e(f)$).*

THEOREM 2. *Let $\mathcal{L}$ be a $3r$ by $3r$ partial Latin square with filled cells which lie in two disjoint r by r squares. Then $\mathcal{L}$ can be completed.*

PROOF. If the two r by r squares lie in the same set of r rows, then the result follows immediately from Theorem A.

If otherwise, then we can assume that the r by r squares have no rows in common (possibly by transposing the roles of row and column). Observe then that the columns can be reordered so that in each row a cell is filled in precisely one of columns i and $i+r$, for each $i = 1, \ldots, r$, and so that all the empty columns are collected on the right-hand side. Thus, by Theorem 1, it is enough to simply fill the empty cells of $\mathcal{T}$.

Let G be the bipartite graph with bipartition (R, C), where R and C are the sets of rows and columns of $\mathcal{T}$, respectively. In G we join a row ρ to a column c by an edge $e_{\rho c}$ precisely when the cell (ρ, c) in $\mathcal{T}$ is empty. We also give each edge $e_{\rho c}$ a list $L_e(e_{\rho c})$ of the symbols which appear in neither row r nor column c. Consider two different types of cell. Firstly, there are cells which have a square below or above, and have a square to the left or right. Such a cell corresponds to an edge $e_{\rho c}$ in G, and by construction we see that $d_G(\rho) = d_G(c) = r$ and $|L_e(e_{\rho c})| \geq r$. Secondly, there are possibly cells which lie in an empty column. These have only a square to their left and so correspond to an edge $e'_{\rho c}$ where $L_e(e'_{\rho c}) = 2r$, $d_G(\rho) = r$ and $d_G(c) = 2r$.

Thus for each $e_{\rho c} \in E(G)$ we have $|L_e(e_{\rho c})| = \max\{d_G(\rho), d_G(c)\}$ and so Theorem B ensures that G has an L_e-list colouring, which exactly corresponds to a valid filling of the empty cells of $\mathcal{T}$. The result follows. □

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