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A q-analog of the Seidel generation of Genocchi numbers

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Abstract

A new q-analog of Genocchi numbers is introduced through a q-analog of Seidel's triangle associated with Genocchi numbers. It is then shown that these q-Genocchi numbers have interesting combinatorial interpretations in the classical models for Genocchi numbers such as alternating pistols, alternating permutations, non-intersecting lattice paths and skew Young tableaux.

1. Introduction

The *Genocchi numbers* G_{2n} can be defined through their relation with Bernoulli numbers $G_{2n} = 2(2^{2n} - 1)B_n$ or by their exponential generating function [16, p. 74–75]:

$$\frac{2t}{e^t + 1} = t - \frac{t^2}{2!} + \frac{t^4}{4!} - 3\frac{t^6}{6!} + \cdots + (-1)^n G_{2n} \frac{t^{2n}}{(2n)!} + \cdots.$$

However it is not straightforward to see from the above definition that G_{2n} should be *integers*. It was Seidel [14] who first gave a Pascal type triangle for Genocchi numbers in the nineteenth century. Recall that the *Seidel triangle* for Genocchi numbers [4,5,18] is an

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Table 1
q-analog of Seidel's triangle $(g_{i,j}(q))_{i,j\geq 1}$

						$1+2q+3q^2+4q^3+4q^4+2q^5+q^6$	4
				$1+q+q^2$	$q^2+q^3+q^4$	$1+2q+3q^2+4q^3+4q^4+2q^5+q^6$	3
		1	q	$1+q+q^2$	$q+2q^2+2q^3+q^4$	$1+2q+3q^2+4q^3+3q^4+q^5$	2
1	1	1	$1+q$	$1+q$	$1+2q+2q^2+2q^3+q^4$	$1+2q+2q^2+2q^3+q^4$	1
1	2	3	4	5	6	7	$i \setminus j$

array of integers $(g_{i,j})_{i,j\geq 1}$ such that $g_{1,1} = g_{2,1} = 1$ and

$$\begin{cases} g_{2i+1,j} = g_{2i+1,j-1} + g_{2i,j}, & \text{for } j = 1, 2, \ldots, i+1, \\ g_{2i,j} = g_{2i,j+1} + g_{2i-1,j}, & \text{for } j = i, i-1, \ldots, 1, \end{cases} \tag{1}$$

where $g_{i,j} = 0$ if $j < 0$ or $j > \lceil i/2 \rceil$ by convention. The first values of $g_{i,j}$ for $1 \leq i, j \leq 10$ can be displayed in *Seidel's triangle for Genocchi numbers* as follows:

								155	155	5
						17	17	155	310	4
				3	3	17	34	138	448	3
		1	1	3	6	14	48	104	552	2
1	1	1	2	2	8	8	56	56	608	1
1	2	3	4	5	6	7	8	9	10	$i \setminus j$

The Genocchi numbers G_{2n} and the so-called *median Genocchi numbers* H_{2n-1} are given by the following relations [4]:

$$G_{2n} = g_{2n-1,n}, \qquad H_{2n-1} = g_{2n-1,1}.$$

The purpose of this paper is to show that there is a q-analog of Seidel's algorithm and the resulting q-Genocchi numbers inherit most of the nice results proved by Dumont and Viennot, Gessel and Viennot, and Dumont and Zeng for ordinary Genocchi numbers [4,10,6].

A q-Seidel triangle is an array $(g_{i,j}(q))_{i,j\geq 1}$ of polynomials in q such that $g_{1,1}(q) = g_{2,1}(q) = 1$ and

$$\begin{cases} g_{2i+1,j}(q) = g_{2i+1,j-1}(q) + q^{j-1} g_{2i,j}(q), & \text{for } j = 1, 2, \ldots, i+1, \\ g_{2i,j}(q) = g_{2i,j+1}(q) + q^{j-1} g_{2i-1,j}(q), & \text{for } j = i, i-1, \ldots, 1, \end{cases} \tag{2}$$

where $g_{i,j}(q) = 0$ if $j < 0$ or $j > \lceil i/2 \rceil$ by convention. The first values of $g_{i,j}(q)$ are given in Table 1.

Define the q-Genocchi numbers $G_{2n}(q)$ and *q-median Genocchi numbers* $H_{2n-1}(q)$ by $G_2(q) = H_1(q) = 1$ and for all $n \geq 2$:

$$G_{2n}(q) = g_{2n-1,n}(q), \qquad H_{2n-1}(q) = q^{n-2} g_{2n-1,1}(q). \tag{3}$$

Thus, the sequences for $G_{2n}(q)$ and $H_{2n-1}(q)$ start with $1, 1, 1+q+q^2$ and $1, 1, q+q^2$, respectively.

						X		4
							X	3
		X						2
X	X		X	X	X			1
1	2	3	4	5	6	7	8	$i \setminus j$

Fig. 1. An alternating pistol $p = 11211143$.

Note that using the difference operator Gandhi [8] gave another algorithm for computing Genocchi numbers, which has inspired Dumont to give the first combinatorial interpretation of Genocchi numbers [2,3]. Some different q-analogs of Genocchi numbers have been investigated from both combinatorial and algebraic points of view [11,13] in the literature. In particular, Han and Zeng [11] have found an interesting q-analog of Gandhi's algorithm [8] by using the q-difference operator instead of the difference operator and proved that the ordinary generating function of these q-Genocchi numbers has a remarkable continued fraction expansion. Finally other refinements of the Genocchi numbers have been proposed by Sundaram [17] and Ehrenborg and Steingrímsson [7].

This paper is organized as follows. In Sections 2 and 3 we generalize the combinatorial results of Dumont and Viennot [4] by first interpreting $g_{i,j}(q)$ (and in particular the two kinds of q-Genocchi numbers) in the model of alternating pistols and then derive the interpretation $G_{2n}(q)$ as generating polynomials of *alternating permutations*. In Section 4 we give the q-version of the results of Gessel and Viennot [10] and Dumont and Zeng [5]. In Section 4, by extending the matrix of q-binomial coefficients to *negative indices* we obtain a q-analog of results of Dumont and Zeng [6]. Finally, in Section 6, we show that there is a remarkable triangle of q-integers containing the two kinds of q-Genocchi numbers and conjecture that the terms of this triangle refine the classical q-secant numbers, generalizing a result of Dumont and Zeng [5].

2. Alternating pistols

An *alternating pistol* (resp. *strict alternating pistol*) on $[m] = \{1, \dots, m\}$ is a mapping $p : [m] \to [m]$ such that for $i = 1, 2, \dots, \lceil m/2 \rceil$:

(1) $p(2i) \le i$ and $p(2i-1) \le i$,

(2) $p(2i-1) \ge p(2i)$ and $p(2i) \le p(2i+1)$ (resp. $p(2i) < p(2i+1)$).

We can illustrate an alternating pistol on $[m]$ by an array $(T_{i,j})_{1 \le i,j \le m}$ with a cross at (i, j) if $p(i) = j$. For example, the alternating pistol $p = p(1)p(2)\dots p(8) = 11211143$ can be illustrated as in Fig. 1.

For all $i \ge 1$ and $1 \le j \le \lceil i/2 \rceil$, let $\mathcal{AP}_{i,j}$ (resp. $\mathcal{SAP}_{i,j}$) be the set of alternating pistols p (resp. strict alternating pistols) on $[i]$ such that $p(i) = j$. Dumont and Viennot [4] proved that the entry $g_{i,j}$ of Seidel's triangle is the cardinality of $\mathcal{AP}_{i,j}$. Hence G_{2n} (resp. H_{2n+1}) is the number of alternating pistols (resp. strict alternating pistols) on $[2n]$.

To obtain a q-version of Dumont and Viennot's result, we define the *charge* of a pistol p by

$$\mathrm{ch}(p) = (p_1 - 1) + (p_2 - 1) + \cdots + (p_m - 1).$$

In other words the charge of a pistol p amounts to the number of cells below its crosses. For example, the charge of the pistol in Fig. 1 is $\mathrm{ch}(p) = 1 + 3 + 2 = 6$.

Proposition 1. *For $i \geq 1$ and $1 \leq j \leq \lceil i/2 \rceil$, $g_{i,j}(q)$ is the generating function of alternating pistols p on $[i]$ such that $p(i) = j$, with respect to the charge, i.e.,*

$$g_{i,j}(q) = \sum_{p \in \mathcal{AP}_{i,j}} q^{\mathrm{ch}(p)-j+1}.$$

Proof. We proceed by double inductions on i and j, where $1 \leq j \leq \lceil i/2 \rceil$:

- If $i = 1$, then $p(1) = 1$ and $\mathrm{ch}(p) = 0$, so $g_{1,1}(q) = 1$,
- Let $p \in \mathcal{AP}_{2k+1,j}$ and suppose the recurrence is true for all elements of $\mathcal{AP}_{2k'+1,j'}$ with $k' < k$, or $k' = k$ and $j' < j$.
  (1) If $j > p(2k)$, let $p' \in \mathcal{AP}_{2k+1,j-1}$ such that p and p' have the same restrictions to $[2k]$. Then $\mathrm{ch}(p) = \mathrm{ch}(p')$,
  (2) If $j = p(2k)$ then the charge of the restriction of p to $[2k]$ is $\mathrm{ch}(p) - j + 1$.
  Summing over all elements of $\mathcal{AP}_{2k+1,j}$, we obtain the first equation of (2).
- Let $p \in \mathcal{AP}_{2k,j}$ and suppose the recurrence true for all elements of $\mathcal{AP}_{2k',j'}$ with $k' < k$, or $k' = k$ and $j' > j$.
  (1) If $j < p(2k-1)$, let $p' \in \mathcal{AP}_{2k,j+1}$ such that p and p' have same restrictions to $[2k-1]$. Then $\mathrm{ch}(p) = \mathrm{ch}(p')$.
  (2) If $j = p(2k-1)$ then the charge of the restriction of p to $[2k-1]$ is $\mathrm{ch}(p) - j + 1$.

  Summing over all elements of $\mathcal{AP}_{2k,j}$, we obtain the second equation of (2). □

In order to interpret the q-median Genocchi numbers $H_{2n-1}(q)$, it is convenient to introduce another array $(h_{i,j}(q))_{i,j\geq 1}$ of polynomials in q such that $h_{1,1}(q) = h_{2,1}(q) = 1$, $h_{2i+1,1}(q) = 0$ and

$$\begin{cases} h_{2i+1,j}(q) = h_{2i+1,j-1}(q) + q^{j-2} h_{2i,j-1}(q), \\ h_{2i,j}(q) = h_{2i,j+1}(q) + q^{j-1} h_{2i-1,j}(q), \end{cases} \tag{4}$$

where by convention $h_{i,j}(q) = 0$ if $j < 0$ or $j > \lceil i/2 \rceil$. The first values of $h_{i,j}(q)$ are given in Table 2. Similarly we can prove the following:

Proposition 2. *For all $i \geq 1$ and $1 \leq j \leq \lceil i/2 \rceil$, we have*

$$h_{i,j}(q) = \sum_{\sigma \in \mathcal{SAP}_{i,j}} q^{\mathrm{ch}(\sigma)-j+1}.$$

Notice that

$$G_{2n+2}(q) = g_{2n+1,n+1}(q) = \sum_{1\leq k\leq n} q^{k-1} g_{2n,k}(q),$$

and since $h_{2n-1,n}(q) = q^{n-2} g_{2n-1,1}(q)$, we have also

$$H_{2n+1}(q) = h_{2n+1,n+1}(q) = \sum_{1\leq k\leq n} q^{k-1} h_{2n,k}(q).$$

Table 2
First values of $h_{i,j}(q)$

						$q^2+2q^3+2q^4+2q^5+q^6$	$q^5+2q^6+2q^7+2q^8+q^9$	4
				$q+q^2$	q^3+q^4	$q^2+2q^3+2q^4+q^5$	$q^4+3q^5+4q^6+3q^7+2q^8+q^9$	3
		1	q	q	$q^2+q^3+q^4$	$q^2+q^3+q^4$	$q^3+2q^4+4q^5+4q^6+3q^7+2q^8+q^9$	2
1	1	0	q	0	$q^2+q^3+q^4$	0	$q^3+2q^4+4q^5+4q^6+3q^7+2q^8+q^9$	1
1	2	3	4	5	6	7	8	$i \setminus j$

The above observations and propositions imply immediately the following result.

Proposition 3. *For all $n \geq 1$, the q-Genocchi number $G_{2n+2}(q)$ (resp. q-median Genocchi numbers $H_{2n+1}(q)$) is the generating function of alternating pistols (resp. are the strict alternating pistols) on $[2n]$ with respect to the statistics charge, i.e.,*

$$G_{2n+2}(q) = \sum_{p \in \mathcal{AP}_{2n}} q^{\mathrm{ch}\, p}, \qquad H_{2n+1}(q) = \sum_{p \in \mathcal{SAP}_{2n}} q^{\mathrm{ch}\, p}.$$

Dumont and Viennot [4, Section 3] also gave a combinatorial interpretation of Genocchi numbers with alternating permutations. In the next section we show that one can translate the statistics *charge* through all the bijections involved in their proof and interpret the q-Genocchi numbers as a q-counting of alternating permutations.

3. Alternating permutations

For any $\sigma \in S_n$ and $i \in [n]$, the *inversion table* of σ is a mapping $f_\sigma : [n] \to [0, n-1]$ defined by

$$\forall i \in [n], \quad f_\sigma(i) \text{ is the number of indices } j \text{ such that } j < i \text{ and } \sigma(j) < \sigma(i).$$

The mapping f_σ is an *subexceedant function* on $[n]$, that is a mapping $f_\sigma : [n] \to [0, n-1]$ such that $0 \leq f_\sigma(i) < i$ for every $i \in [n]$. It is well known [15, p. 21] that the correspondence $\ell : \sigma \mapsto I_\sigma$ is a bijection between the set of permutations of $[n]$ and the set of subexceedant functions on $[n]$. Note that in [15] the *inversion table* of σ is the mapping $I_\sigma : [n] \to [n-1]$ defined by $I_\sigma(i) = i - 1 - f_\sigma(i)$ for all $i \in [n]$ and the inversion number of a permutation of σ is defined as the following:

$$\mathrm{inv}\, \sigma = \sum_{i=1}^{n} (i - 1 - f_\sigma(i)) = \frac{n(n-1)}{2} - \sum_{i=1}^{n} f_\sigma(i). \tag{5}$$

For example, let $\sigma = 8\,3\,9\,4\,5\,1\,6\,2\,7 \in S_9$; then the inversion table is $f_\sigma = 002120416$ and the inversion number is $\mathrm{inv}\, \sigma = 20$.

A permutation σ of $[2n+1]$ is said to be *alternating* if

$$\forall i \in [n], \quad \sigma(2i-1) > \sigma(2i) \quad \text{and} \quad \sigma(2i) < \sigma(2i+1).$$

Let $\mathcal{F}_{2n+1}$ be the set of alternating permutations on $[2n+1]$ with even inversion table.

Proposition 4. *The q-Genocchi number $G_{2n+2}(q^2)$ is the generating function of $\mathcal{F}_{2n+1}$ with respect to* inv $- n$, *i.e.,*

$$G_{2n+2}(q) = \sum_{\sigma \in \mathcal{F}_{2n+1}} q^{\frac{1}{2}(\operatorname{inv}\sigma - n)}.$$

Proof. As in [4], we define the mapping $\alpha : p \mapsto p'$ from $\mathcal{AP}_{2n}$ to $\mathcal{AP}_{2n+1}$ by

$$p'(1) = 1, \quad p'(2i) = i + 1 - p(2i - 1), \quad p'(2i + 1) = i + 2 - p(2i), \ \forall i \in [n].$$

Note that $\operatorname{ch}(p') = n^2 - \operatorname{ch}(p)$. Then we can construct an even subexceedant function $\phi(p') = f$ on $[2n + 1]$ via the following:

$$f(i) = 2(p'(i) - 1), \qquad \forall i \in [2n + 1].$$

Let $\sigma = \ell^{-1}(f)$ be the permutation whose inversion table is f; it is easily verified (cf. [4]) that p is an alternating pistol on $[2n]$ if and only if σ is an alternating permutation $[2n+1]$. Finally, it follows from (5) that

$$\operatorname{ch}(p) = \frac{1}{2}(\operatorname{inv}\sigma - n).$$

For example, for the alternating pistol $p = 11211143 \in \mathcal{AP}_8$ in Fig. 1, we have $p' = 112133413 \in \mathcal{AP}_9$, $f = 002044604$ and $\sigma = 436287915 \in \mathcal{F}_9$. □

4. Disjoint lattice paths

The q-shifted factorials $(x; q)_n$ are defined by

$$(x; q)_n = (1 - x)(1 - xq) \cdots (1 - xq^{n-1}), \qquad \forall n \ge 0.$$

They can be used to define the q-binomial coefficients $\begin{bmatrix} m \\ n \end{bmatrix}_q$ as

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{(q^{m-n+1}; q)_n}{(q; q)_n} \qquad \forall m \in \mathbb{Z} \quad \text{and} \quad n \in \mathbb{N}.$$

Let $G_q^{-1} = ((-1)^{i-j} c_{i,j}(q))_{i,j\ge 1}$ be the inverse matrix of

$$G_q = \left(\begin{bmatrix} i \\ 2i - 2j \end{bmatrix}_q q^{(i-j-1)(i-j)} \right)_{i,j\ge 1}. \tag{6}$$

The first values of $c_{i,j}(q)$ are given in Table 3.

$c_{k,l}(q)$ is a polynomial in q with non-negative integer coefficients, using Gessel and Viennot's theory [9,10].

Let A and B be two points in the plan $\Pi = \mathbb{N} \times \mathbb{N}$ of coordinates (a, b) and (c, d), respectively. A *lattice path* from A to B is a sequence of points $((x_i, y_i))_{0\le i\le k}$ such that $(x_0, y_0) = (a, b)$, $(x_k, y_k) = (c, d)$ and each step is either *east* or *north*, i.e., $x_i - x_{i-1} = 1$ and $y_i - y_{i-1} = 0$ or $x_i - x_{i-1} = 0$ and $y_i - y_{i-1} = -1$ for $1 \le i \le k$. Clearly there is a path from A to B if and only if $a \le c$ and $b \ge d$.

Two lattice paths are said to be *disjoint* if they are vertex-disjoint. With each path w from A to B with l vertical steps of abscissa $x_1, x_2, \ldots, x_l$, arranged in decreasing order,

Table 3
First values of $c_{i,j}(q)$

$i \setminus j$	1	2	3	4
1	1	0	0	0
2	1	1	0	0
3	q^2+q+1	q^2+q+1	1	0
4	$q^6+2q^5+4q^4+4q^3+3q^2+2q+1$	$q^6+2q^5+4q^4+4q^3+3q^2+2q+1$	$\left(q^2+q+1\right)\left(q^2+1\right)$	1

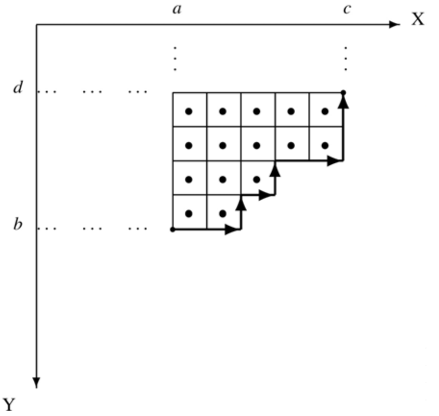

Fig. 2. A lattice path from (a, b) to (c, d) and its associated Ferrers diagram.

we can associate a partition of integers $\lambda_w = (x_1 - a, x_2 - a, \ldots, x_l - a)$. Actually the Ferrers graph of λ_w corresponds to the area of the region limited by the lines $x = a$, $y = d$ and the horizontal and vertical steps of w. The weight of the partition λ_w is defined by

$$|\lambda_w| = (x_1 - a) + (x_2 - a) + \cdots + (x_l - a).$$

For example, for the lattice path w in Fig. 2, we have $|\lambda_w| = 5 + 5 + 3 + 2 = 15$. Define the weight of a n-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of lattice paths by

$$\psi(\gamma) = q^{|\lambda_{\gamma_1}|+\cdots+|\lambda_{\gamma_n}|}.$$

We need the following result, which can be easily verified.

Lemma 1. *Let $(a_{ij})_{i,j=0,\ldots,m}$ be an invertible lower triangular matrix, and let $(b_{ij})_{i,j} = (a_{ij})^{-1}_{i,j}$. Then for $0 \le k \le n \le m$, we have*

$$b_{n,k} = \frac{(-1)^{n-k}}{a_{k,k}a_{k+1,k+1}\cdots a_{n,n}} \left|a_{k+i,k+j-1}\right|_{i,j=1,\ldots,n-k}.$$

Let $\Gamma_{k,l}$ be the set of n-tuples of non-intersecting lattice paths $\gamma = (\gamma_1, \ldots, \gamma_n)$ such that

- γ_i goes from $A_i(i-1, 2i-1)$ to $B_i(2i-1, 2i-1)$ for $1 \le i < l$ or $k < i \le n$ and from $A_{i+1}(i, 2i+1)$ to $B_i(2i-1, 2i-1)$ for $l \le i < k$.

Theorem 1. *For integers $k, l \geq 1$ the coefficient $c_{k,l}(q)$ is the generating function of $\Gamma_{k,l}$ with respect to the weight ψ, i.e.,*

$$c_{k,l}(q) = \sum_{\gamma \in \Gamma_{k,l}} q^{\psi(\gamma)}.$$

Proof. By Lemma 1, for $1 \leq l \leq k$ and $n \geq k$, we have

$$\begin{aligned} c_{k,l}(q) &= \left| \begin{bmatrix} l+i \\ 2i-2j+2 \end{bmatrix}_q q^{(i-j)(i-j+1)} \right|_{i,j=1}^{k-l} \\ &= \left| \begin{bmatrix} l+i+1 \\ 2i-2j+2 \end{bmatrix}_q q^{(i-j)(i-j+1)} \right|_{i,j=0}^{k-l-1} \\ &= \sum_{\sigma \in S_n} (-1)^{\mathrm{inv}(\sigma)} \prod_{i=1}^{n} \begin{bmatrix} l+i+1 \\ 2i-2\sigma(i)+2 \end{bmatrix}_q q^{(i-\sigma(i))(i-\sigma(i)+1)}. \end{aligned}$$

For any $\sigma \in S_n$ denote by $C(\sigma, k, l)$ the set of n-tuples of lattice paths $\gamma = (\gamma_1, \ldots, \gamma_n)$, where γ_i goes from A_i to $B_{\sigma(i)}$ for $1 \leq i < l$ or $k < i \leq n$, and from A_{i+1} to $B_{\sigma(i)}$ for $l \leq i < k$.

Let $f : S_n \to \mathbb{Z}$ be a mapping defined by

$$\forall \sigma \in S_n, \qquad f(\sigma) = \sum_{i=1}^{n} (i - \sigma(i))(i - \sigma(i) + 1).$$

Since the q-binomial coefficient has the following interpretation [1, p. 33]:

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \sum_{\gamma} q^{|\lambda_\gamma|},$$

where the sum is over all lattice paths γ from $(0, m)$ to $(n, 0)$, we derive immediately

$$c_{k,l}(q) = \sum_{\sigma \in S_n} \sum_{\gamma \in C(\sigma,k,l)} (-1)^{\mathrm{inv}(\sigma)} q^{\psi(\gamma)+f(\sigma)}. \tag{7}$$

For any n-tuple of lattice paths $(\gamma_1, \ldots, \gamma_n)$, if there is at least one intersecting point, we can define the *extreme intersecting point* $(i, j) \in \Pi$ to be the greatest intersecting point by the lexicographic order of their coordinates. It is easy to see that this point must be an intersecting point of two lattice paths w_i and w_{i+1} of consecutive indices. We apply the Gessel–Viennot method by "switching the tails", i.e., exchanging the parts of w_i and w_{i+1} starting from the extreme point. Let $\phi : \gamma \mapsto \gamma'$ be the corresponding transformation on the n-tuple of lattice paths with at least one intersecting point. This transformation does not keep the value ψ of intersecting paths as illustrated in Fig. 3. However, it is easy to see that f is the unique mapping on S_n satisfying $f(id) = 0$ and

$$f(\sigma) - f(\sigma \circ (i, i+1)) = 2(\sigma(i) - \sigma(i+1)), \qquad \text{for any } \sigma \in S_n.$$

Hence, for any $\sigma \in S_n$ and $\gamma \in C(\sigma, k, l)$, we have

$$q^{\psi(\gamma)+f(\sigma)}(-1)^{\mathrm{inv}(\sigma)} = -q^{\psi(\phi(\gamma))+f(\sigma\circ(i,i+1))}(-1)^{\mathrm{inv}(\sigma\circ(i,i+1))}.$$

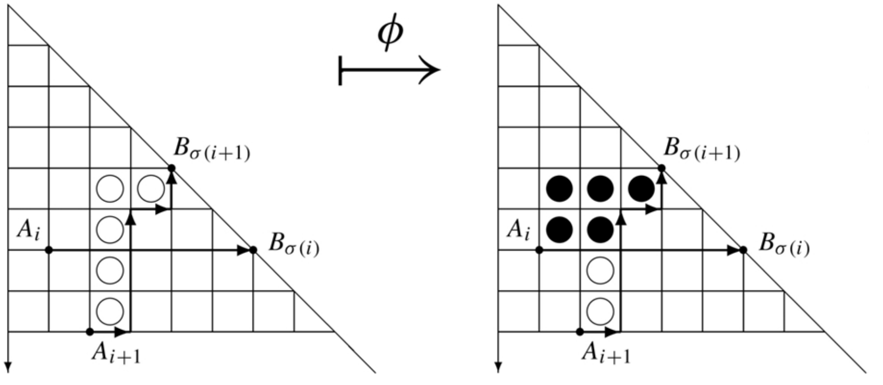

Fig. 3. Change of weight after switching tails.

This means that ϕ is a *weight-preserving–sign-reversing* involution on the set of n-tuples of intersecting lattice paths in $\cup_{\sigma\in S_n} C(\sigma,k,l)$. $\gamma \in C(\sigma,k,l)$ is non-intersecting only if σ is an identity permutation; that is $\gamma \in C(id,k,l)$. The result follows then from Eq. (7). □

Notice that for $1 \le i < l$ or $k < i \le n$, there is only one lattice path from A_i to B_i; the others have two vertical steps. With each vertical step of γ_i we can associate the number $v = x_0 - i + 1$ between 1 and i, where x_0 is the abscissa of the vertical step. We define the function $p : [2n-2] \longrightarrow [0, n-1]$ as follows:

$$p(i) = \begin{cases} 0 & \text{if there is no vertical steps between the lines } y = i, y = i+1; \\ v & \text{if } v \text{ is the number associated with the vertical step.} \end{cases}$$

For example, for the preceding configuration, we have

$$p(1) = \cdots = p(4) = 0,\ p(5) = 2,\ p(6) = 1,\ p(7) = p(8) = p(10) = 3,\ p(9) = 5.$$

By construction, $p(2i-1) \ge p(2i)$ for all $i \in [n-1]$. Now the condition of non-intersecting paths is equivalent to $p(2i) \le p(2i+1)$ for all $i \in [k-2] \setminus [l-1]$; and the value of w is $\psi(w) = -2(n-k) + \sum_i p(i)$.

Then we obtain a bijection between the configurations of Proposition 5 and those that we can call *truncated alternating pistols*. More precisely we have the following result:

Theorem 2. *For $0 \le l \le k$ and $n \ge k$, the coefficient $c_{k+1,l+1}(q)$ is the generating function of alternating pistols of $[2k]$, weighted by ch' and truncated at the index $2l$, i.e. the weight of mappings $p : [2k] \longrightarrow [0,k]$ satisfying the three conditions:*

(1) $p(2i-1) = p(2i) = 0$ *for* $1 \le i \le l$,
(2) $p(2i-1) \le i$ *and* $p(2i) \le i$ *for* $l < i \le k$,
(3) $p(2i-1) \ge p(2i) \le p(2i+1)$ *for* $1 \le i < k$.

For example, the array $(g'_{i,j})$ with $5 \le i \le 8$ and $1 \le j \le 4$, corresponding to the truncated alternating pistols using for counting the coefficient $c_{5,3}(q) = \sum_{k=1}^{4} q^{k-1} g'_{8,k}$, is given in Table 4.

Table 4
Computation of $c_{5,3}(q)$

		$1+q+2q^2+q^3+q^4$	$q^3+q^4+2q^5+q^6+q^7$	4
1	q^2	$1+q+2q^2+q^3+q^4$	$q^2+2q^3+3q^4+3q^5+2q^6+q^7$	3
1	$q+q^2$	$1+q+2q^2+q^3$	$q+2q^2+4q^3+4q^4+3q^5+2q^6+q^7$	2
1	$1+q+q^2$	$1+q+q^2$	$1+2q+3q^2+4q^3+4q^4+3q^5+2q^6+q^7$	1
5	6	7	8	$i \setminus j$

In particular we recover the alternating pistol in the case $l = 0$, and then we obtain the following result:

Corollary 1. *For $n \geq 1$, the coefficient $c_{n,1}(q)$ of the inverse matrix of G_q is the q-Genocchi number $G_{2n}(q)$.*

Now we give a last combinatorial interpretation of the q-Genocchi numbers. Some definitions concerning *integer partitions* are needed. A *paritition* $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a finite nonincreasing sequence of nonnegative integers, called the *parts* of λ. The *diagram* of λ is an arrangements of squares with λ_i squares, left justified, in the ith row. A partition $\mu = (\mu_1, \mu_2, \ldots)$ is said to *smaller* than another partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ if and only if all the parts of μ are smaller than those of λ. If $\mu \leq \lambda$ we define a skew hook of shape $\lambda \setminus \mu$ as the diagram obtained from that of λ by removing the diagram of μ. Finally, a row-strict plane partition T of $\lambda \setminus \mu$ is a skew hook of shape $\lambda \setminus \mu$ where we associate with the jth cell (from left to right) of the ith line (from top to bottom) a positive integer $p_{i,j}(T)$ such that, $\forall i \in [k], \forall j \in [\lambda_i - \mu_i]$,

$$p_{i,j}(T) > p_{i,j+1}(T) \qquad \text{and} \qquad p_{i,j}(T) \geq p_{i+1,j}(T). \tag{8}$$

A reverse plane partition is obtained by reversing all the inequalities of (8).

Now, let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be one of the configuration counted by $c_{k,l}(q)$, $n \geq k \geq l$. Then we can associate with this configuration two partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ defined by λ_i (resp. μ_i) equal to $n + i - 1$ for $i < l$ (resp. $i < k$) and $n + i + 1$ otherwise. By construction, λ is larger than μ and then we can construct a row-strict plane partition T where each case of $\lambda \setminus \mu$ is labelled in the following way:

If the vertical steps of ω_{l+i-1} $(1 \leq i \leq k - l)$ have $x_{i,1}$ and $x_{i,2}$ for the abscissa from left to right, so $x_{i,1} \leq x_{i,2}$, define

$$p_{i,j}(T) = 2l + 2i - j - x_{i,j} \qquad \text{for } j = 1, 2.$$

For example, the row-strict plane partition corresponding to the configuration of five paths in Fig. 5 is

		4	2
	3	2	
4	1		

Let $T_{k,l}$ be the set of row-strict plane partitions of form $(k-l+1, k-l, \ldots, 2) - (k-l-1, k-l-2, \ldots, 0)$ such that the largest entry in row i is at most $l+i$. For any $T \in T_{k,l}$

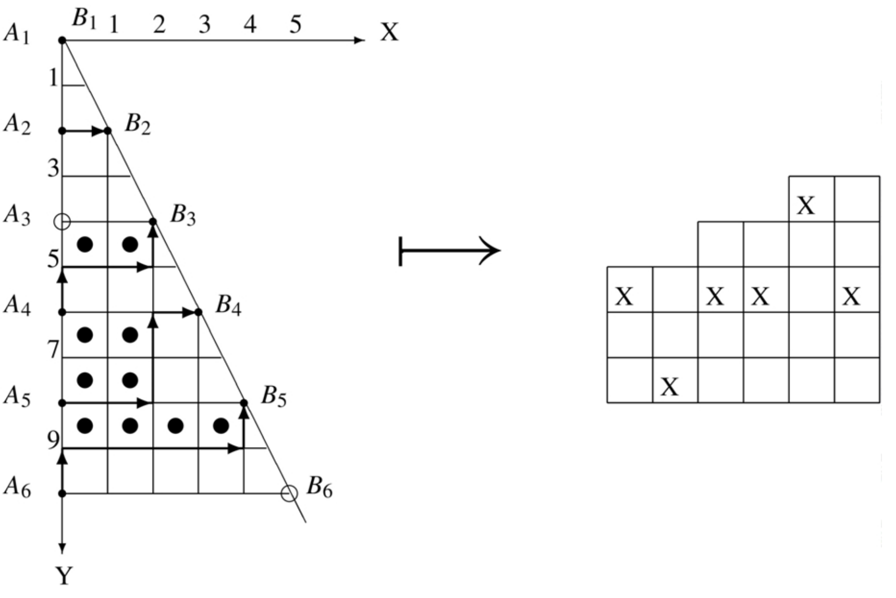

Fig. 4. One of the 493 configurations counted by $d_{6,3}(1)$ and its associated truncated pistol.

define the value of T by

$$|T| = \sum_{i=1}^{k-l} (p_{i,1}(T) + p_{i,2}(T));$$

then we have the following result, which is a q-analog of a result of Gessel and Viennot [10, Theorem 31].

Theorem 3. *For $k \geq l \geq 1$, the entry $c_{k,l}(q)$ is the following generating function of $T_{k,l}$:*

$$c_{k,l}(q) = \sum_{T \in T_{k,l}} q^{k^2 - l^2 - |T|}.$$

5. Extension to negative indices and median q-Genocchi numbers

As in [6], we can extend the matrix G_q to the negative indices as follows:

$$H_q = \left(\begin{bmatrix} -j \\ 2i - 2j \end{bmatrix}_q q^{(i-j)(2i-1)} \right)_{i,j \geq 1} = \left(\begin{bmatrix} 2i - j - 1 \\ j - 1 \end{bmatrix}_q \right)_{i,j \geq 1},$$

and its inverse:

$$H_q^{-1} = ((-1)^{i-j} d_{i,j}(q))_{i,j \geq 1}.$$

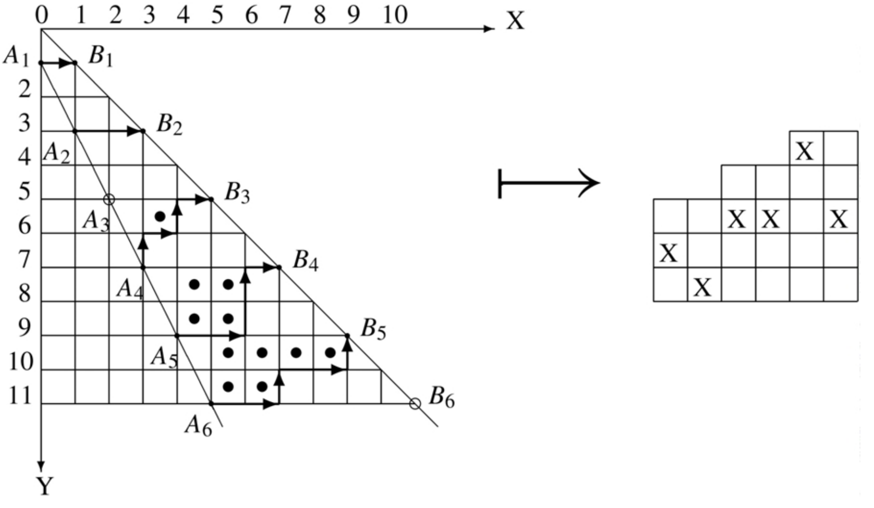

Fig. 5. One of the 736 configurations counted by $c_{6,3}(1)$ and its associated truncated pistol.

Table 5
First values of $d_{i,j}(q)$

$i \setminus j$	1	2	3	4
1	1	0	0	0
2	1	1	0	0
3	q^2+q	q^2+q+1	1	0
4	$q^6+2q^5+2q^4+2q^3+q^2$	$q^6+2q^5+3q^4+3q^3+3q^2+q$	$\left(q^2+q+1\right)\left(q^2+1\right)$	1

Using the result of Lemma 1, for $1 \le l \le k$ and $n \ge k$, the coefficient $d_{k,l}(q)$ is equal to

$$d_{k,l}(q) = \left| \begin{bmatrix} l+2i-j \\ 2i-2j+2 \end{bmatrix}_q \right|_{i,j=1}^{k-l}. \tag{9}$$

The first values of $d_{i,j}(q)$ are given in Table 5.

As in the previous section, we then derive from (9) the following result.

Theorem 4. *For integers $k, l \ge 1$ the coefficient $d_{k,l}(q)$ is the generating function of the configuration of lattice path $\Omega = (\omega_1, \ldots, \omega_n)$, weighted by ψ, satisfying the following two conditions:*

(1) *ω_i joins $A_i(0, 2i-2)$ to $B_i(i-1, 2i-2)$ for $1 \le i < l$ or $k < i \le n$ and ω_i joins $A_{i+1}(0, 2i)$ to $B_i(i-1, 2i-2)$ for $l \le i < k$;*
(2) *the paths $\omega_1, \ldots, \omega_n$ are disjoint.*

Similarly to in the preceding section, remark that for $1 \le i < l$ or $k < i \le n$, there is only a lattice path from A_i to B_i and the others have two vertical steps. With each of the

Table 6
Computation of $d_{5,3}(q)$

		$1+q+2q^2+q^3+q^4$	$q^3+q^4+2q^5+q^6+q^7$	4
1	q^2	$1+q+2q^2+q^3$	$q^2+2q^3+3q^4+3q^5+q^6+q^7$	3
1	$q+q^2$	$1+q+q^2$	$q+2q^2+3q^3+3q^4+3q^5+q^6+q^7$	2
1	$1+q+q^2$	0	$q+2q^2+3q^3+3q^4+3q^5+q^6+q^7$	1
5	6	7	8	$i \setminus j$

vertical steps of ω_i, we associate a number $v = x_0 + 1$ between 1 and i where x_0 is the abscissa of this vertical step. Then we can define a function $p : [2n-2] \longrightarrow [0, n-1]$ as follows:

$$p(i) = \begin{cases} 0 & \text{if there is no vertical steps between the lines } y = i-1, y = i, \\ v & \text{if } v \text{ is the number associated with the vertical step.} \end{cases}$$

For example, for the preceding configuration, we have $p(1) = p(2) = p(3) = p(4) = 0$, $p(5) = p(7) = p(8) = 3$, $p(6) = p(10) = 1$, $p(9) = 5$. By construction, $p(2i-1) \geq p(2i)$ for all $i \in [n-1]$ and the condition of non-intersecting paths is equivalent to $p(2i) < p(2i+1)$ for all $i \in [k-2] \setminus [l-1]$. The value of w is $\psi(w) = -2(n-k) + \sum_i p(i)$. Then we obtain a bijection between the configurations of Theorem 4 and those that we can call *truncated alternating pistols*. More precisely we state the following result:

Proposition 5. *For $0 \leq l \leq k$ and $n \geq k$, the coefficient $d_{k+1,l+1}(q)$ is the generating function of alternating pistols of $[2k]$, weighted by ch' and truncated at the index $2l$, i.e. the mappings $p : [2k] \longrightarrow [0, k]$ satisfying the three conditions:*

(1) $p(2i-1) = p(2i) = 0$ *for* $1 \leq i \leq l$,
(2) $p(2i-1) \leq i$ *and* $p(2i) \leq i$ *for* $l < i \leq k$,
(3) $p(2i-1) \geq p(2i) < p(2i+1)$ *for* $1 \leq i < k$.

The array for the computation of $d_{5,3}(q)$ is given in Table 6.

In particular we recover the alternating pistol when $l = 0$, and then we obtain the following result:

Corollary 2. *For $n \geq 1$, the coefficient $d_{n,1}(q)$ of the inverse matrix of H_q is the median q-Genocchi number $H_{2n+1}(q)$.*

Now, let $\Omega = (\omega_1, \ldots, \omega_n)$ be one of the configurations counted by $d_{k,l}(1)$, $n \geq k \geq l$. Then we can associate with this configuration two partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ defined by λ_i (resp. μ_i) equal to $n + i - 2$ for $i < l$ (resp. $i < k$) and $n+i$ otherwise. By construction, λ is bigger than μ and then we can construct an array T where each case of $\lambda \setminus \mu$ is labelled in the following way:

If the vertical steps of ω_{l+i-1} $(1 \leq i \leq k-l)$ have respectively $x_{i,1}$ and $x_{i,2}$ for the abscissa, $(x_{i,1} \leq x_{i,2})$, then $p_{i,j}(T) = x_{i,j} + 1$ for $j = 1, 2$.

For example, the row-strict plane partition corresponding to the configuration of five paths in Fig. 4 is

		1	3
	3	3	
1	5		

.

Similarly we have the following:

Theorem 5. *For* $k \geq l \geq 1$,

$$d_{k,l}(q) = \sum_{T \in \widetilde{T}_{k,l}} q^{-2(k-l)+|T|},$$

where $\widetilde{T}_{k,l}$ *is the set of column-strict reverse plane partitions of* $(k-l+1, k-l, \ldots, 2) - (k-l-1, k-l-2, \ldots, 0)$ *with positive integer entries in which the largest entry in row* i *is at most* $l+i-1$.

6. A remarkable triangle of q-numbers refining q-Euler numbers

Recall that the Euler numbers E_{2n} are the coefficients in the Taylor expansion of the function $\frac{1}{\cos x}$:

$$\frac{1}{\cos x} = \sum_{n \geq 0} E_{2n} \frac{x^{2n}}{(2n)!}.$$

Let $c_{i,j} = c_{i,j}(1)$. Then Dumont and Zeng [5] proved that there is a triangle of positive integers $k_{n,j}$ $(1 \leq j \leq n-1)$ featuring the two kinds of Genocchi numbers and refining Euler numbers as follows:

$$k_{n,1} + k_{n,2} + \cdots + k_{n,n-1} = E_{2n-2}, \qquad k_{n,1} = G_{2n} \quad \text{and} \quad k_{n,n-1} = H_{2n-1}.$$

Moreover,

$$\sum_{j \geq 0} c_{n+j,j+1} x^{j+1} = \frac{k_{n,1}x + k_{n,2}x^2 + \cdots + k_{n,n-1}x^{n-1}}{(1-x)^{2n-1}}.$$

The first values of $k_{n,j}$ $(1 \leq j \leq n-1)$ are tabulated as follows:

$n \setminus j$	1	2	3	4	5	$\sum_j k_{n,j} = E_{2n-2}$
1	1					1
2	1					1
3	3	2				5
4	17	36	8			61
5	155	678	496	56		1385
6	2073	15820	23576	8444	608	50521

We show now there is a q-analog of the above triangle. Following Jackson [12] the q-secant numbers $E_{2n}(q)$ are defined by

$$\sum_{n\geq 0} E_{2n}(q)\frac{u^{2n}}{(q;q)_{2n}} = \left(\sum_{n\geq 0}(-1)^n\frac{u^{2n}}{(q;q)_{2n}}\right)^{-1}.$$

Let $[x]=(q^x-1)/(q-1)$ and $[x]_n=[x][x-1]\cdots[x-n+1]$ for $n\geq 0$. Then $([x]_n)$ is a basis of $C[q^x]$. For any integer $n\geq 0$ we define a linear q-difference operator δ_q^n on $C[q^x]$ as follows: For $f(x)\in C[q^x]$,

$$\delta_q^0 f(x) = f(x), \qquad \delta_q^{n+1} f(x) = (E-q^n I)\,\delta_q^n f(x). \tag{10}$$

That is,

$$\delta_q^n f(x) = (E-q^{n-1}I)(E-q^{n-2}I)\cdots(E-I)\,f(x).$$

In view of the q-binomial formula [1, p. 36]:

$$(x;q)_n = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n\\ k\end{bmatrix}_q q^{\binom{k}{2}}x^k, \tag{11}$$

we have

$$\delta_q^n f(x) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n\\ k\end{bmatrix}_q q^{\binom{k}{2}} f(x+n-k).$$

Lemma 2. *For all non-negative integers n, m we have*

$$\delta_q^n[x]_m = \begin{cases}[m]_n[x]_{m-n}q^{n(x+n-m)} & \text{if } n\leq m\\ 0 & \text{if } n>m.\end{cases}$$

Hence $\delta_q^n f(x)=0$ if $f(x)$ is a polynomial in q^x of degree $<n$. It follows from the q-binomial identity (11) that

$$\begin{aligned}(x;q)_{2n-1}\sum_{j\geq 0}c_{n+j,j+1}(q)x^{j+1} &= \sum_{m\geq 0}x^{m+1}\sum_{k\geq 0}(-1)^k\begin{bmatrix} 2n-1\\ k\end{bmatrix}_q \\ &\quad\times q^{\binom{k}{2}}c_{n+m-k,m-k+1}(q),\\ &= \sum_{m\geq 0}x^{m+1}\delta_q^{2n-1}f(m),\end{aligned}$$

where $f(m)$ denotes the following determinant:

$$f(m) = \left|\begin{bmatrix} m-2(n-1)+i\\ 2i-2j+2\end{bmatrix}_q q^{(i-j)(i-j+1)}\right|_{i,j=1}^{n-1},$$

is a polynomial in q^m of degree $2(n-1)$ when $m\geq 2n-3$. Hence the preceding expression is a polynomial in x of degree $d\leq 2n-1$, i.e., we have

$$\sum_{j\geq 0}c_{n+j,j+1}(q)x^{j+1} = \frac{\alpha_0(q)+\cdots+\alpha_{d-1}(q)x^d}{(x;q)_{2n-1}}. \tag{12}$$

Applying a well-known result about rational functions [15, p. 202–210], we derive from (12) that

$$\sum_{j\geq 1} c_{n-j,-j+1}(q)x^j = -\frac{\alpha_0+\alpha_1 x^{-1}+\cdots+\alpha_{d-1}x^{-d}}{(1/x;q)_{2n-2}} = -\frac{\alpha_0 x^{2n-1}+\cdots+\alpha_{d-1}x^{2n-d}}{(x;q)_{2n-2}}.$$

But the coefficient $c_{n-j,-j+1}(q)$ is null for all $1 \leq j \leq n$ because the determinant formula of $c_{k,l}(q)$ contains a row with only zeros. So $d \leq n-1$.

Summarizing all the above we get the following theorem, which is a q-analog of a result of Dumont and Zeng [6, Proposition 7].

Theorem 6. *For $n \geq 2$, $\forall j \in [n-1]$, there are polynomials $k_{n,j}(q)$ in q such that*

$$\sum_{j\geq 0} c_{n+j,j+1}(q)x^{j+1} = \frac{\sum_{i=1}^{n-1} q^{(i-1)i}k_{n,i}(q)x^i}{(x;q)_{2n-1}}. \tag{13}$$

$$\sum_{j\geq 0} d_{n+j,j+1}(q)x^{j+1} = \frac{\sum_{i=1}^{n-1} q^{(i-1)i}k_{n,n-i}(q)x^i}{(x;q)_{2n-1}}. \tag{14}$$

Moreover, we have $k_{n,1}(q) = G_{2n}(q)$, $k_{n,n-1}(q) = H_{2n-1}(q)$ and

$$E_{2n-2}(q) = \sum_{i=1}^{n-1} q^{(i-1)i}k_{n,n-i}(q).$$

Proof. Eqs. (13) and (14) have been proved previously. In view of Corollaries 1 and 2 we derive from (13) and (14) that

$$k_{n,1}(q) = c_{n,1}(q) = G_{2n}(q),$$
$$k_{n,n-1}(q) = d_{n,1}(q) = H_{2n-1}(q).$$

Recall that for any sequence $(a_n)_n$ in $\mathbb{C}[[q]]$, we have $\lim_{q\to 1}(1-x)\sum_{n\geq 0} a_n q^n = \lim_{n\to\infty} a_n$, provided the latter limit exists. Hence we derive from (14) that

$$\sum_{i=1}^{n-1} q^{(i-1)i}k_{n,n-i}(q) = \lim_{x\to 1}(x;q)_{2n-1}\sum_{j\geq 0} d_{n+j,j+1}(q)x^{j+1} = (q;q)_{2n-2}\lim_{j\to\infty} d_{n+j,j+1}(q).$$

As $\lim_{n\to+\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{1}{(q;q)_k}$ it follows from (9) that

$$\sum_{i=1}^{n-1} q^{(i-1)i}k_{n,n-i}(q) = (q,q)_{2n-2}\left|\frac{1}{(q;q)_{2i-2j+2}}\right|_{i,j=1}^{n-1}. \tag{15}$$

Now, using the inclusion–exclusion principle we can show (see [15, p. 70]) that the right-hand side of (15) is the enumerating polynomial of up–down permutations on $[2n-2]$, i.e., whose descent set is $\{2, 4, \ldots, 2n-4\}$, with respect to inversion numbers, and it is also known (see [15, p. 148]) that this enumerating polynomial is equal to the q-Euler polynomial $E_{2n-2k}(q)$. □

It is not difficult to derive from Theorem 6 the following result.

Corollary 3. *For $n \geq 2$, for all $i \in [n-1]$, we have*

$$q^{(i-1)i} k_{n,i}(q) = \sum_{l=0}^{i-1} (-1)^l q^{\binom{k}{2}} \begin{bmatrix} 2n-1 \\ l \end{bmatrix}_q c_{n+i-l-1,i-l}(q),$$

and

$$q^{(i-1)i} k_{n,n-i}(q) = \sum_{l=0}^{i-1} (-1)^l q^{\binom{k}{2}} \begin{bmatrix} 2n-1 \\ l \end{bmatrix}_q d_{n+i-l-1,i-l}(q).$$

Finally, for $n = 2, 3$, Eq. (13) reads as follows:

$$\frac{x}{(x;q)_3} = x + (1+q+q^2)x^2 + (1+q+2q^2+q^3+q^4)x^3 + \cdots,$$

$$\frac{(1+q+q^2)x + q^2(q+q^2)x^2}{(x;q)_5} = (1+q+q^2)x + (1+2q+3q^2+4q^3+4q^4+2q^5+q^6)x^2 + \cdots.$$

So $k_{3,1}(q) = 1+q+q^2$ and $k_{3,2}(q) = q+q^2$, while the five up–down permutations on [4] are

1 3 2 4, 1 4 2 3, 2 3 1 4, 2 3 1 4, 3 4 1 2.

Therefore $E_4(q) = q+2q^2+q^3+q^4$ and we can check that $E_4(q) = k_{3,2}(q) + q^2 k_{3,1}(q)$.

For $n = 4$ the values of $k_{4,j}(q)$, $1 \leq j \leq 3$, are given by

$$k_{4,1}(q) = 1 + 2q + 3q^2 + 4q^3 + 4q^4 + 2q^5 + q^6,$$
$$k_{4,2}(q) = q(1+q)(1+q^2)(1+q+q^2)^2,$$
$$k_{4,3}(q) = q^2(q^2+1)(q+1)^2.$$

It seems that the coefficients of the polynomial $k_{n,i}(q)$ in q are *non-negative integers* and it would be interesting to find a combinatorial interpretation for $k_{n,i}(q)$ for the case where the above conjecture is true.

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