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A strong antidiamond principle compatible with CH

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ABSTRACT

A strong antidiamond principle ($\star_c$) is shown to be consistent with CH. This principle can be stated as a "P-ideal dichotomy": *every P-ideal on ω_1 (i.e. an ideal that is σ-directed under inclusion modulo finite) either has a closed unbounded subset of ω_1 locally inside of it, or else has a stationary subset of ω_1 orthogonal to it.* We rely on Shelah's theory of *parameterized properness* for NNR iterations, and make a contribution to the theory with a method of constructing the properness parameter simultaneously with the iteration. Our handling of the application of the NNR iteration theory involves definability of forcing notions in third order arithmetic, analogous to Souslin forcing in second order arithmetic.

1. Introduction

It is a remarkable fact (i.e. theorem of ZFC and the existence of some large cardinals) that if ϕ and ψ are two $\Pi_2(\mathrm{NS})$ sentences in the language of set theory, both of which can individually be forced to hold in the structure $(H_{\aleph_2}, \in, \mathrm{NS})$ (NS denotes the ideal on nonstationary subsets of ω_1), then their conjunction can also be forced to hold in this structure. Indeed Woodin has constructed a canonical model $\mathbb{P}_{\max}$ where the Π_2 theory over $(H_{\aleph_2}, \in, \mathrm{NS})$ is maximal (cf. [13]). In this model Cantor's Continuum Hypothesis (CH) is false. The question of whether the Π_2 theory can be maximized over structures $(H_{\aleph_2}, \in, \mathrm{NS})$ satisfying CH, is a major obstacle to further progress on the Continuum Hypothesis. This is closely related to the question of whether there are forcing axioms analogous to the Proper Forcing Axiom (PFA) or Martin's Maximum (MM) that are consistent with CH.

Specifically, there is the test question of Shelah and Woodin given below asking whether the above-mentioned remarkable fact still holds if we take the conjunctions of ϕ and ψ with CH: Let $\Pi_2(\mathrm{NS})$ denote the collection of all Π_2 sentences in the language of set theory (i.e. of the form $\forall x\, \exists y\ \ \varphi(x, y)$ where φ has no unbounded quantifiers) with the added unary predicate NS.

Question 1. *Are there $\Pi_2(\mathrm{NS})$ sentences ϕ and ψ such that both*

(1) $(H_{\aleph_2}, \in, \mathrm{NS}) \models \ulcorner \phi \wedge \mathrm{CH} \urcorner$ *and*
(2) $(H_{\aleph_2}, \in, \mathrm{NS}) \models \ulcorner \psi \wedge \mathrm{CH} \urcorner$

can be forced, yet provably $(H_{\aleph_2}, \in, \mathrm{NS}) \models \ulcorner \phi \wedge \psi \rightarrow \neg\mathrm{CH} \urcorner$?

Woodin has conjectured a positive answer, which would indicate that the $\Pi_2(\mathrm{NS})$ theory over $H_{\aleph_2}$ cannot be maximized for models of CH, and thus there are "disjoint" Π_2-rich models of CH.

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There has been much work done on maximizing the Π_2 theory in the presence of CH, where the idea to show that some 'strong' Π_2 statement is consistent with CH. A major breakthrough in this line of research was the Abraham–Todorčević P-ideal dichotomy $(*)$ that implies many well-known Π_2 consequences of PFA, and yet was shown to be consistent with CH ([2]). In the present paper, we push the boundary of maximizing the Π_2 theory over CH, by proving that the strengthening $(\star_c)_{\omega_1}$ (see below) of $(*)_{\omega_1}$ is consistent with CH. (Technically speaking, $(\star_c)$ is a variant of $(*)$, but once we can obtain a model of $(\star_c)$ with CH we can easily obtain $(*)$ simultaneously, whereas the converse is false.)

We believe there are good indications that $(\star_c)_{\omega_1}$ is strong enough to serve as one of the two $\Pi_2(\mathrm{NS})$ statements in giving a positive answer to Question 1. In particular, there is Example 3, which tells us that in an iterated forcing construction, given a P-ideal $\mathcal{I}$ where the second alternative of $(\star_c)$ fails, depending on the initial stages of the iteration, we may or may not be able to force the first alternative while at the same time making sure we do not add reals. And this suggests serious difficulties in obtaining a forcing axiom consistent with CH that would imply $(\star_c)$.

Consider now the following dichotomy of Eisworth, based on the P-ideal dichotomy $(*)$.

$(\star_c)$ For every ordinal θ of uncountable cofinality, every σ-directed downward closed (i.e. under subsets) subfamily $\mathcal{I}$ of $([\theta]^{\aleph_0}, \subseteq^*)$ has either
   (1) a closed uncountable subset of θ locally in $\mathcal{I}$,
   (2) a stationary subset of θ orthogonal to $\mathcal{I}$,

where $C \subseteq \theta$ is *locally in* $\mathcal{I}$ means $[C]^{\aleph_0} \subseteq \mathcal{I}$, and $S \subseteq \theta$ is *orthogonal to* $\mathcal{I}$ means $S \cap x$ is finite for all $x \in \mathcal{I}$. For some fixed ordinal θ, $(\star_c)_\theta$ denotes the restriction of $(\star_c)$ to θ. The original principle $(*)$ is also a dichotomy, where in the first alternative (1), "closed uncountable" is weakened to "uncountable"; and the second alternative (2) is strengthened to the existence of a countable decomposition of θ into pieces orthogonal to $\mathcal{I}$. Other similar variations are possible such as the principle $(\circledast_s)$ studied in [3] (actually this is a weakening of $(\star_c)$ optimal with respect to permitting the existence of a nonspecial Aronszajn tree).

The main result of this research is that $(\star_c)_{\omega_1}$ is consistent with CH.

Theorem 1. $(\star_c)_{\omega_1}$ *is consistent with* CH *relative to* ZFC.

This answers Shelah's question [10, Question 2.17]. The methods here can also be modified in the straightforward manner to obtain the consistency of the unrestricted principle $(\star_c)$ with CH relative to a supercompact cardinal.

It was already known that $(\star_c)$ is consistent with the failure of CH. The following theorem is due to Eisworth, at least in the case $\theta = \omega_1$, and is proved in [4].

Theorem 2. PFA *implies* $(\star_c)$.

The principle $(*)$ is already very powerful with applications to uncountable objects appearing in other areas of mathematics such as measure theory. The principle $(\star_c)$ moreover brings into play the most significant structural property of $H_{\aleph_2}$, as compared to $H_{\aleph_1}$; thus, unlike $(*)$, it is not a Π_2 statement (i.e. without the predicate NS). Let us briefly consider a couple of examples of how such principles are applied to combinatorial objects.

Example 1. As demonstrated in [2], to any tree $\mathcal{T} = (T, \leq_T)$ we can associate the ideal $\mathcal{T}^\perp$ of all countable subsets x of T perpendicular to the tree, i.e. every node has at most finitely many predecessors in x. Then, for example, if $\mathcal{T}$ has all levels countable then $\mathcal{T}^\perp$ is a P-ideal. And $\mathcal{T}^\perp$ has an uncountable set orthogonal to it iff $\mathcal{T}$ has an uncountable branch.

Example 2. If $\vec{x} = (x_\delta : \delta < \theta \text{ with } \mathrm{cof}(\delta) = \omega)$ is a sequence where each $x_\delta \subseteq \delta$ is a cofinal subset of order type ω, then we can associate an ideal $\vec{x}^\perp$ of all countable $y \subseteq \theta$ orthogonal to $\vec{x}$, i.e. $x_\delta \cap y$ is finite for all δ. Then $\vec{x}^\perp$ is a P-ideal, with no orthogonal subset of θ of order type ω^2. And $\vec{x}$ is a club-guessing sequence, in the weak sense, iff it has no closed unbounded subset of θ locally in $\vec{x}^\perp$. See e.g. [4], [8, Ch. XVIII, Problem 1.9].

Let us mention some of the challenges that need to be overcome to prove Theorem 1. First of all, it is known that $(\star_c)_{\omega_1}$ negates the relatively weak consequence of $\diamondsuit$ that there is a club-guessing sequence on ω_1 (see Example 2, [4]). Therefore, we cannot use a-proper forcing to obtain Theorem 1. Moreover, $(\star_c)_{\omega_1}$ implies that all Aronszajn trees are special [4], and thus there are significant difficulties in using Shelah's theory in [8, Ch. XVIII, Section 2] that he developed for negating club guessing with CH. We use his newest NNR (no new reals) iteration theory from [9], called parameterized properness, which was developed in order to obtain the negation of club-guessing sequences together with all Aronszajn trees being special simultaneously with CH. This involved devising new techniques for constructing the properness parameters. We also discuss the possibility of using the NNR iteration theory in [8, Ch. XVIII, Section 2] (cf. Section 4.4).

We say that two families $\mathcal{H}, \mathcal{I} \subseteq [\theta]^{\aleph_0}$ are *orthogonal*, written $\mathcal{H} \perp \mathcal{I}$ if $x \cap y$ is finite for all $x \in \mathcal{H}$ and $y \in \mathcal{I}$. The following example can be obtained by a straightforward construction of an (ω_1, ω_1) gap in $([\omega_1]^{\aleph_0}, \subseteq^*)$.

Example 3 ($\diamondsuit$). There exist two σ-directed subfamilies $\mathcal{H}$ and $\mathcal{I}$ of $([\omega_1]^{\aleph_0}, \subseteq^*)$ such that $\mathcal{H} \perp \mathcal{I}$ and neither has a stationary set orthogonal to it.

It follows from this (see [9]) that we cannot obtain a model of $(\star_c)_{\omega_1} + \mathrm{CH}$ by a straightforward iteration, where at each stage a club is forced locally inside a P-ideal with no stationary subset of ω_1 orthogonal to it. The above example shows that we will run into the so-called "disjoint clubs".

Many forcing notions, such as Cohen forcing and random forcing, can be represented as sets of reals that have simple definitions. This fact has been well used to obtain results in the descriptive set theory of the reals. For example, in [7, Section 5] the simplicity of the representations of random forcing and amoeba forcing as sets of reals, respectively, is used (rather spectacularly) to construct a nonmeasurable Σ^1_3-set of reals from some real computing ω_1 over L. The property of being simply definable as a set of reals is particularly useful in the iteration of such forcing notions. Judah–Shelah gave a systematic treatment of these forcing notions in [5], where they are named *Souslin forcing*, with the emphasis on the iteration of Souslin forcing notions.

In overcoming the "disjoint clubs" obstacle by constructing a properness parameter suitable for our iteration, we entered an analogous situation but in the realm of third order arithmetic, instead of the second order realm of set theory of the reals. We used the fact that our forcing notions can be represented as simply definable subsets of $\mathcal{P}(\omega_1)$ to establish nice properties of their iterations. For example, we were able to show that our forcing notions, which have cardinality $2^{\aleph_1}$, satisfy properties such as commutativity, of both their iterations and their generic objects (analogous to the fact that if r is a random real over V and s is a random real over $V[r]$ then r is a random real over $V[s]$).

1.1. Terminology

We use standard order theoretic notation and terminology. Thus for a family $\mathcal{F}$ of subsets of some fixed set S, we let $\downarrow\mathcal{F}$ denote the downwards closure in the inclusion order, i.e. $\downarrow\mathcal{F} = \bigcup_{x\in\mathcal{F}} \mathcal{P}(x)$. The definition of the upwards closure $\uparrow\mathcal{F}$ is symmetric. When want to take the downwards closure with respect to some other quasi-ordering $\lesssim$ of $\mathcal{P}(S)$, we write $\downarrow(\mathcal{F}, \lesssim)$. For example, we will consider the *almost inclusion* quasi-ordering $\subseteq^*$, where $x \subseteq^* y$ means that $x \setminus y$ is finite. A *P-ideal* is an ideal that is also σ-directed in the $\subseteq^*$-ordering; furthermore, a P-ideal on some specified set S is always assumed to contain every finite subset of S. A subset $A \subseteq Q$ of a quasi-order $(Q, \lesssim)$ is *cofinal* if every $q \in Q$ has an $a \in A$ with $q \leq a$. While a subset $A \subseteq P$ of a strict partial order $(P, <)$ is *cofinal* if every $p \in P$ has an $a \in A$ with $p < a$, e.g. we will consider cofinal subsets of some structure $(M, \in)$.

A subfamily of $\mathcal{H} \subseteq [S]^{\aleph_0}$ is called *cofinal* if it is cofinal in the inclusion ordering, i.e. for all $a \in [S]^{\aleph_0}$ there exists $b \in \mathcal{H}$ with $a \subseteq b$. It is *closed* if whenever $a_0 \subseteq a_1 \subseteq \cdots$ is a sequence of elements of $\mathcal{H}$ then so is $\bigcup_{n<\omega} a_n \in \mathcal{H}$, and *stationary* if it intersects every closed set.

We write $q \geq p$ for q extends p, i.e. carries more information than p. This is clearly more natural than, the perhaps more common, $q \leq p$, especially in the context of a-properness and more generally parameterized properness (cf. Definition 11). As usual, $\mathrm{Gen}(M, P)$ denotes the family of ideals $G \subseteq P$ that are generic over M, while $\mathrm{gen}(M, P)$ is the set of all (M, P)-generic elements of P. And $\mathrm{Gen}^+(M, P)$ is the subfamily of all $G \in \mathrm{Gen}(M, P)$ that have a common extension in P, and $\mathrm{gen}^+(M, P)$ is set of all *completely* (M, P)-*generic* elements q of P, meaning q extends some member of D for every dense $D \subseteq P$ in M. Every $q \in \mathrm{gen}^+(M, P)$ uniquely determines a member of $\mathrm{Gen}^+(M, P)$, namely $\{p \in P \cap M : p \leq q\}$, which we denote as $\dot{G}_P[M, q]$. *Complete properness* has the same formulation as properness using countable elementary submodels, but with (M, P)-generic replaced by completely (M, P)-generic. Thus a forcing notion is completely proper iff it is proper and adds no new reals. For more about proper forcing see e.g. [1].

Unless otherwise stated, for some function f and some subset $X \subseteq \mathrm{dom}(f)$, we write $f[X]$ for the image $\{f(x) : x \in X\}$ of X under f. Hopefully this will not cause confusion, because we also use square brackets for generic interpretations.

2. Parameters for properness

Although the following definition looks slightly different, it is almost the same as the definition of a "reasonable parameter" from [9, Section 1]. The main difference is that we only require cofinality in H_λ (cf. (ii)); indeed, it is noted in Remark 1.10(2) of that article that this is sufficient.

Definition 4. For a regular cardinal λ, a λ-*parameter for properness* is a pair $(\vec{\mathcal{A}}, \mathfrak{D})$ for which there exists a sequence of regular cardinals $\vec{\mu} = (\mu_\alpha : \alpha < \omega_1)$ with $\mu_0 \geq \lambda$ and $H_{\mu_\alpha} \in H_{\mu_\beta}$ for all $\alpha < \beta$, such that

(i) $\vec{\mathcal{A}}$ is an ω_1 sequence where $\mathcal{A}_\alpha \subseteq [H_{\mu_\alpha}]^{\aleph_0}$ is stationary for all $\alpha < \omega_1$, and for every $M \in \mathcal{A}_\alpha$,
  (a) $M \prec H_{\mu_\alpha}$,
  (b) $(\vec{\mu} \restriction \alpha, \vec{\mathcal{A}} \restriction \alpha) \in M$,

$\vec{\mathcal{A}}$ is called the *skeleton* of the parameter, and the rank function on $\varinjlim \mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$ is defined by

$$M \in \mathcal{A}_{\mathrm{rank}(M)}. \tag{1}$$

We use the notation $\mathcal{A}_{<\alpha} = \bigcup_{\xi<\alpha} \mathcal{A}_\xi$.

(ii) For all $M \in \mathcal{A}$, if $\mathrm{rank}(M) = 0$ then $\mathfrak{D}(M) = \{0\}$; and if $\mathrm{rank}(M) > 0$ then $\mathfrak{D}(M)$ is a *nonempty* collection of subsets of $\mathcal{A}_{<\mathrm{rank}(M)} \cap M$ so that for each element $X \in \mathfrak{D}(M)$, every $\xi < \mathrm{rank}(M)$ and every $a \in M \cap H_\lambda$ has a $K \in X$ such that
  (a) $\mathrm{rank}(K) \geq \xi$,
  (b) $X \cap K \in \mathfrak{D}(K)$,
  (c) $a \in K$.

Proposition 5. $\mathrm{rank}(M) < \omega_1 \cap M$.

Proof. By (i(b)). □

Proposition 6. $\operatorname{rank}(K) \le \operatorname{rank}(M)$ *for all* $K \in \varinjlim \mathcal{A} \cap M$.

Proof. By (i). □

Proposition 7. *For all* $M \in \varinjlim \mathcal{A}$ *with* $\operatorname{rank}(M) > 0$, $\mathfrak{D}(M)$ *is closed under supersets in* $\mathcal{P}(\mathcal{A}_{<\operatorname{rank}(M)} \cap M)$.

Proof. A simple induction on $\operatorname{rank}(M)$. □

The properness parameter is often utilized through the following game. It is a simplification of the game in [9, Definition 1.5], that appears to serve the same purpose. In any case, the two games are equivalent for the properness parameters we will be using (cf. Definition 16).

Definition 8. Let $(\vec{\mathcal{A}}, \mathfrak{D})$ be a properness parameter. For each $M \in \varinjlim \mathcal{A}$ of positive rank, the *Chooser Game* $\Game(M) = \Game(\vec{\mathcal{A}}, \mathfrak{D})(M)$ is defined as follows. It is a two player game of length ω, where the *challenger* moves first against the *chooser*. On the kth move (setting $X_{-1} = \mathcal{A}_{<\operatorname{rank}(M)} \cap M$):

- The challenger plays $X_k \subseteq X_{k-1}$ in $\mathfrak{D}(M)$.
- The chooser plays $K_k \in X_k$, and $Y_k \subseteq X_k$ in $\mathfrak{D}(K_k)$.

The chooser wins the game if $\bigcup_{k<\omega} Y_k \cup \{K_k\} \in \mathfrak{D}(M)$. Otherwise the challenger wins.

We say that the chooser has a *global winning* (*nonlosing*) *strategy in the game* $\Game(\vec{\mathcal{A}}, \mathfrak{D})$ if the chooser has a winning (nonlosing) strategy in the game $\Game(\vec{\mathcal{A}}, \mathfrak{D})(M)$ for all $M \in \varinjlim \mathcal{A}$ with $\operatorname{rank}(M) > 0$.

Note that the chooser always has a valid move:

Lemma 9. *Every* $X \in \mathfrak{D}(M)$ *has a* $Y \subseteq X$ *in* $\mathfrak{D}(M)$ *such that* $Y \cap K \in \mathfrak{D}(K)$ *for all* $K \in Y$.

Proof. The proof is by induction on $\operatorname{rank}(M)$. For each $a \in M \cap H_\lambda$ and each $\xi < \operatorname{rank}(M)$, there exists $K_{a\xi} \in X$ with $a \in K_{a\xi}$, $X \cap K_{a\xi} \in \mathfrak{D}(K_{a\xi})$ and $\operatorname{rank}(K_{a\xi}) \ge \xi$. Then by the induction hypothesis, there exist $Y_{a\xi} \subseteq X \cap K_{a\xi}$ in $\mathfrak{D}(K_{a\xi})$ such that $Y_{a\xi} \cap J \in \mathfrak{D}(J)$ for all $J \in Y_{a\xi}$. Now $\bigcup_{a\in M\cap H_\lambda} \bigcup_{\xi<\operatorname{rank}(M)} Y_{a\xi} \cup \{K_{a\xi}\}$ is in $\mathfrak{D}(M)$ and satisfies the desired conclusion, by Proposition 7. □

Example 10. The present example corresponds to a *standard parameter* in [9]. Given a skeleton $\vec{\mathcal{A}}$ for a properness parameter, define $\mathfrak{D}(M)$ by recursion on $\operatorname{rank}(M)$ as follows. Let $\mathfrak{D}(M) = \{0\}$ if $\operatorname{rank}(M) = 0$, and $\mathfrak{D}(M)$ consist of all subsets of $\mathcal{A}_{<\operatorname{rank}(M)} \cap M$ satisfying (ii) otherwise. Condition (i) on the skeleton ensures that $\mathfrak{D}(M) \ne \emptyset$ for all $M \in \varinjlim \mathcal{A}$, and thus this does indeed define a parameter for properness. The chooser has a global winning strategy in the corresponding Chooser Game, because given $M \in \varinjlim \mathcal{A}$ of positive rank, if $(a_n : n < \omega)$ enumerates $M \cap H_\lambda$ and $\lim_{n\to\omega} \xi_n + 1 = \operatorname{rank}(M)$ (cf. Remark 17), then playing K_n with $a_0, \dots, a_n \in K_n$ and $\operatorname{rank}(K_n) \ge \xi_n$ (and Y_k arbitrary) defines a winning strategy for the chooser in the game $\Game(\vec{\mathcal{A}}, \mathfrak{D})(M)$.

The following definition is the case $\alpha = \beta$ of Shelah [9, Definition 2.8].

Definition 11. Let $(\vec{\mathcal{A}}, \mathfrak{D})$ be a λ-parameter for properness. A poset P is *proper for the parameter* $(\vec{\mathcal{A}}, \mathfrak{D})$ or $(\vec{\mathcal{A}}, \mathfrak{D})$*-proper* if $P \in H_\lambda$ [1] and there exists $a \in H_\lambda$ such that for all $M \in \varinjlim \mathcal{A}$ with $a, P \in M$:

($\mathfrak{p}$) for all $p \in P \cap M$ and all $X \in \mathfrak{D}(M)$, there is an (M, P)-generic extension q of p such that

$$\mathcal{M}(q) \cap X \in \mathfrak{D}(M), \tag{2}$$

where $\mathcal{M}(q) = \{M \in [H_\lambda]^{\aleph_0} : q \in \operatorname{gen}(M, P)\}$.

Note that Eq. (2) is trivial when $\operatorname{rank}(M) = 0$.

Example 12. Suppose P is an a-proper forcing notion with $\mathcal{P}(P) \in H_\lambda$. Then P is $(\vec{\mathcal{A}}, \mathfrak{D})$-proper for every λ-parameter for properness.

2.0.1. Tails

Definition 13. For any $X \in \mathfrak{D}(M)$, a *tail* of X is a subset of the form $\{K \in X : a \in K\}$ where $a \in M \cap H_\lambda$.

Proposition 14. *The intersection of two tails of* X *is itself a tail of* X.

Proposition 15. *For all* $X \in \mathfrak{D}(M)$ *and all* $J \in \varinjlim \mathcal{A} \cap M$, *there exists a tail* Y *of* X *such that* $K \notin J$ *for all* $K \in Y$.

Any 'reasonable' parameter has each $\mathfrak{D}(M)$ closed under taking tails, but this is not a requirement.

[1] In [9] it is required that in fact $\mathcal{P}(P) \in H_\lambda$. Although it is harmless to ask for this, we have left it out.

2.1. Properness parameters for shooting clubs

When forcing a club subset of θ, if p is generic over M then p forces that $\sup(\theta \cap M)$ is in the club. This explains why a-properness is unsuitable for purposes such as destroying a club-guessing sequence, because it can be used to guess the generic club in the ground model. The following class of properness parameters is designed to handle this difficulty by putting restrictions on these suprema. For any family $\mathcal{M}$, the *trace of the suprema of* $\mathcal{M}$ *on* θ is

$$\operatorname{tr}\sup_\theta(\mathcal{M}) = \{\sup(\theta \cap M) : M \in \mathcal{M}\}. \tag{3}$$

Definition 16. Suppose θ is an ordinal of uncountable cofinality and $\vec{\mathcal{A}}$ is a skeleton of a λ-parameter for properness for some λ. For each $M \in \varinjlim \mathcal{A}$, let there a countable family $\Omega(M) \subseteq \mathcal{P}(\theta)$ (normally, $\Omega(M) \subseteq \mathcal{P}(\theta \cap M)$). For each $M \in \varinjlim \mathcal{A}$, $\mathfrak{D}_\Omega(\vec{\mathcal{A}})(M) = \mathfrak{D}_\Omega(M)$ is defined by recursion on $\operatorname{rank}(M)$. If $\operatorname{rank}(M) = 0$ then define $\mathfrak{D}_\Omega(M) = \{\emptyset\}$; otherwise, it is defined as the family of subsets of $\mathcal{A}_{<\operatorname{rank}(M)} \cap M$ containing a subset of the form

$$X = \bigcup_{n<\omega} X_n \cup \{K_n\} \tag{4}$$

where

(i) $K_0 \in K_1 \in \cdots$ is cofinal in $(M \cap H_\lambda, \in)$ with $\lim_{n\to\omega} \operatorname{rank}(K_n) + 1 = \operatorname{rank}(M)$,
(ii) $X_n \in \mathfrak{D}_\Omega(\vec{\mathcal{A}})(K_n)$ for all n,
(iii) every $x \in \Omega(M)$ has a tail Y of X with $\operatorname{tr}\sup_\theta(Y) \subseteq x$.

Condition (iii) is a geometrical restriction on the trace of the suprema.

Remark. The limit in condition (i) has its usual topological meaning. Thus for any $f : \omega \to \mathrm{On}$, $\lim_{n\to\omega} f(n) = \alpha$ iff every $\xi < \alpha$ has a $k < \omega$ such that $f(n) \in (\xi, \alpha]$ for all $n \geq k$. Also, $\limsup_{n\to\omega} f(n)$ should be interpreted as $\lim_{n\to\omega} \sup_{i\geq n} f(i)$.

Lemma 18. *When* $\operatorname{rank}(M) > 0$, $\mathfrak{D}_\Omega(M)$ *is closed under taking tails.*

Proof. The proof is by induction on $\operatorname{rank}(M)$. Suppose $X \in \mathfrak{D}_\Omega(M)$ satisfies (i)–(iii), and let $Y = \{K \in X : a \in K\}$ be a tail of X for some $a \in M \cap H_\lambda$. Condition (i) holds because $K_n \in Y$ for all but finitely many n; condition (ii) holds for Y by the induction hypothesis; and condition (iii) is by Proposition 14. □

Proposition 19. *If* $X \in \mathfrak{D}_\Omega(M)$, *and* $Y \subseteq X$ *and* $Y^K \subseteq X$ $(K \in Y)$ *satisfy*

(a) $Y^K \in \mathfrak{D}_\Omega(K)$,
(b) *for all* $a \in M \cap H_\lambda$ *and all* $\xi < \operatorname{rank}(M)$, *there exists* $K \in Y$ *with* $a \in K$ *and* $\operatorname{rank}(K) \geq \xi$,

then $Y \cup \bigcup_{K\in Y} Y^K \in \mathfrak{D}_\Omega(M)$.

Proposition 20. $(\vec{\mathcal{A}}, \mathfrak{D}_\Omega)$ *is a properness parameter whenever* $\mathfrak{D}_\Omega(M) \neq \emptyset$ *for all* $M \in \varinjlim \mathcal{A}$.

An arbitrary mapping Ω may fail to define a properness parameter, i.e. the $\mathfrak{D}_\Omega(M)$ may be empty for some M. We provide a general construction to avoid this.

Definition 21. A map $\Lambda : \varinjlim \mathcal{A} \to \mathcal{P}([\theta]^{\aleph_0})$ is said to *instantiate* Ω if every $M \in \varinjlim \mathcal{A}$ with $\operatorname{rank}(M) > 0$, every finite $A \subseteq \Omega(M)$, every $y \in \Lambda(M)$, every $a \in M \cap H_\lambda$ and every $\xi < \operatorname{rank}(M)$ has a $K \in \mathcal{A}_{<\operatorname{rank}(M)} \cap M$ such that

(i) $a \in K$,
(ii) $\operatorname{rank}(K) \geq \xi$,
(iii) $\sup(\theta \cap K) \in y \cap \bigcap A$,
(iv) $y \cap \bigcap A \in \Lambda(K)$.

Remark. In all of our applications of instantiations, we will have $\Lambda(M) = \uparrow\Omega(M)$ for all M, and thus we can omit the 'y' in (iii) and (iv) in the verification. See e.g. Example 24.

Lemma 23. *If there exists a map instantiating* Ω, *then* $(\vec{\mathcal{A}}, \mathfrak{D}_\Omega)$ *is a* λ*-properness parameter, i.e.* $\mathfrak{D}_\Omega(M) \neq \emptyset$ *for all* $M \in \varinjlim \mathcal{A}$.

Proof. Assume that Λ instantiates Ω. We proceed by induction on $\operatorname{rank}(M)$, with the induction hypothesis that for all $y \in \Lambda(M)$, there exists $X \in \mathfrak{D}_\Omega(M)$ with $\operatorname{tr}\sup(X) \subseteq y$. Suppose $M \in \mathcal{A}_\alpha$ where $\alpha > 0$, and $y \in \Lambda(M)$. Let $(x_n : n < \omega)$ enumerate $\Omega(M)$, letting $x_n = \theta \cap M$ in case $\Omega(M) = \emptyset$. Let $(a_n : n < \omega)$ enumerate $M \cap H_\lambda$, and fix a sequence $\xi_n < \alpha$ $(n < \omega)$ such that $\limsup_{n\to\omega} \xi_n + 1 = \alpha$. We recursively choose $K_{n+1} \ni K_n$ in $\mathcal{A}_{<\alpha} \cap M$ with $a_n \in K_n$, $\operatorname{rank}(K_n) \geq \xi_n$, $\sup(\theta \cap K_n) \in y \cap \bigcap_{i=0}^n x_i$ and $y \cap \bigcap_{i=0}^n x_i \in \Lambda(K_n)$. This is possible by (i)–(iv). And for each n, there exists $X_n \in \mathfrak{D}_\Omega(K_n)$ with $\operatorname{tr}\sup(X_n) \subseteq y \cap \bigcap_{i=0}^n x_i$ by the induction hypothesis. Then putting $X = \bigcup_{n<\omega} X_n \cup \{K_n\}$, conditions (i)–(iii) of Definition 16 are clearly satisfied, i.e. $X \in \mathfrak{D}_\Omega(M)$, and also $\operatorname{tr}\sup(X) \subseteq y$, completing the induction. □

Example 24. Suppose $\mathcal{E} \subseteq \{M \in [H_\lambda]^{\aleph_0} : M \prec H_\lambda\}$ is stationary, and $\vec{\mathcal{A}}$ is a skeleton of a λ-properness parameter with $M \cap H_\lambda \in \mathcal{E}$ for all $M \in \varinjlim \mathcal{A}$. Suppose $\mathcal{H} \subseteq [\theta]^{\aleph_0}$ has no stationary orthogonal set. If $y_M \in \downarrow\mathcal{H}$ $(M \in \mathcal{E})$ satisfies $x \subseteq^* y_M$ for all $x \in \mathcal{H} \cap M$, then defining $\Omega : \varinjlim \mathcal{A} \to [[\theta]^{\aleph_0}]^{\le\aleph_0}$ by

$$\Omega(M) = \{y_M \setminus s : s \subseteq y_M \text{ is finite}\}, \tag{5}$$

we have that $(\vec{\mathcal{A}}, \mathfrak{D}_\Omega)$ is a λ-properness parameter. This is instantiated by Λ, where $\Lambda(M) = \uparrow\Omega(M)$ for all M.

Proof. We apply Lemma 23. Thus given $M \in \mathcal{A}_\alpha$ with $\alpha > 0$, a nonempty finite $A \subseteq \Omega(M)$, $a \in M \cap H_\lambda$ and $\xi < \alpha$, we need to show that there exists $K \in \mathcal{A}_{<\alpha} \cap M$ satisfying (i)–(iv).

Put $\mathcal{B} = \{K \in \mathcal{A}_\xi : a \in K\}$. Since $\mathcal{B} \in M$ and $\mathcal{B}$ stationary, $\operatorname{tr}\sup(\mathcal{B}) \in M$ is a stationary subset of θ. Thus there exists $\delta \in \operatorname{tr}\sup(\mathcal{B}) \cap \bigcap A \cap M$ by elementarity, since $\mathcal{H}$ has no stationary orthogonal set. And by elementarity, we can find $K \in \mathcal{B} \cap M$ with $\sup(\theta \cap K) = \delta$, and hence K satisfies (i)–(iii). Condition (iv) is satisfied because $y_K \subseteq^* y_M$. □

Proposition 25. *Assume that Ω does define a properness parameter in Definition 16. Then the chooser has a global winning strategy in the game $\eth(\vec{\mathcal{A}}, \mathfrak{D}_\Omega)$.*

Proof. It is immediate from the definition of $\mathfrak{D}_\Omega(M)$ and the payout of the game, that the chooser wins so long as $\lim_{n\to\omega} \operatorname{rank}(K_n) + 1 = \operatorname{rank}(M)$ and $(K_n : n < \omega)$ is cofinal in $M \cap H_\lambda$, where (K_n, Y_n) denotes the chooser's nth move. The chooser can guarantee this in the obvious manner. □

2.1.1. Direction constraints

In addition to the limitations imposed on the trace of the suprema, we shall want additional control over the 'direction' in which the members of each $\mathfrak{D}(M)$ can grow. This simply means that we want the set in Eq. (4) to be contained in some subfamily of $\varinjlim \mathcal{A}$, but what is more, this family lives in some forcing extension (and is thus 'imaginary').

Definition 26. Let $\mathcal{Z}$ be a collection of pairs of the form $(P, \dot{\mathcal{B}})$, where P is a forcing notion and $\dot{\mathcal{B}}$ is a P-name for a subset of $\varinjlim \mathcal{A}$. Then we define subfamilies $\mathfrak{D}_\Omega(\mathcal{Z})(M) = \mathfrak{D}_\Omega(\vec{\mathcal{A}}; \mathcal{Z})(M) \subseteq \mathfrak{D}_\Omega(\vec{\mathcal{A}})(M)$ by recursion on $\operatorname{rank}(M)$ as follows. If $\operatorname{rank}(M) = 0$ then $\mathfrak{D}_\Omega(\vec{\mathcal{A}}; \mathcal{Z})(M) = \{\emptyset\}$; otherwise, it is the family of all members of $\mathfrak{D}_\Omega(\vec{A})(M)$ such that the set X in Eq. (4) additionally satisfies

(iv) every $p \in P \cap M$ has a $q \in \operatorname{gen}(M, P, p)$ and a tail $Y \subseteq X$ such that $q \Vdash Y \subseteq \dot{\mathcal{B}}$,

for all $(P, \dot{\mathcal{B}}) \in \mathcal{Z} \cap M$.

Lemma 27. *When $\operatorname{rank}(M) > 0$, $\mathfrak{D}_\Omega(\vec{\mathcal{A}}; \mathcal{Z})(M)$ is closed under taking tails.*

Proof. By Lemma 18. □

Typically, we have $\dot{\mathcal{B}}$ a subset of the models over which $\dot{G}_P$ is generic, in which case we automatically have that P is $\Omega(\vec{\mathcal{A}}; \mathcal{Z})$-proper.

Corollary 28. *Suppose that $\mathfrak{D}_\Omega(\vec{\mathcal{A}}; \mathcal{Z})$ is a properness parameter, i.e. $\mathfrak{D}_\Omega(\vec{\mathcal{A}}; \mathcal{Z})(M) \neq \emptyset$ for all $M \in \varinjlim \mathcal{A}$. If $(P, \dot{\mathcal{B}}) \in \mathcal{Z}$ and $P \Vdash \dot{\mathcal{B}} \subseteq \{M \in \varinjlim \mathcal{A} : \dot{G}_P \in \operatorname{Gen}(M, P)\}$, then P is $\mathfrak{D}_\Omega(\vec{\mathcal{A}}; \mathcal{Z})$-proper.*

We generalize Lemma 23 to the present setting.

Lemma 29. *Assume that for every $M \in \varinjlim \mathcal{A}$ with $\operatorname{rank}(M) > 0$, every $(P, \dot{\mathcal{B}}) \in \mathcal{Z} \cap M$ has a $q^M_{P,\dot{\mathcal{B}}}(p) \in \operatorname{gen}(M, P, p)$ for each $p \in P \cap M$, such that every finite $A \subseteq \Omega(M)$, every finite sequence $(P_0, \dot{\mathcal{B}}_0), \ldots, (P_{m-1}, \dot{\mathcal{B}}_{m-1})$ in $\mathcal{Z} \cap M$, all finite $O_i \subseteq P_i \cap M$ $(i = 0, \ldots, m-1)$, every $a \in M \cap H_\lambda$ and every $\xi < \operatorname{rank}(M)$ has a $K \in \mathcal{A}_{<\operatorname{rank}(M)} \cap M$ satisfying*

(i) *$a, (P_0, \dot{\mathcal{B}}_0), \ldots, (P_{m-1}, \dot{\mathcal{B}}_{m-1}) \in K$,*
(ii) *$\operatorname{rank}(K) \ge \xi$,*
(iii) *$\sup(\theta \cap K) \in \bigcap A$,*
(iv) *$\bigcap A \in \uparrow\Omega(K)$,*
(v) *$q^M_{P_i,\dot{\mathcal{B}}_i}(p) \Vdash K \in \dot{\mathcal{B}}_i$ for all $p \in O_i$ for all $i = 0, \ldots, m-1$,*
(vi) *$q^M_{P_i,\dot{\mathcal{B}}_i}(p) \ge q^K_{P_i,\dot{\mathcal{B}}_i}(p)$ for all $p \in O_i$ for all $i = 0, \ldots, m-1$.*

Then $\mathfrak{D}_\Omega(\vec{\mathcal{A}}; \mathcal{Z})$ is a properness parameter.

Proof. We establish the lemma by induction on $\operatorname{rank}(M)$, with the induction hypothesis that there exists $X \in \mathfrak{D}_\Omega(\mathcal{Z})(M)$ such that $q^M_{P,\dot{\mathcal{B}}}(p)$ $(p \in P_i \cap M)$ from the hypothesis of the lemma witnesses Definition 26(iv) for X, for all $(P, \dot{\mathcal{B}}) \in \mathcal{Z} \cap M$.

Suppose then that $M \in \varinjlim \mathcal{A}$ with $\alpha = \operatorname{rank}(M) > 0$. Let $(x_n : n < \omega)$ enumerate $\Omega(M)$, letting $x_n = \theta \cap M$ in case $\Omega(M) = \emptyset$, let $(a_n : n < \omega)$ enumerate $M \cap H_\lambda$, let $(P_n, \dot{\mathcal{B}}_n : n < \omega)$ enumerate $\mathcal{Z} \cap M$, and let $(p_{ni} : i < \omega)$ enumerate $P_n \cap M$ for each n. Fix a sequence $\xi_n < \operatorname{rank}(M)$ $(n < \omega)$ such that $\limsup_{n\to\omega} \xi_n + 1 = \alpha$. We recursively choose $K_{n+1} \ni K_n$ in $\mathcal{A}_{<\alpha} \cap M$ with $a_n, (P_0, \dot{\mathcal{B}}_0), \ldots, (P_n, \dot{\mathcal{B}}_n) \in K_n$, $\xi_n \le \operatorname{rank}(K_n)$, $\sup(\theta \cap K_n) \in \bigcap_{i=0}^n x_i$, $\bigcap_{i=0}^n x_i \in \uparrow\Omega(K_n)$ and $q^M_{P_i,\dot{\mathcal{B}}_i}(p_{ij}) \Vdash K_n \in \dot{\mathcal{B}}_i$ and $q^M_{P_i,\dot{\mathcal{B}}_i}(p_{ij}) \ge q^{K_n}_{P_i,\dot{\mathcal{B}}_i}(p_{ij})$ for all $i, j = 0, \ldots, n$. This is possible by (i)–(vi). And for each n, there exists X_n in $\mathfrak{D}_\Omega(\mathcal{Z})(K_n)$ as in the induction hypothesis. Furthermore, by going to a tail of X_n, we may assume that $\operatorname{tr}\sup(X_n) \subseteq \bigcap_{i=0}^n x_i$ and that $q^{K_n}_{P_i,\dot{\mathcal{B}}_i}(p_{ij}) \Vdash X_n \subseteq \dot{\mathcal{B}}_i$ for all $i, j = 0, \ldots, n$. Then putting $X = \bigcup_{n<\omega} X_n \cup \{K_n\}$, conditions (i)–(iii) of Definition 16 are clearly satisfied. And for condition (iv), given $(P, \dot{\mathcal{B}}) \in \mathcal{Z} \cap M$ and $p \in P \cap M$, say $(P, \dot{\mathcal{B}}) = (P_i, \dot{\mathcal{B}}_i)$ and $p = p_{ij}$, $q^M_{P_i,\dot{\mathcal{B}}_i}(p) \Vdash \bigcup_{n \ge \max(i,j)} X_n \cup \{K_n\} \subseteq \dot{\mathcal{B}}_i$. This proves that $X \in \mathfrak{D}_\Omega(\mathcal{Z})(M)$ is as needed, completing the induction. □

2.2. The iteration

Notation 30. *Let $\mathcal{E} \subseteq [H_\lambda]^{\aleph_0}$. For a poset P with $P \in H_\lambda$, if $G \subseteq P$ is a generic ideal over V, in $V[G]$ we define*

$$\mathcal{E}[G] = \{M[G] : M \in \mathcal{E}, P \in M \text{ and } G \in \operatorname{Gen}(V, P)\}. \tag{6}$$

Proposition 31. *Let P be a forcing notion that adds no new ω-sequences of ground model elements (e.g. P completely proper). If $\mathcal{E} \subseteq [H_\kappa]^{\aleph_0}$ is closed and cofinal then so is $\mathcal{E}[G]$.*

A λ-properness parameter $(\vec{\mathcal{A}}, \mathfrak{D})$ can be interpreted in a forcing extension $V[G]$ by some forcing notion $P \in H_\lambda$ as $(\vec{\mathcal{B}}, \mathfrak{E})$ where $\mathcal{B}_\alpha = \mathcal{A}_\alpha[G]$ for all $\alpha < \omega_1$ and $\mathfrak{E}(M[G]) = \mathfrak{D}(M)$ for all $M \in \varinjlim \mathcal{A}$ with $M[G] \in \varinjlim \mathcal{B}$. Thus when we say that $P \Vdash \ulcorner\dot{Q}$ is $(\vec{\mathcal{A}}, \mathfrak{D})$-proper$\urcorner$, we mean that $V[G] \models \ulcorner\dot{Q}[G]$ is $(\vec{\mathcal{B}}, \mathfrak{E})$-proper$\urcorner$.

Let us now address the issue of "long properness". The following is essentially [9, Definition 1.8(c)], but without the requirement of complete properness.

Definition 32. Let $(\vec{\mathcal{A}}, \mathfrak{D})$ be a λ-properness parameter. A countable support iteration $(P_\xi : \xi \le \delta)$ is called *long $\mathfrak{D}$-proper* if $P_\delta \in H_\lambda$ and there exists $a \in H_\lambda$ such that for all $M \in \varinjlim \mathcal{A}$ with $a, P_\delta \in M$, for all $\xi < \delta$ in M, if

(i) $q \in \operatorname{gen}(M, P_\xi)$,
(ii) $X = \mathcal{M}(q) \cap M \in \mathfrak{D}(M)$,
(iii) $\dot{p}$ is a P_ξ-name such that
  (a) $q \Vdash \dot{p} \in P_\delta \cap M$,
  (b) $q \Vdash \dot{p} \restriction \xi \in \dot{G}_{P_\xi}$,

then there exists $r \in \operatorname{gen}(M, P_\delta)$ such that

(iv) $r \restriction \xi = q$,
(v) $\mathcal{M}(r) \cap X \in \mathfrak{D}(M)$,
(vi) $r \Vdash \dot{p} \in \dot{G}_{P_\delta}$.

The following lemma says that the iteration is long $\mathfrak{D}$-proper when each iterand is $\mathfrak{D}$-proper. It is proved in [9, page 17, "Proof of clause (c)"].

Lemma 33. *Assume that the chooser has a global winning strategy in the game $\supset(\mathfrak{D})$. Suppose $(P_\xi, \dot{Q}_\xi : \xi < \delta)$ is a countable support iteration such that $P_\xi \Vdash \ulcorner\dot{Q}_\xi$ is $\mathfrak{D}$-proper$\urcorner$ for all $\xi < \delta$. Then $(P_\xi : \xi \le \delta)$ is long $\mathfrak{D}$-proper.*

Although [9] is the first place we read the phrase "long properness", it is a familiar concept used for example in the proof of the preservation of properness under countable support iterations. Indeed Lemma 33 corresponds to the "Properness Extension Lemma" of [1, Lemma 2.8] and the "a-Extension Property" of [1, Lemma 5.6].

The following is a simplified, and somewhat weakened, version of [9, Main Claim 1.9], which is the basic NNR iteration theorem for parameterized properness.

Theorem 3 (*Shelah*). *Let $(\vec{\mathcal{A}}, \mathfrak{D})$ be a properness parameter, where the chooser has a global winning strategy in the game $\supset(\mathfrak{D})$. Suppose $(P_\xi, \dot{Q}_\xi : \xi < \delta)$ is a countable support iteration such that*

(a) $P_\xi \Vdash \dot{Q}_\xi$ *is $\mathfrak{D}$-proper for all $\xi < \delta$,*
(b) $P_\xi \Vdash \dot{Q}_\xi$ *is $\mathbb{D}$-complete for all $\xi < \delta$.*

Then the limit P_δ of the iteration does not add new reals.

We do not refer to the following strengthening of Theorem 3 as a ''theorem'' because, unlike Theorem 3, it does not stand alone in the sense that the needed hypothesis is preserved at limits. That is, there is no conclusion from the hypothesis that the limit P_δ is $\mathfrak{D}_\delta$-proper for some properness parameter $(\vec{\mathcal{A}}, \mathfrak{D}_\delta)$; although in our application of Lemma 34 this will be the case. In fact Lemma 34 below is in the same spirit as the fact that an iteration of proper $\mathbb{D}$-complete forcings of length less than ω^2 adds no new reals.

Lemma 34. *Let $(\vec{\mathcal{A}}, \mathfrak{D}_\xi : \xi < \delta)$ be a sequence of properness parameters such that the chooser has a global winning strategy in the game $\eth(\mathfrak{D}_\xi)$ for all $\xi < \delta$. Suppose that $(P_\xi, \dot{Q}_\xi : \xi < \delta)$ is a countable support iteration satisfying*

(a) *P_ξ is long $\mathfrak{D}_\xi$-proper for all $\xi < \delta$,*
(b) *$P_\xi \Vdash \dot{Q}_\xi$ is $\mathbb{D}$-complete for all $\xi < \delta$.*

Then the limit P_δ does not add new reals.

Proof (*Sketch of Proof*). The proof is based on Abraham's proof in [1, Section 5] of Shelah's fundamental NNR iteration theorem that countable support iterations of forcing notions that are both a-proper and $\mathbb{D}$-complete do not add new reals.

There is a function $\mathbb{E}$ that is implicitly assumed to exist in e.g. the proof in [8, Ch. V, Section 7], and is thankfully made explicit in [1]. It takes arguments of the form $(M, \vec{M}, (P_\xi, \dot{Q}_\xi : \xi < \gamma), G, p)$ as input, where $M \prec H_\lambda$ is countable containing $(P_\xi, \dot{Q}_\xi : \xi < \gamma)$, $\vec{M} = (M_\eta : \eta < \alpha)$ is an $\in$-tower of countable elementary submodels with $M_0 = M$, $G \in \operatorname{Gen}(M, P_{\gamma_0}, p \restriction \gamma_0) \cap M_1$ for some $\gamma_0 < \gamma$ in M and $p \in P_\gamma \cap M$. The value $\mathbb{E}(M, \vec{M}, (P_\xi, \dot{Q}_\xi : \xi < \gamma), G, p)$ returned is an element $H \in \operatorname{Gen}(M, P_\gamma, p)$ extending G, i.e. $p \restriction \gamma_0 \in G$ for all $p \in H$. The whole point of introducing $\mathbb{E}$ is that it is definable from some parameters, and thus the generic output by $\mathbb{E}$ can be found inside a suitable elementary submodel.

It is then shown in [1, Lemma 5.21] that if the tower $\vec{M}$ is high enough, if $(P_\xi, \dot{Q}_\xi : \xi < \gamma)$ is a countable support iteration of a-proper and $\mathbb{D}$-complete forcing notions, and if $G \in \operatorname{Gen}^+(M, P_{\gamma_0})$ is also generic over all members of the tower, then $\mathbb{E}(M, \vec{M}, (P_\xi, \dot{Q}_\xi : \xi < \gamma), G, p)$ is completely generic over M, proving that P_γ does not add new reals.

By making the corresponding changes to the definition of $\mathbb{E}$, the exactly analogous proof works for iterations $(P_\xi, \dot{Q}_\xi : \xi < \gamma)$ of $\mathbb{D}$-complete forcing notions that are long $\mathfrak{D}$-proper for some properness parameter $(\vec{\mathcal{A}}, \mathfrak{D})$; we obtain a completely generic $\mathbb{E}(M, \vec{M}, (P_\xi, \dot{Q}_\xi : \xi < \gamma), G, p)$ whenever the range of $\vec{M}$ is in $\mathfrak{D}(N)$ for some $N \in \varinjlim \mathcal{A}$ of big enough rank.

Now consider the iteration from the hypothesis of the lemma. By the hypotheses (a) and (b), for every $\gamma < \delta$, $\mathbb{E}(M, \vec{M}, (P_\xi, \dot{Q}_\xi : \xi < \gamma), G, p)$ is completely generic for all suitable $\vec{M}$ and G. To show that P_δ does not add reals, we want to find a completely (M, P_δ)-generic ideal. Although we cannot take $\mathbb{E}(M, \vec{M}, (P_\xi, \dot{Q}_\xi : \xi < \delta), G, p)$ since we do not have a $\mathfrak{D}_\delta$, we can still go through the proof of [1, Lemma 5.21] to obtain complete genericity, by using $\vec{M} = \vec{M}^0 {}^\frown \vec{M}^1 {}^\frown \cdots$, where $\xi_0 < \xi_1 < \cdots$ is cofinal in $\delta \cap M$, $N_0 \in N_1 \in \cdots$ in $\varinjlim \mathcal{A}$ are of big enough rank, and the range of $\vec{M}^n$ is in $\mathfrak{D}_{\xi_n}(N_n)$ for all $n < \omega$. $\square$

3. The forcing notions

In the context of an ordinal θ of uncountable cofinality, κ will always denote a regular cardinal $\kappa \geq (|\theta|^{\aleph_0})^+$; and in the context of a cardinal κ, we let λ denote a regular cardinal that is sufficiently large, by which we mean $\lambda \geq |H_\kappa|^+ = (2^{<\kappa})^+$. Thus in the most important case $\theta = \omega_1$, taking κ and λ to be the least sufficiently large regular cardinals and assuming CH and $2^{\aleph_1} = \aleph_2$ (e.g. assuming GCH),

$$|\theta| = \aleph_1 \qquad \kappa = (\aleph_1^{\aleph_0})^+ = \aleph_2 \qquad \lambda = (2^{\aleph_1})^+ = \aleph_3. \tag{7}$$

Definition 35. Let S be a set. For two families $\mathcal{F} \subseteq \mathcal{H} \subseteq \mathcal{P}(S)$, we let

$$\partial_{\mathcal{H}}(\mathcal{F}) = \{x \subseteq \theta : \uparrow x \cap \mathcal{F} \text{ is cofinal in } (\mathcal{H}, \subseteq^*)\}, \tag{8}$$

i.e. the set of all x such that $\{y \in \mathcal{F} : x \subseteq y\}$ is $\subseteq^*$-cofinal in $\mathcal{H}$. We write $\partial(\mathcal{H})$ for $\partial_{\mathcal{H}}(\mathcal{H})$, and we write $\alpha \in \partial_{\mathcal{H}}(\mathcal{F})$ to indicate that $\{\alpha\} \in \partial_{\mathcal{H}}(\mathcal{F})$.

Proposition 36. *$\partial_{\mathcal{H}}(\mathcal{F}) \subseteq \downarrow\mathcal{F}$ whenever $\mathcal{H}$ is nonempty.*

3.1. *Forcing notion for shooting clubs*

The following forcing notion is equivalent to the forcing notion $\mathcal{R}(\mathcal{H}, \mathcal{C}(\downarrow\mathcal{H}))$ from [3]. Thus the forcing notion in Definition 37 is a special case of a more general class of forcing notions studied there. Many of the main results here, with the exception of the new result in Lemma 59, follow from the general theory developed in [3]. We will provide direct proofs for most of the results.

Definition 37. For some ordinal θ of uncountable cofinality, let $\mathcal{H} \subseteq [\theta]^{\aleph_0}$ be nonempty. Then define $\mathcal{Q}(\mathcal{H})$ to be the poset consisting of all pairs $p = (x_p, \mathcal{X}_p)$ where

(i) $x_p \in \partial(\mathcal{H})$ is a closed subset of θ,

(ii) $\mathcal{X}_p$ is a countable family of subsets of $\mathcal{H}$ with

$$x_p \in \partial_{\mathcal{H}}(\mathcal{J}) \quad \text{for all } \mathcal{J} \in \mathcal{X}_p,$$

ordered by q extends p if

(iii) $x_q \sqsupseteq x_p$ (i.e. x_q end-extends x_p with respect to the ordinal ordering),
(iv) $\mathcal{X}_q \supseteq \mathcal{X}_p$.

For an ideal $G \subseteq \mathcal{Q}(\mathcal{H})$, we write $C_G = \bigcup_{p \in G} x_p$. And we write $0_{\mathcal{Q}(\mathcal{H})}$ for the condition $(\emptyset, \emptyset)$.

Our poset forces the following desired result.

Proposition 38. $\mathcal{Q}(\mathcal{H}) \Vdash \ulcorner C_{\dot{G}_{\mathcal{Q}(\mathcal{H})}}$ is locally in $\mathcal{H}\urcorner$.

Proof. By Proposition 36. □

Lemma 39. *Suppose $\mathcal{J} \subseteq \mathcal{H}$ is cofinal in $(\mathcal{H}, \subseteq^*)$. Then $\mathcal{Q}(\mathcal{H}) \Vdash \ulcorner\exists y \in \mathcal{H}\ C_{\dot{G}_{\mathcal{Q}(\mathcal{H})}} \setminus y$ is locally in $\mathcal{J}\urcorner$.*

Proof. Observe that the set of all $p \in \mathcal{Q}(\mathcal{H})$ containing $\mathcal{K} \in \mathcal{X}_p$ of the form $\mathcal{K}_y = \{x \cup y : x \in \mathcal{J}\}$ for some $y \in \downarrow\mathcal{H}$ is dense. That is, given $p \in \mathcal{Q}(\mathcal{H})$, $(x_p, \mathcal{X}_p \cup \{\mathcal{K}_{x_p}\}) \in \mathcal{Q}(\mathcal{H})$ since $x_p \in \partial_{\mathcal{H}}(\mathcal{K}_{x_p})$. □

Proposition 40. *The class $\mathcal{Q}$ is provably equivalent to a Δ_0-formula.*

Proposition 41. *If p and q are two conditions in $\mathcal{Q}(\mathcal{H})$ such that $x_q \sqsubseteq x_p$ and every $\mathcal{J} \in \mathcal{X}_q$ has a $\mathcal{K} \subseteq \mathcal{J}$ in $\mathcal{X}_p$, then $\mathcal{Q}(\mathcal{H}) / \sim_{\mathrm{sep}} \models q \leq p$.*

Proposition 42. *p and q are compatible in $\mathcal{Q}(\mathcal{H})$ iff x_p and x_q are comparable under end-extension and $x_p \cup x_q \in \partial_{\mathcal{H}}(\mathcal{J})$ for all $\mathcal{J} \in \mathcal{X}_p \cup \mathcal{X}_q$.*

The following game is equivalent to the game $\circleddash_{\mathrm{gen}}(M, y, \mathcal{H}, \mathcal{C}(\downarrow\mathcal{H}), p)$ from [3, Definition 3.11], except for the requirement that $p_0 = p$. It is used to establish the various properties of our forcing notion.

Notation 43. *For a centered subset C of some forcing notion P, we let $\langle C\rangle$ denote the ideal on P generated by C.*

Definition 44. For any $M \prec H_\kappa$, with $\mathcal{H} \subseteq [\theta]^{\aleph_0}$ nonempty and in M, for $y \in [\theta]^{\aleph_0}$ and $p \in \mathcal{Q}(\mathcal{H}) \cap M$, we define the game $\circleddash_{\mathrm{gen}}(M, y, \mathcal{H}, p)$ with players *Extender* and *Complete* of length ω. Extender plays first and on move 0 must play p_0 so that

- $p_0 = p$.

On Extender's $(k+1)$th move:

- Extender plays $p_{k+1} \in \mathcal{Q}(\mathcal{H}) \cap M$ satisfying
  (1) p_{k+1} extends p_k,
  (2) $x_{p_{k+1}} \setminus x_{p_k} \subseteq y \setminus \bigcup_{i=0}^{k} s_i$.

On Complete's kth move:

- Complete plays a finite $s_k \subseteq y$.

This game has three possible outcomes, determined as follows:

(i) Extender loses (i.e. Complete wins) if $\langle p_k : k < \omega\rangle \notin \mathrm{Gen}(M, \mathcal{Q}(\mathcal{H}))$,
(ii) the game is drawn (i.e. a tie) if $\langle p_k : k < \omega\rangle \in \mathrm{Gen}^+(M, \mathcal{Q}(\mathcal{H}))$,
(iii) Extender wins the game if $\langle p_k : k < \omega\rangle \in \mathrm{Gen}(M, \mathcal{Q}(\mathcal{H}))$ but (ii) fails.

The game $\circleddash_{\mathrm{gen}}(M, y, \mathcal{H}, p)$ is especially interesting for us because a draw in this game corresponds precisely with complete genericity.

Proposition 45. *Let p_k denote Extender's kth move in the game $\circleddash_{\mathrm{gen}}(M, y, \mathcal{H}, p)$. Then the game results in a draw iff $\langle p_k : k < \omega\rangle \in \mathrm{Gen}^+(M, \mathcal{Q}(\mathcal{H}), p)$.*

Proposition 46. *At the end of the game $\circleddash_{\mathrm{gen}}(M, y, \mathcal{H}, p)$, for every $k < \omega$,*

$$\bigcup_{n<\omega} x_{p_n} \setminus x_{p_k} \subseteq y \setminus \bigcup_{i=0}^{k} s_i. \tag{9}$$

Proposition 47. *Suppose Φ is a nonlosing strategy for Complete in the game $\circleddash_{\mathrm{gen}}(M, y, \mathcal{H}, p)$. Then Complete does not lose if it plays $s_k \supseteq \Phi(P_k)$, where P_k is the position after Extender's kth move.*

Proof. This is by general principles. If Complete plays as in the hypothesis, then it is restricting Extender's moves. Since the outcome of the game is determined solely on Extender's sequence of moves, this is beneficial to Complete. □

Similarly:

Proposition 48. *For all $y \subseteq z$, if Φ is a nonlosing strategy for Complete in the game $\circlearrowleft_{\text{gen}}(M, z, \mathcal{H}, p)$ then $P_k \mapsto \Phi(P_k) \cap y$ is a nonlosing strategy for Complete in the game $\circlearrowleft_{\text{gen}}(M, y, \mathcal{H}, p)$.*

Before proceeding, recall that whenever $(\mathcal{H}, \subseteq^*)$ is σ-directed, the ideal of noncofinal subsets of $\mathcal{H}$ forms a σ-ideal, i.e. is closed under countable unions (see e.g. [3, Lemma 2.2]).

Lemma 49. *Suppose $\mathcal{H}$ is a σ-directed subfamily of $[\theta]^{\aleph_0}$ with no stationary orthogonal set. For every countable $M \prec H_\kappa$ with $\mathcal{H} \in M$, if $\mathcal{J} \subseteq [\theta]^{\aleph_0}$ in M is cofinal in $(\mathcal{H}, \subseteq^*)$, then $\sup(\theta \cap M) \in \partial_{\mathcal{H}}(\mathcal{J})$.*

Proof. Let Z be the set of all $\alpha < \theta$ such that $\alpha \notin \partial_{\mathcal{H}}(\mathcal{J})$. Supposing towards a contradiction that the lemma fails, $\sup(\theta \cap M) \in Z$, and thus Z is stationary because $Z \in M$. By the assumption on $\mathcal{H}$, Z is not orthogonal to $\mathcal{H}$, say $x \in [Z]^{\aleph_0}$ with $x \subseteq y$ for some $y \in \mathcal{H}$. Since $\{z \in \mathcal{J} : y \subseteq^* z\}$ is cofinal in $(\mathcal{H}, \subseteq^*)$ as $\mathcal{H}$ is directed, there must exist a finite $s \subseteq x$ such that $\{z \in \mathcal{J} : x \setminus s \subseteq z\}$ is cofinal because $\mathcal{H}$ is σ-directed. We have now arrived at the contradiction that $\alpha \in \partial_{\mathcal{H}}(\mathcal{J})$ for all $\alpha \in z \setminus s$. □

The following corollary implies that $\mathcal{Q}(\mathcal{H})$ forces a closed cofinal subset of θ, although it remains to show that $\mathcal{Q}(\mathcal{H})$ does not collapse $\aleph_1$.

Corollary 50. *Assume $\mathcal{H}$ is as in Lemma 49. For every $\xi < \theta$,*

$$\mathcal{D}_\xi = \{p \in \mathcal{Q}(\mathcal{H}) : \max(x_p) \geq \xi\} \tag{10}$$

is a dense subset of $\mathcal{Q}(\mathcal{H})$.

Proof. Given $p \in \mathcal{Q}(\mathcal{H})$, find a countable elementary $M \prec H_\kappa$ with $\mathcal{H}, p, \xi \in M$, set $\delta = \sup(\theta \cap M)$. For each $\mathcal{J} \in \mathcal{X}_p$, let $\mathcal{K}_{\mathcal{J}} = \{y \in \mathcal{J} : x_p \subseteq y\}$. Each $\mathcal{K}_{\mathcal{J}}$ is cofinal as $x_p \in \partial_{\mathcal{H}}(\mathcal{J})$, and thus by Lemma 49, $\delta \in \partial_{\mathcal{H}}(\mathcal{K}_{\mathcal{J}})$ for all $\mathcal{J} \in \mathcal{X}_p$. This implies that $x_p \cup \{\delta\} \in \partial_{\mathcal{H}}(\mathcal{J})$ for all $\mathcal{J} \in \mathcal{X}_p$, and therefore $q = (x_p \cup \{\delta\}, \mathcal{X}_p) \in \mathcal{Q}(\mathcal{H})$. Since $\delta > \xi$, the proof is complete. □

Corollary 51. *If $\mathcal{H} \subseteq [\theta]^{\aleph_0}$ is σ-directed under $\subseteq^*$ and has no stationary orthogonal set, then $\mathcal{Q}(\mathcal{H}) \Vdash C_{\dot{G}_{\mathcal{Q}(\mathcal{H})}}$ is a closed cofinal subset of θ.*

Terminology 52. *Henceforth, we let $\varphi(\theta, \mathcal{H})$ abbreviate the statement:* $\mathcal{H}$ is a σ-directed subfamily of $([\theta]^{\aleph_0}, \subseteq^*)$ with no stationary orthogonal set.

Corollary 53. *Assume $\varphi(\theta, \mathcal{H})$. Let $\mathcal{H} \in M \prec H_\kappa$, $y \in [\theta]^{\aleph_0}$ and $p \in \mathcal{Q}(\mathcal{H}) \cap M$, and let p_k denote Extender's kth move in the game $\circlearrowleft_{\text{gen}}(M, y, \mathcal{H}, p)$. Suppose that Extender does not lose the game. Then the following are equivalent:*

(a) *The game $\circlearrowleft_{\text{gen}}(M, y, \mathcal{H}, p)$ is drawn.*
(b) $\left(\bigcup_{k<\omega} x_{p_k} \cup \{\sup(\theta \cap M)\}, \bigcup_{k<\omega} \mathcal{X}_{p_k}\right) \in \mathcal{Q}(\mathcal{H})$.
(c) $\bigcup_{k<\omega} x_{p_k} \cup \{\sup(\theta \cap M)\} \in \partial_{\mathcal{H}}(\mathcal{J})$ *for all* $\mathcal{J} \in \mathcal{X}_q$.

Proof. Put $\delta = \sup(\theta \cap M)$. By Corollary 50, for every $\xi < \theta$ in M, $\mathcal{D}_\xi \in M$ (cf. Eq. (10)) is dense, and thus $x_{p_k} \in \mathcal{D}_\xi$ for some k since Extender did not lose. Hence $\bigcup_{k<\omega} x_{p_k}$ is unbounded in δ.

(a)→(b): $\{p_k : k < \omega\}$ has a common extension, say q, by Proposition 45. Since we must have $\delta \in x_q$, and obviously $\bigcup_{k<\omega} \mathcal{X}_{p_k} \subseteq \mathcal{X}_q$, it clearly follows that the pair defined in (b) is a condition of $\mathcal{Q}(\mathcal{H})$.

The implication (b)→(c) is trivial by definition; the implication (c)→(b) is because the set is closed; and the implication (b)→(a) is an immediate consequence of Proposition 45. □

Lemma 54. *Let $\mathcal{H} \subseteq [\theta]^{\aleph_0}$ be σ-directed. Suppose $M \prec H_\lambda$ is countable with $\mathcal{H} \in M$, and $y \in [\theta]^{\aleph_0}$ satisfies $x \subseteq^* y$ for all $x \in \mathcal{H} \cap M$. Then every $p \in \mathcal{Q}(\mathcal{H}) \cap M$ and every dense $D \subseteq \mathcal{Q}(\mathcal{H})$ in M, has an extension q of p in $D \cap M$ such that $x_q \setminus x_p \subseteq y$.*

Proof. Suppose to the contrary that there is no $q \geq p$ in $D \cap M$ with $x_q \setminus x_p \subseteq y$. Define $\mathcal{F}$ as the set of all $K \in [H_\kappa]^{\aleph_0}$ for which there exists $y_K \in [\theta]^{\aleph_0}$ such that $x_p \subseteq y_K$, there is no $q \geq p$ in D with $x_q \subseteq y_K$, and $x \subseteq^* y_K$ for all $x \in \mathcal{H} \cap K$. Then $\mathcal{F} \in M$.

Take $K \in [H_\kappa]^{\aleph_0} \cap M$. Since $\{z \in \mathcal{H} : x_p \subseteq z\}$ is cofinal by condition (i) of the forcing notion, and since $\mathcal{H}$ is σ-directed, there exists $z \in \mathcal{H} \cap M$ such that $x_p \subseteq z$ and $x \subseteq^* z$ for all $x \in \mathcal{H} \cap K$. Then there exists a finite $s \subseteq z \setminus x_p$ such that $z \setminus s \subseteq y \cup x_p$. There can be no $q \geq p$ in D with $x_q \subseteq z \setminus s$, because otherwise by elementarity we could find such a $q \in D \cap M$, contradicting our supposition. Hence $y_K = z \setminus s$ witnesses that $K \in \mathcal{F}$.

By elementarity, we have proved that $\mathcal{F} = [H_\kappa]^{\aleph_0}$, and thus $\mathcal{J} = \{y_K : K \in \mathcal{F}\} \subseteq \mathcal{H}$ is cofinal in $(\mathcal{H}, \subseteq^*)$ with $x_p \in \partial_{\mathcal{H}}(\mathcal{J})$. Hence

$$q = (x_p, \mathcal{X}_p \cup \{\mathcal{J}\}) \in \mathcal{Q}(\mathcal{H}). \tag{11}$$

Since D is dense, there exists $q' \geq q$ in D. But then $x_{q'} \in \partial_{\mathcal{H}}(\mathcal{J})$, and in particular $x_{q'} \subseteq y_K$ for some $K \in \mathcal{F}$, contradicting the choice of y_K. □

Corollary 55. *Let $\mathcal{H}$ be a σ-directed subfamily of $([\theta]^{\aleph_0}, \subseteq^*)$. Suppose $M \prec H_\lambda$ with $\mathcal{H} \in M$, $y \in [\theta]^{\aleph_0}$ and $p \in \mathcal{Q}(\mathcal{H}) \cap M$. If $x \subseteq^* y$ for all $x \in \mathcal{H} \cap M$, then Extender has a nonlosing strategy in the game $\eth_{\mathrm{gen}}(M, y, \mathcal{H}, p)$.*

Proof. Let $(D_k : k < \omega)$ enumerate all of the dense subsets of $\mathcal{Q}(\mathcal{H})$ in M. Suppose the position in the game is $(p_0, s_0), \ldots, (p_k, s_k)$ after the kth move. By the assumption on y,

$$x \subseteq^* y \setminus \bigcup_{i=0}^{k} s_i \quad \text{for all } x \in \mathcal{H} \cap M. \tag{12}$$

On move $k+1$, by Lemma 54, Extender can thus play $p_{k+1} \geq p_k$ in $D_k \cap M$ such that $x_{p_{k+1}} \setminus x_{p_k} \subseteq y \setminus \bigcup_{i=0}^{k} s_i$. Clearly then $\langle p_k : k < \omega \rangle \in \mathrm{Gen}(M, \mathcal{Q}(\mathcal{H}))$. □

Lemma 56. *Assume $\varphi(\theta, \mathcal{H})$. Let $M \prec H_\lambda$ be a countable elementary submodel with $\mathcal{H} \in M$, let $y \in [\theta]^{\aleph_0}$ and let $p \in \mathcal{Q}(\mathcal{H}) \cap M$. If $x \subseteq^* y$ for all $x \in \mathcal{H} \cap M$, then Complete has a nonlosing strategy in the game $\eth_{\mathrm{gen}}(M, y, \mathcal{H}, p)$.*

Proof. Suppose that $x \subseteq^* y$ for all $x \in \mathcal{H} \cap M$. Then since $\mathcal{H}$ is σ-directed, we can find $z \in \mathcal{H}$ such that $x \subseteq^* z$ for all $x \in \mathcal{H} \cap M$. Therefore, by Proposition 48, replacing y with $y \cap z$ we can assume that $y \in \downarrow\mathcal{H}$.

Set $\delta = \sup(\theta \cap M)$. At the end of the game $\eth_{\mathrm{gen}}(M, y, \mathcal{H}, p)$, where Extender has played p_k on its kth move, we will set $x_q = \bigcup_{k<\omega} x_{p_k} \cup \{\delta\}$. The aim of the Complete's strategy is to ensure that $x_q \in \partial_{\mathcal{H}}(\mathcal{J})$ for all $\mathcal{J} \in \mathcal{X}_{p_k}$, for all $k < \omega$.

We know that $\bigcup_{k<\omega} \mathcal{X}_{p_k}$ will be countable, and thus we can arrange a diagonalization $(\mathcal{J}_k : k < \omega)$ in advance, and since the $\mathcal{X}_{p_k}$'s will be increasing with k, we can also insist that $\mathcal{J}_k \in \mathcal{X}_{p_k}$ for all k. After Extender plays p_k on move k, we take care of some $\mathcal{J}_k \in \mathcal{X}_{p_k}$ according to the diagonalization. Set

$$\mathcal{K}_k = \{z \in \mathcal{J}_k : x_{p_k} \subseteq z\}. \tag{13}$$

Then $\mathcal{K}_k$ is $\subseteq^*$-cofinal in $\mathcal{H}$ because $x_{p_k} \in \partial_{\mathcal{H}}(\mathcal{J}_k)$ by the definition of the poset. Thus as $p_k \in M$, by Lemma 49,

$$\mathcal{K}'_k = \{z \in \mathcal{K}_k : \delta \in z\} \tag{14}$$

is cofinal too.

Claim 56.1. *There exists a finite $s_k \subseteq y$ such that $y \setminus s_k \in \partial_{\mathcal{H}}(\mathcal{K}'_k)$.*

Proof. Since $y \in \downarrow\mathcal{H}$, and $\mathcal{H}$ is directed, $\{z \in \mathcal{K}'_k : y \subseteq^* z\}$ is cofinal. It then follows from the fact that $\mathcal{H}$ is σ-directed that there exists a finite $s_k \subseteq y$ such that $y \setminus s_k \in \partial_{\mathcal{H}}(\mathcal{K}'_k)$. □

Complete plays s_k as in the claim on its kth move. This describes the strategy for Complete.

If Extender loses then Complete wins, and thus we may assume that Extender does not lose. Put $x_q = \bigcup_{k<\omega} x_{p_k} \cup \{\delta\}$ and $\mathcal{X}_q = \bigcup_{k<\omega} \mathcal{X}_{p_k}$. It remains to show that the game is drawn, and thus it suffices to show that $x_q \in \partial_{\mathcal{H}}(\mathcal{J})$ for all $\mathcal{J} \in \mathcal{X}_q$ by Corollary 53. But every $\mathcal{J} \in \mathcal{X}_q$ appears as $\mathcal{J}_k$ for some k, and thus as $\bigcup_{n<\omega} x_{p_n} \setminus x_{p_k} \subseteq y \setminus s_k$ by Proposition 46, $y \setminus s_k \in \partial_{\mathcal{H}}(\mathcal{K}'_k)$ implies that $\{y \in \mathcal{J}_k : x_q \subseteq y\}$ is cofinal by Eqs. (13) and (14), proving $x_q \in \partial_{\mathcal{H}}(\mathcal{J}_k)$. □

The following lemma implies that our poset does not collapse $\aleph_1$.

Lemma 57. *Assume that $\mathcal{H}$ is a σ-directed subfamily of $([\theta]^{\aleph_0}, \subseteq^*)$ with no stationary orthogonal set (i.e. $\varphi(\theta, \mathcal{H})$). Then $\mathcal{Q}(\mathcal{H})$ is completely proper.*

Proof. Let $M \prec H_\lambda$ be countable with $\mathcal{H} \in M$. Since $\mathcal{H}$ is σ-directed there exists $y \in \mathcal{H}$ such that $x \subseteq^* y$ for all $x \in \mathcal{H} \cap M$. Let $p \in \mathcal{Q}(\mathcal{H}) \cap M$ be given. Then the hypotheses of Corollary 55 and Lemma 56 are satisfied. Therefore both Extender and Complete have nonlosing strategies in the game $\eth_{\mathrm{gen}}(M, y, \mathcal{H}, p)$. The game is played with both Extender and Complete playing according to their respective strategies. Since the game results in a draw, there exists $q \in \mathrm{gen}^+(M, \mathcal{Q}(\mathcal{H}), p)$ by Proposition 45. This proves that $\mathcal{Q}(\mathcal{H})$ is completely proper. □

Terminology 58. *We let $\varphi_*(M, \mathcal{H}, y)$ abbreviate the statement: $x \subseteq^* y$ for all $x \in \mathcal{H} \cap M$.*

Lemma 59. *Let $(\vec{\mathcal{A}}, \mathfrak{D}_\Omega)$ be a λ-properness parameter determined by $\Omega : \varinjlim \mathcal{A} \to [[\theta]^{\aleph_0}]^{\leq \aleph_0}$ (cf. Definition 16). Assume $\varphi(\theta, \mathcal{H})$. If every $M \in \varinjlim \mathcal{A}$ of positive rank with $\mathcal{H} \in M$ has a $y \in \Omega(M) \cap \downarrow\mathcal{H}$ satisfying $\varphi_*(M, \mathcal{H}, y)$, then $\mathcal{Q}(\mathcal{H})$ is $(\vec{\mathcal{A}}, \mathfrak{D}_\Omega)$-proper.*

Proof. This is proved by induction on $\mathrm{rank}(M)$, where M is from the set of all $M \in \varinjlim \mathcal{A}$ with $\mathcal{H} \in M$. The induction hypothesis is that for all $p \in \mathcal{Q}(\mathcal{H}) \cap M$, and all $X \in \mathfrak{D}_\Omega(M)$, for every $y_M \in \downarrow\mathcal{H}$ satisfying $\varphi_*(M, \mathcal{H}, y_M)$ and moreover $y_M \in \Omega(M)$ when $\mathrm{rank}(M) > 0$, and every finite $t \subseteq y_M$, there exists $q \in \mathrm{gen}(M, \mathcal{Q}(\mathcal{H}), p)$ such that $\mathcal{M}(q) \cap X \in \mathfrak{D}_\Omega(M)$ and

$$x_q \setminus x_p \subseteq (y_M \setminus t) \cup \{\sup(\theta \cap M)\}. \tag{15}$$

This will in particular entail that $\mathcal{Q}(\mathcal{H})$ is proper for the desired parameter, because by the hypothesis on Ω there always exists such a y_M and hence $a = \mathcal{H}$ will witness that $(\mathfrak{p})$ holds.

For the base case $\operatorname{rank}(M) = 0$, it suffices to show that every $p \in \mathcal{Q}(\mathcal{H}) \cap M$ and every $z \in [\theta]^{\aleph_0}$ satisfying $\varphi_*(M, \mathcal{H}, z)$ have an $(M, \mathcal{Q}(\mathcal{H}))$-generic extension $q \geq p$ with $x_q \setminus x_p \subseteq z \cup \{\sup(\theta \cap M)\}$. And Extender and Complete both have nonlosing strategies in the game $\Game_{\mathrm{gen}}(M, z, \mathcal{H}, p)$ by Corollary 55 and Lemma 56. After the game is played according to these respective strategies, with p_k denoting Extender's kth move, we obtain $q \in \operatorname{gen}^+(M, \mathcal{Q}(\mathcal{H}), p)$ with $x_q = \bigcup_{k<\omega} x_{p_k} \cup \{\sup(\theta \cap M)\}$ by Corollary 53. And then $x_q \setminus x_p \subseteq z \cup \{\sup(\theta \cap M)\}$ by Proposition 46 with $k = 0$ since $p_0 = p$.

Suppose now that $\operatorname{rank}(M) > 0$ with $\mathcal{H} \in M$, and we are given $p \in \mathcal{Q}(\mathcal{H}) \cap M$, $X \in \mathfrak{D}_\Omega(M)$ and $y_M \in \Omega(M) \cap \downarrow\mathcal{H}$ satisfying $\varphi_*(M, \mathcal{H}, y_M)$ and a finite $t \subseteq y_M$. By going to a subset of X, we can assume that $X \cap K \in \mathfrak{D}_\Omega(K)$ for all $K \in X$ by Lemma 9. Moreover, since $y_M \in \Omega(M)$, by going to a tail of X, we can assume that

$$\operatorname{tr}\sup_\theta(X) \subseteq y_M \setminus t. \tag{16}$$

Let $(a_k : k < \omega)$ enumerate $M \cap H_\lambda$ and let $(\xi_k : k < \omega)$ satisfy $\lim_{k\to\omega} \xi_k + 1 = \operatorname{rank}(M)$.

The game $\Game_{\mathrm{gen}}(M, y_M \setminus t, \mathcal{H}, p)$ shall be played with (p_k, s_k) denoting the kth move. Since $\varphi_*(M, \mathcal{H}, y_M \setminus t)$, Complete has a nonlosing strategy in this game, which it plays by. After the kth move has been played, we can find $K_k \in X$ such that $a_k, p_k \in K_k$, $\operatorname{rank}(K_k) \geq \xi_k$ and moreover

$$\sup(\theta \cap K_k) \notin \bigcup_{i=0}^{k} s_i. \tag{17}$$

Since $y_{K_k} \in \downarrow\mathcal{H}$, we can find a finite $u_k \subseteq y_{K_k}$ such that $y_{K_k} \setminus u_k \subseteq y_M$. Now by the induction hypothesis, there exists $p_{k+1} \geq p_k$ in $\operatorname{gen}(K_k, \mathcal{Q}(\mathcal{H}))$ such that

$$Y_k = \mathcal{M}(p_{k+1}) \cap X \in \mathfrak{D}_\Omega(K_k), \tag{18}$$

and $x_{p_{k+1}} \setminus x_{p_k} \subseteq \left(y_{K_k} \setminus \left(\bigcup_{i=0}^{k} s_i \cup t \cup u_k\right)\right) \cup \{\sup(\theta \cap K_k)\} \subseteq y_M \setminus \left(\bigcup_{i=0}^{k} s_i \cup t\right) \cup \{\sup(\theta \cap K_k)\}$. Then in fact $x_{p_{k+1}} \setminus x_{p_k} \subseteq y_M \setminus \left(\bigcup_{i=0}^{k} s_i \cup t\right)$ by (16) and (17), and thus p_{k+1} is a valid move for Extender.

At the end of the game, since $(K_k : k < \omega)$ is cofinal in $M \cap H_\lambda$ and each p_{k+1} is $(K_k, \mathcal{Q}(\mathcal{H}))$-generic, it follows that the ideal $\langle p_k : k < \omega \rangle$ is in $\operatorname{Gen}(M, \mathcal{Q}(\mathcal{H}))$, and thus Extender does not lose. Since Complete also does not lose, the conditions $\{p_k : k < \omega\}$ have a common extension q. Now $\bigcup_{k<\omega} Y_k \cup \{K_k\} \subseteq \mathcal{M}(q) \cap X$, and clearly $\bigcup_{k<\omega} Y_k \cup \{K_k\} \in \mathfrak{D}_\Omega(M)$. Moreover we can assume that $x_q = \bigcup_{k<\omega} x_{p_k} \cup \{\sup(\theta \cap M)\}$ by Corollary 53, and thus $x_q \setminus x_p \subseteq (y_M \setminus t) \cup \{\sup(\theta \cap M)\}$, completing the induction. □

For definitions of $\mathbb{D}$-completeness we refer the reader to [3] and/or [1]. In the present paper we say that a poset is $\mathbb{D}$-complete, if it has a simply definable $\aleph_1$-complete completeness system. To avoid a complicated proof we only prove that $\mathcal{Q}(\mathcal{H})$ has a simply definable $\aleph_0$-complete completeness system. If an $\aleph_1$-complete system is desired, one can use the notion of a *forward strategy* introduced there; in particular, Lemma 60 can be obtained as an application of [3, Lemma 3.39].

Lemma 60. *Let $\mathcal{H} \subseteq [\theta]^{\aleph_0}$ be σ-directed with no stationary orthogonal set (i.e. $\varphi(\theta, \mathcal{H})$). Then $\mathcal{Q}(\mathcal{H})$ is $\mathbb{D}$-complete.*

Proof (*Proof for $\aleph_0$-completeness*). The completeness system receives as input a countable $M \prec H_\lambda$, a family $\mathcal{H} \subseteq [\theta]^{\aleph_0}$ in M and $p \in \mathcal{Q}(\mathcal{H}) \cap M$. We fix a suitably definable way of coding

- a subset y_Z of $\theta \cap M$,
- a partial function Φ_Z on M with $\Phi_Z(a) \in [\theta \cap M]^{<\aleph_0}$ for all $a \in \operatorname{dom}(\Phi_Z)$,

by subsets $Z \subseteq M$. The second order formula φ defining the family of subsets of $\operatorname{Gen}(M, \mathcal{Q}(\mathcal{H}), p)$ is given by ⌜if

(a) $x \subseteq^* y_Z$ for all $x \in \mathcal{H}$,

there exists[2] a sequence $(p_k : k < \omega)$ of conditions in $\mathcal{Q}(\mathcal{H})$ and a sequence $(s_k : k < \omega)$ of finite subsets of θ such that

(b) (p_k, s_k) is valid for move k of the game $\Game_{\mathrm{gen}}(M, y_Z, \mathcal{H}, p)$,
(c) $\vec{a} = \left((p_0, s_0), \dots, (p_{k-1}, s_{k-1}), p_k\right) \in \operatorname{dom}(\Phi_Z)$ and $s_k \supseteq \Phi_Z(\vec{a})$⌝.

Thus the family coded by some $Z \subseteq M$ is

$$\mathcal{G}_Z = \{G \in \operatorname{Gen}(M, \mathcal{Q}(\mathcal{H}), p) : M \models \varphi(G, Z; \mathcal{H}, p)\}. \tag{19}$$

First we verify $\aleph_0$-completeness. Let $Z_0, \dots, Z_{n-1}$ be given subsets of M. We can assume without loss of generality that condition (a) holds for all $j = 0, \dots, n-1$. The game $\Game_{\mathrm{gen}}\left(M, \bigcap_{j=0}^{n-1} y_{Z_j}, \mathcal{H}, p\right)$ is played with (p_k, s_k) being the kth move. By condition (a), $x \subseteq^* \bigcap_{j=0}^{n-1} y_{Z_j}$ for all $x \in \mathcal{H} \cap M$, and hence by Corollary 55 Extender has a nonlosing strategy in this game, which it plays by. For each $j = 0, \dots, n-1$, we recursively choose for each $k < \omega$, t_{jk} so that $\vec{a}_{jk} = \left((p_0, t_{j0}), \dots, (p_{k-1}, t_{j(k-1)})\right)$ is a valid position in the game $\Game_{\mathrm{gen}}(M, y_{Z_j}, \mathcal{H}, p)$; its definition is $t_{jk} = \Phi_{Z_j}(\vec{a}_{jk})$. On move

[2] Note that this is a second order quantifier, so that e.g. the sequence $(p_k : k < \omega)$ need not be an element of M.

k, Complete plays $s_k = \left(\bigcup_{j=0}^{n-1} t_{jk}\right) \cap \bigcap_{j=0}^{n-1} y_{Z_j}$, which ensures that Extender's move p_{k+1} in the former game is also a valid move in each of the games $\eth_{\mathrm{gen}}(M, y_{Z_j}, \mathcal{H}, p)$ $(j = 0, \ldots, n-1)$. Let $G = \langle p_k : k < \omega \rangle$. Then $G \in \mathrm{Gen}(M, \mathcal{Q}(\mathcal{H}), p)$ because Extender does not lose. And thus for each j, $M \models \varphi(G, Z_j; \mathcal{H}, p)$ as witnessed by $(p_0, t_{j0}), (p_1, t_{j1}), \ldots$; hence, $\bigcap_{j=0}^{n-1} \mathcal{G}_{Z_j} \neq \emptyset$ as wanted.

For $\mathbb{D}$-completeness, it remains to find a $Z \subseteq M$ such that $\mathcal{G}_Z \subseteq \mathrm{Gen}^+(M, \mathcal{Q}(\mathcal{H}), p)$. However, choosing any $y \in [\theta]^{\aleph_0}$ satisfying $\varphi_*(M, \mathcal{H}, y)$, Complete has a nonlosing strategy Φ in the game $\eth_{\mathrm{gen}}(M, y, \mathcal{H}, p)$ by Lemma 56. Find $Z \subseteq M$ such that $y_Z = y$ and $\Phi_Z = \Phi$. Now suppose that $G \in \mathcal{G}_Z$, witnessed by $(p_k : k < \omega)$ and $(s_k : k < \omega)$. Then by (b) and (c), and Proposition 47, Complete does not lose the game $\eth_{\mathrm{gen}}(M, y, \mathcal{H}, p)$ where (p_k, y_k) is played on move k. Since Complete does not lose, and we already know that $G \in \mathrm{Gen}(M, \mathcal{Q}(\mathcal{H}))$, we must have $G \in \mathrm{Gen}^+(M, \mathcal{Q}(\mathcal{H}))$. □

In the case $\theta = \omega_1$, assuming CH, our poset $\mathcal{Q}(\mathcal{H})$ clearly has the $\aleph_2$-cc and thus does not collapse cardinals. However, if we want to avoid using an inaccessible cardinal, we need that iterated forcing constructions using our poset also have the $\aleph_2$-cc, which is not in general preserved under countable support iterations. The usual approach in this situation is to use the *properness isomorphism condition*, and apply the theory from [8, Ch. VIII, Section 2]. By the properness isomorphism condition, we mean the $\aleph_2$-pic there; and there is a theorem that under CH, a countable support iteration of length at most ω_2 of posets satisfying the $\aleph_2$-pic has the $\aleph_2$-cc. As an alternative to using Lemma 61, one can always iterate up to a strongly inaccessible cardinal μ since our posets will all have the μ-cc.

We will not give the actual definition of the properness isomorphism condition here, but instead refer the reader to either Shelah's book, [3] or [1]. We also do not provide a proof of the following lemma, because as is usual with this property, it is a straightforward modification of the proof of properness. One can also obtain Lemma 61 by applying [3, Corollary 3.54.1].

Lemma 61. *Assume that $\mathcal{H}$ is a σ-directed subfamily of $([\omega_1]^{\aleph_0}, \subseteq^*)$ with no stationary orthogonal set (i.e. $\varphi(\omega_1, \mathcal{H})$). Then $\mathcal{Q}(\mathcal{H})$ satisfies the properness isomorphism condition.*

3.2. Forcing notion for shooting non-clubs

The following is perhaps the most natural forcing notion for forcing an uncountable set locally in some σ-directed subfamily of $([S]^{\aleph_0}, \subseteq^*)$, for some set S.

Definition 62. For $\mathcal{H} \subseteq \mathcal{P}(S)$, let $\mathcal{R}(\mathcal{H})$ be the poset consisting of all pairs $p = (x_p, \mathcal{X}_p)$ where

(i) $x_p \in \partial(\mathcal{H})$,
(ii) $\mathcal{X}_p$ is a countable family of cofinal subsets of $(\mathcal{H}, \subseteq^*)$ with

$$x_p \in \partial_{\mathcal{H}}(\mathcal{J}) \quad \text{for all } \mathcal{J} \in \mathcal{X}_p, \tag{20}$$

ordered by q extends p if

(iii) $x_q \supseteq x_p$,
(iv) $\mathcal{X}_q \supseteq \mathcal{X}_p$.

For an ideal $G \subseteq \mathcal{R}(\mathcal{H})$, we write $X_G = \bigcup_{p \in G} x_p$. We write $0_{\mathcal{R}(\mathcal{H})}$ for the condition $(\emptyset, \emptyset)$.

Proposition 63. $\mathcal{R}(\mathcal{H}) \Vdash \ulcorner X_{\dot{G}_{\mathcal{R}(\mathcal{H})}}$ is locally in $\downarrow\mathcal{H}\urcorner$.

Proposition 64. *The class $\mathcal{R}$ is provably equivalent to a Δ_0-formula.*

Proposition 65. *If p and q are two conditions in $\mathcal{R}(\mathcal{H})$ such that $x_q \subseteq x_p$ and every $\mathcal{J} \in \mathcal{X}_q$ has a $\mathcal{K} \subseteq \mathcal{J}$ in $\mathcal{X}_p$, then $\mathcal{R}(\mathcal{H}) / \sim_{\mathrm{sep}} \models q \le p$.*

The only significant difference between Definition 62 and the forcing notion called "$\mathcal{R}(\mathcal{H})$" in [3, Definition 3.1], which is itself very closed based on the original forcing notion from [2], is that condition (iii) is not required to be end-extension as in $\mathcal{Q}(\mathcal{H})$. This weakens the compatibility relation as follows.

Proposition 66. *p and q are compatible in $\mathcal{R}(\mathcal{H})$ iff $x_p \cup x_q \in \partial_{\mathcal{H}}(\mathcal{J})$ for all $\mathcal{J} \in \mathcal{X}_p \cup \mathcal{X}_q$.*

Condition (iii), however, leaves many properties unaffected. For example, the following facts can be established with exactly the same proofs as in [3]. Assume for now that $\mathcal{H}$ is a σ-directed subfamily of $([S]^{\aleph_0}, \subseteq^*)$, where S is some uncountable set.

Lemma 67. *$\mathcal{R}(\mathcal{H})$ is* a*-proper.*

Note that, unlike with $\mathcal{Q}(\mathcal{H})$, we do not need any additional requirements on $\mathcal{H}$ for properness as in Lemma 57.

Lemma 68. *$\mathcal{R}(\mathcal{H})$ is $\mathbb{D}$-complete.*

Lemma 69. *$\mathcal{R}(\mathcal{H})$ satisfies the properness isomorphism condition.*

Lemma 70. *If S cannot be decomposed into countably many pieces that are orthogonal to $\mathcal{H}$, then $\mathcal{R}(\mathcal{H})$ forces that $X_{\dot{G}_{\mathcal{R}(\mathcal{H})}}$ is uncountable.*

The following is established in [3].

Lemma 71. *Let θ be an ordinal of uncountable cofinality, and let $\mathcal{H}$ be a σ-directed subfamily of $([\theta]^{\aleph_0}, \subseteq^*)$. Let $S \subseteq \theta$ be stationary. If S has no stationary subset orthogonal to $\mathcal{H}$, then $\mathcal{R}(\mathcal{H})$ forces that $X_{\dot{G}_{\mathcal{R}(\mathcal{H})}} \cap S$ is stationary.*

4. Absolute antichains

Suppose $\mathcal{H}$ is a subfamily of $[\theta]^{\aleph_0}$. Suppose that W is an outer model of V. Since $\mathcal{Q}$ and $\mathcal{R}$ are considered as classes, we can interpret $\mathcal{Q}(\mathcal{H})$ and $\mathcal{R}(\mathcal{H})$ in W. And by Propositions 40 and 64, respectively, we have

$$\mathcal{Q}(\mathcal{H})^V \subseteq \mathcal{Q}(\mathcal{H})^W \quad \text{and} \quad \mathcal{R}(\mathcal{H})^V \subseteq \mathcal{R}(\mathcal{H})^W, \tag{21}$$

and thus for $\mathcal{O} = \mathcal{Q}, \mathcal{R}$, $\mathcal{O}(\mathcal{H})^V$ is a suborder of $\mathcal{O}(\mathcal{H})^W$ (recall that P is a *suborder* of Q when $P \subseteq Q$ and $p \leq_P q \leftrightarrow p \leq_Q q$ for all $p, q \in P$). Since $(x_p \cup x_q, \mathcal{X}_p \cup \mathcal{X}_q)$ is a common extension of any two compatible conditions p and q, it follows that '$p \perp q$' is absolute between V and W, for either forcing notion. Here we are interested in having $\mathcal{O}(\mathcal{H})^V$ generically included in $\mathcal{O}(\mathcal{H})^W$ – see Definition 72 where we obtain an approximation to this property – and therefore we are interested in the upwards absoluteness of ⌜A is a maximal antichain of $\mathcal{O}(\mathcal{H})$⌝ for both classes of forcing notions $\mathcal{O} = \mathcal{Q}, \mathcal{R}$.

This is a familiar scenario. The concept of *Souslin forcing* was introduced in [5], concerning a certain class of forcing notions that can be represented as definable subsets of the reals, or more generally, definable over $(H_{\aleph_1}, \in)$. These Souslin forcing notions can thus be interpreted in any outer model, and they enjoy many nice absoluteness properties, which are particularly useful in the iteration of Souslin forcing notions. For example, the maximality of antichains of Souslin ccc forcing notions is upwards absolute. In our case, say for $\theta = \omega_1$ and assuming CH, our forcing notions are $\aleph_2$-cc and representable as subsets of $\mathcal{P}(\omega_1)$, and simply definable over $(H_{\aleph_2}, \in)$ by Propositions 40 and 64, respectively; and we also want to establish the upwards absoluteness of antichains. However, in the present case we shall rely on combinatorial arguments rather than absoluteness results of second order arithmetic. In the process, we shall observe that $\mathcal{R}(\mathcal{H})$ and $\mathcal{Q}(\mathcal{H})$ have very nice properties, such as commutativity, that are typically associated with certain Souslin forcing notions.

4.1. Embeddings

We write $P \preccurlyeq Q$ to specify that a forcing notion P *generically embeds* into a forcing notion Q, which as usual we mean that for all $G \in \mathrm{Gen}(V, Q)$, $V[G] \models$ ⌜$\mathrm{Gen}(V, P) \neq \emptyset$⌝, i.e. every generic for Q induces a generic for P. A *generic embedding* between two forcing notions has the usual meaning, i.e. they are called *complete embeddings* in [6, Ch. VII, Section 7]. We write $P \cong Q$ to indicate that P and Q are isomorphic as forcing notions, i.e. $P \preccurlyeq Q$ and $Q \preccurlyeq P$. Recall that $P \preccurlyeq Q$ iff there exists a *generic embedding* $e : P / \sim_{\mathrm{sep}} \to Q$, where the separative quotient is indicated in the domain. Also recall that when P is a suborder of Q, the inclusion map $i : P \to Q$ is a generic embedding iff $i[A]$ is a maximal antichain of Q for every maximal antichain A of P. We want to generalize the notion $P \preccurlyeq Q$, where Q is allowed to be outside of some universe.

Definition 72. Let M be a model (typically transitive), and let P and Q be forcing notions with $P \in M$. We say that P *generically embeds into Q over M* if for all $G \in \mathrm{Gen}(V, Q)$, $V[G] \models$ ⌜$\mathrm{Gen}(M, P) \neq \emptyset$⌝. We write $P \preccurlyeq_M Q$. We say that P is *generically included in Q over M* if P is a suborder of Q that generically embeds over M. We write $P \preccurlyeq^i_M Q$. And $e : P \to Q$ is a *generic embedding over M* if it is order preserving, i.e. $q \geq p$ implies $e(q) \geq e(p)$, $q \perp p$ implies $e(q) \perp e(p)$ and for every maximal antichain A of P in M, $e[A]$ is a maximal antichain of Q.

Thus $V \models$ ⌜$P \preccurlyeq_V Q$⌝ iff $V \models$ ⌜$P \preccurlyeq Q$⌝.

Proposition 73. *$P \preccurlyeq^i_M Q$ iff the inclusion map i is a generic embedding over M.*

Notation 74. *For a separative poset P, let $\bar{P}$ denote its completion.*

Lemma 75. *Let M be a transitive model of enough of* ZFC. *Then $P \preccurlyeq_M Q$ iff there exists $e : P \to \overline{Q / \sim_{\mathrm{sep}}}$ such that e is a generic embedding over M.*

Proof. If $e : P \to \overline{Q / \sim_{\mathrm{sep}}}$ is a generic embedding over M, and $G \in \mathrm{Gen}(V, \overline{Q / \sim_{\mathrm{sep}}})$, then $e^{-1}[G] \subseteq P$ is downwards closed and upwards linked (i.e. pairwise compatible), and intersects every maximal antichain of P in M. It thus follows from the well-known facts that $e^{-1}[G]$ generates a member of $\mathrm{Gen}(M, P)$. Thus $P \preccurlyeq_M \overline{Q / \sim_{\mathrm{sep}}} \cong Q$.

Conversely, if $P \preccurlyeq_M Q$ then there is a Q-name $\dot{G}$ for a member of $\mathrm{Gen}(M, P)$. Then $e : P \to \overline{Q / \sim_{\mathrm{sep}}}$ defined by

$$e(p) = \| p \in \dot{G} \| \tag{22}$$

is a generic embedding over M. □

Proposition 76. *Suppose that $P, Q \in M$ and M is a model of enough of* ZFC. *If $e : P \to Q$ is a generic embedding over M, and $H \in \mathrm{Gen}(M, Q)$, then $e^{-1}[H] \in \mathrm{Gen}(M, P)$.*

Recall the following basic fact.

Proposition 77. *If $e : P \to Q$ is a generic embedding, then every $q \in Q$ has a $p_q \in P$ such that $e(p')$ is compatible with q for all $p' \geq p_q$.*

Lemma 78. *Let P and Q be separative forcing notions with $P \in M$, where M is a transitive model of enough of* ZFC. *If $e : P \to Q$ is a generic embedding over M, then so is $\tilde{e} : P / \sim_{\text{sep}} \to Q / \sim_{\text{sep}}$ given by $\tilde{e}([p]) = [e(p)]$.*

Proof. First we observe that if $P / \sim_{\text{sep}} \models p \geq q$ then $Q / \sim_{\text{sep}} \models e(p) \geq e(q)$. For suppose to the contrary that $Q / \sim_{\text{sep}} \models e(p) \not\geq e(q)$. Then there exists $r \geq e(p)$ in Q that is Q-incompatible with $e(q)$. In M, let A be a maximal antichain with $q \in A$. Then since $e[A]$ is a maximal antichain, there exists $q' \in A$ such that $e(q')$ is Q-compatible with r. If $q' \perp q$ then $q' \perp p$ as $P / \sim_{\text{sep}} \models p \geq q$, and hence $e(q') \perp e(p)$ implies $e(q') \perp r$. But then $q' \in A$ implies $q' = q$, contradicting that $e(q) \perp r$.

The preceding observation obviously implies that $\tilde{e}$ is well defined and order preserving. That $\tilde{e}$ preserves maximal antichains follows immediately from the fact that $p \perp q$ implies $[p] \perp [q]$. □

Proposition 79. *Let P and Q be separative forcing notions with $P \in M$, where M is a transitive model of enough of* ZFC. *If $e : P \to Q$ is a generic embedding over M, then so is $\bar{e} : \bar{P} \to \bar{Q}$ given by $\bar{e}(\bar{p}) = \bigwedge\{e(p) : p \in P, p \geq \bar{p}\}$.*

The notion of a *projection* is used in [1] as a map from Q into P witnessing $P \preccurlyeq Q$. We weaken the requirements on projections here for brevity, but only use them as inverses of generic embeddings (noting that it would have been better to do it the former way).

Definition 80. We say that $\pi : Q \to P$ is a *projection* if π is an order preserving surjection.

Definition 81. Let κ be a cardinal. A forcing notion P is said to be *κ-semicomplete* if every $A \subseteq P$ of cardinality $|A| < \kappa$, with an upper bound in P, has a minimal upper bound in P. It is *semicomplete* if it is κ-semicomplete for all cardinals κ.

In the case were P is a poset (and not just a quasi-order), minimal upper bounds are suprema. Then semicomplete is precisely the order theoretic notion of a complete semilattice. Also note that complete semilattices always have a minimum element, namely $\bigvee \emptyset$. Recall that a poset P is *pointed* if it has a minimum element, which denote as 0_P.

Example 82. *$\mathcal{Q}(\mathcal{H})$ and $\mathcal{R}(\mathcal{H})$ are both complete semilattices.* If $A \subseteq \mathcal{R}(\mathcal{H})$ and $a \leq p$ for all $a \in A$, then clearly $\bigvee A = \left(\bigcup_{a\in A} x_a, \bigcup_{a\in A} \mathcal{X}_a\right)$. Similarly for $\mathcal{Q}(\mathcal{H})$, but taking the closure of $\bigcup_{a\in A} x_a$.

Definition 83. Recall that a subset A of a poset P *upwards order closed* if whenever $B \subseteq A$ is nonempty, if $a = \bigvee B$ exists when taken in P, then $a \in A$.

The above notion is not to be confused with an *upwards closed* subset, also called an *upper set*.

Recall that $e : P \to q$ is an *order embedding* between to quasi-orders if it is both order preserving and reflecting, i.e. $p \leq q$ iff $e(p) \leq e(q)$ for all $p, q \in P$. For a poset, this means that e is isomorphic to its range.

Lemma 84. *Suppose that P is a pointed poset and Q is a complete semilattice. If $e : P \to Q$ is an order embedding with an upwards order closed range, then it has a projection $\pi : Q \to P$ for a left inverse, given by*

$$\pi(q) = \bigvee\{p \in P : e(p) \leq q\}. \tag{23}$$

Proof. To check that the supremum always exists, take $q \in Q$. Put $A = \{p \in P : e(p) \leq q\}$. In the case $A = \emptyset$, the supremum is 0_P. Otherwise, since q is an upper bound of $e[A]$, $a = \bigvee e[A] = \bigvee_{p\in A} e(p)$ exists in Q since it is a complete semilattice; and then since $e[P]$ is upwards order closed, there exists $p' \in P$ such that $e(p') = a$. Then p' is an upper bound of A since e is order reflecting. And if $r \in P$ is an upper bound of A, then $e(r)$ is an upper bound of $e[A]$ since e is order preserving, and thus $e(p') = a \leq e(r)$ implies $p' \leq r$, proving that p' is the least upper bound.

It is clear that π is an order preserving left inverse of e. (And obviously π is a surjection when it is a left inverse.) □

Example 85. Now let us see how this applies to say $\mathcal{Q}(\mathcal{H})$. Let W be a transitive outer model of V which has the same countable sequences of ordinals as V. $\mathcal{Q}(\mathcal{H})$ is a complete semilattice in V, and thus, in W, $\mathcal{Q}(\mathcal{H})^V$ is also a complete semilattice by the assumption on W, because in Example 82 we showed that the suprema are given by countable unions which thus remain in V. The fact that suprema remain in V, also implies that $\mathcal{Q}(\mathcal{H})^V$ is upwards order closed in $\mathcal{Q}(\mathcal{H})^W$. Therefore, Lemma 84 applies in W to the identity mapping $i : \mathcal{Q}(\mathcal{H})^V \to \mathcal{Q}(\mathcal{H})^W$, yielding a projection $\pi : \mathcal{Q}(\mathcal{H})^W \to \mathcal{Q}(\mathcal{H})^V$ in W that is the identity on $\mathcal{Q}(\mathcal{H})^V$. Exactly analogous facts hold for $\mathcal{R}(\mathcal{H})$.

Notation 86. *For any map $e : P \to Q$ let e^* be the corresponding mapping of P-names to Q-names (cf. e.g. [6, Ch. VII, 7.12]).*

Proposition 87. *Let $P \subseteq Q$. If $i : P \to Q$ is the inclusion map then i^* is an inclusion map.*

Proposition 88. *Let $M \prec H_\lambda$ with $P, Q \in M$. Suppose that $e : P \to Q$ is a generic embedding in M*[3] *and that Eq. (23) defines $\pi : Q \to P$ as a left inverse of e. If $q \in \text{gen}(M, Q)$, then $\pi(q) \in \text{gen}(M, P)$; and if $q \in \text{gen}^+(M, Q)$, then $\pi(q) \in \text{gen}^+(M, P)$.*

[3] *Not* over M.

Lemma 89. *Let $M \prec H_\lambda$ with $P, Q \in M$ and λ sufficiently large and regular. Suppose that $e : P \to Q$ is a generic embedding in M and Eq. (23) defines a left inverse $\pi : Q \to P$ of e, $\varphi(v_0, \dots, v_{n-1})$ is a formula which is absolute for transitive models of* ZFC *and $\dot{x}_0, \dots, \dot{x}_{n-1} \in M$ are P-names. Then for all $q \in \operatorname{gen}(M, Q)$, $\pi(q) \in \operatorname{gen}(M, P)$ and*

$$\pi(q) \Vdash \varphi(\dot{x}_0, \dots, \dot{x}_{n-1}) \quad \text{iff} \quad q \Vdash \varphi(e^*(\dot{x}_0), \dots, e^*(\dot{x}_{n-1})). \tag{24}$$

Proof. See e.g. [6, Ch. VII, 7.13]. □

Next we recall some basic forcing facts on maximal antichains and generic embeddings.

Proposition 90. *For $A \subseteq P \star \dot{Q}$, let A / P be the P-name $\{\dot{q} : (p, \dot{q}) \in A, p \in \dot{G}_P\}$ for a subset of $\dot{Q}$. Then the following are equivalent:*

(a) *$A \subseteq P \star \dot{Q}$ is a maximal antichain.*
(b) *$P \Vdash A / P$ is a maximal antichain of $\dot{Q}$.*

Proposition 91. *Suppose $R \Vdash \dot{Q} \preccurlyeq \dot{P}$. Then $\dot{P} / \dot{Q} \cong (R \star \dot{P}) / (R \star \dot{Q})$.*

Remark. Both sides of the equivalence in Proposition 91 are $R \star \dot{Q}$-names, and thus the equivalence should of course be interpreted as $R \star \dot{Q} \Vdash \dot{P} / \dot{Q} \cong (R \star \dot{P}) / (R \star \dot{Q})$. We shall apply the equivalent statement

$$R \star \dot{P} \cong R \star \dot{Q} \star (\dot{P} / \dot{Q}). \tag{25}$$

Notation 93. *For a forcing notion P and $p \in P$, we let P_p denote the principle filter $\{q \in P : q \geq p\}$.*

Proposition 94. *Assume $Q \Vdash P \preccurlyeq_V \dot{R}$. Then $P \preccurlyeq Q \star \dot{R}$. Moreover if Q has a minimum element 0_Q and $Q \Vdash \ulcorner \dot{e} : P \to \dot{R}$* is a generic embedding over $V\urcorner$*, then $p \mapsto (0_Q, \dot{e}(p))$ defines a generic embedding; hence, if $A \subseteq P$ is a maximal antichain then*

$$(Q \star \dot{R}) / P \cong \coprod_{p \in A} \left(Q \star \dot{R}_{\dot{e}(p)}\right) / P_p. \tag{26}$$

Proof. Let $G \in \operatorname{Gen}(V, P)$ and $H \in \operatorname{Gen}(V[G], \dot{R}[G])$. In $V[G][H]$, $\operatorname{Gen}(V, P) \neq \emptyset$, because $P \preccurlyeq_V \dot{R}[G]$. Hence $P \preccurlyeq Q \star \dot{R}$ by definition. It is immediate from Proposition 90 that the defined mapping is a generic embedding. □

The following lemma is well known, at least for the case $M = V$.

Lemma 95. *Let $(P_\xi, \dot{Q}_\xi : \xi < \alpha)$ and $(P'_\xi, \dot{Q}'_\xi : \xi < \alpha)$ both be iterated forcing constructs with resulting forcing notions P_α and P'_α, respectively; and let M be a transitive model of enough of* ZFC*. If $P_\xi \preccurlyeq^i_M P'_\xi$ for all $\xi < \alpha$, then $P_\alpha \preccurlyeq^i_M P'_\alpha$.*

Definition 96. When we say that a forcing notion P is *densely included* in a forcing notion Q, we mean that P is a predense suborder of Q.

Proposition 97. *If P is a suborder of Q and $P \cong Q$ then P is densely included in Q.*

Lemma 98. *Let $(P_\xi, \dot{Q}_\xi : \xi < \alpha)$ and $(P'_\xi, \dot{Q}'_\xi : \xi < \alpha)$ both be iterated forcing constructs with resulting forcing notions P_α and P'_α, respectively. If P_ξ is densely included in P'_ξ for all $\xi < \alpha$, then $P_\alpha \cong P'_\alpha$.*

The next lemma is important for computing quotients.

Lemma 99. *Let O and $P \star \dot{Q}$ be forcing notions such that $O \preccurlyeq P \star \dot{Q}$. Suppose that for all $G \in \operatorname{Gen}(V, O)$, there exists $f : (P \star \dot{Q}) / G \to P$ such that*

(a) *$f(p, \dot{q}) \geq p$,*
(b) *$(p', \dot{q}) \in (P \star \dot{Q}) / G$ for all $p' \geq f(p, \dot{q})$,*
(c) *$(f(p, \dot{q}), \dot{q})$ and $(f(p', \dot{q}'), \dot{q}')$ are $(P \star \dot{Q})$-compatible whenever $f(p, \dot{q})$ and $f(p', \dot{q}')$ are compatible.*

Then $(P \star \dot{Q}) / O \cong P$.

Proof. Let $G \in \operatorname{Gen}(V, O)$. Let $D = \{(f(p, \dot{q}), \dot{q}) : (p, \dot{q}) \in (P \star \dot{Q}) / G\}$. Then D is a dense subset of $(P \star \dot{Q}) / G$ by (a) and (b). And by (b) and (c), for all $(p, \dot{q}), (p', \dot{q}') \in (P \star \dot{Q}) / G$, $(f(p, \dot{q}), \dot{q})$ is $(P \star \dot{Q}) / G$-compatible with $(f(p', \dot{q}'), \dot{q}')$ iff $f(p, \dot{q})$ is P-compatible with $f(p', \dot{q}')$. Therefore, for all $(p, \dot{q}), (p', \dot{q}') \in D$, $(p, \dot{q})$ and $(p', \dot{q}')$ are $(P \star \dot{Q}) / G$-compatible iff p and p' are compatible. Hence, $(p, \dot{q}) \mapsto p$ is an isomorphism between $D / \sim_{\text{sep}}$ and $P / \sim_{\text{sep}}$. Thus, as D is dense, $((P \star \dot{Q}) / G) / \sim_{\text{sep}} \cong D / \sim_{\text{sep}} \cong P / \sim_{\text{sep}}$. Since G is arbitrary, this proves that $O \Vdash (P \star \dot{Q}) / O \cong P$, as required. □

4.2. *Analysis of $\mathcal{Q}(\mathcal{H})$ and $\mathcal{R}(\mathcal{H})$*

To obtain generic embeddings of e.g. $\mathcal{Q}(\mathcal{H})^V$ into $\mathcal{Q}(\mathcal{H})^W$, we shall analyze the maximal antichains of $\mathcal{Q}(\mathcal{H})$ and $\mathcal{R}(\mathcal{H})$. Clearly, if $A \subseteq \mathcal{Q}(\mathcal{H})$ is a maximal antichain then $\pi[A] = \{x_p : p \in A\}$ must be predense in $(\mathcal{H}, \subseteq)$; however, it need not form an antichain. For example, suppose that $\mathcal{H}$ is a P-ideal (and thus closed under addition of finite sets), $y \in \partial_{\mathcal{H}}(\mathcal{J})$ is closed and countable, and $\alpha > \max(y)$ is not in $\partial_{\mathcal{H}}(\mathcal{J})$ but $\mathcal{K}$ is a $\subseteq^*$-cofinal subset of $\mathcal{H}$ with $\alpha \in \partial_{\mathcal{H}}(\mathcal{K})$. Then $(y, \{\mathcal{J}\})$ and $(y \cup \{\alpha\}, \{\mathcal{K}\})$ are incompatible conditions of $\mathcal{Q}(\mathcal{H})$ even though $y \sqsubseteq y \cup \{\alpha\}$. Indeed, in analyzing the sets $\pi[A]$, the difficulty is when y is in the set and we want to determine whether some $z \sqsubset y$ is also present in $\pi[A]$.

For any $\mathcal{H} \subseteq [\theta]^{\aleph_0}$, the following auxiliary family of countable subsets of $\mathcal{H}$ allows us to analyze the maximality of antichains in $\mathcal{Q}(\mathcal{H})$ and $\mathcal{R}(\mathcal{H})$.

Definition 100. Let S be a set, and $\mathcal{H} \subseteq \mathcal{P}(S)$. Define a subcollection $\Psi(\mathcal{H}) \subseteq [\mathcal{H}]^{\le\aleph_0}$ of all $Z \in [\mathcal{H}]^{\le\aleph_0}$ for which there exists $y \in \mathcal{H}$ such that every finite $s \subseteq y$ has a finite $A_s \subseteq Z$ such that

$$\bigcup Z \setminus A_s \subseteq y \setminus s. \tag{27}$$

In particular, whenever $\mathcal{H}$ is an ideal, $\bigcup Z \in \mathcal{H}$ for all $Z \in \Psi(\mathcal{H})$.

Lemma 101. *Let $\mathcal{H}$ be a σ-directed subfamily of $([S]^{\aleph_0}, \subseteq^*)$. Then $\Psi(\mathcal{H})$ is a P-ideal on $\mathcal{H}$.*

Proof. Assume that $\mathcal{H}$ is σ-directed. To verify that $\Psi(\mathcal{H})$ is σ-directed, let Z_n $(n < \omega)$ enumerate a subset of $\Psi(\mathcal{H})$, with $y_n \in \mathcal{H}$ witnessing $Z_n \in \Psi(\mathcal{H})$ for each $n < \omega$. Since $\mathcal{H}$ is σ-directed, there exists $z \in \mathcal{H}$ with $y_n \subseteq^* z$ for all $n < \omega$. Choose finite subsets $t_n \subseteq S$ $(n < \omega)$ so that

(28) $y_n \setminus t_n \subseteq z$,
(29) $\bigcup_{n<\omega} t_n \supseteq z$,

and put $s_n = y_n \cap \bigcup_{i=0}^n t_i$ for each n. Let A_{s_n} be the finite subset from Eq. (27) so that $\bigcup Z_n \setminus A_{s_n} \subseteq y_n \setminus s_n \subseteq z$. Putting $Y = \bigcup_{n<\omega} Z_n \setminus A_{s_n}$, $Z_n \subseteq^* Y$ for all $n < \omega$. And $Y \in \Psi(\mathcal{H})$, because for any finite $u \subseteq z$, we can find n so that $u \subseteq \bigcup_{i=0}^n t_i$, and then $\bigcup\big(Y \setminus \bigcup_{i=0}^n A_{y_i \cap u}\big) \subseteq z \setminus u$, where each $A_{y_i\cap u}$ is from Eq. (27) with $y := y_i$ and $Z := Z_i$.

Moreover, $\Psi(\mathcal{H})$ is obviously downwards closed, and it is an ideal because $\mathcal{H}$ is directed. □

Lemma 102. *Let $\mathcal{H}$ be a downwards closed σ-directed subfamily of $[S]^{\aleph_0}$. Then for $\mathcal{K} \subseteq \mathcal{H}$, the following are equivalent:*

(a) *There exists a countable decomposition of $\mathcal{K}$ into pieces orthogonal to $\Psi(\mathcal{H})$.*
(b) *There exists a countable family $\mathcal{X}$ of cofinal subsets of $(\mathcal{H}, \subseteq^*)$ such that $\mathcal{K} \cap \bigcap_{\mathcal{J}\in\mathcal{X}} \partial_{\mathcal{H}}(\mathcal{J}) \subseteq \{\emptyset\}$.*

Proof. (a)→(b): Let $\mathcal{K} = \bigcup_{n<\omega} \mathcal{L}_n$ with each $\mathcal{L}_n \perp \Psi(\mathcal{H})$. Observe that every $y \in \mathcal{H}$ has a finite $s_{yn} \subseteq y$ such that $\downarrow(y \setminus s_{yn}) \cap \mathcal{L}_n \subseteq \{\emptyset\}$: Otherwise, letting $(\alpha_k : k < \omega)$ enumerate y, there exists $z_k \in \mathcal{L}_n$ with $\emptyset \ne z_k \subseteq y \setminus \{\alpha_0, \dots, \alpha_k\}$ for all $k < \omega$, and then $\{z_k : k < \omega\} \in [\mathcal{L}_n]^{\aleph_0} \cap \Psi(\mathcal{H})$, contrary to the fact that $\mathcal{L}_n \perp \Psi(\mathcal{H})$. Now for each n, let $\mathcal{J}_n = \{y \setminus s_{yn} : y \in \mathcal{H}\}$. Then every $\mathcal{J}_n$ is a cofinal subset of $\mathcal{H}$ as $\mathcal{H}$ is downwards closed, and $\mathcal{K} \cap \bigcap_{n<\omega} \partial_{\mathcal{H}}(\mathcal{J}_n) \subseteq \mathcal{K} \cap \bigcap_{n<\omega} \downarrow\mathcal{J}_n \subseteq \{\emptyset\}$.

(b)→(a): Let $\mathcal{X} = \{\mathcal{J}_0, \mathcal{J}_1, \dots\}$ be as in (b). Since $\mathcal{H}$ is σ-directed, the noncofinal subsets of $\mathcal{H}$ form a σ-ideal; therefore, each n and each $y \in \mathcal{H}$ has a finite $s_{yn} \subseteq y$ such that $y \setminus s_{yn} \in \partial_{\mathcal{H}}(\mathcal{J}_n)$. Then putting $\mathcal{K}_n = \{y \setminus s_{yn} : y \in \mathcal{H}\}$ $(n < \omega)$ we get $\mathcal{K} \cap \bigcap_{n<\omega} \downarrow\mathcal{K}_n \subseteq \{\emptyset\}$. For each n, let $\mathcal{L}_n = \mathcal{K} \setminus \downarrow\mathcal{K}_n$. Then $\mathcal{K} \setminus \{\emptyset\} = \bigcup_{n<\omega} \mathcal{L}_n$, and each $\mathcal{L}_n \perp \Psi(\mathcal{H})$, because $\mathcal{K}_n$ is cofinal, and hence if there were a $Z \in [\mathcal{L}_n]^{\aleph_0} \cap \Psi(\mathcal{H})$ witnessed by some $y \in \mathcal{H}$, then for any $z \in \mathcal{K}_n$ with $y \subseteq^* z$ we arrive at the contradiction that $\mathcal{L}_n \cap \downarrow z \ne \emptyset$. □

Notation 103. *For any $\mathcal{S} \subseteq \mathcal{P}(\theta)$ and $x \subseteq \theta$, denote $\mathcal{S}_x = \{y \in \mathcal{S} : x \sqsubseteq y\}$ and $\mathcal{S}_{[x]} = \{y \setminus x : y \in \mathcal{S}_x\}$. Analogously, we denote $\mathcal{S}_{(x)} = \{y \setminus x : y \in \mathcal{S} \cap \uparrow x\}$ $(\mathcal{S} \cap \uparrow x = \{y \in \mathcal{S} : x \subseteq y\})$.*

Proposition 104. *If $\mathcal{H}$ is a P-ideal then $\mathcal{H}_{[x]}$ and $\mathcal{H}_{(x)}$ are both P-ideals for all x.*

Remark. $\mathcal{H}_{[x]}$ and $\mathcal{H}_{(x)}$ may fail to be σ-directed if $\mathcal{H}$ is not a P-ideal, even if $\mathcal{H}$ is σ-directed.

Next we see how to 'freeze' a maximal antichain A, so that the statement ⌜A is a maximal antichain of $\mathcal{Q}(\mathcal{H})$⌝ is upwards absolute.

We are assuming, until Section 4.2.1, that $\mathcal{H}$ is a P-ideal on θ.

Corollary 106. *Let A be a maximal antichain of $\mathcal{Q}(\mathcal{H})$, and suppose $W \supseteq V$ is an outer model with $([\theta]^{\aleph_0})^V = ([\theta]^{\aleph_0})^W$. If for every $x \in \mathcal{H}$, either*

(a) *there exists in V, a countable decomposition of $\pi[A]_{[x]}$ into pieces orthogonal to $\Psi(\mathcal{H}_{[x]})$, or*
(b) *there does not exist in W, a countable decomposition of $\pi[A]_{[x]}$ into pieces orthogonal to $\Psi(\mathcal{H}_{[x]})$,*

then A is a maximal antichain of $\mathcal{Q}(\mathcal{H})^W$.

Proof. In W: We have already observed that A is an antichain of $\mathcal{Q}(\mathcal{H})^W$, and hence it remains to establish its maximality. By assumption, $x_p \in V$ for all $p \in \mathcal{Q}(\mathcal{H})^W$. Fix $p \in \mathcal{Q}(\mathcal{H})^W$, and take any $\bar{p} \in \mathcal{Q}(\mathcal{H})^V$ with $x_{\bar{p}} = x_p$.

Case 1. $x_q \sqsubseteq x_p$ for some $q \in A$ compatible with $\bar{p}$.

Then from the definition of $\mathcal{Q}(\mathcal{H})$, $x_p \in \partial_{\mathcal{H}}(\mathcal{J})$ for all $\mathcal{J} \in \mathcal{X}_q$, and conversely this proves that p is also compatible with q, as required.

Case 2. There is no $q \in A$ compatible with $\bar{p}$ such that $x_q \sqsubseteq x_p$.

Observe that, in V, there is no countable decomposition of $\pi[A]_{[x_p]}$ into pieces orthogonal to $\Psi(\mathcal{H}_{[x_p]})$: For if there was, then by Lemma 102, and also Proposition 104 using the fact that $\mathcal{H}$ is a P-ideal, there would be a countable family $\mathcal{X}$ of cofinal subsets of $\mathcal{H}_{[x_p]}$ with $\pi[A]_{[x_p]} \cap \bigcap_{\mathcal{J}\in\mathcal{X}} \partial_{\mathcal{H}_{[x_p]}}(\mathcal{J}) \subseteq \{\emptyset\}$; then $p' = \left(x_p, \mathcal{X}_{\bar{p}} \cup \left\{\{x_p \cup y : y \in \mathcal{J}\} : \mathcal{J} \in \mathcal{X}\right\}\right)$ is a condition of $\mathcal{Q}(\mathcal{H})$ extending $\bar{p}$. Since we are in Case 2, by maximality in V there would exist $q \in A$ compatible with p' with $x_p \sqsubset x_q$. However, this would entail that $x_q \setminus x_p \in \pi[A]_{[x_p]} \cap \bigcap_{\mathcal{J}\in\mathcal{X}} \partial_{\mathcal{H}_{[x_p]}}(\mathcal{J})$, contradicting the fact that $x_q \setminus x_p \neq \emptyset$.

Therefore, by the hypothesis of the corollary, in W there can be no countable decomposition of $\pi[A]_{[x_p]}$ into pieces orthogonal to $\Psi(\mathcal{H}_{[x_p]})$. Therefore, Lemma 102 implies that

$$\pi[A]_{[x_p]} \cap \bigcap_{\mathcal{J}\in\mathcal{X}_p} \partial_{\mathcal{H}_{[x_p]}}(\mathcal{J}_{[x_p]}) \neq \emptyset, \tag{30}$$

say y is in the intersection. Then $y \in \pi[A]_{[x_p]}$ means that there exists $q \in A$ such that $x_p \sqsubseteq x_q$ and $y = x_q \setminus x_p$. And Eq. (30) implies that $x_q \in \partial_{\mathcal{H}}(\mathcal{J})$ for all $\mathcal{J} \in \mathcal{X}_p$, and thus q is compatible with p as required, because $x_p \sqsubseteq x_q$. □

Next we establish the analogous result for $\mathcal{R}(\mathcal{H})$.

Corollary 107. *Let A be a maximal antichain of $\mathcal{R}(\mathcal{H})$, and suppose $W \supseteq V$ is an outer model with $([\theta]^{\aleph_0})^V = ([\theta]^{\aleph_0})^W$. If for every $x \in \mathcal{H}$, either*

(a) *there exists in V, a countable decomposition of $\pi[A]_{(x)}$ into pieces orthogonal to $\Psi(\mathcal{H}_{(x)})$, or*
(b) *there does not exist in W, a countable decomposition of $\pi[A]_{(x)}$ into pieces orthogonal to $\Psi(\mathcal{H}_{(x)})$,*

then A is a maximal antichain of $\mathcal{R}(\mathcal{H})^W$.

Proof. In W: We have already observed that A is an antichain of $\mathcal{R}(\mathcal{H})^W$, and hence it remains to establish its maximality. By assumption, $x_p \in V$ for all $p \in \mathcal{R}(\mathcal{H})^W$. Fix $p \in \mathcal{R}(\mathcal{H})^W$, and take any $\bar{p} \in \mathcal{R}(\mathcal{H})^V$ with $x_{\bar{p}} = x_p$.

Case 1. $x_q \subseteq x_p$ for some $q \in A$ compatible with $\bar{p}$.

Then applying Proposition 66 to $\bar{p}$ and q, $x_p \in \partial_{\mathcal{H}}(\mathcal{J})$ for all $\mathcal{J} \in \mathcal{X}_q$, and conversely this proves that p is also compatible with q, as required.

Case 2. There is no $q \in A$ compatible with $\bar{p}$ such that $x_q \subseteq x_p$.

Observe that, in V, there is no countable decomposition of $\pi[A]_{(x_p)}$ into pieces orthogonal to $\Psi(\mathcal{H}_{(x_p)})$: For if there was, then by Lemma 102 there would be a countable family $\mathcal{X}$ of cofinal subsets of $\mathcal{H}_{(x_p)}$ with $\pi[A]_{(x_p)} \cap \bigcap_{\mathcal{J}\in\mathcal{X}} \partial_{\mathcal{H}_{(x_p)}}(\mathcal{J}) \subseteq \{\emptyset\}$; then $p' = \left(x_p, \mathcal{X}_{\bar{p}} \cup \left\{\{x_p \cup y : y \in \mathcal{J}\} : \mathcal{J} \in \mathcal{X}\right\}\right)$ is a condition of $\mathcal{R}(\mathcal{H})$ extending $\bar{p}$. Since we are in Case 2, by maximality in V there would exist $q \in A$ compatible with p' with $x_q \nsubseteq x_q$. However, this would entail that $x_q \setminus x_p \in \pi[A]_{(x_p)} \cap \bigcap_{\mathcal{J}\in\mathcal{X}} \partial_{\mathcal{H}_{(x_p)}}(\mathcal{J})$, contradicting the fact that $x_q \setminus x_p \neq \emptyset$.

Therefore, by the hypothesis of the corollary, in W there can be no countable decomposition of $\pi[A]_{(x_p)}$ into pieces orthogonal to $\Psi(\mathcal{H}_{(x_p)})$. Therefore, Lemma 102 implies that

$$\pi[A]_{(x_p)} \cap \bigcap_{\mathcal{J}\in\mathcal{X}_p} \partial_{\mathcal{H}_{(x_p)}}(\mathcal{J}_{(x_p)}) \neq \emptyset, \tag{31}$$

say y is in the intersection. Then $y \in \pi[A]_{(x_p)}$ means that there exists $q \in A$ such that $x_p \subseteq x_q$ and $y = x_q \setminus x_p$. And Eq. (31) implies that $x_q \in \partial_{\mathcal{H}}(\mathcal{J})$ for all $\mathcal{J} \in \mathcal{X}_p$, and thus it follows from Proposition 66 that q is compatible with p, as required. □

Corollary 108 $(V \models (*))$. *Suppose that $W \supseteq V$ is an outer model with $([\theta]^{\aleph_0})^V = ([\theta]^{\aleph_0})^W$. Then $\mathcal{Q}(\mathcal{H})^V \preccurlyeq_V \mathcal{Q}(\mathcal{H})^W$ (and $\mathcal{R}(\mathcal{H})^V \preccurlyeq_V \mathcal{R}(\mathcal{H})^W$) via the inclusion map. Furthermore, in the case $\theta = \omega_1$, we can weaken the assumption to* CH $+ (*)_{\omega_1}$.

Proof. Lemma 101 and Corollary 106. In the case $\theta = \omega_1$, under CH, $\pi[A]$ has cardinality at most $\aleph_1$ for every antichain A. □

In particular, if $\mathcal{H}$ has no stationary orthogonal set, so that $\mathcal{Q}(\mathcal{H})$ is completely proper and thus adds no new countable subsets of θ, then assuming $(*)$, $\mathcal{Q}(\mathcal{H}) \Vdash \mathcal{Q}(\mathcal{H})^V \preccurlyeq_V \mathcal{Q}(\mathcal{H})$. Note that this is weaker than the statement $\mathcal{Q}(\mathcal{H}) \Vdash \mathcal{Q}(\mathcal{H})^V \preccurlyeq \mathcal{Q}(\mathcal{H})$, that would in particular imply $\mathcal{Q}(\mathcal{H}) \times \mathcal{Q}(\mathcal{H})$ is proper by Lemma 57, because $\mathcal{Q}(\mathcal{H})$ (in particular) forces that $\mathcal{H}$ has no stationary orthogonal set. This latter property, that is the square being proper, is the essence behind Shelah's NNR theory in [8, Ch. XVIII, Section 2]. We have thus been led to the following notion.

Definition 109. Let $\mathcal{C}$ be some class. Suppose that P is a forcing notion definable from some parameter a. We say that $P(a)$ is $\mathcal{C}$-*frozen* over a transitive model $M \ni P(a)$ of (enough of) ZFC, if for every outer model $N \supseteq M$ satisfying

(i) $N \models \ulcorner \mathrm{Gen}(M, P(a)^M) \neq \emptyset \urcorner$,
(ii) $\mathcal{C}^M = \mathcal{C}^N$,

we have $N \models \ulcorner P(a)^M \preccurlyeq^i_M P(a)^N \urcorner$.

In other words, in every outer model N extending some generic extension of M by P and preserving $\mathcal{C}$, $P(a)^M$ generically embeds via the identity into $P(a)^N$ over M.

Example 110. $(*)$ implies that $\mathcal{Q}(\mathcal{H})$ is $[\theta]^{\aleph_0}$-frozen over V, for any σ-directed $\mathcal{H} \subseteq [\theta]^{\aleph_0}$ with no stationary orthogonal set. In fact, Corollary 108 says that this is true for every outer model preserving $[\theta]^{\aleph_0}$, and not just those also satisfying (i).

To obtain a model of $(\star_c)_{\omega_1}$ with CH it is necessary (at least in our approach) that for every forcing notion appearing in our iteration, of the form $\mathcal{Q}(\mathcal{H})$, the property that $\ulcorner A$ is a maximal antichain of $\mathcal{Q}(\mathcal{H})\urcorner$ is upwards absolute for forcing extensions derived in various ways from the iteration (of course this will be made precise). Note that this entails preserving maximal antichains at every stage, because once the maximality of an antichain is lost it can never be restored.

So far we have demonstrated that it is possible to freeze antichains of $\mathcal{Q}(\mathcal{H})$ by forcing uncountable sets locally in the appropriate $\Psi(\mathcal{H}_{[x]})$. In fact, one can prove that $\mathcal{Q}(\mathcal{H})$ itself forces an uncountable set locally in each of the required $\Psi(\mathcal{H}_{[x]})$, and similarly for $\mathcal{R}(\mathcal{H})$. Thus we can strengthen Example 110 by eliminating $(*)$, as follows, although it should be noted that $(*)$ cannot be eliminated from Corollary 108.

Corollary 111. *If $\mathcal{H}$ is a P-ideal on θ with stationary orthogonal set then $\mathcal{Q}(\mathcal{H})$ is $[\theta]^{\aleph_0}$-frozen over V. Similarly, $\mathcal{R}(\mathcal{H})$ is $[\theta]^{\aleph_0}$-frozen over V for all P-ideals $\mathcal{H}$ on θ.*

However, this approach cannot even handle two-stage iterations. By this we mean that it may not be possible to freeze all antichains of say $\mathcal{R}(\mathcal{H}) \star \mathcal{R}(\dot{\mathcal{I}})$. This is in spite of Corollary 111: For suppose $A \subseteq \mathcal{R}(\mathcal{H}) \star \mathcal{R}(\dot{\mathcal{I}})$ is a maximal antichain. Let $G \star H \in \text{Gen}(V, \mathcal{R}(\mathcal{H}) \star \mathcal{R}(\dot{\mathcal{I}}))$. Then applying Corollary 111 in $V[G]$, $A\,/\,G$ (cf. Proposition 90) is frozen, which means that $A\,/\,G$ is a maximal antichain of $\mathcal{R}(\dot{\mathcal{I}}[G])$ in every outer model of $V[G]$ preserving $[\theta]^{\aleph_0}$. This does not however mean that the maximality of A is preserved because outer models of V need not contain G. What is needed, is an $\mathcal{R}(\mathcal{H})$-name for an uncountable set locally in $\Psi(\dot{\mathcal{I}}_{(x)})$, and we believe that this is generally impossible to obtain.

What has been achieved in this section with Corollaries 106 and 107, is that the problem of preserving the maximality of antichains of $\mathcal{Q}(\mathcal{H})$ and $\mathcal{R}(\mathcal{H})$ has been reduced to preserving the property that certain P-ideals have no countable decompositions of their underlying set into pieces orthogonal to them.

4.2.1. Products of P-ideals

Notation 112. *Let $(\mathcal{H}_i : i \in I)$ be an indexed family where each $\mathcal{H}_i$ is a family $\mathcal{H}_i \subseteq \mathcal{P}(S_i)$ of subsets of some fixed set S_i. Define*

$$\bigotimes_{i\in I} \mathcal{H}_i = \left\{ \coprod_{i\in I} x(i) : \vec{x} \in \prod_{i\in I} \mathcal{H}_i \right\}, \tag{32}$$

where $\coprod$ denotes the coproduct, i.e. disjoint union. Notice that $\bigotimes_{i\in I} \mathcal{H}_i \subseteq \mathcal{P}\left(\coprod_{i\in I} S_i\right)$.

Definition 113. Let $(X_i : i \in I)$ be an indexed family of sets, where each set has a zero $0_i \in X_i$. For $\vec{x} \in \prod_{i\in I} X_i$, write $\text{supp}(\vec{x}) = \{i \in I : x(i) \neq 0_i\}$. The *$\Sigma$-product* of $(X_i : i \in I)$ has the usual meaning:

$$\sum\left(\prod_{i\in I} X_i\right) = \left\{\vec{x} \in \prod_{i\in I} X_i : \text{supp}(\vec{x}) \text{ is countable}\right\}. \tag{33}$$

Notation 114. *Suppose that $\mathcal{H}_i \subseteq \mathcal{P}(S_i)$, and moreover that $\emptyset \in \mathcal{H}_i$, for all $i \in I$. Taking $0_i = \emptyset$ for all $i \in I$, we extend the Σ-product notation as follows*

$$\sum\left(\bigotimes_{i\in I} \mathcal{H}_i\right) = \left\{\coprod_{i\in I} x(i) : \vec{x} \in \sum\left(\prod_{i\in I} \mathcal{H}_i\right)\right\}. \tag{34}$$

Proposition 115. *Let $\mathcal{H}_i \subseteq [S_i]^{\aleph_0}$ for every $i \in I$. Then $\sum\left(\bigotimes_{i\in I} \mathcal{H}_i\right) \subseteq \left[\coprod_{i\in I} S_i\right]^{\aleph_0}$.*

Proposition 116. *Let $\mathcal{H}_0, \dots, \mathcal{H}_{n-1}$ be a finite sequence of σ-directed subfamilies of $([S_i]^{\aleph_0}, \subseteq^*)$ for each $i = 0, \dots, n-1$. Then $\bigotimes_{i=0}^{n-1} \mathcal{H}_i$ is a σ-directed subfamily of $([\coprod_{i=0}^{n-1} S_i]^{\aleph_0}, \subseteq^*)$.*

We can do better with P-ideals:

Lemma 117. *Let $(\mathcal{I}_i : i \in I)$ be an indexed family of P-ideals for some arbitrary I. Then $\sum\left(\bigotimes_{i\in I} \mathcal{I}_i\right)$ is a P-ideal.*

Proof. Let $(y_n : n < \omega)$ be an enumeration of members of $\sum\left(\bigotimes_{i\in I} \mathcal{I}_i\right)$, say each $y_n = \coprod_{i\in I} x_n(i)$ for some $\vec{x}_n \in \sum\left(\prod_{i\in I} \mathcal{I}_i\right)$. Then $J = \bigcup_{n<\omega} \text{supp}(\vec{x}_n)$ is countable, say $J = \{i_k : k < \omega\}$. For each $i \in J$, since $\mathcal{I}_i$ is σ-directed, there exists $z_i \in \mathcal{I}_i$ such that $x_n(i) \subseteq^* z_i$ for all $n < \omega$. Hence, as $\mathcal{I}_i$ is a P-ideal, $\bigcup_{n\in A} x_n(i) \cup z_i \in \mathcal{I}_i$ for every finite $A \subseteq \omega$. Therefore, $\vec{w} \in \sum\left(\prod_{i\in I} \mathcal{I}_i\right)$, where $\text{supp}(\vec{w}) \subseteq J$ is given by

$$w(i_k) = \bigcup_{n=0}^{k-1} x_n(i_k) \cup z_{i_k} \tag{35}$$

for each $k < \omega$. And clearly $y_n \subseteq^* \coprod_{i\in I} w(i)$ for all $n < \omega$. $\square$

Lemma 118. *Suppose that $\mathcal{H}_0, \ldots, \mathcal{H}_{n-1}$ are σ-directed with each $\mathcal{H}_i \subseteq [S_i]^{\aleph_0}$, and each S_i uncountable. Let J be the set of all $i = 0, \ldots, n-1$ for which S_i has no countable decomposition into pieces orthogonal to $\mathcal{H}_i$. Then $\mathcal{R}\big(\bigotimes_{i=0}^{n-1} \mathcal{H}_i\big)$ forces that there exists $X \subseteq \coprod_{i=0}^{n-1} S_i$ locally in $\bigotimes_{i=0}^{n-1} \mathcal{H}_i$ such that $X \cap S_i$ is uncountable for all $i \in J$. Similarly for Σ-products of P-ideals.*

Proof. Essentially the same as for Lemma 70. □

Proposition 119. *Let $\mathcal{H}_0, \ldots, \mathcal{H}_{n-1}$ be a finite sequence with each $\mathcal{H}_i \subseteq \mathcal{P}(S_i)$. Then $(\mathcal{H}_0 \otimes \cdots \otimes \mathcal{H}_{n-1}, \subseteq^*)$ is order isomorphic to $\mathcal{H}_0 \times \cdots \times \mathcal{H}_{n-1}$ with the product order obtained from $(\mathcal{H}_i, \subseteq^*)$.*

Recall the notion from [12], where a map $f : D \to E$ between two directed posets is called *convergent* if every $e \in E$ has a $d \in D$ such that $f(a) \geq e$ for all $a \geq d$. Notice that f is convergent iff it maps cofinal subsets of D to cofinal subsets of E. We say that D is *cofinally finer* than E, written $E \lesssim D$, if there exists a convergent map from D into E. It was established by Tukey (in [12]) that $E \lesssim D$ is equivalent to the existence of a map $g : E \to D$ that maps unbounded subsets of E to unbounded subsets of D. Then $\lesssim$ is a quasi-ordering of the class directed posets, which we refer to as the *Tukey order*. For two directed quasi-orders A and B, we use the same definition of convergent maps. Then the existence of a convergent map from A into B is equivalent to the existence of a convergent map from the poset $A / \sim_{\text{asym}}$ into the poset $B / \sim_{\text{asym}}$, i.e. the antisymmetric quotient. Thus the Tukey ordering $\lesssim$ also makes sense between directed quasi-orders. The notation $D \cong E$ indicates that D is *cofinally equivalent* to E, i.e. $D \lesssim E$ and $E \lesssim D$. Then $\cong$ is an equivalent relation, and the equivalence classes are called *cofinal types*.

A basic result on this is as follows.

Lemma 120 (*Tukey*). *For any finite sequence $D_0, \ldots, D_{n-1}$ of directed sets, $D_0 \times \cdots \times D_{n-1}$ is their least upper bound in the Tukey order.*

Example 121. 1, ω, ω_1, $\omega \times \omega_1$ and $[\omega_1]^{<\aleph_0}$ are five distinct cofinal types, where the first four orders are given by the $\in$ relation and $[\omega_1]^{<\aleph_0}$ is ordered by $\subseteq$. It is proved in [11] that: PFA implies that these five are the only cofinal types of cardinality at most $\aleph_1$, while on the other hand, CH implies that there are $2^{\aleph_1}$ many cofinal types of cardinality $\aleph_1$.

Example 122 (CH). *If $\mathcal{H} \subseteq [\omega_1]^{\aleph_0}$ and $(\mathcal{H}, \subseteq^*)$ is σ-directed, then $(\mathcal{H}, \subseteq^*)$ is of cofinal type either 1 or ω_1. It has cofinal type 1 iff $(\mathcal{H}, \subseteq^*)$ has a maximal element.*

Proposition 123. *If D is a directed set and $\kappa \lesssim D$ for some infinite cardinal κ (ordered by $\in$), then no bounded subset of κ can be mapped onto a cofinal subset of D.*

Proof. If $f : \kappa \to D$ maps a bounded subset of κ onto a cofinal subset of D, then for any convergent $g : D \to \kappa$, $g \circ f$ maps a bounded subset of κ onto a cofinal subset of κ, which is impossible if κ is an infinite cardinal. □

Lemma 124. *For any finite sequence $\mathcal{H}_0, \ldots, \mathcal{H}_{n-1}$ where each $\mathcal{H}_i$ is a directed subfamily of $([S_i]^{\aleph_0}, \subseteq^*)$, $\bigotimes_{i=0}^{n-1} \mathcal{H}_i$ is the $\lesssim$-least upper bound of the sequence, under the almost inclusion order.*

Proof. By Proposition 119 and Lemma 120. □

Corollary 125. *If $\mathcal{H}$ and $\mathcal{I}$ are $\subseteq^*$-directed subfamilies of $[S]^{\aleph_0}$ and $[T]^{\aleph_0}$, respectively, and $\mathcal{I} \lesssim \mathcal{H}$, then $\mathcal{H} \cong \mathcal{H} \otimes \mathcal{I}$.*

Proof. Since $\mathcal{H} \otimes \mathcal{I}$ is the least upper bound of $\mathcal{H}$ and $\mathcal{I}$ by Lemma 124. □

We need something more specific.

Lemma 126. *Suppose that $\mathcal{H}$ and $\mathcal{I}$ are directed subfamilies of $([S]^{\aleph_0}, \subseteq^*)$ and $([T]^{\aleph_0}, \subseteq^*)$, respectively, and both $\mathcal{H}$ and $\mathcal{I}$ have cofinal type κ for some infinite cardinal κ. Then every cofinal $\mathcal{K} \subseteq \mathcal{H} \otimes \mathcal{I}$ has a cofinal subset $\mathcal{L} \subseteq \mathcal{K}$ such that for every cofinal subset $\mathcal{J} \subseteq \pi[\mathcal{L}] = \{x \in \mathcal{H} : x \amalg y \in \mathcal{L}$ for some $y \in \mathcal{I}\}$, $(\mathcal{J} \otimes \mathcal{I}) \cap \mathcal{L}$ is cofinal in $\mathcal{H} \otimes \mathcal{I}$.*

Proof. Since $\mathcal{H} \otimes \mathcal{I} \cong \kappa$ by Lemma 124, there is a convergent map $g : \kappa \to \mathcal{H} \otimes \mathcal{I}$. For each $\alpha < \kappa$, since $\mathcal{K}$ is cofinal we can find $x_\alpha \amalg y_\alpha \in \mathcal{K}$ such that

$$g(\alpha) \subseteq^* x_\alpha \amalg y_\alpha. \tag{36}$$

We claim that $\mathcal{L} = \{x_\alpha \amalg y_\alpha : \alpha < \kappa\}$ satisfies the conclusion: $\mathcal{L}$ is cofinal because g is convergent. Suppose $\mathcal{J} \subseteq \pi[\mathcal{L}]$ is cofinal. Then $\mathcal{J}$ is cofinal in $\mathcal{H}$ as $\pi[\mathcal{L}]$ is by Proposition 119. Thus $\mathcal{J} = \{x_\alpha : \alpha \in A\}$ for some cofinal $A \subseteq \kappa$, because $g[B]$ is noncofinal for all bounded $B \subseteq \kappa$ by Proposition 123 as $\kappa \lesssim \mathcal{H}$. And $(\mathcal{J} \otimes \mathcal{I}) \cap \mathcal{L} \supseteq \{x_\alpha \amalg y_\alpha : \alpha \in A\}$, which is cofinal by (36) since g is convergent. □

Lemma 127. *Let $\mathcal{H}$ and $\mathcal{I}$ be σ-directed subfamilies of $([S]^{\aleph_0}, \subseteq^*)$ and $([T]^{\aleph_0}, \subseteq^*)$, respectively. If both $\mathcal{H}$ and $\mathcal{I}$ have cofinal type κ for some infinite cardinal κ, then $\mathcal{R}(\mathcal{H})$ generically embeds into $\mathcal{R}(\mathcal{H} \otimes \mathcal{I})$.*

Proof. Define $e : \mathcal{R}(\mathcal{H}) \to \mathcal{R}(\mathcal{H} \otimes \mathcal{I})$ by $e(p) = (x_p, \mathcal{Y}_p)$ where

$$\mathcal{Y}_p = \{\mathcal{J} \otimes \mathcal{I} : \mathcal{J} \in \mathcal{X}_p\}. \tag{37}$$

Given a maximal antichain $A \subseteq \mathcal{R}(\mathcal{H})$, we need to show that $e[A]$ is a maximal antichain of $\mathcal{R}(\mathcal{H} \otimes \mathcal{I})$. Take $q \in \mathcal{R}(\mathcal{H} \otimes \mathcal{I})$. Write $x_q = y \amalg z$ ($y \in \mathcal{H}, z \in \mathcal{I}$). For each $\mathcal{K} \in \mathcal{X}_q$, apply Lemma 126 to the cofinal set $\{w \in \mathcal{K} : x_q \subseteq w\}$ to obtain a cofinal subset $\mathcal{L}_\mathcal{K}$ as in the conclusion of that lemma. Then clearly

$$q' = (y, \{\pi[\mathcal{L}_\mathcal{K}] : \mathcal{K} \in \mathcal{X}_q\}) \tag{38}$$

is a condition of $\mathcal{R}(\mathcal{H})$. Hence there must be $p \in A$ compatible with q'. For all $\mathcal{J} \in \mathcal{X}_p$, $x_p \cup y \in \partial_\mathcal{H}(\mathcal{J})$, and therefore $x_p \cup x_q = x_p \cup (y \amalg z) \in \partial_{\mathcal{H}\otimes\mathcal{I}}(\mathcal{J} \otimes \mathcal{I})$; and for all $\mathcal{K} \in \mathcal{X}_q$, $\mathcal{J}_\mathcal{K} = \{x \in \pi[\mathcal{L}_\mathcal{K}] : x_p \cup y \subseteq x\}$ is cofinal, and therefore $(\mathcal{J}_\mathcal{K} \otimes \mathcal{I}) \cap \mathcal{L}_\mathcal{K}$ is cofinal, which implies that $x_p \cup x_q \in \partial_{\mathcal{H}\otimes\mathcal{I}}(\mathcal{K})$, because $x_p \cup (y \amalg z) \subseteq w$ for all $w \in \mathcal{L}_\mathcal{K}$ with $\pi(w) \in \mathcal{J}_\mathcal{K}$. This proves that $e(p)$ is compatible with q, as required. □

Remark. We do not believe that there is any analogue of Lemma 127 for $\mathcal{Q}$. This embeddability of $\mathcal{R}(\mathcal{H})$ is the primary reason we are interested in the forcing notion $\mathcal{R}(\mathcal{H})$ when we are only trying to force clubs with the forcing notion $\mathcal{Q}(\mathcal{I})$. For example, it figures in the analysis of properties of the forcing notion $\mathcal{Q}(\mathcal{I})$ in Corollary 132. There is also a secondary use of the forcing notion $\mathcal{R}(\mathcal{H})$ in Section 5 where it is used to force stationary sets.

Corollary 129 (CH). *Let $\mathcal{H}$ and $\mathcal{I}$ be σ-directed subfamilies of $([\omega_1]^{\aleph_0}, \subseteq^*)$. If $\mathcal{I}$ has no countable decomposition of ω_1 into orthogonal pieces, then $\mathcal{R}(\mathcal{H})$ forces that $\mathcal{I}$ has no countable decomposition of ω_1 into orthogonal pieces.*

Proof. Let $\mathcal{H}$ and $\mathcal{I}$ be as in the hypothesis. By Lemma 118, $\mathcal{R}(\mathcal{H} \otimes \mathcal{I})$ forces that there is an uncountable $X \subseteq \omega_1$ locally in $\mathcal{I}$ (meaning uncountable in the forcing extension, i.e. $\mathcal{R}(\mathcal{H} \otimes \mathcal{I})$ does not collapse $\aleph_1$). In particular, $\mathcal{R}(\mathcal{H} \otimes \mathcal{I})$ forces that there is no countable decomposition of ω_1 into pieces orthogonal to $\mathcal{I}$. By CH and Example 122, we know that both $\mathcal{H}$ and $\mathcal{I}$ are of cofinal type either 1 or ω_1, and by the hypothesis we know further that $\mathcal{I} \cong \omega_1$. In the case $\mathcal{H} \cong 1$, $\mathcal{R}(\mathcal{H})$ is the trivial forcing notion and thus the conclusion of the corollary is trivial. Assume then that $\mathcal{H} \cong \mathcal{I} \cong \omega_1$ in the Tukey order. Then by Lemma 127, $\mathcal{R}(\mathcal{H}) \lessdot \mathcal{R}(\mathcal{H} \otimes \mathcal{I})$, and thus $\mathcal{R}(\mathcal{H})$ cannot introduce a countable decomposition of ω_1 into pieces orthogonal to $\mathcal{I}$. □

Corollary 130 (CH). *Let $\mathcal{H}$ and $\mathcal{I}$ be σ-directed subfamilies of $([\omega_1]^{\aleph_0}, \subseteq^*)$, with $\mathcal{I}$ moreover a P-ideal. Then $\mathcal{R}(\mathcal{H}) \Vdash \mathcal{R}(\mathcal{I})^V \lessdot^i_V \mathcal{R}(\mathcal{I})$; hence, $\mathcal{R}(\mathcal{I}) \lessdot \mathcal{R}(\mathcal{H}) \star \mathcal{R}(\mathcal{I})$.*

Proof. Let $G \in \mathrm{Gen}(V, \mathcal{R}(\mathcal{H}))$, and set $W = V[G]$. In V, let A be a maximal antichain of $\mathcal{R}(\mathcal{I})$. Then for all $x \in \mathcal{I}$, if $\pi[A]_{(x)}$ has no countable decomposition, in V, into pieces orthogonal to $\Psi(\mathcal{I}_{(x)})$, then by Corollary 129 with $\mathcal{I} := \Psi(\mathcal{I}_{(x)})$, which is σ-directed by Proposition 104 and Lemma 101, $\pi[A]_{(x)}$ has no countable decomposition, in W, into pieces orthogonal to $\Psi(\mathcal{I}_{(x)})$. Therefore, A is a maximal antichain of $\mathcal{R}(\mathcal{I})^W$ by Corollary 107.

$\mathcal{R}(\mathcal{I}) \lessdot \mathcal{R}(\mathcal{H}) \star \mathcal{R}(\mathcal{I})$ is immediate from Proposition 94. □

Corollary 131 (CH). *Let $\mathcal{H}$ and $\mathcal{I}$ be σ-directed subfamilies of $([\omega_1]^{\aleph_0}, \subseteq^*)$, with $\mathcal{I}$ moreover a P-ideal. Then $\mathcal{R}(\mathcal{H}) \Vdash \mathcal{Q}(\mathcal{I})^V \lessdot^i_V \mathcal{Q}(\mathcal{I})$; hence, $\mathcal{Q}(\mathcal{I}) \lessdot \mathcal{R}(\mathcal{H}) \star \mathcal{Q}(\mathcal{I})$.*

Proof. This is the same as the proof of Corollary 130 but using Corollary 106. □

Corollary 132 (CH). *Let $\mathcal{H}$ and $\mathcal{I}$ be σ-directed subfamilies of $([\omega_1]^{\aleph_0}, \subseteq^*)$, with $\mathcal{I}$ moreover a P-ideal. Suppose $\mathcal{H}$ has no countable decomposition of ω_1 into orthogonal pieces. If $\mathcal{R}(\mathcal{H})$ forces that $\mathcal{I}$ has no stationary orthogonal subset of ω_1, then $\mathcal{Q}(\mathcal{I})$ forces that $\mathcal{H}$ has no countable decomposition of ω_1 into orthogonal pieces.*

Proof. By Corollary 131, $\mathcal{Q}(\mathcal{I}) \lessdot \mathcal{R}(\mathcal{H}) \star \mathcal{Q}(\mathcal{I})$. Now, if $\mathcal{R}(\mathcal{H})$ forces that $\mathcal{I}$ has no stationary orthogonal set, then $\mathcal{R}(\mathcal{H}) \star \mathcal{Q}(\mathcal{I})$ is proper by Lemmas 57 and 67, and hence does not collapse $\aleph_1$. Therefore, by our assumption on $\mathcal{H}$, $\mathcal{R}(\mathcal{H}) \star \mathcal{Q}(\mathcal{I})$ forces that there exists an uncountable set locally in $\downarrow\mathcal{H}$. Thus $\mathcal{Q}(\mathcal{I})$ cannot force a countable decomposition of ω_1 into pieces orthogonal to $\mathcal{H}$. □

Corollary 133 (CH). *Let $\mathcal{I}$ be a P-ideal on ω_1. Suppose that $\mathcal{R}(\mathcal{H})$ forces that there is no stationary subset of ω_1 orthogonal to $\mathcal{I}$, for every σ-directed subfamily $\mathcal{H}$ of $([\omega_1]^{\aleph_0}, \subseteq^*)$ having no countable decomposition of ω_1 into orthogonal pieces. Then for every P-ideal $\mathcal{J}$ on ω_1, $\mathcal{Q}(\mathcal{I}) \Vdash \mathcal{R}(\mathcal{J})^V \lessdot^i_V \mathcal{R}(\mathcal{J})$; hence, $\mathcal{R}(\mathcal{J}) \lessdot \mathcal{Q}(\mathcal{I}) \star \mathcal{R}(\mathcal{J})$.*

Proof. Let $G \in \mathrm{Gen}(V, \mathcal{Q}(\mathcal{I}))$, and set $W = V[G]$. In V, let A be a maximal antichain of $\mathcal{R}(\mathcal{J})$. Then for all $x \in \mathcal{J}$, if $\pi[A]_{(x)}$ has no countable decomposition, in V, into pieces orthogonal to $\Psi(\mathcal{J}_{(x)})$, then by Corollary 132 with $\mathcal{H} := \Psi(\mathcal{J}_{(x)})$, which is σ-directed by Proposition 104 and Lemma 101, $\pi[A]_{(x)}$ has no countable decomposition, in W, into pieces orthogonal to $\Psi(\mathcal{J}_{(x)})$. Therefore, A is a maximal antichain of $\mathcal{R}(\mathcal{J})^W$ by Corollary 107. □

Similarly:

Corollary 134 (CH). *Let $\mathcal{I}$ be a P-ideal on ω_1. Suppose that $\mathcal{R}(\mathcal{H})$ forces that there is no countable decomposition of ω_1 into pieces orthogonal to $\mathcal{I}$, for every σ-directed subfamily $\mathcal{H}$ of $([\omega_1]^{\aleph_0}, \subseteq^*)$ having no countable decomposition of ω_1 into orthogonal pieces. Then for every P-ideal $\mathcal{J}$ on ω_1, $\mathcal{Q}(\mathcal{I}) \Vdash \mathcal{Q}(\mathcal{J})^V \lessdot^i_V \mathcal{Q}(\mathcal{J})$; hence, $\mathcal{Q}(\mathcal{J}) \lessdot \mathcal{Q}(\mathcal{I}) \star \mathcal{Q}(\mathcal{J})$.*

In Corollaries 130, 131, 133 and 134, all four permutations of $\mathcal{R}(\mathcal{I})$ $(\mathcal{Q}(\mathcal{I}))$ in the extension by $\mathcal{R}(\mathcal{H})$ $[\mathcal{Q}(\mathcal{H})]$ have been considered. The next step is to consider iterations. It is immediate from two applications of Corollary 130, that $\mathcal{R}(\mathcal{H}) \star \mathcal{R}(\dot{\mathcal{I}}) \Vdash \mathcal{R}(\mathcal{J})^V \preccurlyeq^i_V \mathcal{R}(\mathcal{J})$. But this begs the question of whether

$$\mathcal{R}(\mathcal{J}) \Vdash [\mathcal{R}(\mathcal{H}) \star \mathcal{R}(\dot{\mathcal{I}})]^V \preccurlyeq^i_V [\mathcal{R}(\mathcal{H}) \star \mathcal{R}(\dot{\mathcal{I}})]? \tag{39}$$

The key to answering this is to establish that, under certain conditions, the forcing notions $\mathcal{Q}(\mathcal{H})$ and $\mathcal{R}(\mathcal{H})$ commute among themselves.

Notation 135. *For any forcing notion P and any P-name $\dot{A}$, we denote*

$$\dot{A}[p] = \{x : q \Vdash x \in \dot{A} \text{ for some } q \geq p\}, \tag{40}$$

for each $p \in P$.

Proposition 136. *If $P \Vdash \dot{A} \subseteq V$ then $p \Vdash \dot{A} \subseteq \dot{A}[p]$, for all $p \in P$.*

Remark. Note that when we say "p decides $\dot{A}$" this means the same thing as "$p \Vdash \dot{A} = \dot{A}[p]$".

Proposition 138. *Let $\mathcal{H} \subseteq [\theta]^{\aleph_0}$. Suppose P is a forcing notion that adds no new countable subsets of θ. If $(p, \dot{q})$ is a condition of $P \star \mathcal{Q}(\mathcal{H})$ and p decides $x_{\dot{q}}$, then $(x_{\dot{q}}[p], \{\dot{\mathcal{J}}[p] : \dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}\}) \in \mathcal{Q}(\mathcal{H})$. Similarly, for $\mathcal{R}(\mathcal{H})$.*

Proof. By the assumption on P, $P \Vdash \dot{\mathcal{J}} \subseteq V$ for all $\dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}$. The result thus follows from Proposition 136. □

Note that we are implicitly assuming an enumeration of $\mathcal{X}_{\dot{q}}$ by $\aleph_0$ when referring to $\dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}$.

Lemma 139. *Let $\mathcal{H} \subseteq [\theta]^{\aleph_0}$. For every forcing notion P that adds no new countable subsets of θ, if $(p, \dot{q}) \in P \star \mathcal{Q}(\mathcal{H})$ and $D \subseteq \{r \in P : r \geq p\}$ is predense above p then*

$$\overline{(P \star \mathcal{Q}(\mathcal{H})) / \sim_{\text{sep}}} \models (p, \dot{q}) \geq \bigwedge_{d \in D} \big(d, (x_{\dot{q}}, \{\dot{\mathcal{J}}[d] : \dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}\})\big), \tag{41}$$

Similarly, for $\mathcal{R}(\mathcal{H})$.

Remark. Eq. (41) is equivalent to: for every $(p', \dot{q}') \geq (p, \dot{q})$ there exists $d \in D$ such that $(p', \dot{q}')$ is compatible with $\big(d, (x_{\dot{q}}, \{\dot{\mathcal{J}}[d] : \dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}\})\big)$.

Proof (*Proof of Lemma 139*). We establish Eq. (41) using Remark 140. Given $(p', \dot{q}') \geq (p, \dot{q})$, since D is predense above p, there exists $d \in D$ compatible with p', say with common extension d'. By the assumption on P, Proposition 136 applies, and then by Proposition 41,

$$d \Vdash \overline{\mathcal{Q}(\mathcal{H}) / \sim_{\text{sep}}} \models \dot{q} \geq (x_{\dot{q}}, \{\dot{\mathcal{J}}[d] : \dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}\}), \tag{42}$$

which implies that there is an $\dot{s}$ such that $d \Vdash \dot{s}$ is a common extension of $\dot{q}'$ and $(x_{\dot{q}}, \{\dot{\mathcal{J}}[d] : \dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}\})$. Therefore, $(d', \dot{s}) \geq (p', \dot{q}')$ and $(d', \dot{s}) \geq \big(d, (x_{\dot{q}}, \{\dot{\mathcal{J}}[d] : \dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}\})\big)$, concluding the proof. □

Lemma 141. *Let $\mathcal{H} \subseteq [\theta]^{\aleph_0}$. If P is a forcing notion that adds no new countable subsets of θ, and $P \Vdash \mathcal{Q}(\mathcal{H})^V \preccurlyeq^i_V \mathcal{Q}(\mathcal{H})$, then $P \star \mathcal{Q}(\mathcal{H}) / \mathcal{Q}(\mathcal{H}) \cong P$.*

Proof. First we deal with the pathological case where some $q \in \mathcal{Q}(\mathcal{H})$ forces that $C_{\dot{G}_{\mathcal{Q}(\mathcal{H})}}$ is countable. Let A be a antichain maximal with respect to every $q \in A$ having this property. Then let $B \subseteq \mathcal{Q}(\mathcal{H})$ satisfy $A \cup B$ is a maximal antichain. For all $a \in A$, clearly $\mathcal{Q}(\mathcal{H})_a$ is the trivial forcing notion and moreover this is upwards absolute, and thus $P \star \mathcal{Q}(\mathcal{H})_a / \mathcal{Q}(\mathcal{H})_a \cong P \star 1 / 1 \cong P$. Now, by Eq. (26), it suffices to prove that $P \star \mathcal{Q}(\mathcal{H})_b / \mathcal{Q}(\mathcal{H})_b \cong P$ for all $b \in B$. Henceforth, we assume without loss of generality that $\mathcal{Q}(\mathcal{H}) \Vdash C_{\dot{G}_{\mathcal{Q}(\mathcal{H})}}$ is uncountable.

By Proposition 94, the map $e : \mathcal{Q}(\mathcal{H}) \to P \star \mathcal{Q}(\mathcal{H})$ given by

$$e(q) = (0_{\mathcal{Q}(\mathcal{H})}, q) \tag{43}$$

defines a generic embedding of $\mathcal{Q}(\mathcal{H})$ into $P \star \mathcal{Q}(\mathcal{H})$. Let $G \in \text{Gen}(V, \mathcal{Q}(\mathcal{H}))$.

In $V[G]$: The representation of the quotient given by e is

$$(P \star \mathcal{Q}(\mathcal{H})) / G = \{(p, \dot{q}) \in Q \star \dot{R} : (p, \dot{q}) \text{ is } Q \star \dot{R}\text{-compatible with every member of } e[G]\}, \tag{44}$$

with the order inherited from $P \star \mathcal{Q}(\mathcal{H})$. Thus, as $e[G] = \{0_Q\} \times G$, $(p, \dot{q}) \in (P \star \mathcal{Q}(\mathcal{H})) / G$ iff every $r \in G$ has a $p' \geq p$ forcing that r is compatible with $\dot{q}$.

Claim 141.1. *For all $(p, \dot{q}) \in (P \star \mathcal{Q}(\mathcal{H})) / G$, there exists $p' \geq p$ such that p' decides $x_{\dot{q}}$ and $(p'', \dot{q}) \in (P \star \mathcal{Q}(\mathcal{H})) / G$ for all $p'' \geq p'$.*

Proof. Let $(p, \dot{q}) \in (P \star \mathcal{Q}(\mathcal{H})) \,/\, G$ be given. Then letting D be the set of all $d \geq p$ deciding $x_{\dot{q}}$, D is dense above p by our assumption on P. Note that $(x_{\dot{q}}[d], \{\dot{\mathcal{J}}[d] : \dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}\}) \in \mathcal{Q}(\mathcal{H})$ for all $d \in D$ by Proposition 138.

Now assume towards a contradiction that Claim 141.1 fails. Let E be the set of all $d \in D$ for which there is an $r_d \in G$ such that

$$r_d \perp (x_{\dot{q}}[d], \{\dot{\mathcal{J}}[d] : \dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}\}). \tag{45}$$

Subclaim 141.1.1. *E is dense above p.*

Proof. Take $p_0 \geq p$. Pick $d \geq p_0$ in D. By our assumption that the claim fails, there exists $p_1 \geq d$ such that $(p_1, \dot{q}) \notin (P \star \mathcal{Q}(\mathcal{H})) \,/\, G$. Hence there is an $r \in G$ and $p_2 \geq p_1$ forcing that r is incompatible with $\dot{q}$. Since $p_2 \Vdash x_{\dot{q}} = x_{\dot{q}}[d]$, by Proposition 42, either $x_{\dot{q}}[d]$ is not comparable under end-extension with x_r, in which case it is clear that $r_d := r$ witnesses that $d \in E$, or else p_2 forces that there exists $\mathcal{J} \in \mathcal{X}_{\dot{q}} \cup \mathcal{X}_r$ with $x_{\dot{q}}[d] \cup x_r \notin \partial_{\mathcal{H}}(\mathcal{J})$. If it is the case that $x_{\dot{q}}[d] \notin \partial_{\mathcal{H}}(\mathcal{J})$ for some $\mathcal{J} \in \mathcal{X}_r$, then $r_d := r$ witnesses that $d \in E$. Otherwise, in the remaining case there exists $p_3 \geq p_2$ and $\dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}$ such that $p_3 \Vdash x_r \notin \partial_{\mathcal{H}}(\dot{\mathcal{J}})$. Hence, there exists $y \in \mathcal{H}$ and $p_4 \geq p_3$ forcing that there is no $z \supseteq^* y$ in $\dot{\mathcal{J}}$ with $x_r \subseteq z$. Therefore, there is no $z \in \dot{\mathcal{J}}[p_4]$ with $y \subseteq^* z$ and $x_r \subseteq z$, i.e. $x_r \notin \partial_{\mathcal{H}}(\dot{\mathcal{J}}[p_4])$. Now $d := p_4$ and $r_d := r$ witness that $p_4 \in E$. □

Subclaim 141.1.2. *There exists $d \in E$ such that $(x_{\dot{q}}[d], \{\dot{\mathcal{J}}[d] : \dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}\}) \in G$.*

Proof. Suppose not. Then there exists $r \in G$ such that $r \Vdash (x_{\dot{q}}[d], \{\dot{\mathcal{J}}[d] : \dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}\}) \notin \dot{G}_{\mathcal{Q}(\mathcal{H})}$ for all $d \in E$. This means that

$$r \perp (x_{\dot{q}}[d], \{\dot{\mathcal{J}}[d] : \dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}\}) \quad \text{for all } d \in E. \tag{46}$$

However, $(p, \dot{q}) \in (P \star \mathcal{Q}(\mathcal{H}))/G$ implies that there exists $(p', \dot{q}') \geq (p, \dot{q})$ in $P \star \mathcal{Q}(\mathcal{H})$ such that $p' \Vdash \dot{q}' \geq r$. And by Subclaim 141.1.1, and Lemma 139 with $D := E$, there exists $d \in E$ such that $(p', \dot{q}')$ is compatible with $\big(d, (x_{\dot{q}}, \{\dot{\mathcal{J}}[d] : \dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}\})\big) = \big(d, (x_{\dot{q}}[d], \{\dot{\mathcal{J}}[d] : \dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}\})\big)$. This clearly implies that r is compatible with $(x_{\dot{q}}[d], \{\dot{\mathcal{J}}[d] : \dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}\})$, contradicting (46). □

Let d be as in Subclaim 141.1.2. Then as in Eq. (45), there exists $r \in G$ such that $r \perp (x_{\dot{q}}[d], \{\dot{\mathcal{J}}[d] : \dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}\})$. This obviously contradicts $(x_{\dot{q}}, \{\dot{\mathcal{J}}[d] : \dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}) \in G$. □

Claim 141.1 allows us to define $f : (P \star \mathcal{Q}(\mathcal{H})) \,/\, G \to P$ so that for all $(p, \dot{q}) \in (P \star \mathcal{Q}(\mathcal{H})) \,/\, G$, $f(p, \dot{q}) \in D$, $f(p, \dot{q}) \geq p$ and $(p', \dot{q}) \in (P \star \mathcal{Q}(\mathcal{H})) \,/\, G$ for all $p' \geq f(p, \dot{q})$. Thus f satisfies clauses (a) and (b) of Lemma 99 with $O := \mathcal{Q}(\mathcal{H})$ and $\dot{Q} := \mathcal{Q}(\mathcal{H})^{V[\dot{G}_P]}$. Observe that for all $(p, \dot{q}) \in (P \star \mathcal{Q}(\mathcal{H})) \,/\, G$,

$$f(p, \dot{q}) \Vdash \dot{q} \text{ is compatible with } r \quad \text{for all } r \in G. \tag{47}$$

It remains to verify clause (c) that $(f(p, \dot{q}), \dot{q})$ and $(f(p', \dot{q}'), \dot{q}')$ are $P \star \dot{Q}$-compatible whenever $f(p, \dot{q})$ and $f(p', \dot{q}')$ are compatible. Then Lemma 99 will yield $(P \star \mathcal{Q}(\mathcal{H})) \,/\, \mathcal{Q}(\mathcal{H}) \cong \mathcal{Q}(\mathcal{H})$.

Suppose then that $f(p, \dot{q})$ and $f(p', \dot{q}')$ are compatible, say p'' is a common extension. By our assumption that C_G is uncountable, there exists $r \in G$ such that $x_r \not\subseteq x_{\dot{q}}[f(p, \dot{q})]$ and $x_r \not\subseteq x_{\dot{q}'}[f(p', \dot{q}')]$. Therefore, by (47) and Proposition 42, $x_{\dot{q}}[f(p, \dot{q})]$ and $x_{\dot{q}'}[f(p', \dot{q}')]$ are both initial segments of x_r and are thus comparable under end-extension. Again by (47) and Proposition 42, $f(p, \dot{q}) \Vdash x_r \in \partial_{\mathcal{H}}(\dot{\mathcal{J}})$ for all $\dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}}$ and $f(p', \dot{q}') \Vdash x_r \in \partial_{\mathcal{H}}(\dot{\mathcal{J}})$ for all $\dot{\mathcal{J}} \in \mathcal{X}_{\dot{q}'}$. Thus $p'' \Vdash \ulcorner x_{\dot{q}} \cup x_{\dot{q}'} \subseteq x_r \in \partial_{\mathcal{H}}(\mathcal{J})$ for all $\mathcal{J} \in \mathcal{X}_{\dot{q}} \cup \mathcal{X}_{\dot{q}'} \urcorner$, proving that $p'' \Vdash \dot{q}$ and $\dot{q}'$ are compatible, by Proposition 42. □

Lemma 142. *Let $\mathcal{H} \subseteq [\theta]^{\aleph_0}$. If P is a forcing notion that adds no new countable subsets of θ, and $P \Vdash \mathcal{R}(\mathcal{H})^V \preccurlyeq^i_V \mathcal{R}(\mathcal{H})$, then $P \star \mathcal{R}(\mathcal{H}) \,/\, \mathcal{R}(\mathcal{H}) \cong P$.*

Proof. Essentially the same as Lemma 141 but using Proposition 66. □

Corollary 143 (CH). *Let $\mathcal{H}$ and $\mathcal{I}$ be σ-directed subfamilies of $([\omega_1]^{\aleph_0}, \subseteq^*)$, with $\mathcal{I}$ moreover a P-ideal. Then*

(a) $\mathcal{R}(\mathcal{H}) \star \mathcal{R}(\mathcal{I}) \,/\, \mathcal{R}(\mathcal{I}) \cong \mathcal{R}(\mathcal{H})$,
(b) $\mathcal{R}(\mathcal{H}) \star \mathcal{Q}(\mathcal{I}) \,/\, \mathcal{Q}(\mathcal{I}) \cong \mathcal{R}(\mathcal{H})$.

Proof. For conclusion (a), we can apply Lemma 142 with $P := \mathcal{R}(\mathcal{H})$ and $\mathcal{H} := \mathcal{I}$, because $\mathcal{R}(\mathcal{H}) \Vdash \mathcal{R}(\mathcal{I})^V \preccurlyeq^i_V \mathcal{R}(\mathcal{I})$ by Corollary 130.

For (b), we can apply Lemma 141 with $P := \mathcal{R}(\mathcal{H})$, by Corollary 131. □

Corollary 144 (CH). *Let $\mathcal{H}$ and $\mathcal{I}$ be P-ideals on ω_1. Suppose that $\mathcal{R}(\mathcal{J})$ forces that there is no stationary set orthogonal to $\mathcal{H}$ for every σ-directed subfamily $\mathcal{J}$ of $([\omega_1]^{\aleph_0}, \subseteq^*)$ having no countable decomposition of ω_1 into orthogonal pieces. Then*

(a) $\mathcal{Q}(\mathcal{H}) \star \mathcal{R}(\mathcal{I}) \,/\, \mathcal{R}(\mathcal{I}) \cong \mathcal{Q}(\mathcal{H})$,
(b) $\mathcal{Q}(\mathcal{H}) \star \mathcal{Q}(\mathcal{I}) \,/\, \mathcal{Q}(\mathcal{I}) \cong \mathcal{Q}(\mathcal{H})$.

Proof. By the hypothesis, Corollary 133 applies so that $\mathcal{Q}(\mathcal{H}) \Vdash \mathcal{R}(\mathcal{I})^V \preccurlyeq^i_V \mathcal{R}(\mathcal{I})$, and thus conclusion (a) holds by Lemma 142.

Similarly, Corollary 134 applies so that $\mathcal{Q}(\mathcal{H}) \Vdash \mathcal{Q}(\mathcal{I})^V \preccurlyeq^i_V \mathcal{Q}(\mathcal{I})$, and thus conclusion (b) holds by Lemma 141. □

We have now achieved commutativity.

Corollary 145 (CH). *Let $\mathcal{H}$ and $\mathcal{I}$ be P-ideals on ω_1. Then*

$$\mathcal{R}(\mathcal{H}) \star \mathcal{R}(\mathcal{I}) \cong \mathcal{R}(\mathcal{H}) \times \mathcal{R}(\mathcal{I}) \cong \mathcal{R}(\mathcal{I}) \times \mathcal{R}(\mathcal{H}) \cong \mathcal{R}(\mathcal{I}) \star \mathcal{R}(\mathcal{H}). \tag{48}$$

Proof. Corollary 143(a) is equivalent to $\mathcal{R}(\mathcal{H}) \star \mathcal{R}(\mathcal{I}) \cong \mathcal{R}(\mathcal{I}) \times \mathcal{R}(\mathcal{H})$. The remaining equivalences are by the commutativity of products and another application of Corollary 143(a). □

Corollary 146 (CH). *Let $\mathcal{H}$ and $\mathcal{I}$ be P-ideals on ω_1. Suppose that $\mathcal{R}(\mathcal{J})$ forces that there is no stationary set orthogonal to $\mathcal{H}$ for every σ-directed subfamily $\mathcal{J}$ of $([\omega_1]^{\aleph_0}, \subseteq^*)$ having no countable decomposition of ω_1 into orthogonal pieces. Then*

$$\mathcal{Q}(\mathcal{H}) \star \mathcal{R}(\mathcal{I}) \cong \mathcal{Q}(\mathcal{H}) \times \mathcal{R}(\mathcal{I}) \cong \mathcal{R}(\mathcal{I}) \times \mathcal{Q}(\mathcal{H}) \cong \mathcal{R}(\mathcal{I}) \star \mathcal{Q}(\mathcal{H}). \tag{49}$$

Proof. We have $\mathcal{Q}(\mathcal{H}) \star \mathcal{R}(\mathcal{I}) \cong \mathcal{R}(\mathcal{I}) \times \mathcal{Q}(\mathcal{H})$ by Corollary 144(a), and $\mathcal{R}(\mathcal{I}) \star \mathcal{Q}(\mathcal{H}) \cong \mathcal{Q}(\mathcal{H}) \times \mathcal{R}(\mathcal{I})$ by Corollary 143(b). □

Corollary 147 (CH). *Let $\mathcal{H}$ and $\mathcal{I}$ be P-ideals on ω_1. Suppose that $\mathcal{R}(\mathcal{J})$ forces that there is no stationary set orthogonal to $\mathcal{H}$ and that there is no stationary set orthogonal to $\mathcal{I}$ for every σ-directed subfamily $\mathcal{J}$ of $([\omega_1]^{\aleph_0}, \subseteq^*)$ having no countable decomposition of ω_1 into orthogonal pieces. Then*

$$\mathcal{Q}(\mathcal{H}) \star \mathcal{Q}(\mathcal{I}) \cong \mathcal{Q}(\mathcal{H}) \times \mathcal{Q}(\mathcal{I}) \cong \mathcal{Q}(\mathcal{I}) \times \mathcal{Q}(\mathcal{H}) \cong \mathcal{Q}(\mathcal{I}) \star \mathcal{Q}(\mathcal{H}). \tag{50}$$

Proof. By two applications of Corollary 144(b). □

Remark. This is already very significant. For example, by Corollary 145, $\mathcal{R}(\mathcal{H}) \times \mathcal{R}(\mathcal{I}) \cong \mathcal{R}(\mathcal{H}) \star \mathcal{R}(\mathcal{I})$ which is proper. This can easily be extended arbitrary finite products, whence $\mathcal{R}(\mathcal{H}_0) \times \cdots \times \mathcal{R}(\mathcal{H}_{n-1})$ is proper. This strongly suggests that Shelah's NNR theory from [8, Ch. XVIII, Section 2] applies to our classes of forcing notions (see Section 4.4 for more discussion). This would be the first instance we are aware of where the theory applies to forcing notions of cardinality $\aleph_2$ or greater. All of the examples in [8, Ch. XVIII, Sections 1, 2] are forcing notions of cardinality $\aleph_1$.

Note for example that, at least when dealing with P-ideals, Corollary 130 strengthens to:

Corollary 149 (CH). *Let $\mathcal{H}$ and $\mathcal{I}$ be P-ideals on ω_1. Then $\mathcal{R}(\mathcal{H}) \Vdash \mathcal{R}(\mathcal{I})^V \cong \mathcal{R}(\mathcal{I})$; hence, $\mathcal{R}(\mathcal{H}) \Vdash \ulcorner\mathcal{R}(\mathcal{I})^V$* is densely included in $\mathcal{R}(\mathcal{I})\urcorner$.

Proof. By Corollary 145 and Proposition 97. □

We are going to extend e.g. Corollary 129 to countable support iterations. For example, we shall prove:

Theorem 4 (CH). *Suppose that $(P_\xi, \mathcal{R}(\dot{\mathcal{H}}_\xi) : \xi < \delta)$ is a countable support iteration, where each $\dot{\mathcal{H}}_\xi$ is a P_ξ-name for a P-ideal on ω_1, and $\mathcal{I}$ is a P-ideal on ω_1 with no countable decomposition of ω_1 into orthogonal pieces. Let then the limit P_δ of the iteration forces that $\mathcal{I}$ has no countable decomposition of ω_1 into pieces orthogonal to $\mathcal{I}$.*

We do not however obtain a preservation theorem for countable support iterations not decomposing ω_1 into countably many pieces orthogonal to $\mathcal{I}$, and we doubt that this property is preserved under the iteration of any general class of proper forcing notions (as opposed to the specific class $\mathcal{R}$).

4.3. Coding iterations

While the forcing notions $\mathcal{Q}$ and $\mathcal{R}$ are viewed as classes with one parameter, we need to generalize definability to iterations, to also allow iterations of $\mathcal{Q}$ and $\mathcal{R}$ to be interpreted in the relevant model. This is necessary for our analysis of embedability, and will be necessary for our handling of the NNR iteration as well.

Definition 150. Let θ be an ordinal of uncountable cofinality. We describe a coding of those iterations consisting of combinations of the forcing notions $\mathcal{Q}(\mathcal{H})$ and $\mathcal{R}(\mathcal{H})$, with $\mathcal{H} \subseteq [\theta]^{\aleph_0}$. We define a class $\mathbb{C}^\theta$ of sequences, or codes, and forcing notions $P(\vec{a})$ for each $\vec{a} \in \mathbb{C}^\theta$, by recursion on $\xi = \operatorname{len}(\vec{a})$. Let $\mathbb{C}^\theta \restriction 0$ be the singleton containing the null sequence $\langle\rangle$ and let $P(\emptyset)$ be the trivial forcing notion. Having defined $\mathbb{C}^\theta \restriction \xi$, let $\mathbb{C}^\theta \restriction \xi + 1$ be the collection of all sequences of the form $\vec{a}^\frown(\dot{\mathcal{H}}, \mathcal{O})$ where $\vec{a} \in \mathbb{C}^\theta \restriction \xi$, $\dot{\mathcal{H}}$ is a $P(\vec{a})$-name for a σ-directed subfamily of $([\theta]^{\aleph_0}, \subseteq^*)$ and $\mathcal{O}$ is either $\mathcal{Q}$ or $\mathcal{R}$; then let

$$P(\vec{a}^\frown(\dot{\mathcal{H}}, \mathcal{O})) = \begin{cases} P(\vec{a}) \star \mathcal{Q}(\dot{\mathcal{H}}), & \text{if } \mathcal{O} = \mathcal{Q}, \\ P(\vec{a}) \star \mathcal{R}(\dot{\mathcal{H}}), & \text{if } \mathcal{O} = \mathcal{R}. \end{cases} \tag{51}$$

For limit δ, let $\mathbb{C}^\theta \restriction \delta = \varprojlim_{\xi<\delta} \mathbb{C}^\theta \restriction \xi$ be the inverse limit, i.e. all sequences $\vec{a}$ of length δ with $\vec{a} \restriction \xi \in \mathbb{C}^\theta$ for all $\xi < \delta$; then for each $\vec{a} \in \mathbb{C}^\theta \restriction \delta$, we let $P(\vec{a})$ be the corresponding countable support iteration. Thus $P(\vec{a})$ is the limit of $(P_\xi : \xi < \delta)$ of the iterated forcing $(P_\xi, \dot{Q}_\xi : \xi < \delta)$, where each $\dot{Q}_\xi$ is the second iterand in Eq. (51) plugging in $\vec{a} := \vec{a} \restriction \xi$ and $(\dot{\mathcal{H}}, \mathcal{O}) := \vec{a}(\xi)$, inverse limits are taken at limits of countable cofinality and direct limits are taken at limits of uncountable cofinality. Denote the class $\mathbb{C}^\theta = \bigcup_{\xi\in\mathrm{On}} \mathbb{C}^\theta \restriction \xi$. For each $\vec{a} \in \mathbb{C}^\theta$ and each $\xi < \operatorname{len}(\vec{a})$, we let $\dot{\mathcal{H}}(\vec{a}(\xi)) = \dot{\mathcal{H}}$ where $\vec{a}(\xi) = (\dot{\mathcal{H}}, \mathcal{O})$.

Let $\mathbb{P}^\theta \subseteq \mathbb{C}^\theta$ be the set of all codes $\vec{a}$ such that for all $\xi < \operatorname{len}(\vec{a})$, if $\vec{a}(\xi)$ is of the form $(\dot{\mathcal{H}}, \mathcal{Q})$ then $P(a \restriction \xi) \Vdash$ ⌜there is no stationary subset of θ orthogonal to $\dot{\mathcal{H}}$⌝.

For $\vec{a} \in \mathbb{C}^\theta$, let $D(\vec{a})$ be the set of codes generated by the operation $\vec{a}^\frown(\dot{\mathcal{H}}, \mathcal{R})$, where $P(\vec{a})$ forces that $\dot{\mathcal{H}}$ is σ-directed (and thus $\operatorname{len}(\vec{b}) < \operatorname{len}(\vec{a}) + \omega$ for all $\vec{b} \in D(\vec{a})$). Then define $\mathbb{Q}^\theta \subseteq \mathbb{C}^\theta$ as the set of all codes $\vec{a}$ such that for all $\xi < \operatorname{len}(\vec{a})$, if $\vec{a}(\xi)$ is of the form $(\dot{\mathcal{H}}, \mathcal{Q})$, then for all $\vec{b} \in D(\vec{a} \restriction \xi)$, $P(\vec{b})$ forces that there is no stationary subset of θ orthogonal to $\dot{\mathcal{H}}$.

We also define $\mathbb{G}^\theta \subseteq \mathbb{Q}^\theta$ as the set of all codes $\vec{a}$ such that for all $\xi < \operatorname{len}(\vec{a})$, if $\vec{a}(\xi)$ is of the form $(\dot{\mathcal{H}}, \mathcal{Q})$, then for every $\vec{c} \in D(\vec{a} \restriction \xi)$ and every $\vec{b} \in C(\vec{c})$ with $\vec{a} \restriction \xi \subseteq \vec{b}$ (cf. Definition 172), $P(\vec{b})$ forces that there is no stationary subset of θ orthogonal to $\dot{\mathcal{H}}$.

Define $\mathbb{C}^\theta_P \subseteq \mathbb{C}^\theta$ as the set of all codes $\vec{a}$ such that $P(\vec{a} \restriction \xi) \Vdash$ ⌜$\dot{\mathcal{H}}(\vec{a}(\xi))$ is a P-ideal⌝ for all $\xi < \operatorname{len}(\vec{a})$. Let $\mathbb{P}^\theta_P = \mathbb{P}^\theta \cap \mathbb{C}^\theta_P$, $\mathbb{Q}^\theta_P = \mathbb{Q}^\theta \cap \mathbb{C}^\theta_P$ and $\mathbb{G}^\theta_P = \mathbb{G}^\theta \cap \mathbb{C}^\theta_P$.

Proposition 151. *If $\vec{a}, \vec{b} \in \mathbb{C}^\theta$ then $\vec{a}^\frown\vec{b} \in \mathbb{C}^\theta$ and $P(\vec{a}^\frown\vec{b}) = P(\vec{a}) \star P(\vec{b})$. More generally, if $(\vec{a}_\xi : \xi < \mu)$ is a sequence of elements of $\mathbb{C}^\theta$, then the concatenation $\vec{b} = \vec{a}_0{}^\frown\vec{a}_1{}^\frown \cdots {}^\frown\vec{a}_\xi{}^\frown \cdots$ is in $\mathbb{C}^\theta$ and $P(\vec{b})$ is the limit of the countable support iteration determined by $(P(\vec{a}_\xi) : \xi < \mu)$.*

Lemma 152. *For all $\vec{a} \in \mathbb{P}^\theta$, $P(\vec{a})$ is proper.*

Proof. By Lemmas 57 and 67. □

Proposition 153. $\mathbb{G}^\theta \subseteq \mathbb{Q}^\theta \subseteq \mathbb{P}^\theta$.

Proof. For all $\xi < \operatorname{len}(\vec{a})$, $\vec{a} \restriction \xi \in D(\vec{a} \restriction \xi)$ and $\vec{a} \restriction \xi \in C(\vec{a} \restriction \xi)$. □

Proposition 154. *For all $\vec{a} \in \mathbb{C}^\theta$, for all $\xi < \operatorname{len}(\vec{a})$, $P(\vec{a} \restriction \xi) \Vdash \vec{a} \restriction [\xi, \operatorname{len}(\vec{a})) \in \mathbb{C}^\theta$, i.e. we are taking $\mathbb{C}^\theta$ as a class with parameter θ that is being interpreted in the forcing extension by $P(\vec{a} \restriction \xi)$. Similarly, for all $\vec{a} \in \mathbb{P}^\theta$ $(\mathbb{Q}^\theta)$ $[\mathbb{C}^\theta_P]$, for all $\xi < \operatorname{len}(\vec{a})$, $P(\vec{a} \restriction \xi) \Vdash \vec{a} \restriction [\xi, \operatorname{len}(\vec{a})) \in \mathbb{P}^\theta$ $(\mathbb{Q}^\theta)$ $[\mathbb{C}^\theta_P]$.*

Proof. These are immediate from the associativity of iterated forcing. □

Remark. Proposition 154 may fail for $\mathbb{G}^\theta$, because in some forcing extension by $P(\vec{a} \restriction \xi)$ there may be new elements of $C(\dot{\vec{c}})$ that do not correspond to elements of $C(\vec{a})^V$, because for example elements of $C(\dot{\vec{c}})$ include uncountable concatenations.

We also have a converse.

Proposition 156. *For all $\vec{a} \in \mathbb{C}^\theta$, and every $P(\vec{a})$-name $\dot{\vec{c}}$, if $P(\vec{a}) \Vdash \dot{\vec{c}} \in \mathbb{C}^\theta$ then $\vec{a}^\frown\dot{\vec{c}} \in \mathbb{C}^\theta$ (assuming a suitable representation of $\dot{\vec{c}}$). Similarly, for $\mathbb{P}^\theta$, $\mathbb{Q}^\theta$ and $\mathbb{C}^\theta_P$.*

Now we can generalize Corollary 107 using our coding of iterations in the definition of frozen (Definition 109).

Lemma 157 (CH). *Let $\mathcal{I}$ be a P-ideal on ω_1 and let $\vec{a} \in \mathbb{Q}^{\omega_1}_P$. If $P(\vec{a})$ adds no new reals, then all of the following are true*:

(a) *If there is no countable decomposition of ω_1 into pieces orthogonal to $\mathcal{I}$, then $P(\vec{a})$ forces that $\mathcal{I}$ has no countable decomposition of ω_1 into orthogonal pieces.*
(b) $P(\vec{a}) \Vdash \mathcal{R}(\mathcal{I})^V \preccurlyeq^i_V \mathcal{R}(\mathcal{I})$; *hence*, $\mathcal{R}(\mathcal{I}) \preccurlyeq P(\vec{a}) \star \mathcal{R}(\mathcal{I})$.
(c) $\mathcal{R}(\mathcal{I}) \Vdash P(\vec{a})^V \preccurlyeq^i_V P(\vec{a})$; *hence*, $P(\vec{a}) \preccurlyeq \mathcal{R}(\mathcal{I}) \star P(\vec{a})$.
(d) $P(\vec{a}) \star \mathcal{R}(\mathcal{I}) \,/\, \mathcal{R}(\mathcal{I}) \cong P(\vec{a})$.
(e) $P(\vec{a}) \star \mathcal{R}(\mathcal{I}) \cong P(\vec{a}) \times \mathcal{R}(\mathcal{I}) \cong \mathcal{R}(\mathcal{I}) \times P(\vec{a})$.
(f) $P(\vec{a}) \star \mathcal{R}(\mathcal{I}) \cong \mathcal{R}(\mathcal{I}) \star P(\vec{a})$.
(g) $\mathcal{R}(\mathcal{I}) \Vdash$ ⌜$P(\vec{a})^V$ is densely included in $P(\vec{a})$⌝.
(h) $(\mathcal{I}, \mathcal{R})^\frown\vec{a} \in \mathbb{P}^{\omega_1}_P$.
(i) *Let $G \in \operatorname{Gen}(V, \mathcal{R}(\mathcal{I}))$ and $G \star H \in \operatorname{Gen}(V, \mathcal{R}(\mathcal{I}) \star P(\vec{a}))$. Then for every $P(\vec{a})$-name $\dot{\mathcal{H}}$ for a σ-directed subfamily of $[\omega_1]^{\aleph_0}$, if $V[H] \models$* ⌜$\dot{\mathcal{H}}[H]$ has no countable decomposition into orthogonal sets⌝, *then so does $V[G \star H] \models$* ⌜$\dot{\mathcal{H}}[H]$ has no such countable decomposition of ω_1⌝.

Proof. All clauses (a)–(i) are proved simultaneously by induction on $\operatorname{len}(\vec{a})$.

Base case: $\operatorname{len}(\vec{a}) = 0$.

$P(\vec{a})$ is the trivial forcing notion. Thus clauses (a)–(h) are trivial, while (i) reduces to Corollary 129.

Successor case: $\operatorname{len}(\vec{a}) = \xi + 1$.

Then $\vec{a}$ is either of the form $\vec{b}^\frown(\dot{\mathcal{J}}, \mathcal{R})$ or $\vec{b}^\frown(\dot{\mathcal{J}}, \mathcal{Q})$, i.e. $P(\vec{a})$ is either of the form $P(\vec{b}) \star \mathcal{R}(\dot{\mathcal{J}})$ or $P(\vec{b}) \star \mathcal{Q}(\dot{\mathcal{J}})$ (possibly $\vec{b} = \langle\rangle$).

For clause (a), let $G \in \operatorname{Gen}(V, P(\vec{b}))$ and let $G \star H \in \operatorname{Gen}(V, P(\vec{a}))$. By the induction hypothesis, $\mathcal{I}$ has no countable decomposition, in $V[G]$, of ω_1 into orthogonal pieces. In the first case $\vec{a} = \vec{b}^\frown(\dot{\mathcal{J}}, \mathcal{R})$, applying Corollary 129 in $V[G]$, this remains true in $V[G\star H]$. In the other case $\vec{a} = \vec{b}^\frown(\dot{\mathcal{J}}, \mathcal{Q})$. Then, in $V[G]$, $\dot{\mathcal{J}}[G]$ is a P-ideal since $\vec{a} \in \mathbb{C}_P^{\omega_1}$; and $\mathcal{R}(\mathcal{I})$ forces there is no stationary set orthogonal to $\dot{\mathcal{J}}[G]$, because, in V, $\vec{a} \in \mathbb{Q}^{\omega_1}$ and thus $\vec{a}\restriction\xi^\frown(\mathcal{I}, \mathcal{R}) \in D(\vec{a}\restriction\xi)$ implies that $P(\vec{b})\star\mathcal{R}(\mathcal{I})$ forces there is no stationary subset of ω_1 orthogonal to $\dot{\mathcal{J}}$. Therefore, Corollary 132 applies in $V[G]$, establishing that in $V[G\star H]$ there is no countable decomposition of ω_1 into pieces orthogonal to $\mathcal{I}$.

Clause (b) follows from clause (a), just as in the proof of Corollary 130.

For clause (c), given a maximal antichain $A \subseteq P(\vec{a})$, we need to show that $\mathcal{R}(\mathcal{I}) \Vdash \ulcorner A$ is a maximal antichain of $P(\vec{a})\urcorner$. First suppose $\vec{a}$ is of the form $\vec{b}^\frown(\dot{\mathcal{J}}, \mathcal{R})$. Fix $I \in \mathrm{Gen}(V, \mathcal{R}(\mathcal{I}))$. Then take $J \in \mathrm{Gen}(V[I], P(\vec{b})^{V[I]})$, so that $I\star J \in \mathrm{Gen}(V, \mathcal{R}(\mathcal{I})\star P(\vec{b}))$. By Proposition 90, we have that $P(\vec{b})$ forces $A\,/\,P(\vec{b})$ is a maximal antichain of $\mathcal{R}(\dot{\mathcal{J}})$. Therefore, by the induction hypothesis that clause (c) holds for $\vec{b}$, $J \in \mathrm{Gen}(V, P(\vec{b}))$ and hence putting $B = (A\,/\,P(\vec{b}))[J]$, $V[J] \models \ulcorner B$ is a maximal antichain of $\mathcal{R}(\dot{\mathcal{J}}[J])\urcorner$. We apply Corollary 107 with $V := V[J]$, $\mathcal{H} := \dot{\mathcal{J}}[J]$, $A := B$ and $W := V[I\star J]$. For any $x \in \dot{\mathcal{J}}[J]$, applying the induction hypothesis that clause (i) holds for $\vec{b}$, with $\dot{\mathcal{J}} := \Psi(\dot{\mathcal{J}}_{(x)})$, we see that if, in $V[J]$, there is no countable decomposition into sets orthogonal to $\Psi(\dot{\mathcal{J}}_{(x)})[J] = \Psi(\dot{\mathcal{J}}[J]_{(x)})$, then, in $V[I\star J]$, there is also no such decomposition. Therefore, Corollary 107 yields $V[I\star J] \models \ulcorner(A\,/\,P(\vec{b}))[J]$ is a maximal antichain of $\mathcal{R}(\dot{\mathcal{J}}[J])\urcorner$. Since J is arbitrary, this proves that $V[I] \models \ulcorner A$ is a maximal antichain of $\mathcal{R}(\vec{a})\urcorner$ by Proposition 90, as desired. The other case where $\vec{a} = \vec{b}^\frown(\dot{\mathcal{J}}, \mathcal{Q})$, is exactly the same but Corollary 106 is used instead.

For (d), the hypothesis of Lemma 142, with $P := P(\vec{a})$, is satisfied because $P(\vec{a})$ adds no new countable subsets of ω_1 and by clause (b). Then clause (d) is the conclusion of the lemma.

Clause (e) is a restatement of clause (d) together with the fact that products commute.

Clause (f) is proved algebraically. First consider $\vec{a} = \vec{b}^\frown(\dot{\mathcal{J}}, \mathcal{R})$. Since $P(\vec{b})$ adds no new reals, $P(\vec{b}) \Vdash \mathrm{CH}$, and thus

$$P(\vec{b}) \Vdash \mathcal{R}(\dot{\mathcal{J}})\star\mathcal{R}(\mathcal{I}) \cong \mathcal{R}(\mathcal{I})\star\mathcal{R}(\dot{\mathcal{J}}) \tag{52}$$

by applying Corollary 145 in this forcing extension. Now by associativity of iterated forcing for the first equivalence, by Eq. (52) for the second equivalence, and by the induction hypothesis that (f) holds for $\vec{b}$ for the third equivalence,

$$\begin{aligned} P(\vec{a})\star\mathcal{R}(\mathcal{I}) &\cong P(\vec{b})\star[\mathcal{R}(\dot{\mathcal{J}})\star\mathcal{R}(\mathcal{I})] \\ &\cong [P(\vec{b})\star\mathcal{R}(\mathcal{I})]\star\mathcal{R}(\dot{\mathcal{J}}) \\ &\cong \mathcal{R}(\mathcal{I})\star[P(\vec{b})\star\mathcal{R}(\dot{\mathcal{J}})] \\ &= \mathcal{R}(\mathcal{I})\star P(\vec{a}), \end{aligned} \tag{53}$$

as required.

Now we consider the other case $\vec{a} = \vec{b}^\frown(\dot{\mathcal{J}}, \mathcal{Q})$. If $\dot{\mathcal{H}}$ is a $P(\vec{b})$-name for a σ-directed family with no countable decomposition of ω_1 into orthogonal pieces, then $\vec{b}^\frown(\dot{\mathcal{H}}, \mathcal{R}) \in D(\vec{b})$ and hence $P(\vec{b}) \Vdash \mathcal{R}(\dot{\mathcal{H}}) \Vdash \ulcorner$there is no stationary set orthogonal to $\dot{\mathcal{J}}\urcorner$ because $\vec{a} \in \mathbb{Q}^{\omega_1}$. Therefore, the hypothesis of Corollary 146 holds in the extension by $P(\vec{b})$, and hence by the corollary,

$$P(\vec{b}) \Vdash \mathcal{Q}(\dot{\mathcal{J}})\star\mathcal{R}(\mathcal{I}) \cong \mathcal{R}(\mathcal{I})\star\mathcal{Q}(\dot{\mathcal{J}}). \tag{54}$$

Now we can obtain the result in exactly the same manner as Eq. (53).

Clause (g) is an immediate consequence of (e) and (f) and Proposition 97.

For (h), put $\vec{c} = (\mathcal{I}, \mathcal{R})^\frown\vec{a}$. First of all note that $\vec{c} \in \mathbb{C}^{\omega_1}$ by clause (c). Take $\xi < \mathrm{len}(\vec{c})$. We can assume $\xi > 0$ since $\vec{c}(0) = (\mathcal{I}, \mathcal{R})$ is not of the form $(\dot{\mathcal{H}}, \mathcal{Q})$, say $\xi = 1+\eta$. Then $\eta < \mathrm{len}(\vec{a})$. We have to deal with the situation where $\vec{a}(\eta)$ is of the form $(\dot{\mathcal{H}}, \mathcal{Q})$, in which case we must show that $\mathcal{R}(\mathcal{I})\star P(\vec{a}\restriction\eta)$ forces there is no stationary set orthogonal to $\dot{\mathcal{H}}$. Applying the induction hypothesis that clause (f) holds for $\vec{a}\restriction\eta$, $\mathcal{R}(\mathcal{I})\star P(\vec{a}\restriction\eta) \cong P(\vec{a}\restriction\eta)\star\mathcal{R}(\mathcal{I})$. Now $\vec{d} = (\vec{a}\restriction\eta)^\frown(\mathcal{I}, \mathcal{R}) \in D(\vec{a}\restriction\eta)$, and thus $P(\vec{d})$ forces there is no stationary set orthogonal to $\dot{\mathcal{H}}$, because $\vec{a} \in \mathbb{Q}^{\omega_1}$. Since $P(\vec{c}\restriction\xi) = \mathcal{R}(\mathcal{I})\star P(\vec{a}\restriction\eta) \cong P(\vec{d})$, this concludes the proof that $\vec{c} \in \mathbb{P}^{\omega_1}$.

For clause (i), let $\dot{\mathcal{H}}$ be a $P(\vec{a})$-name for a σ-directed family. Let $G \in \mathrm{Gen}(V, \mathcal{R}(\mathcal{I}))$ and $H \in \mathrm{Gen}(V[G], P(\vec{a})^{V[G]})$ (thus $G\star H \in \mathrm{Gen}(V, \mathcal{R}(\mathcal{I})\star P(\vec{a}))$). Then $H \in \mathrm{Gen}(V, P(\vec{a}))$ by clause (c). We assume that, in $V[H]$, there is no countable decomposition into sets orthogonal to $\dot{\mathcal{H}}[H]$. But then by (f), we know that $V[G\star H]$ is an $\mathcal{R}(\mathcal{I})$-generic extension of $V[H]$, and therefore there is no countable decomposition, in $V[G\star H]$, into sets orthogonal to $\dot{\mathcal{H}}[H]$ by Corollary 129.

Limit case: $\mathrm{len}(\vec{a})$ equals some limit ordinal δ.

First we establish clause (c). Let $G \in \mathrm{Gen}(V, \mathcal{R}(\mathcal{H}))$. In $V[G]$: $P(\vec{a})$ is the limit of $(P(\vec{a}\restriction\xi) : \xi < \delta)$ (Proposition 151). And by the induction hypothesis, $P(\vec{a}\restriction\xi)^V \preccurlyeq_V^i P(\vec{a}\restriction\xi)$ for all $\xi < \delta$. Therefore, by Lemma 95, the limit, let us call it Q, of $(P(\vec{a}\restriction\xi)^V : \xi < \delta)$ is generically included in $P(\vec{a})$ over V. Since we are dealing with countable support iterations, and since $\mathcal{R}(\mathcal{H})$ adds no new reals, the limit of $(P(\vec{a}\restriction\xi)^V : \xi < \delta)$ is the same whether taken here in $V[G]$ or in the ground model V. Hence $P(\vec{a})^V = Q \preccurlyeq_V^i P(\vec{a})^{V[G]}$.

For clause (h), we first of all have $(\mathcal{I}, \mathcal{R})^\frown\vec{a} \in \mathbb{C}_P^{\omega_1}$ by clause (c). It then follows immediately from the induction hypothesis that $(\mathcal{I}, \mathcal{R})^\frown P(\vec{a}\restriction\xi) \in \mathbb{P}^{\omega_1}$ for all $\xi < \delta$, that $(\mathcal{I}, \mathcal{R})^\frown P(\vec{a}) \in \mathbb{P}^{\omega_1}$.

Next we deal with clause (a). If there is no countable decomposition of ω_1 into pieces orthogonal to $\mathcal{I}$, then $\mathcal{R}(\mathcal{I})$ forces an uncountable set locally in $\mathcal{I}$. Moreover, $\mathcal{R}(\mathcal{I}) \star P(\vec{a})$ is proper, and in particular does not collapse $\aleph_1$, by Lemma 152, because $(\mathcal{I}, \mathcal{R})^\frown \vec{a} \in \mathbb{P}^{\omega_1}$ by clause (h). Therefore, as $P(\vec{a}) \preccurlyeq \mathcal{R}(\mathcal{I}) \star P(\vec{a})$ by (c), $P(\vec{a})$ cannot force a countable decomposition of ω_1 into pieces orthogonal to $\mathcal{I}$.

Clause (b) follows from clause (a) as before.

Clause (d) follows from clause (b) exactly as in the successor case; similarly for clause (e).

For clause (f), let $G \in \mathrm{Gen}(V, \mathcal{R}(\mathcal{I}))$. In $V[G]$: By the induction hypothesis that clause (g) holds for $\vec{a} \restriction \xi$ for all $\xi < \mathrm{len}(\vec{a})$, we have that $P(\vec{a} \restriction \xi)^V$ is densely included in $P(\vec{a} \restriction \xi)$ for all $\xi < \delta$. Therefore, by Lemma 98, the limit of $(P(\vec{a} \restriction \xi) : \xi < \delta)$, call it Q, is isomorphic as a forcing notion to $P(\vec{a})$. Since $\mathcal{R}(\mathcal{I})$ adds no reals, and the iterations are of countable support, $P(\vec{a})^V = Q \cong P(\vec{a})$. Now, back in V, we have established that $\mathcal{R}(\mathcal{I}) \times P(\vec{a}) \cong \mathcal{R}(\mathcal{I}) \star P(\vec{a})$, and thus the result is now a consequence of (e).

Clause (g) follows as for the successor case.

Clause (i) follows from (c), (f) and Corollary 129 identically as for the successor case. □

Lemma 158 (CH). *Let $\mathcal{I}$ be a P-ideal on ω_1 and let $\vec{a} \in \mathbb{Q}_P^{\omega_1}$. If $P(\vec{a})$ adds no new reals, then all of the following are true*:

(a) $P(\vec{a}) \Vdash \mathcal{Q}(\mathcal{I})^V \preccurlyeq_V^i \mathcal{Q}(\mathcal{I})$; *hence*, $\mathcal{Q}(\mathcal{I}) \preccurlyeq P(\vec{a}) \star \mathcal{Q}(\mathcal{I})$.
(b) *If* $\vec{a}^\frown(\mathcal{I}, \mathcal{Q}) \in \mathbb{Q}^{\omega_1}$ *then* $\mathcal{Q}(\mathcal{I}) \Vdash P(\vec{a})^V \preccurlyeq_V^i P(\vec{a})$; *hence*, $P(\vec{a}) \preccurlyeq \mathcal{Q}(\mathcal{I}) \star P(\vec{a})$.
(c) $P(\vec{a}) \star \mathcal{Q}(\mathcal{I}) \,/\, \mathcal{Q}(\mathcal{I}) \cong P(\vec{a})$.
(d) $P(\vec{a}) \star \mathcal{Q}(\mathcal{I}) \cong P(\vec{a}) \times \mathcal{Q}(\mathcal{I}) \cong \mathcal{Q}(\mathcal{I}) \times P(\vec{a})$.
(e) *If* $\vec{a}^\frown(\mathcal{I}, \mathcal{Q}) \in \mathbb{Q}^{\omega_1}$ *then* $P(\vec{a}) \star \mathcal{Q}(\mathcal{I}) \cong \mathcal{Q}(\mathcal{I}) \star P(\vec{a})$.
(f) *If* $\vec{a}^\frown(\mathcal{I}, \mathcal{Q}) \in \mathbb{Q}^{\omega_1}$ *then* $\mathcal{Q}(\mathcal{I}) \Vdash \ulcorner P(\vec{a})^V$ is densely included in $P(\vec{a})\urcorner$.
(g) *If* $\vec{a}^\frown(\mathcal{I}, \mathcal{Q}) \in \mathbb{Q}^{\omega_1}$ *then* $(\mathcal{I}, \mathcal{R})^\frown \vec{a} \in \mathbb{P}_P^{\omega_1}$.
(h) *Let* $G \in \mathrm{Gen}(V, \mathcal{Q}(\mathcal{I}))$ *and* $G \star H \in \mathrm{Gen}(V, \mathcal{Q}(\mathcal{I}) \star P(\vec{a}))$. *Then for every* $P(\vec{a})$*-name* $\dot{\mathcal{H}}$ *for a* σ*-directed subfamily of* $[\omega_1]^{\aleph_0}$, *if*
   (1) $V[H] \models \ulcorner \dot{\mathcal{H}}[H]$ has no countable orthogonal decomposition$\urcorner$,
   (2) $V[H] \models \ulcorner \mathcal{R}(\dot{\mathcal{H}}[H])$ forces that there is no stationary set orthogonal to $\mathcal{I}\urcorner$,
   then so does $V[G \star H] \models \ulcorner \dot{\mathcal{H}}[H]$ *has no countable orthogonal decomposition of* $\omega_1\urcorner$.

Proof. The proof is by induction on $\mathrm{len}(\vec{a})$. The base case $\mathrm{len}(\vec{a}) = 0$ is completely straightforward, and the limit case is the same as for the proof of Lemma 157. Hence we only deal with the successor case $\mathrm{len}(\vec{a}) = \xi + 1$.

Clause (a) follows from Lemma 157(a), just as in the proof of Corollary 131.

For clause (b), given a maximal antichain $A \subseteq P(\vec{a})$, we need to show that $\mathcal{Q}(\mathcal{I}) \Vdash \ulcorner A$ is a maximal antichain of $P(\vec{a})\urcorner$. First suppose $\vec{a}$ is of the form $\vec{b}^\frown(\dot{\mathcal{J}}, \mathcal{R})$. Fix $I \in \mathrm{Gen}(V, \mathcal{Q}(\mathcal{I}))$. Then take $J \in \mathrm{Gen}(V[I], P(\vec{b})^{V[I]})$, so that $I \star J \in \mathrm{Gen}(V, \mathcal{Q}(\mathcal{I}) \star P(\vec{b}))$. By Proposition 90, we have that $P(\vec{b})$ forces $A \,/\, P(\vec{b})$ is a maximal antichain of $\mathcal{R}(\dot{\mathcal{J}})$. Therefore, by the induction hypothesis that clause (b) holds for $\vec{b}, J \in \mathrm{Gen}(V, P(\vec{b}))$ and hence putting $B = (A \,/\, P(\vec{b}))[J]$, $V[J] \models \ulcorner B$ is a maximal antichain of $\mathcal{R}(\dot{\mathcal{J}}[J])\urcorner$. We apply Corollary 107 with $V := V[J]$, $\mathcal{H} := \dot{\mathcal{J}}[J]$, $A := B$ and $W := V[I \star J]$. For any $x \in \dot{\mathcal{J}}[J]$, suppose that, in $V[J]$, there is no countable decomposition into sets orthogonal to $\Psi(\dot{\mathcal{J}}[J]_{(x)})$. Since $\vec{b}^\frown(\Psi(\dot{\mathcal{J}}_{(x)}), \mathcal{R}) \in D(\vec{b})$, and since the hypothesis on $\mathcal{I}$ clearly entails that $\vec{b}^\frown(\mathcal{I}, \mathcal{Q}) \in \mathbb{Q}^{\omega_1}$, we have that $P(\vec{b}) \star \mathcal{R}(\Psi(\dot{\mathcal{J}}_{(x)}))$ forces there is no stationary set orthogonal to $\mathcal{I}$, i.e. $V[J] \models \ulcorner \mathcal{R}(\Psi(\dot{\mathcal{J}}[J]_{(x)}))$ forces there is no stationary set orthogonal to $\mathcal{I}\urcorner$. Therefore, the induction hypothesis (h) applies to $\vec{b}$ with $\dot{\mathcal{H}} := \Psi(\dot{\mathcal{J}}_{(x)})$, and thus, in $V[I \star J]$, there is also no countable decomposition into sets orthogonal to $\Psi(\dot{\mathcal{J}}[J]_{(x)})$. Therefore, Corollary 107 yields $V[I \star J] \models \ulcorner (A \,/\, P(\vec{b}))[J]$ is a maximal antichain of $\mathcal{R}(\dot{\mathcal{J}}[J])\urcorner$. Since J is arbitrary, this proves that $V[I] \models \ulcorner A$ is a maximal antichain of $\mathcal{R}(\vec{a})\urcorner$ by Proposition 90, as desired. The other case where $\vec{a} = \vec{b}^\frown(\dot{\mathcal{J}}, \mathcal{Q})$, is exactly the same but Corollary 106 is used instead.

Clause (c) is a consequence of Lemma 141 with $P := P(\vec{a})$, by the hypothesis that $P(\vec{a})$ does not add reals and clause (a).

Clause (d) is a restatement of clause (c).

Clause (e) is proved algebraically. First consider $\vec{a} = \vec{b}^\frown(\dot{\mathcal{J}}, \mathcal{R})$. Since $P(\vec{b})$ adds no new reals, $P(\vec{b}) \Vdash$ CH; and for every $P(\vec{b})$-name $\dot{\mathcal{H}}$ for a σ-directed family, it follows from the fact that $\vec{a}^\frown(\mathcal{I}, \mathcal{Q}) \in \mathbb{Q}^{\omega_1}$ that $P(\vec{b}) \Vdash \mathcal{R}(\dot{\mathcal{H}}) \Vdash \ulcorner$there is no stationary set orthogonal to $\mathcal{I}\urcorner$; and thus

$$P(\vec{b}) \Vdash \mathcal{R}(\dot{\mathcal{J}}) \star \mathcal{Q}(\mathcal{I}) \cong \mathcal{Q}(\mathcal{I}) \star \mathcal{R}(\dot{\mathcal{J}}) \tag{55}$$

by applying Corollary 146 the forcing extension by $P(\vec{b})$. Using by Eq. (55) for the second equivalence, and the induction hypothesis (e) for $\vec{b}$ for the third equivalence,

$$\begin{aligned} P(\vec{a}) \star \mathcal{Q}(\mathcal{I}) &\cong P(\vec{b}) \star [\mathcal{R}(\dot{\mathcal{J}}) \star \mathcal{Q}(\mathcal{I})] \\ &\cong [P(\vec{b}) \star \mathcal{Q}(\mathcal{I})] \star \mathcal{R}(\dot{\mathcal{J}}) \\ &\cong \mathcal{Q}(\mathcal{I}) \star [P(\vec{b}) \star \mathcal{R}(\dot{\mathcal{J}})] \\ &= \mathcal{Q}(\mathcal{I}) \star P(\vec{a}), \end{aligned} \tag{56}$$

as required.

Now we consider the other case $\vec{a} = \vec{b}^\frown(\dot{\mathcal{J}}, \mathcal{Q})$. If $\dot{\mathcal{H}}$ is a $P(\vec{b})$-name for a σ-directed family, then $\vec{b}^\frown(\dot{\mathcal{H}}, \mathcal{R}) \in D(\vec{b})$ and hence $P(\vec{b}) \Vdash \mathcal{R}(\dot{\mathcal{H}}) \Vdash \ulcorner$there is no stationary set orthogonal to $\dot{\mathcal{J}}\urcorner$ because $\vec{a} \in \mathbb{Q}^{\omega_1}$; and furthermore, we saw above that $P(\vec{b}) \Vdash \mathcal{R}(\dot{\mathcal{H}}) \Vdash \ulcorner$there is no stationary set orthogonal to $\mathcal{I}\urcorner$. Therefore, the hypothesis of Corollary 147 holds in the extension by $P(\vec{b})$, and hence by the corollary,

$$P(\vec{b}) \Vdash \mathcal{Q}(\dot{\mathcal{J}}) \star \mathcal{Q}(\mathcal{I}) \cong \mathcal{Q}(\mathcal{I}) \star \mathcal{Q}(\dot{\mathcal{J}}). \tag{57}$$

Now we can obtain the result in exactly the same manner as Eq. (56).

Clause (f) is an immediate consequence of Proposition 97.

For (g) is immediate from (e).

For clause (h), let $\dot{\mathcal{H}}$ be a $P(\vec{a})$-name for a σ-directed family. Let $G \in \mathrm{Gen}(V, \mathcal{Q}(\mathcal{I}))$ and $H \in \mathrm{Gen}(V[G], P(\vec{a})^{V[G]})$. Then $H \in \mathrm{Gen}(V, P(\vec{a}))$ by clause (b); and by (e), we know that $V[G \star H]$ is a $\mathcal{Q}(\mathcal{I})$-generic extension of $V[H]$. We assume that (1) and (2) hold, and therefore, in $V[H]$, the hypotheses of Corollary 132 hold with $\mathcal{H} := \dot{\mathcal{H}}[H]$, and hence, in $V[G \star H]$, there is no countable decomposition into sets orthogonal to $\dot{\mathcal{H}}[H]$ by the corollary. □

The following theorem is the absoluteness result we have been working towards.

Theorem 5 (CH). *Let $\vec{a}, \vec{b} \in \mathbb{C}_P^{\omega_1}$. Suppose $\vec{a}^\frown\vec{b} \in \mathbb{Q}^{\omega_1}$. If $P(\vec{a}^\frown\vec{b})$ adds no new reals, then $P(\vec{a}) \Vdash P(\vec{b})^V \preccurlyeq_V^i P(\vec{b})$, and hence $P(\vec{b}) \preccurlyeq P(\vec{a}^\frown\vec{b})$. Moreover,*

$$\begin{aligned} P(\vec{a}^\frown\vec{b}) = P(\vec{a}) \star P(\vec{b}) &\cong P(\vec{a}) \times P(\vec{b}) \\ &\cong P(\vec{b}) \times P(\vec{a}) \cong P(\vec{b}) \star P(\vec{a}) = P(\vec{b}^\frown\vec{a}). \end{aligned} \tag{58}$$

Proof. This is proved by a straightforward induction from Lemmas 157 and 158. □

Let us next describe how Theorem 5 is applied, after introducing notation for concatenating sequences of sequences.

Definition 159. For $X \subseteq \mathrm{On}$ and any sequence $\vec{\vec{x}} = (\vec{x}_\gamma : \gamma \in X)$ of *sequences* (i.e. functions whose domains are ordinals), let $\rho(\vec{\vec{x}})$ be the concatenation under the ordinal ordering, i.e. $\rho(\vec{\vec{x}})$ is a sequence of length $\sum_{\gamma \in X} \mathrm{len}(\vec{x}_\gamma)$ and $\rho(\vec{\vec{x}}) \restriction [\zeta_\gamma, \zeta_{\gamma+1}) = \vec{x}_\gamma$ for all $\gamma \in X$, where $\zeta_\gamma = \sum_{\xi \in X, \xi < \gamma} \mathrm{len}(\vec{x}_\xi)$.

Proposition 160. *Suppose that $\vec{\vec{a}} = (\vec{a}_\gamma : \gamma \in X)$ where each $\vec{a}_\gamma \in \mathbb{C}^\theta$. Then every $p \in P(\rho(\vec{\vec{a}}))$ is of the form $\rho(\vec{p})$ where $\vec{p} = (p_\gamma : \gamma \in X)$ and each $p_\gamma \in P(\vec{a}_\gamma)$.*

Definition 161. For all $\vec{a}, \vec{b} \in \mathbb{C}^\theta$, let $e(\vec{a}, \vec{b}) : P(\vec{b}) \to P(\vec{a}^\frown\vec{b})$ be given by

$$e(\vec{a}, \vec{b})(p) = 0_{P(\vec{a})}{}^\frown p \tag{59}$$

for all $p \in P(\vec{b})$.

More generally, suppose that $\vec{\vec{a}} = (\vec{a}_\gamma : \gamma < \delta)$ is a sequence with each $\vec{a}_\gamma \in \mathbb{C}^\theta$. For $X \subseteq \delta$, let $f(\vec{\vec{a}}, X) : P(\rho(\vec{\vec{a}} \restriction X)) \to P(\rho(\vec{\vec{a}}))$ be given by $f(\vec{\vec{a}}, X)(p) = \rho(\vec{q})$ where $\vec{q} = (q_\gamma : \gamma < \delta)$ is given by

$$q_\gamma = \begin{cases} p_\gamma, & \text{if } \gamma \in X, \\ 0_{P(\vec{a}_\gamma)}, & \text{if } \gamma \notin X, \end{cases} \tag{60}$$

and $p = \rho(\vec{p})$ as in Proposition 160.

The following are corollaries of Theorem 5.

Corollary 162 (CH). *Let $\vec{a}, \vec{b} \in \mathbb{C}_P^{\omega_1}$. Suppose $\vec{a}^\frown\vec{b} \in \mathbb{Q}^{\omega_1}$ and $P(\vec{a}^\frown\vec{b})$ adds no new reals. Then $e(\vec{a}, \vec{b})$ is a generic embedding.*

Corollary 163 (CH). *Let $\vec{a}_\gamma \in \mathbb{C}_P^{\omega_1}$ ($\gamma < \delta$). Suppose that $\rho(\vec{\vec{a}}) \in \mathbb{Q}^{\omega_1}$ and $P(\rho(\vec{\vec{a}}))$ adds no new reals. Then $f(\vec{\vec{a}}, X)$ is a generic embedding for all $X \subseteq \delta$.*

Remark. The argument in Example 85 applies so that the antisymmetric quotient of $P(\rho(\vec{\vec{a}}))$ is a complete semilattice and the map from $P(\rho(\vec{\vec{a}} \restriction X))/\sim_{\mathrm{asym}}$ into $P(\rho(\vec{\vec{a}}))/\sim_{\mathrm{asym}}$ induced by $f(\vec{\vec{a}}, X)$ has an upward order closed range. Hence Lemma 84 justifies the following definition.

Definition 165. Let $\pi(\vec{a}, \vec{b}) : P(\vec{a}^\frown\vec{b}) \to P(\vec{b})$ be the projection defined in Eq. (23) from $e(\vec{a}, \vec{b})$; and let $\nu(\vec{\vec{a}}, X) : P(\rho(\vec{\vec{a}})) \to P(\rho(\vec{\vec{a}} \restriction X))$ be the projection defined in Eq. (23) from $f(\vec{\vec{a}}, X)$.

Proposition 166. *$\pi(\vec{a}, \vec{b})$ is a left inverse of $e(\vec{a}, \vec{b})$.*

Proposition 167. *$\nu(\vec{\vec{a}}, X)$ is a left inverse of $f(\vec{\vec{a}}, X)$.*

The important properties of π, and more generally of ν are:

Lemma 168 (CH). *Let $\vec{a}, \vec{b} \in \mathbb{C}_P^{\omega_1}$. Suppose $\vec{a}^\frown\vec{b} \in \mathbb{Q}^{\omega_1}$ and $P(\vec{a}^\frown\vec{b})$ adds no new reals. If $q \in \operatorname{gen}^+(M, P(\vec{a}^\frown\vec{b}))$ then $\pi(\vec{a}, \vec{b})(q) \in \operatorname{gen}^+(M, P(\vec{b}))$.*

Proof. Proposition 88, Corollary 162 and Proposition 166. □

Lemma 169 (CH). *Suppose that $\vec{\vec{a}}$ is a sequence of members of $\mathbb{C}_P^{\omega_1}$, with $\rho(\vec{\vec{a}}) \in \mathbb{Q}^{\omega_1}$, and $P(\rho(\vec{\vec{a}}))$ adds no new reals. If $q \in \operatorname{gen}^+(M, P(\rho(\vec{\vec{a}})))$, then for all $X \subseteq \delta$, $\rho(\vec{q}) \in \operatorname{gen}^+(M, P(\rho(\vec{\vec{a}} \restriction X)))$ where*

$$q_\gamma = \pi\big(\rho(\vec{\vec{a}} \restriction \gamma), \vec{a}_\gamma\big)\big(q \restriction \operatorname{len}(\rho(\vec{\vec{a}} \restriction \gamma)^\frown\vec{a}_\gamma)\big) \tag{61}$$

for all $\gamma \in X$.

Proof. By Proposition 88, Corollary 163 and Proposition 167, $\nu(\vec{\vec{a}}, X)(q) \in \operatorname{gen}^+(M, P(\rho(\vec{\vec{a}} \restriction X)))$. Eq. (61) is established by verifying that

$$\pi\big(\rho(\vec{\vec{a}} \restriction \gamma), \vec{a}_\gamma\big)\big(q \restriction \operatorname{len}(\rho(\vec{\vec{a}} \restriction \gamma)^\frown\vec{a}_\gamma)\big) \sim_{\text{sep}} p_\gamma \quad \text{for all } \gamma \in X, \tag{62}$$

where $\nu(\vec{\vec{a}}, X)(q) = \rho(\vec{p})$ and $\vec{p} = (p_\gamma : \gamma \in X)$. □

4.4. trind-*properness*

In [8, Ch. XVIII, Definition 2.1], the notation $\operatorname{trind}_\alpha(t)$ is used to denote all labelings $\vec{\beta} = (\beta_x : x \in t)$ of some finite tree t with ordinals at most α, i.e. $\beta_x \le \alpha$ for all $x \in t$, so that

$$x <_t y \quad \text{implies} \quad \beta_x \le \beta_y. \tag{63}$$

An operation is defined on iterations $\vec{P} = (P_\xi, \dot{Q}_\xi : \xi < \alpha)$ of length α by members of $\vec{\beta} \in \operatorname{trind}_\alpha(t)$, where $\vec{P}_{\vec{\beta}}$ is the collection of all sequences $(p_x : x \in t)$ such that

(i) $p_x \in P_{\beta_x}$ for all $x \in t$,
(ii) $x <_t y$ implies $p_y \restriction \beta_x = p_x$.

Thus for example, if t is a finite tree of height 1 then for every $\vec{\beta} \in \operatorname{trind}_\alpha(t)$, $\vec{P}_{\vec{\beta}}$ is a finite product of the form $P_{\beta_0} \times \cdots \times P_{\beta_{n-1}}$ where $\beta_i \le \alpha$ for all $i = 0, \ldots, n-1$.

Then (in [8, Ch. XVIII, Definition 2.2]) the notion of an NNR_2-*iteration* is defined, which in particular entails that the iteration is completely proper. Then the new theorem for iterations not adding new reals is [8, Ch. XVIII, Main Lemma 2.8] stating that if $(P_\xi : \xi \le \delta)$ an iteration where P_ξ is NNR_2 for all ξ less than the limit δ, then P_δ is NNR_2. Without reviewing the details of the definition of NNR_2, we refer to this theory as the trind-*properness* NNR theory.

Unexpectedly, in overcoming the difficulties in constructing a properness parameter suitable for forcing ($\star_c$), we came very close to satisfying the hypotheses for the trind-properness NNR theory. Indeed, using the methods we have already presented, our Theorem 5 can be extended to say: $P(\vec{a})_{\vec{\beta}}$ is proper for all $\vec{a} \in \mathbb{G}_P^{\omega_1}$ and all $\vec{\beta} \in \operatorname{trind}_\alpha(t)$ for every finite tree t. Thus our iteration, which will be of the form $P(\vec{a})$ for some $\vec{a} \in \mathbb{G}_P^{\omega_1}$, is trind-*proper*, i.e. it remains proper after operating on it with members of $\operatorname{trind}_\alpha(t)$.

We think it is most likely that Shelah's above-mentioned theorem can be strengthened to something like: if $(P_\xi, \dot{Q}_\xi : \xi < \delta)$ is a countable support iteration such that $P_\xi \Vdash \ulcorner\dot{Q}_\xi$ is $\mathbb{D}$-complete$\urcorner$ for all $\xi < \delta$, and $(P_\xi, \dot{Q}_\xi : \xi < \delta)$ is trind-proper, then P_δ adds no new reals (probably this would require a slightly more general operation than trind). This seems to agree with his description of the essence of the theory in [8, page 868]; however, at present we do not have a good enough understanding of his proof to make a conjecture.

Such a theorem would result in a better (or at least shorter) proof of Theorem 1 than the one here using properness parameters. However, as it stands, the definition of $\vec{P} = (P_\xi, \dot{Q}_\xi : \xi < \delta)$ being NNR_2 requires the properness of $\vec{P}'_{\vec{\beta}}$ for $\beta \in \operatorname{trind}_\alpha(t)$ where $\vec{P}'$ is some arbitrary completely proper *extension* of some initial segment of $\vec{P}$. Our iteration will not satisfy this requirement of NNR_2.

5. Model of CH

We begin with an arbitrary ground model V (of enough of ZFC) satisfying GCH. Set $\kappa = \aleph_2$ and $\lambda = \aleph_3$ as in Eq. (7).

As usual, $\operatorname{NS}([A]^{\aleph_0})$ denotes the nonstationary ideal on $[A]^{\aleph_0}$ and $\operatorname{NS}^*([A]^{\aleph_0})$ is the dual filter, and thus is generated by the family of closed cofinal subsets of $[A]^{\aleph_0}$.

Definition 170. Whenever $V \models \ulcorner\mathcal{E} \in \operatorname{NS}^*([H_\kappa]^{\aleph_0})\urcorner$, let $\operatorname{NS}^*(\mathcal{E}; V) = \big\{\mathcal{F} \subseteq \mathcal{E} : \mathcal{F} \cap \mathcal{S} \ne \emptyset \text{ for all } \mathcal{S} \in \big(\operatorname{NS}^*([H_\kappa]^{\aleph_0})\big)^V\big\}$. Let $\operatorname{NS}^*(V)$ denote $\operatorname{NS}^*(([H_\kappa]^{\aleph_0})^V, V)$, when H_κ is understood.

Proposition 171. *Suppose P is proper and $\dot{\mathcal{T}}$ is a P-name where $P \Vdash \dot{\mathcal{T}} \subseteq \mathcal{E}[\dot{G}_P]$ (cf. Notation 30). The following are equivalent:*

(a) $P \Vdash \dot{\mathcal{T}} \in \mathrm{NS}^*(\mathcal{E}; V)$.
(b) *For all $M \prec H_\lambda$ with $P, \mathcal{E} \in M$ and $M \cap H_\kappa \in \mathcal{E}$, every $p \in P \cap M$ has an (M, P)-generic extension q such that $q \Vdash M \cap H_\kappa \in \dot{\mathcal{T}}$.*

5.1. The properness parameters

The cardinal sequence

$$\mu_\alpha = \aleph_{2+\alpha}^+ \quad (\alpha < \omega_1) \tag{64}$$

is suitable for a λ-properness parameter, and we let $\vec{\mathcal{A}}$ be any fixed skeleton, e.g. $\mathcal{A}_\alpha = \{M \in [H_{\mu_\alpha}]^{\aleph_0} : M \prec H_{\mu_\alpha}\}$ $(\alpha < \omega_1)$.

We begin by motivating the definitions to follow. Let $\mathcal{H}$ be a σ-directed subfamily of $([\omega_1]^{\aleph_0}, \subseteq^*)$ with no stationary subset of ω_1 orthogonal to it. In Lemma 59 we saw that for a given map Ω on $\varinjlim \mathcal{A}$, a sufficient condition for $\mathcal{Q}(\mathcal{H})$ to be $(\vec{\mathcal{A}}, \mathfrak{D}_\Omega)$-proper is that there exist $y_M \in \Omega(M) \cap {\downarrow}\mathcal{H}$ for all $M \in \varinjlim \mathcal{A}$ of positive rank satisfying $\varphi_*(M, \mathcal{H}, y)$. Conversely, suppose that $X \subseteq \mathcal{M}(q) \cap M$ for some $q \in \mathrm{gen}(M, \mathcal{Q}(\mathcal{H}))$. Since $C_{\dot{G}_{\mathcal{Q}(\mathcal{H})}}$ names a club (cf. Corollary 51), $q \Vdash \operatorname{tr}\sup_{\omega_1}(X) \subseteq C_{\dot{G}_{\mathcal{Q}(\mathcal{H})}}$. Therefore, every initial segment of $\operatorname{tr}\sup_{\omega_1}(X)$ must be in ${\downarrow}\mathcal{H}$.

Consider the next simplest case: an iteration of the form $\mathcal{R}(\mathcal{I}) \star \mathcal{Q}(\dot{\mathcal{H}})$ where $\dot{\mathcal{H}}$ names an $\mathcal{H}$ as above. In order that $\mathcal{R}(\mathcal{I}) \star \mathcal{Q}(\dot{\mathcal{H}})$ is $\mathfrak{D}_\Omega$-proper, we must in particular have for every M of positive rank, that every finite sequence $(p_0, \dot{s}_0), \ldots, (p_{n-1}, \dot{s}_{n-1}) \in \mathcal{R}(\mathcal{I}) \star \mathcal{Q}(\dot{\mathcal{H}}) \cap M$ has $(q_i, \dot{t}_i) \in \mathrm{gen}(M, \mathcal{R}(\mathcal{I}) \star \mathcal{Q}(\dot{\mathcal{H}}), p_i)$ $(i = 0, \ldots, n-1)$ and an $X \in \mathfrak{D}_\Omega(M)$ such that $X \subseteq \bigcap_{i=0}^{n-1} \mathcal{M}(q_i, \dot{t}_i)$. Let us focus on the case $n = 2$. We shall need $y_0, y_1 \in \Omega(M)$ such that $q_i \Vdash y_i \in {\downarrow}\dot{\mathcal{H}}$ and $q_i \Vdash \varphi_*(M, \mathcal{H}, y_i)$ for $i = 0, 1$. Then to ensure that $\mathfrak{D}_\Omega(M) \neq \emptyset$ we would apply Lemma 23.

In particular, to satisfy property (iii) in the definition of instantiation, this means that we must be able to find cofinally many $K \in \varinjlim \mathcal{A} \cap M$ with

$$\sup(\omega_1 \cap K) \in y_0 \cap y_1. \tag{65}$$

This can be achieved as follows. Let $\vec{a}$ be the code for $\mathcal{R}(\mathcal{I})$, let $\vec{b}$ be the code for $\mathcal{R}(\mathcal{I}) \star \mathcal{Q}(\dot{\mathcal{H}})$. Assume that $\vec{b} \in \mathbb{G}^{\omega_1}$. First of all we choose $r \in \mathrm{gen}^+(M, P(\vec{a}^\frown\vec{a}), p_0{}^\frown p_1)$. Then we let $q_0 = \pi(\emptyset, \vec{a})(r \restriction \mathrm{len}(\vec{a})) \in P(\vec{a})$ and $q_1 = \pi(\vec{a}, \vec{a}^\frown\vec{a})(r) \in P(\vec{a})$. It follows from an application of Lemma 169 that $q_0{}^\frown q_1 \in \mathrm{gen}^+(M, P(\vec{a}^\frown\vec{a}))$. Then by extending q_0 and q_1 we may assume that there exist y_0 and y_1 as above. Now for some fixed $b \in M \cap H_\lambda$ and $\xi < \mathrm{rank}(M)$, suppose that we want to find $K \in \mathcal{A}_\xi \cap M$ with $b \in K$ satisfying (65). Since $\mathcal{A}_\xi \in M$ is stationary, $S = \operatorname{tr}\sup(\mathcal{A}_\xi) \subseteq \omega_1$ is a stationary set in M. By the assumption that $P(\vec{a})$ forces that $\dot{\mathcal{H}}$ has no stationary orthogonal set and by Lemma 71, $P(\vec{a}) \star \mathcal{R}(\dot{\mathcal{H}})$ forces that $S \cap X_{\dot{G}_{\mathcal{R}(\dot{\mathcal{H}})}}$ is stationary. Since in particular, $\vec{b} \in \mathbb{Q}^{\omega_1}$, we know that $P(\vec{a}) \star \mathcal{R}(\dot{\mathcal{H}})$ forces there is no stationary set orthogonal to $\dot{\mathcal{H}}$. It then follows from Lemma 89 that $P(\vec{a}) \star \mathcal{R}(\dot{\mathcal{H}})$ forces that there is no stationary set orthogonal to $e^*(\vec{a}, \vec{a})(\dot{\mathcal{H}})$. Hence applying Lemma 71 again, $P(\vec{a}) \star \mathcal{R}(\dot{\mathcal{H}}) \star \mathcal{R}(e^*(\vec{a}, \vec{a})(\dot{\mathcal{H}}))$ forces that

$$S \cap X_{\ddot{G}_{\mathcal{R}(\dot{\mathcal{H}})}} \cap X_{\ddot{G}_{\mathcal{R}(e^*(\vec{a},\vec{a})(\dot{\mathcal{H}}))}} \text{ is stationary.} \tag{66}$$

The whole point of invoking the embedding $e(\vec{a}, \vec{a})$ is that we want the name $\dot{\mathcal{H}}$ to be interpreted according to q_1 (in particular, $e^*(\vec{a}, \vec{a})(\dot{\mathcal{H}})$ is independent of q_0, unlike $\dot{\mathcal{H}} \cap M$ which is determined by q_0). It is now straightforward to produce $K \in \mathcal{A}_\xi \cap M$ with $b \in K$ satisfying (65).

The argument just outlined is a simplified version of our main lemma, Lemma 182. The general case, were $\vec{a}$ codes an initial segment of our iteration, is where we need to use $\mathbb{G}^{\omega_1}$.

Definition 172. Suppose $\vec{a} \in \mathbb{C}^\theta$. Let $C(\vec{a})$ be the set of all codes generated by restriction and concatenations of arbitrary length, i.e. $\vec{b} \in C(\vec{a})$ implies that $b \restriction \xi \in C(\vec{a})$ for all $\xi < \mathrm{len}(\vec{b})$, and $\vec{b}_\gamma \in C(\vec{a})$ $(\gamma < \delta)$ implies that $\vec{c} \in C(\vec{a})$, where

$$\vec{c} = \vec{b}_0{}^\frown \cdots {}^\frown \vec{b}_\gamma{}^\frown \cdots \quad (\gamma < \delta). \tag{67}$$

For κ an infinite cardinal, we let $C(\vec{a}, \kappa)$ be the subfamily of $C(\vec{a})$ generated by restrictions, and concatenations of length less than κ.

Proposition 173. *Let κ be an infinite regular cardinal. Then $C(\vec{a}, \kappa)$ consists of all codes of the form $(\vec{a} \restriction \xi_0)^\frown \cdots {}^\frown (\vec{a} \restriction \xi_\gamma)^\frown \cdots$ $(\gamma < \delta)$ where each $\xi_\gamma < \mathrm{len}(\vec{a})$ and $\delta < \kappa$.*

Notation 174. *For each $\vec{a} \in \mathbb{C}^\theta$, and each ordinal γ, we let $\vec{a}^\gamma$ denote the concatenation $\vec{a}^\frown\vec{a}^\frown \cdots$ iterated γ times, i.e. $\mathrm{len}(\vec{a}^\gamma) = \mathrm{len}(\vec{a}) \cdot \gamma$ and $\vec{a}^\gamma(\mathrm{len}(\vec{a}) \cdot \zeta + \rho) = \vec{a}(\rho)$ for all $\zeta < \gamma$ and $\rho < \mathrm{len}(\vec{a})$ (and considering $P(\vec{a})$-names to also be $P(\vec{a}^\frown\vec{b})$-names).*

Definition 175. Define a (class) function $\psi = \psi_\theta : \mathbb{C}^\theta \times \mathrm{On} \to \mathbb{C}^\theta$ by recursion on $\mathrm{len}(\vec{a})$ by $\psi(\langle\rangle, \gamma) = \langle\rangle$, and

$$\begin{aligned}\psi(\vec{a}, \gamma) &:= \bigcup_{\xi < \mathrm{len}(\vec{a})} \psi(\vec{a} \restriction \xi) \\ &= (\vec{a} \restriction 1)^\gamma{}^\frown(\vec{a} \restriction 2)^\gamma{}^\frown \cdots {}^\frown (\vec{a} \restriction \xi)^\gamma{}^\frown \cdots \quad (\xi < \mathrm{len}(\vec{a})).\end{aligned} \tag{68}$$

Proposition 176. $\psi_\theta(\vec{a}, \gamma) \in C(\vec{a}, \max\{\mathrm{len}(\vec{a}), |\gamma|\}^+)$ *for all* $\vec{a} \in \mathbb{C}^\theta$.

Henceforth, $\theta = \omega_1$.

Definition 177. We let

$$\Phi\big(\vec{a}, (r^M, q^M_{\xi\gamma}, y^M_{\xi\gamma} : M \in \varinjlim \mathcal{A}, \vec{a} \in M, \xi < \mathrm{len}(\vec{a}) \text{ and } \gamma < \omega_1 \text{ are in } M)\big)$$

be a formula expressing the following state of affairs: $\vec{a} \in \mathbb{G}^{\omega_1}$; and for all $M \in \varinjlim \mathcal{A}$ with $\vec{a} \in M$,

(i) $r^M \in \mathrm{gen}^+(M, P(\psi(\vec{a}, \omega_1)))$,

and for all $\xi < \mathrm{len}(\vec{a})$ and all $\gamma < \omega_1$ with $\xi, \gamma \in M$,

(ii) $q^M_{\xi\gamma} \in P(\vec{a} \restriction \xi)$,
(iii) $q^M_{\xi\gamma} \geq \pi\big(\vec{a} \restriction \xi, \bigcup_{\zeta<\xi} \psi(\vec{a} \restriction \zeta)^\frown(\vec{a} \restriction \xi)^\gamma\big)\big(r^M \restriction \mathrm{len}\big(\bigcup_{\zeta<\xi} \psi(\vec{a} \restriction \zeta)^\frown(\vec{a} \restriction \xi)^\gamma\big)\big)$,
(iv) $q^M_{\xi\gamma} \Vdash y^M_{\xi\gamma} \in \dot{\mathcal{H}}(\vec{a}(\xi))$,
(v) $q^M_{\xi\gamma} \Vdash x \subseteq^* y^M_{\xi\gamma}$ for all $x \in \dot{\mathcal{H}}(\vec{a}(\xi)) \cap M$.

We abbreviate the above expression as $\Phi\big(\vec{a}, (\vec{r}, \vec{q}, \vec{y})\big)$.

Assuming $\Phi\big(\vec{a}, (\vec{r}, \vec{q}, \vec{y})\big)$, for each $\vec{c} \in C(\vec{a}, \aleph_0)$, say $\vec{c} = (\vec{a} \restriction \xi_0)^\frown \cdots ^\frown (\vec{a} \restriction \xi_{k-1})$ (cf. Proposition 173), and each $\alpha < \omega_1$, define a $P(\vec{c})$-name

$$\dot{\mathcal{B}}^{\vec{c}}_\alpha = \big\{M \in \mathcal{A}_\alpha : q^M_{\xi_0\gamma_0}{}^\frown \cdots ^\frown q^M_{\xi_{k-1}\gamma_{k-1}} \in \dot{G}_{P(\vec{c})} \text{ for some } \gamma_0 < \cdots < \gamma_{k-1} \text{ in } \omega_1 \cap M\big\}. \tag{69}$$

We also define $\Omega(\vec{y})(M) \in [[\theta]^{\aleph_0}]^{\leq\aleph_0}$ by

$$\Omega(\vec{y})(M) = \big\{y^M_{\xi\gamma} : \xi < \mathrm{len}(\vec{a}), \gamma < \omega_1, \xi, \gamma \in M\big\}, \tag{70}$$

and we put

$$\mathcal{Z}(\vec{a}) = \big\{(P(\vec{c}), \dot{\mathcal{B}}^{\vec{c}}) : \vec{c} \in C(\vec{a}, \aleph_0)\big\}. \tag{71}$$

Lemma 178. *For all* $\vec{c} = (\vec{a} \restriction \xi_0)^\frown \cdots ^\frown (\vec{a} \restriction \xi_{k-1})$ *in* $C(\vec{a}, \aleph_0)$,

$$q^M_{\xi_0\gamma_0}{}^\frown \cdots ^\frown q^M_{\xi_{k-1}\gamma_{k-1}} \in \mathrm{gen}^+(M, P(\vec{c})) \tag{72}$$

for all $\gamma_0 < \cdots < \gamma_{k-1}$ *in* $\omega_1 \cap M$.

Proof. This is a straightforward application of Lemma 169. □

The reason that the codes are repeated ω_1 times (rather than just ω times) is so that we have the following.

Lemma 179. *For all* $\vec{c} = (\vec{a} \restriction \xi_0)^\frown \cdots ^\frown (\vec{a} \restriction \xi_{k-1})$ *in* $C(\vec{a}, \aleph_0)$*, every* $p \in P(\vec{c}) \cap M$ *has* $\gamma_0 < \cdots < \gamma_{k-1}$ *in* $\omega_1 \cap M$ *such that*

$$q^M_{\xi_0\gamma_0}{}^\frown \cdots ^\frown q^M_{\xi_{k-1}\gamma_{k-1}} \geq p. \tag{73}$$

Proof. Standard density argument since we have countable supports with an iteration of uncountable cofinality. □

Lemma 180. $P(\vec{c}) \Vdash \dot{\mathcal{B}}^{\vec{c}}_\alpha \in \mathrm{NS}^*(\mathcal{A}_\alpha, V)$ *for all* $\alpha < \omega_1$.

Proof. We apply Proposition 171. Find $N \prec H_{\mu_{\alpha+1}}$ with $\mathcal{A}_\alpha \in N$ and $M = \underset{\sim}{N} \in \mathcal{A}_\alpha$. Take $p \in P(\vec{c}) \cap M$. Then $q^M_{\xi_0\gamma_0}{}^\frown \cdots ^\frown q^M_{\xi_{k-1}\gamma_{k-1}} \geq p$ for some $\gamma_0 < \cdots < \gamma_{k-1}$ in $\omega_1 \cap M$ by Lemma 179. Then $q^M_{\xi_0\gamma_0}{}^\frown \cdots ^\frown q^M_{\xi_{k-1}\gamma_{k-1}} \Vdash M \in \dot{\mathcal{B}}^{\vec{c}}_\alpha$ as wanted. □

Notation 181. *For an iterated forcing notion of the form* $R = P_0 \star \dot{Q}_0 \star \dot{Q}_1 \star \cdots \star \dot{Q}_n$*, a* R*-name* $\dot{A}$ *and* $r = (p, \dot{q}(0), \ldots, \dot{q}(n)) \in \mathrm{gen}^+(M, R)$*, we let* $\dot{A}[p, \dot{q}(0), \ldots, \dot{q}(n)]$ *denote the interpretation of* $\dot{A}$ *by* $\dot{G}_R[M, r]$ *(cf.* Section 1.1*).*

Lemma 182. $\Phi\big(\vec{a}, \dot{\vec{\mathcal{H}}}, (\vec{r}, \vec{q}, \vec{y})\big)$ *implies that for all* $M \in \varinjlim \mathcal{A}$ *with* $\mathrm{rank}(M) > 0$*, for all* $\vec{c} \in C(\vec{a}, \aleph_0) \cap M$*, say as in* (67)*, for all* $\gamma_0 < \cdots < \gamma_{k-1}$ *in* $\omega_1 \cap M$*, for all* $b \in M \cap H_\lambda$*, for all* $\alpha < \mathrm{rank}(M)$ *there exists* $K \in \mathcal{A}_\alpha \cap M$ *such that*

(a) $b \in K$,
(b) $\sup(\theta \cap K) \in \bigcap_{i=0}^{k-1} y^M_{\xi_i\gamma_i}$,
(c) $K \in \dot{\mathcal{B}}^{\vec{b}}_\alpha\big[q^M_{\xi_0\gamma_0}, \ldots, q^M_{\xi_{k-1}\gamma_{k-1}}\big]$.

Proof. Working in $M[\dot{G}_{P(\vec{c})}[q^M_{\xi_0\gamma_0}, \ldots, q^M_{\xi_{k-1}\gamma_{k-1}}]]$: Lemma 180 in particular implies that $\mathcal{C} = \big\{K \in \dot{\mathcal{B}}^{\vec{c}}_\alpha[q^M_{\xi_0\gamma_0}, \ldots, q^M_{\xi_{k-1}\gamma_{k-1}}] : b \in K\big\}$ is a cofinal subset of $[H_{\mu_\alpha}]^{\aleph_0}$. Let $S = \mathrm{tr}\,\sup_\theta(\mathcal{C})$, which is thus stationary. We define $\dot{S}_n$ and $\vec{d}_n \in D(\langle\rangle)$ by recursion on $n = 0, \ldots, k$ so that $\dot{S}_0 = S$, $\vec{d}_0 = \langle\rangle$ and

(74) $\dot{S}_{n+1}$ is a $P(\vec{c}) \star \mathcal{R}\big(e^*(\vec{d}_0, \vec{a} \restriction \xi_0)(\dot{\mathcal{H}}(\vec{a}(\xi_0)))\big) \star \cdots \star \mathcal{R}\big(e^*(\vec{d}_n, \vec{a} \restriction \xi_n)(\dot{\mathcal{H}}(\vec{a}(\xi_n)))\big)$-name for a stationary subset of $\dot{S}_n$ locally in $e^*(\vec{d}_n, \vec{a} \restriction \xi_n)(\dot{\mathcal{H}}(\vec{a}(\xi_n)))$,

(75) $\vec{d}_{n+1} = \vec{d}_n{}^\frown\big(e^*(\vec{d}_n, \vec{a} \restriction \xi_n)(\dot{\mathcal{H}}(\vec{a}(\xi_n))), \mathcal{R}\big)$.

This is possible by Lemma 71, by the hypothesis that $\vec{a} \in \mathbb{G}^{\omega_1}$, and thus forcing notions as in (74) do not add stationary subsets of ω_1 orthogonal to any $\dot{\mathcal{H}}(\vec{a}(\xi))$, and therefore do not add stationary subsets orthogonal to any $e^*(\vec{d}_n, \vec{a} \restriction \xi_n)(\dot{\mathcal{H}}(\vec{a}(\xi_n)))$ by Lemma 89.

We can find (an infinite) $x \in [\theta]^{\aleph_0}$ and $\vec{p} \in \mathcal{R}\big(e^*(\vec{d}_0, \vec{a} \restriction \xi_0)(\dot{\mathcal{H}}(\vec{a}(\xi_0)))\big) \star \cdots \star \mathcal{R}\big(e^*(\vec{d}_{k-1}, \vec{a} \restriction \xi_{k-1})(\dot{\mathcal{H}}(\vec{a}(\xi_{k-1})))\big)$ so that $\vec{p} \Vdash x \subseteq \dot{S}_k$. Now by equation (74), $x \in e^*(\vec{d}_n, \vec{a}\restriction\xi_0)(\dot{\mathcal{H}}(\vec{a}(\xi_n)))[e(\vec{d}_n, \vec{a}\restriction\xi_0)(q^M_{\xi_n\gamma_n})] = \dot{\mathcal{H}}(\vec{a}(\xi_n))[q^M_{\xi_n\gamma_n}]$ for all $n = 0, \ldots, k-1$. Thus, as $x \in M$ by complete properness, $x \subseteq^* y^M_{\xi_n\gamma_n}$ by equation (v), for all n. Hence there exists $\delta \in x \cap y^M_{\xi_0\gamma_0} \cap \cdots \cap y^M_{\xi_{k-1}\gamma_{k-1}}$. And then by elementarity, there exists $K \in \mathcal{C} \cap M$ with $\sup(\theta \cap K) = \delta$. □

Corollary 183. *$\Phi\big(\vec{a}, \dot{\vec{\mathcal{H}}}, (\vec{r}, \vec{q}, \vec{y})\big)$ implies that $\mathfrak{D}_{\Omega(\vec{y})}(\mathcal{A}; \mathcal{Z}(\vec{a}))$ is a properness parameter.*

Proof. We apply Lemma 29. Let $M \in \varinjlim \mathcal{A}$ with $\operatorname{rank}(M) > 0$ be given. Each $(P, \dot{\mathcal{B}}) \in \mathcal{Z} \cap M$ is of the form $(P(\vec{c}), \dot{\mathcal{B}}^{\vec{c}})$ for some $\vec{c} \in C(\vec{a}) \cap M$, say $\vec{c} = (\vec{a} \restriction \xi_0)^\frown \cdots ^\frown (\vec{a} \restriction \xi_{n^{\vec{c}}-1})$. Using Lemmas 178 and 179, we can find pairwise disjoint sequences $\vec{\gamma}^{\vec{c}}_p \in \omega_1^{n^{\vec{c}}} \cap M$ $(p \in P(\vec{c}) \cap M)$. We can also arrange that (the ranges of) $\vec{\gamma}^{\vec{c}}_p$ and $\vec{\gamma}^{\vec{c}\,'}_{p'}$ are disjoint whenever $\vec{c} \neq \vec{c}\,'$. Define

$$\bar{q}^M_{P(\vec{c}), \dot{\mathcal{B}}^{\vec{c}}}(p) = q^M_{\xi_0\gamma^{\vec{c}}_p(0)} {}^\frown \cdots {}^\frown q^M_{\xi_{n^{\vec{c}}-1}\gamma^{\vec{c}}_p(n^{\vec{c}}-1)} \tag{76}$$

for each $\vec{c} \in C(\vec{a}) \cap M$ and $p \in P(\vec{c}) \cap M$.

To apply Lemma 29, let $A \subseteq \Omega(\vec{y})$ be finite, say $A = \{y^M_{\xi_0\gamma_0}, \ldots, y^M_{\xi_{k-1}\gamma_{k-1}}\}$, $\vec{c}_0, \ldots, \vec{c}_{m-1}$ be codes for members of $\mathcal{Z} \cap M$, let $O_i \subseteq P_i \cap M$ be finite for each $i = 0, \ldots, m-1$, let $b \in M \cap H_\lambda$ and $\xi < \operatorname{rank}(M)$. By extending both A and the subset of $\mathcal{Z} \cap M$, we may assume without loss of generality that $\{(\xi_0, \gamma_0), \ldots, (\xi_{k-1}, \gamma_{k-1})\} = \bigcup_{i=0}^{m-1} \bigcup_{p\in O_i} \{(\xi_0, \gamma^{\vec{c}_i}_p(0)), \ldots, (\xi_{n^{\vec{c}}-1}, \gamma^{\vec{c}_i}_p(n^{\vec{c}_i} - 1))\}$. Then an application of Lemma 182 yields $K \in \mathcal{A}_\xi \cap M$ with $b \in K$, $\sup(\theta \cap K) \in \bigcap_{i=0}^{k-1} y^M_{\xi_i\gamma_i} = \bigcap A$ and $K \in \dot{\mathcal{B}}^{\vec{c}_0 {}^\frown \cdots {}^\frown (\vec{c}_{m-1})}_\xi[q^M_{\xi_0\gamma_0}, \ldots, q^M_{\xi_{k-1}\gamma_{k-1}}]$. It follows that $\bar{q}^M_{P(\vec{c}_i), \dot{\mathcal{B}}^{\vec{c}_i}}(p) \Vdash K \in \dot{\mathcal{B}}^{\vec{c}_i}_\xi$ for all $i = 0, \ldots, m-1$ and all $p \in O_i$. We have therefore found K witnessing (i), (ii), (iii) and (v) of Lemma 29. Moreover, conditions (iv) and (vi) automatically follow from the definitions of $\Omega(\vec{y})$ and $\dot{\mathcal{B}}^{\vec{c}_i}_\xi$. □

Corollary 184. *$\Phi\big(\vec{a}, (\vec{r}, \vec{q}, \vec{y})\big)$ implies that $P(\vec{c})$ is $\mathfrak{D}_{\Omega(\vec{y})}(\vec{\mathcal{A}}; \mathcal{Z}(\vec{a}))$-proper for all $\vec{c} \in C(\vec{a})$.*

Proof. By Corollary 183, Corollary 28, Eq. (71), the definition of $\dot{\mathcal{B}}^{\vec{c}}_\alpha$ and Proposition 178. □

Remark. What we actually need (see the proof of Lemma 187), is that $P(\vec{c})$ is *long* $\mathfrak{D}_{\Omega(\vec{y})}(\vec{\mathcal{A}}; \mathcal{Z}(\vec{a}))$-proper. This can be proved using the ideas already presented.

The following says that Φ is "preserved" at successors.

Lemma 186. *Assume $\Phi\big(\vec{a}, (\vec{r}, \vec{q}, \vec{y})\big)$. If $\vec{a}^\frown(\dot{\mathcal{H}}, \mathcal{Q}) \in \mathbb{G}^{\omega_1}$, then there exists $(\vec{r}_*, \vec{q}_*, \vec{y}_*)$ such that $\Phi\big(\vec{a}, (\vec{r}_*, \vec{q}_*, \vec{y}_*)\big)$ holds.*

Proof (*Sketch of Proof*). Set $\vec{b} = \vec{a}^\frown(\dot{\mathcal{H}}, \mathcal{Q})$. By Corollary 184 and Theorem 3, we can find $r^M_p \in \operatorname{gen}^+(M, P(\psi(\vec{a}, \omega_1)), p)$ for all $M \in \varinjlim \mathcal{A}$ with $\vec{a} \in M$, and all $p \in P(\psi(\vec{a}, \omega_1)) \cap M$. Then for each p, we can find $q^M_p \geq r^M_p$ and $y^M_p \in [\omega_1]^{\aleph_0}$ such that $q^M_p \Vdash y^M_p \in \dot{\mathcal{H}}$ and $x \subseteq^* y^M_p$ for all $x \in \dot{\mathcal{H}} \cap M$. For each $\alpha < \omega_1$, define a $P(\psi(\vec{a}, \omega_1))$-name

$$\dot{\mathcal{C}}_\alpha = \big\{M \in \mathcal{A}_\alpha : q^M_p \in \dot{G}_{P(\psi(\vec{a},\omega_1))} \text{ for some } p \in P(\psi(\vec{a}, \omega_1))\big\}. \tag{77}$$

Let $G \in \operatorname{Gen}(V, P(\psi(\vec{a}, \omega_1)))$. It is easy to see that $\dot{\mathcal{C}}_\alpha[G]$ is stationary for all $\alpha < \omega_1$. Then defining $\Omega(M) \in [[\omega_1]^{\aleph_0}]^{\leq\aleph_0}$ by $\Omega(M) = \Omega(\vec{y})(M) \cup \{y^M_p : p \in P(\psi(\vec{a}, \omega_1)) \cap M\}$, $\mathcal{Q}(\dot{\mathcal{H}}[G])$ is $(\vec{\mathcal{C}}[G], \mathfrak{D}_\Omega)$-proper by Lemma 59. This proves that $P(\vec{b})$ is $(\vec{\mathcal{A}}, \mathfrak{D}_\Omega)$-proper.

Now this allows us to use the parameterized properness theory to find $r^M_* \in \operatorname{gen}^+(M, \psi(\vec{b}, \omega_1))$ for all M. It is then clear how to find $\vec{q}_*$ and $\vec{y}_*$ so that $\Phi\big(\vec{a}, (\vec{r}_*, \vec{q}_*, \vec{y}_*)\big)$ holds. □

The following says that Φ is "preserved" at limits.

Lemma 187. *Let $\vec{a} \in \mathbb{C}^\theta$. If for all $\xi < \operatorname{len}(\vec{a})$, there exists $(\vec{r}_\xi, \vec{q}_\xi, \vec{y}_\xi)$ satisfying $\Phi\big(\vec{a} \restriction \xi, (\vec{r}_\xi, \vec{q}_\xi, \vec{y}_\xi)\big)$, then there exists $(\vec{r}, \vec{q}, \vec{y})$ satisfying $\Phi\big(\vec{a}, (\vec{r}, \vec{q}, \vec{y})\big)$.*

Proof. This is a straightforward application of Lemma 34. □

Proof (*Proof of Theorem 1*). We are going to recursively define an iterated forcing construct $(P_\xi, \dot{Q}_\xi : \xi < \omega_2)$ of length ω_2 with countable supports, and let P_{ω_2} denote the limit of the iteration. At the same time, we are going to choose $\vec{a}_\xi \in \mathbb{G}^{\omega_1}$ such that

(i) $\operatorname{len}(\vec{a}_\xi) < \omega_2$,
(ii) $P_\xi = P(\vec{a}_\xi)$,
(iii) $a_\xi \subseteq a_\eta$ for all $\xi \leq \eta$;

we will also find $(\vec{r}_\xi, \vec{q}_\xi, \vec{y}_\xi)$ as in Definition 177, so that

(iv) $\Phi\big(\vec{a}_\xi, (\vec{r}_\xi, \vec{q}_\xi, \vec{y}_\xi)\big)$ holds.

Observe that from this information we can already deduce that

(v) P_ξ has the $\aleph_2$-cc for all $\xi \le \omega_2$,
(vi) P_ξ has a dense suborder of cardinality at most $\aleph_2$ for all $\xi \le \omega_2$,
(vii) $P_\xi \Vdash 2^{\aleph_1} = \aleph_2$ for all ξ,
(viii) P_ξ is completely proper for all $\xi \le \omega_2$.

This is so because (i) and (ii) imply that P_ξ is an iteration of length at most ω_2, where each iterand satisfies the properness isomorphism condition by Lemmas 61 and 69; hence, we can conclude condition (v). Conditions (vi) and (vii) are established simultaneously by induction as usual: If $P_\xi \Vdash 2^{\aleph_1} = \aleph_2$, then $P_\xi \Vdash |\dot{Q}_\xi| = |P(\vec{a}_{\xi+1}) / P(\vec{a}_\xi)| \le \aleph_2$, and therefore by the $\aleph_2$-cc, $P_{\xi+1}$ satisfies (vi) and (vii). Condition (viii) is of course by the parameterized properness theory: By (iv) and Corollaries 183 and 184, $\mathfrak{D}_{\Omega(\vec{y}_\xi)}(\vec{\mathcal{A}}; \mathcal{Z}(\vec{a}_\xi))$ is a properness parameter for which P_ξ is proper. Since $P_\xi = P(\vec{a}_\xi)$ is an iteration with $\mathbb{D}$-complete iterands by Lemmas 60 and 68, P_ξ adds no new reals by the NNR theorem (Theorem 3).

Using conditions (v)–(vii), by standard bookkeeping, and regarding P_ξ-names as also being P_η-names for $\xi \le \eta$, we can arrange an enumeration $(\dot{\mathcal{H}}_\xi : \xi < \omega_2)$ of P_ξ-names in advance such that, for every $\xi < \omega_2$ and every P_ξ-name $\dot{\mathcal{H}}$ for a σ-directed subfamily of $[\omega_1]^{\aleph_0}$,

(ix) there exists $\aleph_2$ many $\eta \ge \xi$ such that $P_\eta \Vdash \dot{\mathcal{H}}_\eta = \dot{\mathcal{H}}$.

Now we describe the construction. First we deal with the successor stage $\xi + 1$ of the construction. We separate into two cases:

Case 1 $\vec{a}_\xi {}^\frown (\dot{\mathcal{H}}_\xi, \mathcal{Q}) \in \mathbb{G}^{\omega_1}$.
Case 2 $\vec{a}_\xi {}^\frown (\dot{\mathcal{H}}_\xi, \mathcal{Q}) \notin \mathbb{G}^{\omega_1}$.

In Case 1, we put $\vec{a}_{\xi+1} = \vec{a}_\xi {}^\frown (\dot{\mathcal{H}}_\xi, \mathcal{Q})$. Therefore,

$$P_{\xi+1} \text{ forces that there exists a club locally in } \dot{\mathcal{H}}_\xi \tag{78}$$

by Lemma 39. And there exists $(\vec{r}_{\xi+1}, \vec{q}_{\xi+1}, \vec{y}_{\xi+1})$ satisfying $\Phi\big(\vec{a}_{\xi+1}, (\vec{r}_{\xi+1}, \vec{q}_{\xi+1}, \vec{y}_{\xi+1})\big)$ by Lemma 186.

In Case 2, there exists $\vec{c} \in D(\vec{a})$ and $\vec{b} \in C(\vec{c})$ with $\operatorname{len}(\vec{b}) < \omega_2$, with a condition $p \in P(\vec{b})$ such that

$$p \Vdash \text{there exists a stationary set orthogonal to } \dot{\mathcal{H}}_\xi. \tag{79}$$

We set $\vec{a}_{\xi+1} = \vec{b}$. By Corollary 184, $P(\vec{b})$ is $\mathfrak{D}_{\Omega(\vec{y})}$-proper and thus we can take $(\vec{r}_{\xi+1}, \vec{q}_{\xi+1}, \vec{y}_{\xi+1}) = (\vec{r}_\xi, \vec{q}_\xi, \vec{y}_\xi)$.

At limit stages δ, we let $\vec{a}_\delta = \bigcup_{\xi<\delta} \vec{a}_\xi$. Then there exists $(\vec{r}_\delta, \vec{q}_\delta, \vec{y}_\delta)$ satisfying (iv) by Lemma 187.

Having completed the construction, let $G \in \operatorname{Gen}(V, P_{\omega_2})$. Then $\aleph_1$ is not collapsed, i.e. $\aleph_1^{V[G]} = \aleph_1$, and $V[G] \models \mathrm{CH}$ by condition (viii). Since $V \models \mathrm{CH}$, by the $\aleph_2$-cc and by condition (ix), every σ-directed family $\mathcal{H}$ of $([\omega_1]^{\aleph_0}, \subseteq^*)$ is equal to $\dot{\mathcal{H}}_\xi[G]$ for cofinally many $\xi < \omega_2$. Then assuming standard bookkeeping, we can ensure that there exists $\xi < \omega_2$ such that $\mathcal{H} = \dot{\mathcal{H}}_\xi[G]$ and either Eq. (78) holds, or else there exists $p \in G$ as in Eq. (79). Therefore, $V[G] \models \ulcorner(\star_c)_{\omega_1}\urcorner$. □

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