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Multiple solutions for a class of semilinear elliptic problems with Robin boundary condition [☆]

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ABSTRACT

In this paper, we show the existence of at least four nontrivial solutions for a class of semilinear elliptic problems with Robin boundary condition and jumping nonlinearities. Solvability of oscillating equations with Robin boundary condition is also investigated. We prove the conclusions by using sub-super-solution method, Fučík spectrum theory, mountain pass theorem in order intervals and Morse theory.

1. Introduction

We consider the following elliptic equation with Robin boundary condition:

$$\begin{cases} -\Delta u + \alpha u = f(u), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + b(x)u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $\alpha > 0$, $b(x) \in L^\infty(\partial\Omega)$, $b(x) \geqslant 0$ and $b(x) \not\equiv 0$ on $\partial\Omega$.

Denote by $\sigma(-\Delta) = \{0 < \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots\}$ the eigenvalues of the following problem:

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + b(x)u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Assume that f satisfies the following assumptions:

(f_1) $f \in C^1(\mathbb{R}\backslash\{0\}, \mathbb{R})$, $f(0) = 0$ and there exist $M_1 > 0$ and $M_2 > 0$ such that $f(M_1) = f(-M_2) = 0$.
(f_2) $f'_-(0) \neq f'_+(0)$ and $\min\{f'_+(0), f'_-(0)\} > \lambda_1 + \alpha$ where α is the same as in (1.1).
(f_3) There exists $m > 0$ such that $f(t) + mt$ is increasing for all $t \in \mathbb{R}$.

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It is well known that the Fučík spectrum of $-\Delta$ is defined as the set Σ of those points $(a,b)\in\mathbb{R}^2$ for which

$$\begin{cases} -\Delta u = bu^+ - au^-, & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \nu} + b(x)u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

has nontrivial solutions, here $u^{\pm}(x)=\max\{\pm u(x),0\}$, see [17]. The usual spectrum of $-\Delta$ corresponds to the case that $a=b$. It is known that Σ consists, at least locally, of curves emanating from the points (λ_l,λ_l), see for example [6].

It was shown in Garbuza [7] and Schechter [20] that Σ contains two continuous and strictly decreasing curves C_{l_1}, C_{l_2} passing through (λ_l,λ_l) such that in the square $Q_l=(\lambda_{l-1},\lambda_{l+1})^2$ the points that are either below the lower curve C_{l_1} or above the upper curve C_{l_2} are not in Σ, while the points on the curves are in Σ. We denote by II_l the regions between the curves, then the points in II_l may or may not belong to Σ (when they do not coincide). Denote by I_{l_1} the region below the lower curve C_{l_1} and I_{l_2} the region above the upper curve C_{l_2}. We set $I_l = I_{l_1}\cup I_{(l-1)_2}$.

We assume that f also satisfies:

(f$_4$) Let $(a,b)=(f'_-(0)-\alpha, f'_+(0)-\alpha)$, then $(a,b)\in I_l$, $l\geqslant 3$.

Obviously, (1.1) has the trivial solution $u=0$. We are interested in the question whether (1.1) has nontrivial solutions. For Dirichlet problems, the authors in [15,16,18,23] have obtained the nontrivial solutions. For more general results on (1.1) for $-\Delta$ with Neumann boundary condition, the case that $f'(x)$ exists for some special points has been considered in [9,11]. But for general cases, the existence and multiplicity of solutions is not considered. In this paper, we consider the more general problem with a jumping nonlinearity at some special points. Our result is motivated by earlier ones in [11,19,20].

We recall the Sobolev space $W^{1,2}(\Omega)$ with norm $\|u\|^2=\int_\Omega|\nabla u|^2\,dx+\int_\Omega u^2\,dx$, inner product $\langle u,v\rangle=\int_\Omega \nabla u\nabla v\,dx+\int_\Omega uv\,dx$ for $u,v\in W^{1,2}(\Omega)$. From the variational point of view, solutions of (1.1) are critical points of the following functional defined on the space $W^{1,2}(\Omega)$

$$J(u)=\frac12\int_\Omega|\nabla u|^2\,dx+\frac{\alpha}{2}\int_\Omega u^2\,dx+\frac12\int_{\partial\Omega}b(x)u^2\,ds-\int_\Omega F(u)\,dx,$$

where $F(u)=\int_0^u f(s)\,ds$ and the Frechét derivative of J is defined as

$$\langle J'(u),\varphi\rangle=\int_\Omega\nabla u\nabla\varphi\,dx+\alpha\int_\Omega u\varphi\,dx+\int_{\partial\Omega}b(x)u\varphi\,ds-\int_\Omega f(x)\varphi\,dx,\quad \forall\varphi\in W^{1,2}(\Omega).$$

Then we have the main result of this paper:

Theorem 1.1. *Suppose that f satisfies* (f$_1$)–(f$_4$), *then there exist at least four nontrivial solutions of problem* (1.1).

A stronger result can be obtained if the following assumption is imposed:

(F$_1$) Let φ_1 and φ_2 be the eigenfunctions of (1.2) corresponding to λ_1 and λ_2 respectively such that $\|\varphi_1\|_{W^{1,2}(\Omega)}=\|\varphi_2\|_{W^{1,2}(\Omega)}=1$. For every $\varepsilon_0>0$ and M large enough such that for $u\in E_2=\{u\in W^{1,2}(\Omega)\colon\ u=k\varphi_1+t\varphi_2,\ k,t\in\mathbb{R}\}$, we have $F(u)>[(\lambda_2+\alpha+\varepsilon_0)/2+\widetilde{C}]u^2$, where $\widetilde{C}>\frac{C^2}{2}\|b(x)\|_{L^\infty(\partial\Omega)}$, and C is the imbedding constant, i.e., $T: X\to L^2(\partial\Omega)$ is the trace operator, then $\|Tu\|_{L^2(\partial\Omega)}\leqslant C\|u\|_X$ for all $u\in X$ with the constant C depending on Ω by Sobolev Trace Theorem (see [5]).

Theorem 1.2. *Suppose f satisfies* (f$_1$)–(f$_4$) *and* (F$_1$), *then there exist infinitely many sign-changing solutions of* (1.1) *which are mountain pass type or not mountain pass type but with positive local degree.*

Next we consider a related oscillating problem. Assume that:

(f$_5$) There exist sequences $\{a_i\}$ and $\{b_i\}$, where $a_i,b_i\in\mathbb{R}$, $i=1,2,\ldots,$ which satisfy $a_i>0$, $b_i<0$ and $a_i\nearrow+\infty$, $b_i\searrow-\infty$ as $i\to\infty$, and

$$f(a_i)=\alpha a_i,\qquad f(b_i)=\alpha b_i.$$

Let $a_0=b_0=0$, $f(t)<\alpha t$ if $t\in(a_i,a_{i+1})$, where i is an odd number, $i\geqslant1$; $f(t)>\alpha t$ if $t\in(a_i,a_{i+1})$, where i is an even number, $i\geqslant0$; $f(t)<\alpha t$ if $t\in(b_{i+1},b_i)$, where i is an even number, $i\geqslant0$; $f(t)>\alpha t$ if $t\in(b_{i+1},b_i)$, where i is an odd number, $i\geqslant1$.

(F$_2$) For ε_0 and M the same as (F$_1$) such that for $|t|\geqslant M$ and $\|\varphi_1\|=1$, we have $\int_\Omega F(t\varphi_1)\,dx\geqslant(\frac{\lambda_1+\alpha+\varepsilon_0}{2}+\widetilde{C})t^2\int_\Omega\varphi_1^2\,dx$.

Then we have:

Theorem 1.3. *Under the assumptions* (f_1)–(f_5) *and* (F_2), *there exist infinitely many nontrivial solutions of problem* (1.1), *some of them are local minimizers and others are mountain pass type solutions.*

In this paper, our technique includes constructing sub-super-solutions, mountain pass theorem in an order interval and Fučík spectrum theory. We will recall some basic notions and known results on critical points theory in Section 2, and we prove Theorems 1.1, 1.2 and 1.3 in Section 3.

2. Preliminaries

Now let us recall the notion of critical groups of an isolated critical point u of a C^1 functional J briefly. Assume that E is a Hilbert space, $J^a = \{u \in E \mid J(u) \leqslant a\}$, $K = \{u \in E \mid J'(u) = 0\}$, $K_c = \{u \in K\colon\ J(u) = c\}$, $c \in \mathbb{R}$. Let U be a neighborhood of u such that there is no critical point of J in $U \setminus \{u\}$. The critical groups of u are defined as

$$C_q(J, u) = H_q\big(J^c \cap U, \big(J^c \setminus \{u\}\big) \cap U; G\big), \quad q = 0, 1, \ldots,$$

where $c = J(u)$, $H_q(A, B; G)$ are the qth singular relative homology groups of the topological pair (A, B) with a coefficient group G. For the details, we refer to [1,3]. They are independent of the choices of U, hence are well defined. If $C_1(J, u) \neq 0$, then we call an isolated critical point u of J a mountain pass point.

Assume that $J \in C^2(E, \mathbb{R})$, for u, a critical point of J, $J''(u)$ is a self-adjoint linear operator, the dimension of the largest negative space of $J''(u)$ is called the Morse index of J at u, denoted by $ind(J, u)$; the dimension of the kernel of $J''(u)$ is called the nullity of J at u. u is called nondegenerate if and only if the nullity of J at u is zero.

We have the following basic facts on the critical groups for an isolated critical point of J (see [2]). They are fundamental in the existence and multiplicity results by applying the Morse theory to (1.1).

(1) Let u be an isolated minimum point of J, then $C_q(J, u) \cong \delta_{q0} G$.
(2) Let u be a nondegenerate critical point of J with Morse index j, then $C_q(J, u) \cong \delta_{qj} G$.

Definition 2.1. Let $c \in \mathbb{R}$ be fixed. If any sequence $\{u_k\}$ which satisfies $J(u_k) \to c$ and $J'(u_k) \to 0$ $(k \to \infty)$ has a convergent subsequence, then we say that J satisfies the $(PS)_c$ condition. If J satisfies $(PS)_c$ condition for all $c \in \mathbb{R}$, then we say that J satisfies the (PS) condition.

Definition 2.2. Assume that $J \in C^1(E, \mathbb{R})$, $c \in \mathbb{R}$, for every $\forall \varepsilon^* > 0$ and any closed neighborhood N of K_c, there exist $\varepsilon \in (0, \varepsilon^*)$ and a continuous map $\eta : [0, 1] \times E \to E$, such that

(i) $\eta(0, \cdot) = id$,
(ii) $\eta(t, u) = u$, $\forall u \in E \setminus J^{-1}[c - \varepsilon^*, c + \varepsilon^*]$, $t \in [0, 1]$,
(iii) $J(\eta(\cdot, u))$ is nonincreasing, $u \in E$,
(iv) $\eta(1, J^{c+\varepsilon} \setminus N) \subset J^{c-\varepsilon}$,

then we say that J satisfies deformation property.

By using the famous deformation theorem (see [24]), we know that J satisfies deformation property if J satisfies the (PS) condition.

Let $P_E \subset E$ be a closed convex cone, and let X be densely imbedded in E. Assume that $P = X \cap P_E$ and P has nonempty interior $\dot{P}$. Let $[u_1, u_2] = \{u \in X \mid u_1 \leqslant u \leqslant u_2,\ \ x \in \Omega\}$ be the order interval in X. We assume that any order interval is bounded in any finite-dimensional subspace of X.

In the following we recall some notations and definitions introduced in [14]. Let $\sigma(t, u) \in \mathbb{R} \times E$, we denote the negative gradient flow for J given by

$$\begin{cases} \dfrac{d\sigma(t, u)}{dt} = -\dfrac{\nabla J(\sigma(t, u))}{1 + \|\nabla J(\sigma(t, u))\|}, \\ \sigma(0, u) = u. \end{cases}$$

Definition 2.3. With the flow σ, we call a subset $A \subset E$ an invariant set if $\sigma(t, A) \subset A$, for $t \geqslant 0$.

Definition 2.4. Let $W \subset X$ be an invariant set under σ. We say W is an admissible invariant set for J if

(a) W is the closure of an open set in X, i.e., $W = \dot{W} \cup \partial W$;
(b) If $u_n = \sigma(t_n, v)$ for some $v \notin W$ satisfying as $t_n \to +\infty$, $u_n \to u (n \to +\infty)$ in E for some $u \in K$, it holds $u_n \to u$ in X;

(c) If $u_n \in K \cap W$ such that $u_n \to u$ in E, it holds $u_n \to u$ in X;
(d) For any $u \in \partial W \backslash K$, $\sigma(t, u) \in \dot{W}$ for $t \geqslant 0$.

The functional $J : E \to \mathbb{R}$ satisfies the following conditions:

(J_1) $J \in C^2(E, \mathbb{R})$ and it satisfies the (*PS*) condition in E and the deformation property in X. J only has finitely many isolated critical points.
(J_2) $\nabla J = id - K_E$, where $K_E : E \to E$ is compact. $K_E(X) \subset X$ and the restriction $K = K_E|_X : X \to X$ is of class C^1 and strongly preserving, i.e., $u \gg v \Leftrightarrow u - v \in \dot{P}$.
(J_3) J is bounded from below on any order interval in X.

Lemma 2.1. *(See [13].) Suppose J satisfies (J_1)–(J_3) and $\{u_1, u_2\}$ is a pair of sub-super-solutions of $\nabla J = 0$ in X, then $[u_1, u_2]$ is positively invariant under the negative gradient flow of J and $-\nabla J$ points inward in $[u_1, u_2]$. Moreover, if $\{u_1, u_2\}$ is a pair of strict sub-super-solutions of $\nabla J = 0$ in X, then* $\deg(id - K, [u_1, u_2], 0) = 1$.

We recall the following well-known mountain pass theorem in order intervals [13] and mountain pass theorem in half-order intervals, sup-solutions case [10].

Lemma 2.2. *(See [13].) Suppose J satisfies (J_1)–(J_3) and $\{v_1, v_2\}$, $\{\omega_1, \omega_2\}$ are two pairs of strict sub-super-solutions of $\nabla J = 0$ in X with $v_1 < \omega_2$, $[v_1, v_2] \cap [\omega_1, \omega_2] = \emptyset$. Then J has a mountain pass point u_0, $u_0 \in [v_1, \omega_2] \setminus ([v_1, v_2] \cup [\omega_1, \omega_2])$. More precisely, let v_0 be the maximal minimizer of J in $[v_1, v_2]$ and ω_0 be the minimal minimizer of J in $[\omega_1, \omega_2]$. Then $v_0 \ll u_0 \ll \omega_0$. Moreover, $C_1(J, u_0)$, the critical group of J at u_0, is nontrivial.*

Remark 2.1. Lemma 2.2 still holds if $J \in C^1(E, \mathbb{R})$ and K is of class C^0 or J has infinitely many isolated critical points.

Lemma 2.3. *(See [10].) Suppose J satisfies (J_1)–(J_3), $v_1 < v_2$ is a pair of strict super-solutions of $\nabla J = 0$ and v_0 ($< v_1$) is a sub-solution of $\nabla J = 0$. Suppose that $[v_0, v_1]$ and $[v_0, v_2]$ are admissible invariant sets for J. If J has a local strict minimizer w in $[v_0, v_2] \backslash [v_0, v_1]$. Then J has a mountain pass points u_0 in $[v_0, v_2] \backslash [v_0, v_1]$.*

Here we revise the known results on Fučík spectrum and the computation of the critical groups. Consider the problem

$$\begin{cases} -\Delta u = bu^+ - au^-, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + b(x)u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The corresponding functional is

$$I(u) = I(u, a, b) = \frac{1}{2} \int_{\Omega} \left[|\nabla u|^2 - a\left(u^-\right)^2 - b\left(u^+\right)^2 \right] dx + \frac{1}{2} \int_{\partial\Omega} b(x) u^2 \, ds.$$

If (a, b) does not belong to Σ, 0 is the trivial solution of (2.1), i.e., 0 is an isolated critical point of I, then $C_q(I, 0)$ is well defined for $q = 0, 1, 2, \ldots$. Denote $Q_l = (\lambda_{l-1}, \lambda_{l+1})^2$ for $l \geqslant 2$. The main results in [4,17] are as follows (see Theorem 1 in [4]):

Lemma 2.4. *(See [4].) Let $(a, b) \in Q_l \setminus \Sigma$ and let d_l denote the dimension of the subspace N_l spanned by the eigenfunctions corresponding to $\lambda_1, \ldots, \lambda_l$.*

(i) *If $(a, b) \in I_l$, then* $C_q(I, 0) = \begin{cases} \mathbb{Z}, & q = d_{l-1}, \\ 0, & q \neq d_{l-1}. \end{cases}$
(ii) *If $(a, b) \in II_l$, then $C_q(I, 0) = 0$ for $q \leqslant d_{l-1}$ or for $q \geqslant d_l$.*

In particular, $C_q(I, 0) = 0$ for all q when λ_l is a simple eigenvalue.

Moreover, set $A_l = I - \lambda_l(-\Delta)^{-1}$, let $N_{l-1}, E(\lambda_l), M_l$ denote the negative, zero and positive subspaces of A_l, respectively, and for p, let $I_p = I(\cdot, p)$,

$$I_p(v + \omega_0) = \inf_{\omega \in M_l} I_p(v + \omega), \quad v \in N_l, \tag{2.2}$$

$$I_p(v_0 + \omega) = \sup_{v \in N_{l-1}} I_p(v + \omega), \quad \omega \in M_{l-1}. \tag{2.3}$$

It was shown in Schechter [21] that there are continuous and positive homogeneous functions

$$\tau_l : N_l \to M_l, \qquad \gamma_{l-1} : M_{l-1} \to N_{l-1}$$

such that $\omega_0 = \tau_l(v)$, $v_0 = \gamma_{l-1}(\omega)$ are the unique solutions of (2.2), (2.3), respectively.

Let

$$T_l = \{v + \tau_l(v):\ v \in N_l\}, \qquad R_{l-1} = \{\gamma_{l-1}(\omega) + \omega:\ \omega \in M_{l-1}\},$$
$$S_l = T_l \cap R_{l-1}, \qquad \widehat{S}_l = \{u \in S_l:\ \|u\| = 1\}.$$

Lemma 2.5. *(See [4].)*

$$C_q(I,0) \cong \begin{cases} H^{d_l-q-1}(\widehat{S}_l^+), & q \neq d_{l-1}, \\ H^0(\widehat{S}_l^+)/\mathbb{Z}, & q = d_{l-1}, \end{cases}$$

where $\widehat{S}_l^+ = \{u \in \widehat{S}_l:\ I(u) > 0\}$, *for* $(a,b) \in II_l \setminus \Sigma$.

3. The proof of the main theorems

3.1. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. From the variational point of view, solutions of (1.1) are the critical points of the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\alpha}{2}\int_\Omega u^2\,dx + \frac{1}{2}\int_{\partial\Omega} b(x)u^2\,ds - \int_\Omega F(u)\,dx,$$

defined on $X := W^{1,2}(\Omega)$, where $F(u) = \int_0^u f(s)\,ds$.

(1) We shall apply Lemma 2.2 to functional J. It is easy to show that J belongs to $C^1(X,\mathbb{R})$. In fact, define a functional

$$I(u) = \frac{1}{2}\int_{\partial\Omega} b(x)u^2\,ds,$$

and we only need to prove $I \in C^1(X,\mathbb{R})$. Let $u, v \in X$, $0 < |t| < 1$, then

$$\big[I(u+tv) - I(u)\big]/t = \int_{\partial\Omega} b(x)uv\,ds + \frac{t}{2}\int_{\partial\Omega} b(x)v^2\,ds \to \int_{\partial\Omega} b(x)uv\,ds \quad (t \to 0).$$

So I has a Gateaux derivative and $\langle I'(u), v\rangle = \int_{\partial\Omega} b(x)uv\,ds$.

Let $u_n \to u$ in X,

$$\begin{aligned}\big|\langle I'(u_n) - I'(u), v\rangle\big| &= \left|\int_{\partial\Omega} b(x)(u_n - u)v\,ds\right| \\ &\leqslant \|b\|_{L^\infty(\partial\Omega)}\big\|T(u_n - u)\big\|_{L^2(\partial\Omega)}\|Tv\|_{L^2(\partial\Omega)} \\ &\leqslant C\|b\|_{L^\infty(\partial\Omega)}\|u_n - u\|_X\|v\|_X\end{aligned}$$

where $T : X \to L^2(\partial\Omega)$ is the trace operator and $\|Tu\|_{L^2(\partial\Omega)} \leqslant C\|u\|_X$ for all $u \in X$ with the constant C depending on Ω by Sobolev Trace Theorem (see [5]). Then we obtain

$$\big\|I'(u_n) - I'(u)\big\| \leqslant C\|b\|_{L^\infty(\partial\Omega)}\|u_n - u\|_X \to 0 \quad (n \to \infty).$$

So $I'(u)$ is continuous and $I \in C^1(X,\mathbb{R})$.

By using a truncation trick, we consider the functions

$$\tilde{f}(t) = \begin{cases} 0, & t \leqslant -M_2, \\ f(t), & -M_2 \leqslant t \leqslant M_1, \\ 0, & t \geqslant M_1 \end{cases}$$

and the corresponding functional

$$\widetilde{J}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\alpha}{2}\int_\Omega u^2\,dx + \frac{1}{2}\int_{\partial\Omega} b(x)u^2\,ds - \int_\Omega \widetilde{F}(u)\,dx,$$

where $\widetilde{F}(u) = \int_0^u \tilde{f}(s)\,ds$.

It follows from [9] that $\widetilde{J}(u)$ satisfies coercive condition on X so $\widetilde{J}$ satisfies (PS) condition and $\widetilde{J} \in C^1(X, \mathbb{R})$. Then from the deformation theorem, we know that $\widetilde{J}$ satisfies deformation property.

(2) We construct sub-super-solutions of (1.1). It is easy to see that M_1 is a constant super-solution of (1.1) and $-M_2$ is a constant sub-solution. Moreover, we consider $\varepsilon\varphi_1$ for all $\varepsilon > 0$ small enough. From [22] we know that $\varphi_1(x) > 0$, $x \in \Omega$. In fact, with $\underline{u} := \varepsilon\varphi_1$, by (f_2) we have

$$-\Delta\underline{u} + \alpha\underline{u} - f(\underline{u}) = \varepsilon\varphi_1(x)\big[(\lambda_1 + \alpha) - f'_+(0) + o\big(\|\varepsilon\varphi_1\|\big)\big] \leqslant 0, \quad \text{for small } \varepsilon.$$

Furthermore, $\frac{\partial u}{\partial \nu} + b(x)\underline{u} = 0$. From the above discussion, we have a pair of strict sub-super-solutions $\{\varepsilon\varphi_1, M_1\}$ of (1.1). By a similar argument we can find that $\{-M_2, -\varepsilon\varphi_1\}$ is a pair of strict sub-super-solutions.

Now we study the order interval $[-M_2, M_1]$ in X which includes two intervals $[-M_2, -\varepsilon\varphi_1]$ and $[\varepsilon\varphi_1, M_1]$. Then there exist weak solutions of (1.1) (relative minimum points) u_2, u_3 in $[-M_2, -\varepsilon\varphi_1]$ and $[\varepsilon\varphi_1, M_1]$ respectively. We can infer that $\widetilde{J}(u)$ is bounded from below on $[-M_2, M_1]$, so we get a mountain pass point $u_1 \in [-M_2, M_1] \setminus ([-M_2, -\varepsilon\varphi_1] \cup [\varepsilon\varphi_1, M_1])$ according to Lemma 2.2 and $C_1(\widetilde{J}, u_1)$ is nontrivial.

(3) We claim that u_1 is nontrivial. In fact, from assumption (f_2), we know that the left and the right derivatives of $\tilde{f}$ at 0 are different, we consider the problem

$$\begin{cases} -\Delta u = \tilde{f}(u) - \alpha u, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + b(x)u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\tilde{f} \in C(\overline{\Omega})$ and as $u \to 0$ we have

$$\tilde{f}(u) - \alpha u = \big(\tilde{f}'_+(0) - \alpha\big)u^+ - \big(\tilde{f}'_-(0) - \alpha\big)u^- + o(u).$$

We take $a = \tilde{f}'_-(0) - \alpha$, $b = \tilde{f}'_+(0) - \alpha$, from (f_4), we know that $(a, b) \notin \Sigma$.

It follows from Lemma 2.4, if $(a, b) \in I_l$, $l \geqslant 3$, $d_{l-1} > 1$, then

$$C_q(\widetilde{J}, 0) = \begin{cases} \mathbb{Z}, & q = d_{l-1}, \\ 0, & q \neq d_{l-1}, \end{cases}$$

then we obtain $C_q(\widetilde{J}, 0) \ncong C_q(\widetilde{J}, u_1)$, so $u_1 \neq 0$.

If $(a, b) \in II_l \backslash \Sigma$, $l \geqslant 1$, then from Lemma 2.5, we have

$$C_q(\widetilde{J}, 0) \cong \begin{cases} H^{d_l - q - 1}(\widehat{S}_l^+), & q \neq d_{l-1}, \\ H^0(\widehat{S}_l^+)/\mathbb{Z}, & q = d_{l-1}. \end{cases}$$

If $l = 2$, then $d_{l-1} = d_1 = 1$, so for $q = 1$, we have $C_1(\widetilde{J}, 0) \cong H^0(\widehat{S}_2^+)/\mathbb{Z}$. Furthermore, for a point p, $H^q(p; G) \cong \delta_{q0}G$. Then we have $C_1(\widetilde{J}, 0) \cong 0 \ncong C_1(\widetilde{J}, u_1)$, so $u_1 \neq 0$. If $l > 2$, then $d_{l-1} > 1$, from Lemma 2.4, $C_1(\widetilde{J}, 0) \cong 0$. So we get $C_q(\widetilde{J}, 0) \ncong C_q(\widetilde{J}, u_1)$, $u_1 \neq 0$.

(4) We claim the existence of the fourth solution. Now, we further discuss the solutions in $[-M_2, M_1]$. Since u_1 is a mountain pass point, for the Leray–Schauder degree of $id - K$, we have calculation formula

$$\deg\big(id - K, B(u_1, r), 0\big) = -1,$$

where $r > 0$ is small enough, $K = K_E|_X = (-\Delta + (m + \alpha)id)^{-1} f^*|_X : X \to X$ is of class C^0 and strongly order preserving, $f^*(u) = f(u) + mu$ (see Hofer [8]). Then according to Poincaré–Hopf formula for C^1 case (see [12]) and the computation of $C_q(J, 0)$, we have

$$index(J, 0) = (-1)^{d_{l-1}}.$$

Furthermore, for minimum points u_2 and u_3,

$$C_q(J, u_2) \cong \delta_{q0}G, \qquad C_q(J, u_3) \cong \delta_{q0}G.$$

From the additivity of Leray–Schauder degree and Theorem 1.1 in [13], we can get

$$\begin{aligned} 1 &= \deg\big(id - K, [-M_2, M_1], 0\big) \\ &= \deg\big(id - K, [-M_2, -\varepsilon\varphi_1], 0\big) + \deg\big(id - K, [\varepsilon\varphi_1, M_1], 0\big) + \deg\big(id - K, B(0, r), 0\big) + \deg\big(id - K, B(u_1, r), 0\big) \\ &= 1 + 1 + (-1)^{d_{l-1}} + (-1), \end{aligned}$$

which is impossible. From the above discussion, we conclude that there must exist another critical point $u_1^* \in [-M_2, M_1]$, which satisfies $u_1^* \neq u_1$ and is nontrivial.

This completes the proof of Theorem 1.1. □

The proof of Theorem 1.2 is the same as that of Theorem 3.5 of [11], which we omit here.

Remark 3.1. In Theorem 1.1, we can deal with the case in which $(a,b) \in I_l$, $l > 2$, and $(a,b) \in II_l$, $l \geqslant 1$, but, when $(a,b) \in I_2$, then

$$C_q(J,0) = \begin{cases} \mathbb{Z}, & q = 1 \\ 0, & q \neq 1 \end{cases} = C_q(J, u_1),$$

we cannot distinguish u_1 from 0.

3.2. Proof of Theorem 1.3

Proof of Theorem 1.3. By the truncation trick, we consider the function

$$f_i(t) = \begin{cases} 0, & t < 0, \\ f(t), & 0 \leqslant t \leqslant a_i, \\ f(a_i), & t > a_i. \end{cases}$$

The corresponding functional is

$$J_i(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\alpha}{2}\int_\Omega u^2\,dx + \frac{1}{2}\int_{\partial\Omega} b(x)u^2\,ds - \int_\Omega F_i(u)\,dx,$$

where $F_i(u) = \int_0^u f_i(s)\,ds$, $i = 1, 2, \ldots.$

When $0 \leqslant u(x) \leqslant a_i$, the solution of (1.1) is also a solution of the following equation:

$$\begin{cases} -\Delta u + \alpha u = f_i(u), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + b(x)u = 0, & \text{on } \partial\Omega. \end{cases}$$

Applying Lemma 2.1 to $J_i(u)$, by the standard argument we know that J_i satisfies (J_1)–(J_3) and the order interval consisted of sub-super-solutions is admissible invariant set of J_i. Taking $v_0 = \varepsilon\varphi_1, v_1 = a_1 > 0$, then $J_i(u)$ has a minimizer $u_1 \in [v_0, v_1]$. By assumption (F_2) there exists a $t_1 > 0$ such that

$$\begin{aligned} J(t_1\varphi_1) &= \frac{t_1^2}{2}\int_\Omega |\nabla\varphi_1|^2\,dx + \frac{\alpha}{2}t_1^2\int_\Omega {\varphi_1}^2\,dx + \frac{t_1^2}{2}\int_{\partial\Omega} b(x){\varphi_1}^2\,ds - \int_\Omega F(t_1\varphi_1)\,dx \\ &\leqslant \frac{(\lambda_1+\alpha)t_1^2}{2}\int_\Omega {\varphi_1}^2\,dx - \frac{(\lambda_1+\alpha+\varepsilon_0)t_1^2}{2}\int_\Omega {\varphi_1}^2\,dx < J(u_1). \end{aligned}$$

If we take $v_2 = a_{n_1} > t_1\varphi_1$, where $n_1 < i$, then

$$J_i(t_1\varphi_1) = J(t_1\varphi_1) < J_i(u_1),$$

which implies that $J_i(u)$ has a minimizer $u_2 \in [v_0, v_2] \backslash [v_0, v_1]$ such that $J_i(u_2) < J_i(u_1)$. By Lemma 2.3 we get a mountain pass point u_3. Moreover, $v_0 < u_i < v_2$ and u_i are positive, $i = 1, 2, 3$.

Next, we take $v_1 = a_{n_1}, v_0 = \varepsilon\varphi_1$. Then $J_i(u)$ has a minimizer $u_2 \in [v_0, v_1]$. By assumption (F_2) there is a $t_2 > 0$ such that

$$J(t_2\varphi_1) < J(u_2).$$

If we take $v_2 = a_{n_2} > t_2\varphi_1$, where $n_2 < i$, then

$$J_i(t_2\varphi_1) = J(t_2\varphi_1) < J_i(u_2),$$

which implies that $J_i(u)$ has a minimizer $u_4 \in [v_0, v_2] \backslash [v_0, v_1]$ such that $J_i(u_4) < J_i(u_2)$. By Lemma 2.3 we get a mountain pass point u_5. Moreover, $v_0 < u_i < v_2$, and u_i are all positive, $i = 1, 2, 3, 4, 5$. Continue making the procedure we obtain the result. The proof is complete. □

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