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Multiple positive solutions for singular nth-order nonlocal boundary value problems in Banach spaces

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ABSTRACT

In this paper, we consider a class of singular nth-order nonlocal boundary value problems in Banach spaces. The existence of multiple positive solutions for the problem is obtained by using the fixed point index theory of strict set contraction operators. To demonstrate the applications of our results, two examples are also given in the paper.

1. Introduction

In the past few years, the theory of ordinary differential equations in abstract spaces has become an important new branch [1,2]. Recently, much attention has been focused on investigating the existence and multiplicity of positive solutions for nonlocal boundary value problems in scalar spaces [3–18] or in abstract spaces [19–25]. However, to the best of our knowledge, the corresponding results for singular higher-order nonlocal boundary value problems in Banach spaces are rarely seen (see, for example, [26,27] and the references therein). To fill the gap, we discuss a class of singular higher-order nonlocal boundary value problems in Banach space in this paper. We should mention here that our work unifies and extends some known results both for multi-point boundary value problems [22,25] and for integral boundary value problems [19], and other relevant results in the literature to some degree. The main technique used in the analysis is the fixed point index theory of strict set contraction operators.

Let E be a real Banach space with norm $\|\cdot\|$ and P be a cone of E, and let θ denote the zero element of E, $I=[0,1]$. The purpose of this paper is to investigate the multiplicity of positive solutions for the following singular nth-order nonlocal boundary value problem (BVP) in Banach spaces:

$$\begin{cases} x^{(n)}(t)+f(t,x(t),x'(t),\dots,x^{(n-2)}(t))=\theta, \quad t\in(0,1),\\ x^{(i)}(0)=\theta, \quad 0\le i\le n-3,\\ ax^{(n-2)}(0)-bx^{(n-1)}(0)=\displaystyle\int_0^1 x^{(n-2)}(s)\mathrm{d}A(s),\\ cx^{(n-2)}(1)+\mathrm{d}x^{(n-1)}(1)=\displaystyle\int_0^1 x^{(n-2)}(s)\mathrm{d}B(s),\end{cases} \tag{1.1}$$

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where $a, b, c, d \geq 0$ with $\rho = ac + ad + bc > 0$, $f \in C[(0, 1) \times P^{n-1}, P]$ and f may be singular at $t = 0$ and/or $t = 1$. A and B are right continuous on $[0, 1)$, left continuous at $t = 1$, and nondecreasing on $[0, 1]$, with $A(0) = B(0) = 0$; $\int_0^1 x^{(n-2)}(s)\mathrm{d}A(s)$ and $\int_0^1 x^{(n-2)}(s)\mathrm{d}B(s)$ denote the Riemann–Stieltjes integrals of $x^{(n-2)}$ with respect to A and B, respectively.

If A and B are step functions on $[0, 1]$ (either A or B may be identical to 0), then BVP (1.1) becomes a generic multi-point BVP, some special cases of which have been extensively studied. When $n = 2$, $f(t, x(t)) = a(t)F(x(t))$, $A(t) = \begin{cases} 0, & t \in [0, \xi), \\ \mu_1, & t \in [\xi, 1], \end{cases}$ $B(t) = \begin{cases} 0, & t \in [0, \eta), \\ \mu_2, & t \in [\eta, 1], \end{cases}$ BVP (1.1) reduces to the generalized Sturm–Liouville four-point BVP

$$\begin{cases} x''(t) + a(t)F(x(t)) = \theta, \quad 0 < t < 1, \\ ax(0) - bx'(0) = \mu_1 x(\xi), \qquad cx(1) + \mathrm{d}x'(1) = \mu_2 x(\eta). \end{cases} \tag{1.2}$$

For the case where nonlinearity is continuous, Liu [25] studied the existence of at least one or two positive solutions to BVP (1.2) by using the fixed point theorem of cone expansion and compression of strict set contractions. When $b = d = 0, c = 1$, $A \equiv 0$, $B(t) = \begin{cases} 0, & t \in [0, \eta), \\ \rho, & t \in [\eta, 1], \end{cases}$ BVP (1.1) reduces to the nth-order three-point BVP

$$\begin{cases} x^{(n)}(t) + f(t, x(t), x'(t), \dots, x^{(n-2)}(t)) = \theta, \quad t \in I, \\ x^{(i)}(0) = \theta, \quad 0 \leq i \leq n-2, \\ x^{(n-2)}(1) = \rho x^{(n-2)}(\eta). \end{cases} \tag{1.3}$$

For the nonsingular case, Zhang et al. [22] established some existence, nonexistence and multiplicity results of positive solutions for the BVP (1.3) by using the fixed point principle in cone and the fixed point index theory for strict set contraction operators.

In addition, for the special case $n = 2, a = c = 1, b = d = 0$, $\int_0^1 x(s)\mathrm{d}A(s) = \int_0^1 g(s)x(s)\mathrm{d}s$, $B \equiv 0$ or $A \equiv 0$, $\int_0^1 x(s)\mathrm{d}B(s) = \int_0^1 g(s)x(s)\mathrm{d}s$, f is nonsingular, Feng et al. [19] investigated the existence and multiplicity of positive solutions for BVP (1.1) by using the fixed point index theory in a cone for strict set contraction operators.

Motivated by the above works, in the present paper, by using the fixed point index theory for strict set contractions, we prove the multiplicity results for the BVP (1.1) in Banach spaces. The results obtained in this paper unify and extend some results in [19,22,25] and other relevant papers to some degree.

The rest of this paper is organized as follows. We shall introduce some lemmas and notations in the rest of this section. In Section 2, we provide some preliminary lemmas. In Section 3, the main results will be stated and proved. Finally, we give two examples to illustrate the applications of our results in Section 4.

It is well known that E is partially ordered by cone P, i.e., $x \leq y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$ (the smallest N is called the normal constant of P) and P is said to be solid if its interior $\mathring{P}$ is not empty. When P is solid, we denote $x \ll y$ if $y - x \in \mathring{P}$. For details on cone theory, see [28].

Let $C[I, E]$ denote the Banach space of all continuous mapping x from I into E with norm $\|x\|_C = \max_{t \in I} \|x(t)\|$. Clearly, $Q = \{x \in C[I, E] : x(t) \geq \theta\}$ is a cone of $C[I, E]$. For any $r > 0$, set $P_r = \{x \in P : \|x\| \leq r\}$, $Q_r = \{x \in Q : \|x\|_C \leq r\}$. $x \in C^{n-1}[I, E] \cap C^n[(0, 1), E]$ is called a solution of BVP (1.1) if it satisfies (1.1). x is a positive solution of (1.1) if x is a solution of (1.1) and $x(t) \geq \theta$, $x(t) \not\equiv \theta$.

For a bounded set V in Banach space E, we denote by $\alpha(V)$ the Kuratowski measure of noncompactness. The operator $A : D \to E$ $(D \subset E)$ is said to be a k-set contraction if $A : D \to E$ is continuous and bounded and there is a constant $k \geq 0$ such that $\alpha(A(S)) \leq k\alpha(S)$ for any bounded $S \subset D$; a k-set contraction with $k < 1$ is called a strict set contraction. Let $\alpha(\cdot)$ and $\alpha_C(\cdot)$ denote Kuratowski's measure of noncompactness in E and $C(I, E)$, respectively. For details on the definition and properties of the measure of noncompactness, the reader is referred to [29].

Throughout this paper, we set

$$\phi_1(t) = \frac{at + b}{\rho}, \qquad \phi_2(t) = \frac{-ct + c + d}{\rho}, \qquad \mathbb{R}_+ = [0, +\infty),$$

$$\kappa_1 = \int_0^1 \phi_1(s)\mathrm{d}B(s), \qquad \kappa_2 = \int_0^1 \phi_2(s)\mathrm{d}A(s), \qquad \Lambda = \frac{1}{1 - \kappa_1},$$

$$\kappa_3 = \int_0^1 \phi_2(s)\mathrm{d}B(s), \qquad \kappa_4 = \int_0^1 \phi_1(s)\mathrm{d}A(s), \qquad \Gamma = \frac{1}{1 - \kappa_2},$$

$$\kappa_5 = \int_0^1 \mathrm{d}B(s), \qquad \kappa_6 = \int_0^1 \mathrm{d}A(s), \qquad M_0 = \frac{(a + b)(c + d)}{\rho},$$

$$M = 1 + \frac{\Lambda[(a + b) + \Gamma\kappa_4(c + d)]\kappa_5}{\rho(1 - \Lambda\Gamma\kappa_3\kappa_4)} + \frac{\Gamma[\Lambda\kappa_3(a + b) + (c + d)]\kappa_6}{\rho(1 - \Lambda\Gamma\kappa_3\kappa_4)}.$$

Lemma 1.1 ([29]). *If $H \subset C[I, E]$ is bounded and equicontinuous, then*

$$\alpha_C(H) = \alpha(H(I)) = \max_{t\in I} \alpha(H(t)),$$

where $H(I) = \{u(t) : u \in H,\ t \in I\},\ H(t) = \{u(t) : u \in H\}$.

We also need the following lemma which is concerned with the fixed point index of strict set contractions [29].

Lemma 1.2. *Let K be a cone in a real Banach space E and Ω be a nonempty bounded open convex subset of K. Suppose that $A : \overline{\Omega} \to K$ is a strict set contraction and $A(\overline{\Omega}) \subset \Omega$, where $\overline{\Omega}$ denotes the closure of Ω in K. Then*

$$i(A, \Omega, K) = 1.$$

2. The preliminary lemmas

To establish the existence of multiple positive solutions of BVP (1.1), let us list the following assumptions:

(H_1) $f(t, u_1, u_2, \ldots, u_{n-1}) \in C((0, 1) \times P^{n-1}, P)$, for every $[\alpha, \beta] \subset (0, 1)$ and any $r > 0$, f is uniformly continuous on $[\alpha, \beta] \times P_r^{n-1}$ with respect to t.

(H_2) For any $r > 0$,

$$0 < \int_0^1 G(t, t) f_r(t)\mathrm{d}t < +\infty,$$

where $G(t, s)$ will be given in (2.3), and for $t \in (0, 1)$,

$$f_r(t) = \sup\left\{\|f(t, u_1, u_2, \ldots, u_{n-1})\| : (u_1, u_2, \ldots, u_{n-1}) \in P_r^{n-1}\right\}.$$

(H_3) There exist nonnegative constants L_k $(k = 1, 2, \ldots, n-1)$ with

$$2M_0 M\left[\sum_{k=1}^{n-2} \frac{L_k}{(n-k-2)!} + L_{n-1}\right] < 1$$

such that

$$\alpha(f(t, B_1, B_2, \ldots, B_{n-1})) \le \sum_{k=1}^{n-1} L_k \alpha(B_k)$$

for any $t \in (0, 1)$ and bounded sets $B_k \subset P$ $(k = 1, 2, \ldots, n-1)$.

(H_4) $\kappa_1, \kappa_2 \in [0, 1),\ \Lambda\Gamma\kappa_3\kappa_4 \in [0, 1)$.

In order to overcome the difficulty due to the dependence of f on derivatives, we first consider the following singular second-order nonlinear integro-differential equation:

$$\begin{cases} y''(t) + f\left(t, A_{n-2}y(t), \ldots, A_1 y(t), A_0 y(t)\right) = \theta, & t \in (0, 1),\\ ay(0) - by'(0) = \displaystyle\int_0^1 y(s)\mathrm{d}A(s), \qquad cy(1) + \mathrm{d}y'(1) = \displaystyle\int_0^1 y(s)\mathrm{d}B(s), \end{cases} \tag{2.1}$$

where A_0 is the identity operator, and

$$(A_j y)(t) = \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} y(s)\mathrm{d}s, \quad j = 1, 2, \ldots, n-2.$$

For the proof of our main results, we will make use of the following lemmas.

Lemma 2.1. *The nth-order nonlocal BVP (1.1) has a solution if and only if the nonlinear second-order integro-differential equation (2.1) has a solution.*

Proof. If x is a solution of the nth-order nonlocal BVP (1.1), let $y(t) = x^{(n-2)}(t)$, then it follows from the boundary conditions of the BVP (1.1) and also by exchanging the integral sequence that

$$(A_1 y)(t) = x^{(n-3)}(t), \ldots, (A_{n-3}y)(t) = x'(t), \qquad (A_{n-2}y)(t) = x(t).$$

Thus $y(t) = x^{(n-2)}(t)$ is a solution of the second-order integro-differential equation (2.1).

Conversely, if y is a solution of the second-order integro-differential equation (2.1), let $x(t) = (A_{n-2}y)(t)$, then we have

$$x'(t) = A_{n-3}y(t), \ldots, x^{(n-3)}(t) = A_1 y(t), \qquad x^{(n-2)}(t) = y(t),$$

which imply that $x(0) = \theta,\ x'(0) = \theta, \ldots, x^{(n-3)}(0) = \theta$. Consequently, $x(t) = A_{n-2}y(t)$ is a solution of the nth-order nonlocal BVP (1.1). $\square$

Lemma 2.2. *Assume that $\Lambda\Gamma\kappa_3\kappa_4 \neq 1$ holds. Then for any $g \in C[(0,1),E]$, the problem*

$$\begin{cases} y''(t) + g(t) = \theta, \quad t \in (0,1), \\ ay(0) - by'(0) = \int_0^1 y(s)\mathrm{d}A(s), \\ cy(1) + \mathrm{d}y'(1) = \int_0^1 y(s)\mathrm{d}B(s), \end{cases} \tag{2.2}$$

has a unique solution

$$y(t) = \int_0^1 H(t,s)g(s)\mathrm{d}s,$$

where

$$\begin{gathered} H(t,s) = G(t,s) + \gamma_1(t)\int_0^1 G(\tau,s)\mathrm{d}B(\tau) + \gamma_2(t)\int_0^1 G(\tau,s)\mathrm{d}A(\tau), \\ G(t,s) = \rho\begin{cases} \phi_1(t)\phi_2(s), & 0 \le t \le s \le 1, \\ \phi_1(s)\phi_2(t), & 0 \le s \le t \le 1, \end{cases} \\ \gamma_1(t) = \frac{\Lambda\phi_1(t) + \Lambda\Gamma\kappa_4\phi_2(t)}{1 - \Lambda\Gamma\kappa_3\kappa_4}, \qquad \gamma_2(t) = \frac{\Lambda\Gamma\kappa_3\phi_1(t) + \Gamma\phi_2(t)}{1 - \Lambda\Gamma\kappa_3\kappa_4}. \end{gathered} \tag{2.3}$$

Proof. The proof is similar to Lemma 2.1 of [26], so we omit it. □

Lemma 2.3. *Suppose that* (H_4) *is satisfied, we have*

$$0 \le G(t,s) \le G(s,s) \le M_0, \qquad 0 \le H(t,s) \le MG(s,s), \quad \forall t,s \in I,$$

and for any $\sigma \in \left(0, \frac{1}{2}\right)$,

$$G(t,s) \ge \gamma G(s,s), \quad t \in [\sigma, 1-\sigma],\ s \in I, \tag{2.4}$$

where

$$\gamma = \min\left\{\frac{a\sigma + b}{a+b}, \frac{c\sigma + d}{c+d}\right\}.$$

By Lemmas 2.2 and 2.3, we can obtain the following lemma.

Lemma 2.4. *Let* (H_4) *be satisfied. If $g \in Q$, then the unique solution y of problem (2.2) satisfies $y(t) \ge \theta$, that is $y \in Q$.*

Define an operator $T : Q \to C[I,E]$ by

$$(Ty)(t) = \int_0^1 H(t,s)f\left(s, A_{n-2}y(s), \ldots, A_1y(s), y(s)\right)\mathrm{d}s, \quad t \in I. \tag{2.5}$$

It is easy to see that if $y \in Q \setminus \{\theta\}$ is a fixed point of operator equation $y = Ty$, then $y = y(t)$ is the positive solution of BVP (2.1).

Lemma 2.5. *Suppose that* (H_1)–(H_4) *are satisfied. Then, for any $r > 0$, the operator $T : Q_r \to Q$ is a strict set contraction.*

Proof. For any $y \in Q_r$ and $t \in (0,1)$, by (2.5) and (H_2), we have

$$\|(Ty)(t)\| \le M\int_0^1 G(s,s)f_r(s)\mathrm{d}s < +\infty, \tag{2.6}$$

and thus $T(Q_r) \subset Q$ is bounded. Now we show that T is continuous. For any $y \in Q_r$ and $t_1,\ t_2 \in I$,

$$\begin{aligned} \|(Ty)(t_1) - (Ty)(t_2)\| \le & \int_0^1 |G(t_1,s) - G(t_2,s)|f_r(s)\mathrm{d}s + \kappa_5|\gamma_1(t_1) - \gamma_1(t_2)|\int_0^1 G(s,s)f_r(s)\mathrm{d}s \\ & + \kappa_6|\gamma_2(t_1) - \gamma_2(t_2)|\int_0^1 G(s,s)f_r(s)\mathrm{d}s. \end{aligned}$$

Then for every $V \subset Q_r,\ (TV)(t)$ is equicontinuous on I.

Let $y_p,\ y \in Q_r$ with $\|y_p - y\|_C \to 0$ as $p \to +\infty$, i.e., $\|y_p(t) - y(t)\| \to 0$ as $p \to +\infty$ for $t \in I$. From the Lebesgue dominated convergence theorem and (2.6), it follows that

$$\|(Ty_p)(t) - (Ty)(t)\| \to 0 \quad \text{as } p \to +\infty,\ t \in I. \tag{2.7}$$

Thus, $\{(Ty_p)(t)\}$ is relatively compact for every $t \in I$, and it follows by the equicontinuity of $\{(Ty_p)(t)\}$ and the Ascoli–Arzela theorem that $\{Ty_p\}$ is relatively compact in Q.

Next we show that $\|Ty_p - Ty\|_C \to 0$ as $p \to +\infty$. In fact, if this is not true, then there is a constant $\epsilon_0 > 0$ and a subsequence $\{y_{p_i}\} \subset \{y_p\}$ such that

$$\|Ty_{p_i} - Ty\|_C \geq \epsilon_0, \quad i = 1, 2, 3, \ldots. \tag{2.8}$$

Since $\{Ty_p\}$ is relatively compact, there is a subsequence of $\{Ty_{p_i}\}$ which converges to some $z \in Q$. Without loss of generality, we may assume that $\{Ty_{p_i}\}$ itself converges to z, that is

$$\|Ty_{p_i} - z\|_C \to 0, \quad i \to +\infty. \tag{2.9}$$

By virtue of (2.7) and (2.9) we have $z = Ty$, and so, (2.9) contradicts (2.8). Therefore, T is continuous.

Finally, we show that $T : Q_r \to Q$ is a strict set contraction. Let $V \subset Q_r$ be given arbitrarily, as we have shown in the above that the functions $\{Ty : y \in V\}$ are uniformly bounded and equicontinuous, by Lemma 1.1, $\alpha_C(TV) = \max_{t\in I} \alpha((TV)(t))$. For any $y \in V$, we define

$$(T_p y)(t) = \int_{\frac{1}{p}}^{1-\frac{1}{p}} H(t,s) f(s, A_{n-2}y(s), \ldots, A_1 y(s), y(s))\,\mathrm{d}s, \quad t \in I.$$

By (2.6) we know that

$$(T_p y)(t) \to (Ty)(t) \quad \text{as } p \to +\infty \text{ for } y \in S,\ t \in I \text{ uniformly}.$$

So, for any $\varepsilon > 0$, there exists $X > 0$, for $p > X$, $t \in I$, $y \in V$ such that $\|(T_p y)(t) - (Ty)(t)\| < \varepsilon$. Thus for any $p > X$ and $y \in V,\ t \in I$, we obtain

$$d((T_p y)(t), (TV)(t)) = \inf_{y\in V}\{\|(T_p y)(t) - (Ty)(t)\|\} \leq \|(T_p y)(t) - (Ty)(t)\| < \varepsilon,$$

then

$$\sup_{y\in V} d((T_p y)(t), (TV)(t)) \leq \varepsilon, \quad t \in I.$$

Similarly,

$$\sup_{y\in V} d((T_p V)(t), (Ty)(t)) \leq \varepsilon, \quad t \in I.$$

Hence, we have

$$d_H((T_p V)(t), (TV)(t)) = \max\left\{\sup_{y\in V} d((T_p y)(t), (TV)(t)),\ \sup_{y\in V} d((T_p V)(t), (Ty)(t))\right\} \leq \varepsilon, \tag{2.10}$$

where $d_H(\cdot,\cdot)$ denotes the Hausdorff metric. Thus, by (2.10) we obtain for $p > X$,

$$|\alpha((T_p V)(t)) - \alpha((TV)(t))| < 2 d_H((T_p V)(t), (TV)(t)) \leq 2\varepsilon, \quad t \in I.$$

Hence, we have

$$\lim_{p\to+\infty} \alpha((T_p V)(t)) = \alpha((TV)(t)) \quad \text{for } t \in I. \tag{2.11}$$

In what follows, we estimate $\alpha((T_p V)(t))$ for each $t \in I$. It follows by the formula

$$\int_{\frac{1}{p}}^{1-\frac{1}{p}} y(t)\mathrm{d}t \in \left(1 - \frac{2}{p}\right) \overline{\mathrm{co}} \left\{y(t) : t \in \left[\frac{1}{p}, 1 - \frac{1}{p}\right]\right\}$$

and (H_3) that

$$\begin{aligned}\alpha((T_p V)(t)) &= \alpha\left\{\int_{\frac{1}{p}}^{1-\frac{1}{p}} H(t,s) f(s, A_{n-2}y(s), \ldots, A_1 y(s), y(s))\,\mathrm{d}s : y \in V\right\}\\ &\leq \left(1 - \frac{2}{p}\right) M_0 M \alpha\left(\overline{\mathrm{co}}\left\{f(s, A_{n-2}y(s), \ldots, A_1 y(s), y(s)) : s \in I_p,\ y \in V\right\}\right)\end{aligned}$$

$$\leq M_0 M\alpha\left(f\left(I_p, (A_{n-2}V)(I_p), \dots, (A_1V)(I_p), V(I_p)\right)\right)$$
$$\leq M_0 M\left[\sum_{k=1}^{n-1} L_k\alpha((A_{n-k-1}V)(I_p))\right], \quad t \in I,$$

where $I_p = \left[\frac{1}{p}, 1-\frac{1}{p}\right]$, and then

$$\alpha_C(T_pV) \leq M_0 M\left[\sum_{k=1}^{n-1} L_k\alpha((A_{n-k-1}V)(I_p))\right]. \tag{2.12}$$

Similarly,

$$\alpha((A_kV)(I_p)) \leq \frac{1}{(k-1)!}\alpha(V(I_p)), \quad k = 1, 2, \dots, n-2. \tag{2.13}$$

On the other hand, using a similar method as in the proof of Lemma 2 in [25], we can get that

$$\alpha(V(I_p)) \leq 2\alpha_C(V). \tag{2.14}$$

Therefore, it follows from (2.11)–(2.14) that

$$\alpha_C(TV) \leq 2M_0M\left[\sum_{k=1}^{n-2}\frac{L_k}{(n-k-2)!} + L_{n-1}\right]\alpha_C(V).$$

Noticing that $2M_0M\left[\sum_{k=1}^{n-2}\frac{L_k}{(n-k-2)!} + L_{n-1}\right] < 1$, we claim that T is a strict set contraction. $\square$

3. Main results

In the following, we give the main results of this paper.

Theorem 3.1. *Assume that* (H_1)–(H_4) *are satisfied,* P *is normal and solid, and the following four conditions are satisfied:*

(i) *For* $u_i \in P$ $(i = 1, 2, \dots, n-1)$, *there exist* $q_1, q_2 \in L[(0, 1), \mathbb{R}_+]$ *and* $F \in C[\mathbb{R}_+^{n-1}, \mathbb{R}_+]$ *such that*

$$\|f(t, u_1, \dots, u_{n-1})\| \leq q_1(t) + q_2(t)F(\|u_1\|, \|u_2\|, \dots, \|u_{n-1}\|), \quad t \in (0, 1).$$

(ii) *For* $u_i \in P$ $(i = 1, 2, \dots, n-1)$, *there exists* $q_3 \in L[(0, 1), \mathbb{R}_+]$ *such that*

$$\lim_{\sum_{k=1}^{n-1}\|u_k\|\to+\infty}\frac{\|f(t, u_1, \dots, u_{n-1})\|}{q_3(t)\sum_{k=1}^{n-1}\|u_k\|} = 0$$

uniformly for $t \in (0, 1)$.

(iii) *For* $u_i \in P$ $(i = 1, 2, \dots, n-1)$, *there exists* $q_4 \in L[(0, 1), \mathbb{R}_+]$ *such that*

$$\lim_{\sum_{k=1}^{n-1}\|u_k\|\to 0}\frac{\|f(t, u_1, \dots, u_{n-1})\|}{q_4(t)\sum_{k=1}^{n-1}\|u_k\|} = 0$$

uniformly for $t \in (0, 1)$.

(iv) *There exist* $v \gg \theta$, $t \in [\sigma, 1-\sigma]$ *and* $h(t) \in C([\sigma, 1-\sigma], \mathbb{R}_+)$ *such that*

$$f(t, u_1, \dots, u_{n-1}) \geq h(t)v, \quad t \in [\sigma, 1-\sigma], \qquad u_{n-1} \geq v, \qquad u_i \geq \theta \quad (i = 1, 2, \dots, n-2), \tag{3.1}$$

and

$$\gamma\int_\sigma^{1-\sigma} G(s, s)h(s)\mathrm{d}s > 1, \tag{3.2}$$

where γ *is given in Lemma 2.3. Then the BVP (1.1) has at least two positive solutions.*

Proof. Let T be the cone preserving, strict set contraction that was defined by (2.5). Let

$$q_i^* = \int_0^1 G(s, s)q_i(s)\mathrm{d}s, \quad (i = 1, 2, 3, 4), \qquad \varepsilon_1 = \frac{1}{2Mq_3^*\left[1 + \sum_{i=1}^{n-2}\frac{1}{(i-1)!}\right]}.$$

By virtue of conditions (i) and (ii), given the above $\varepsilon_1 > 0$, there exists $r_1 > 0$ such that

$$\|f(t, u_1, u_2, \ldots, u_{n-1})\| \le \varepsilon_1 q_3(t) \sum_{i=1}^{n-1} \|u_i\|, \quad t \in (0,1),\ u_i \in P,\ \sum_{i=1}^{n-1} \|u_i\| \ge r_1,$$

and

$$\|f(t, u_1, u_2, \ldots, u_{n-1})\| \le q_1(t) + M_1 q_2(t), \quad t \in (0,1),\ u_i \in P,\ \sum_{i=1}^{n-1} \|u_i\| \le r_1$$

where

$$M_1 = \max\{F(x_1, x_2, \ldots, x_{n-1}) : 0 \le x_i \le r_1,\ i = 1, 2, \ldots, n-1\}.$$

Hence

$$\|f(t, u_1, u_2, \ldots, u_{n-1})\| \le \varepsilon_1 q_3(t) \sum_{i=1}^{n-1} \|u_i\| + q_1(t) + M_1 q_2(t), \quad t \in (0,1),\ u_i \in P. \tag{3.3}$$

Choose

$$R > \max\left\{2M(q_1^* + M_1 q_2^*),\ \frac{2}{\sigma}\|v\|\right\} \tag{3.4}$$

and set $U_1 = \{y \in Q : \|y\|_C < R\}$, then $\overline{U}_1 = \{y \in Q : \|y\|_C \le R\}$. For any $y \in \overline{U}_1$, by (3.3), we have

$$\begin{aligned} \|(Ty)(t)\| &\le M \int_0^1 G(s,s) \left[\varepsilon_1 q_3(s) \sum_{i=0}^{n-2} \|(A_i y)(s)\| + q_1(s) + M_1 q_2(s)\right] \mathrm{d}s \\ &\le M\varepsilon_1 q_3^* \left[1 + \sum_{i=1}^{n-2} \frac{1}{(i-1)!}\right] \|y\|_C + M(q_1^* + M_1 q_2^*) \\ &\le \frac{1}{2}\|y\|_C + M(q_1^* + M_1 q_2^*) < \|y\|_C, \quad t \in I, \end{aligned} \tag{3.5}$$

i.e.,

$$\|Ty\|_C < \|y\|_C, \quad \forall y \in \overline{U}_1,$$

which implies that

$$T(\overline{U}_1) \subset U_1. \tag{3.6}$$

Let

$$\varepsilon_2 = \frac{1}{2\left[1 + \sum_{i=1}^{n-2} \frac{1}{(i-1)!}\right] M q_4^*}.$$

By (iii), for $\varepsilon_2 > 0$, there exists $r_2 > 0$ such that

$$\|f(t, u_1, u_2, \ldots, u_{n-1})\| \le \varepsilon_2 q_4(t) \sum_{i=1}^{n-1} \|u_i\|, \quad t \in (0,1),\ u_i \in P,\ \sum_{i=1}^{n-1} \|u_i\| \le r_2.$$

Choose

$$0 < r < \min\left\{r_2 \left[1 + \sum_{i=1}^{n-2} \frac{1}{(i-1)!}\right]^{-1},\ \frac{\|v\|}{N},\ R\right\}$$

and set $U_2 = \{y \in Q : \|y\|_C < r\}$. Then $\overline{U}_2 = \{y \in Q : \|y\|_C \le r\}$, and, for any $y \in \overline{U}_2$, we have

$$\begin{aligned} \|(Ty)(t)\| &\le M\varepsilon_2 \int_0^1 G(s,s) q_4(s) \sum_{i=0}^{n-2} \|(A_i y)(s)\| \mathrm{d}s \\ &\le \varepsilon_2 M q_4^* \left[1 + \sum_{i=1}^{n-2} \frac{1}{(i-1)!}\right] \|y\|_C \\ &\le \frac{1}{2}\|y\|_C < \|y\|_C, \quad t \in I, \end{aligned}$$

i.e.,

$$\|Ty\|_C < \|y\|_C, \quad \forall y \in \overline{U}_2,$$

which implies that

$$T(\overline{U}_2) \subset U_2. \tag{3.7}$$

Let

$$U_3 = \{y \in Q : \|y\|_C < R,\ y(t) \gg v,\ \forall t \in [\sigma, 1-\sigma]\}.$$

As in the proof of Theorem 1 in [30], we can show that U_3 is an open set of Q. Let $e(t) = \frac{2}{\sigma} tv$, it is easy to see that $e \in Q$, $\|e\|_C \le \frac{2}{\sigma}\|v\| < R$ and $e(t) \ge 2v \gg v$ for $t \in [\sigma, 1-\sigma]$. Hence, $e \in U_3$, and so, $U_3 \ne \emptyset$. Let $y \in U_3$, by (3.5), we have $\|Ty\|_C < R$. On the other hand, (2.4), (3.1) and (3.2) imply

$$\begin{aligned}
(Ty)(t) &= \int_0^1 H(t,s) f(s, A_{n-2}y(s), \ldots, A_1 y(s), y(s))\,ds \\
&\ge \int_\sigma^{1-\sigma} G(t,s) f(s, A_{n-2}y(s), \ldots, A_1 y(s), y(s))\,ds \\
&\ge \left(\gamma \int_\sigma^{1-\sigma} G(s,s)h(s)ds\right) v \\
&\gg v, \quad t \in [\sigma, 1-\sigma].
\end{aligned}$$

Hence,

$$T(\overline{U}_3) \subset U_3. \tag{3.8}$$

Since U_1, U_2 and U_3 are nonempty bounded convex open sets of Q, it follows from (3.6)–(3.8) and Lemma 1.2 that

$$i(T, U_i, Q) = 1, \quad i = 1, 2, 3. \tag{3.9}$$

On the other hand, for $y \in U_3$, we have $y(\sigma) \gg v$, and so

$$\|y\|_C \ge \|y(\sigma)\| \ge N^{-1}\|v\|.$$

Consequently,

$$U_2 \subset U_1, \qquad U_3 \subset U_1, \qquad U_2 \cap U_3 = \emptyset. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$i(T, U_1 \setminus (\overline{U_2 \cup U_3}), Q) = i(T, U_1, Q) - i(T, U_2, Q) - i(T, U_3, Q) = -1.$$

Therefore, the operator T has two fixed points $y^* \in U_3$ and $y^{**} \in U_1 \setminus (\overline{U}_2 \cup \overline{U}_3)$ with $y^*(t) \gg v$, $t \in [\sigma, 1-\sigma]$, $\|y^{**}\|_C > r$, and hence $y^*(t) \not\equiv \theta$ and $y^{**}(t) \not\equiv \theta$. This and Lemma 2.1 complete the proof. □

Remark 3.1. Condition (iii) and the continuity of f imply that $f(t, \theta, \theta, \ldots, \theta) = \theta$ for $t \in (0, 1)$. Hence, under the conditions of Theorem 3.1, BVP (1.1) has the trivial solution $y(t) \equiv \theta$ in addition to two positive solutions y^* and y^{**}.

Theorem 3.2. *Let conditions* (H_1)–(H_4), (i) *and* (ii) *be satisfied. Assume that there exist* $v > \theta$, $t \in [\sigma, 1-\sigma]$ *and* $h(t) \in C([\sigma, 1-\sigma], \mathbb{R}_+)$ *such that* (3.1) *holds and*

$$\gamma \int_\sigma^{1-\sigma} G(s,s)h(s)ds \ge 1.$$

Then the BVP (1.1) *has at least one positive solution.*

Proof. Choose R satisfying (3.4) and let

$$U_4 = \{y \in Q : \|y\|_C \le R,\ y(t) \ge v,\ \forall t \in [\sigma, 1-\sigma]\}.$$

It is clear that U_4 is a bounded closed convex set in Q. $U_4 \ne \emptyset$ because $e \in U_4$. Let $y \in U_4$, by (3.5) we have $\|Ty\|_C < R$. On the other hand, as in the proof of Theorem 3.1, we can show that

$$(Ty)(t) \ge v, \quad \forall t \in [\sigma, 1-\sigma],\ y \in U_4.$$

Hence,

$$T(U_4) \subset U_4.$$

Then, the Schauder fixed point theorem implies that T has at least one fixed point $u^* \in U_4$ with $u^*(t) \ge v$, $t \in [\sigma, 1-\sigma]$. The theorem is proved. □

4. Examples

In this section, in order to illustrate the applications of our results, we consider two examples.

Example 4.1. Consider the following four-point boundary value problem for a finite system of third-order scalar differential equations

$$\begin{cases} x'''(t) + \dfrac{16}{\sqrt{t(1-t)}}\left(\dfrac{x(t)+x'(t)}{1+x(t)+x'(t)}\right)^2 = 0, \quad t \in (0,1), \\ x(0) = 0, \qquad x'(0) - x''(0) = \dfrac{1}{4}x'\left(\dfrac{1}{3}\right) + \dfrac{1}{9}x'\left(\dfrac{2}{3}\right), \\ x'(1) + x''(1) = \dfrac{1}{2}x'\left(\dfrac{1}{3}\right) + x'\left(\dfrac{2}{3}\right). \end{cases} \tag{4.1}$$

Conclusion. BVP (4.1) has at least two positive solutions x^* and x^{**}.

Proof. Let $E = \mathbb{R}$ and $P = \mathbb{R}_+$. Then P is a normal and solid cone in E and BVP (4.1) can be regarded as a boundary value problem of the form of (1.1) in E. In this situation, $a = b = c = d = 1$, and

$$A(t) = \begin{cases} 0, & t \in \left[0, \dfrac{1}{3}\right), \\ \dfrac{1}{4}, & t \in \left[\dfrac{1}{3}, \dfrac{2}{3}\right), \\ \dfrac{13}{36}, & t \in \left[\dfrac{2}{3}, 1\right], \end{cases} \qquad B(t) = \begin{cases} 0, & t \in \left[0, \dfrac{1}{3}\right), \\ \dfrac{1}{2}, & t \in \left[\dfrac{1}{3}, \dfrac{2}{3}\right), \\ \dfrac{3}{2}, & t \in \left[\dfrac{2}{3}, 1\right], \end{cases}$$

$$f(t, u_1, u_2) = \frac{16}{\sqrt{t(1-t)}}\left(\frac{u_1+u_2}{1+u_1+u_2}\right)^2, \quad t \in (0,1),\ u_1,\ u_2 \geq 0.$$

Evidently, $f \in C[(0,1) \times P^2, P]$ and is singular at $t = 0$ and $t = 1$. (H_1) is obviously satisfied and (H_3) holds automatically when E is finite dimensional (here, $E = \mathbb{R}$). Note that for $t \in (0,1)$ and $r > 0$,

$$f_r(t) \leq \frac{16}{\sqrt{t(1-t)}},$$

and so (H_2) is satisfied. By calculations, we get

$$\rho = 3, \qquad \phi_1(t) = \frac{1+t}{3}, \qquad \phi_2(t) = \frac{2-t}{3}, \qquad \kappa_1 = \frac{7}{9}, \qquad \kappa_2 = \frac{61}{324}, \qquad \kappa_3 = \frac{13}{18}, \qquad \kappa_4 = \frac{14}{81}.$$

So, (H_4) is satisfied. On the other hand, conditions (i)–(iii) are satisfied for

$$q_1(t) = 0, \qquad q_2(t) = q_3(t) = q_4(t) = \frac{16}{\sqrt{t(1-t)}}, \qquad F(y_1, y_2) = \left(\frac{y_1+y_2}{1+y_1+y_2}\right)^2.$$

Choosing $\sigma = \frac{1}{4}$, then $\gamma = \frac{5}{8}$. For $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$ and $u_1 \geq 0,\ u_2 \geq 1$, we have

$$f(t, u_1, u_2) \geq \frac{16}{\sqrt{t(1-t)}}\left(\frac{1}{2}\right)^2 = \frac{4}{\sqrt{t(1-t)}} = h(t),$$

and

$$\gamma \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,t)h(t)\mathrm{d}t = \frac{5}{6}\int_{\frac{1}{4}}^{\frac{3}{4}} \frac{(1+t)(2-t)}{\sqrt{t(1-t)}}\mathrm{d}t \geq \frac{175}{96}\int_{\frac{1}{4}}^{\frac{3}{4}} \frac{1}{\sqrt{t(1-t)}}\mathrm{d}t = \frac{175}{96} \cdot \frac{\pi}{3} > 1,$$

so condition (iv) is satisfied. Thus, our conclusion follows from Theorem 3.1. □

Example 4.2. Consider the infinite system of scalar fourth-order differential equations

$$\begin{cases} x_n^{(4)}(t) + \dfrac{2\sqrt{2}}{\sqrt{n}\sqrt{t(1-t)}}(1 + x_n - \cos x'_{2n} + x''_{n+1})^{\frac{1}{2}} = 0, \quad t \in (0,1), \\ x_n(0) = x'_n(0) = 0, \qquad x''_n(0) - x'''_n(0) = \dfrac{1}{4}x''_n\left(\dfrac{1}{3}\right) + \dfrac{1}{9}x''_n\left(\dfrac{2}{3}\right), \\ x''_n(1) + x'''_n(1) = \dfrac{1}{2}x''_n\left(\dfrac{1}{3}\right) + x''_n\left(\dfrac{2}{3}\right), \quad n = 1, 2, 3, \dots. \end{cases} \tag{4.2}$$

Conclusion. BVP (4.2) has at least one positive solution.

Proof. Let $E = c_0 = \{x = (x_1, x_2, \dots, x_n, \dots) : x_n \to 0\}$ with the norm $\|x\| = \sup_n |x_n|$, and $P = \{x = (x_1, x_2, \dots, x_n, \dots) \in c_0 : x_n \geq 0,\ n = 1, 2, 3, \dots\}$. Then P is a normal cone in E. Now, BVP (4.2) can be regarded as a boundary value problem of the form of (1.1) in E. In this situation, $a = b = c = d = 1$, A, B are as in Example 4.1, $u = (u_1, u_2, \dots, u_n, \dots)$, $\xi = (\xi_1, \xi_2, \dots \xi_n, \dots)$, $\zeta = (\zeta_1, \zeta_2, \dots \zeta_n, \dots)$, $f = (f_1, f_2, \dots, f_n, \dots)$, in which

$$f_n(t, u, \xi, \zeta) = \frac{2\sqrt{2}}{\sqrt{n}\sqrt{t(1-t)}}(1 + u_n - \cos \xi_{2n} + \zeta_{n+1})^{\frac{1}{2}}, \quad \forall t \in (0,1),\ u, \xi, \zeta \in P.$$

Evidently, $f \in C[(0,1) \times P^3, P]$ and is singular at $t = 0$ and $t = 1$. (H_1) is obviously satisfied. Note that for $t \in (0,1)$ and $r > 0$,

$$f_r(t) \leq \frac{4}{\sqrt{t(1-t)}}(1 + r)^{\frac{1}{2}},$$

and so (H_2) is satisfied. For any $\psi = (\psi_1, \psi_2, \dots, \psi_n, \dots) \in f(t, P_r^3)$, we have

$$0 \leq \psi_n \leq \frac{4}{\sqrt{n}\sqrt{t(1-t)}}(1 + r)^{\frac{1}{2}}, \quad \forall t \in (0,1),\ n = 1, 2, 3, \dots,$$

and the relative compactness of $f(t, P_r^3)$ in c_0 follows directly from a known result (see [31]): a bounded set W of c_0 is relatively compact if and only if

$$\lim_{p \to +\infty} \left\{ \sup_{w \in W} [\max\{|w_m| : m \geq p\}] \right\} = 0.$$

Hence, condition (H_3) is satisfied for $L_k = 0$ $(k = 1, 2, 3)$. As in Example 4.1, condition (H_4) is satisfied. On the other hand,

$$\|f(t, u, \xi, \zeta)\| \leq \frac{2\sqrt{2}}{\sqrt{t(1-t)}}(2 + \|u\| + \|\zeta\|)^{\frac{1}{2}}, \quad t \in (0,1),\ u, \xi, \zeta \in P,$$

which implies that conditions (i)–(ii) are satisfied for

$$q_1(t) = 0, \qquad q_2(t) = q_3(t) = \frac{2\sqrt{2}}{\sqrt{t(1-t)}}, \qquad F(y_1, y_2, y_3) = (2 + y_1 + y_3)^{\frac{1}{2}}.$$

Let $\sigma = \frac{1}{4}$, $\upsilon = \left(1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\right)$, $h(t) = \frac{2\sqrt{2}}{\sqrt{t(1-t)}}$. Then $\gamma = \frac{5}{8}$, $\upsilon > \theta$. For $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$ and $u \geq \theta$, $\xi \geq \theta$, $\zeta \geq \upsilon$, we have

$$f_n(t, u, \xi, \zeta) \geq \frac{2\sqrt{2}}{n\sqrt{t(1-t)}} = \frac{1}{n}h(t),$$

and

$$\gamma \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,t)h(t)\mathrm{d}t \geq \frac{175\sqrt{2}}{192} \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{1}{\sqrt{t(1-t)}}\mathrm{d}t = \frac{175\sqrt{2}}{192} \cdot \frac{\pi}{3} > 1.$$

Thus, our conclusion follows from Theorem 3.2. □

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