Finite Groups Which Possess a Strongly Closed
2-Subgroup of Class at Most Two, II

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8. INTRODUCTION

In Part II of this paper we complete the proof of Theorem A, stated in
[14], which classifies the finite non-abelian simple groups possessing a
nontrivial strongly closed 2-subgroup of class at most two. The section
numbering here continues that of [14], and up to the end of Section 16, $G$
will denote a finite non-abelian simple group containing a strongly closed 2-
subgroup $S$ which is a minimum counterexample to [14, Theorem A] (see
Section 4).

Our first step is to strengthen the statement of Lemma 7.3. This we do by
showing, in Section 9, that $m(S) > 3$. We achieve this mainly by the use of
fusion arguments and frequent appeals to Glauberman's $Z^*-\$-theorem.

Section 10 introduces Hypothesis 10.3 which concerns a quadruple
$(\sigma, A, H, K)$, where $\sigma \in A^*$, $A$ is an elementary abelian subgroup of $S$,
$H \in \mathbb{I}(\sigma)$ and $K$ is a component of $H$ with certain conditions being placed
on $A$, $H$ and $K$ (see Section 10 for the precise details). Hypothesis 10.3 is so
phrased as to avoid some of the complexities inherent in Lemma 3.4(ii).

Using Lemma 3.4(ii) and the sharpened form of Lemma 7.3, the following
result is established.

LEMMA 10.5. Assume Hypothesis 10.3 holds (for $(\sigma, A, H, K)$). Then for
each component $L$ of $H$, $\langle H \rangle = \mathbb{I}(\sigma_n)$ for all $\sigma_n \in \mathbb{I}(C_A(E(H)))E(H) \cap
C_{\sigma}(L))$.

Because of this result our attention, for such $(\sigma, A, H, K)$, is now focused,
almost exclusively, on the components of $H$.

The situation depicted in Hypothesis 10.3 is analysed in Section 11–15,
the combined effect of which is to show that Hypothesis 10.3 is untenable in
a minimal counterexample. Section 16 deals with the outstanding
configurations and produces the final contradiction.

Let $(\sigma, A, H, K)$ satisfy Hypothesis 10.3. The main result of Section 11,
that the number of components of $H$ is at most two, is established by arguing along similar lines to Powell and Thwaites [11], while Section 12–15 deal with the cases when $H$ has one or two components and are somewhat different in nature to Section 11. There the arguments are directed towards obtaining the hypotheses of (2.20), which then gives an immediate contradiction to the minimality of $G$. We also note here that several of our arguments in Section 15 closely parallel those of Gilman and Gorenstein in [2; Sect. 8].

The last section of this paper contains proofs of the corollaries to Theorem A which were stated in Section 1. In the remainder of this section we review a few more results that we shall have recourse to in later sections.

**Lemma 8.1.** Let $K$ be a quasisimple group.

(i) If $K/Z(K) \cong Sz(8)$, then $m(K) = 3 + m(Z(K))$.

(ii) If $K/Z(K) \cong L_2(4)$, then $m(K) = 4 + m(Z(K))$.

(iii) In cases (i) and (ii) above $Z(K)$ is elementary abelian of order at most 4.

**Proof.** See, for example, [2, (2.16)]; [4, (4.2), p. 92].

**Lemma 8.2.** Suppose $H$ is an $L(S)$-group and let $K$ be a component of $H^*$.

(i) If $A$ is a maximal elementary abelian subgroup of $S$ and $A$ normalizes $K$, then $A \cap K \neq 1$.

(ii) If $A \in \mathfrak{A}_4(S)$, then $A \cap K \neq 1$ and either $A \cap K \in \mathfrak{A}_4(S \cap K)$ or $K \cong L_2(q)$ $(q \equiv 3, 5(8))$.

**Proof.** (i) Since $A$ normalizes $K$, we may work in $AK$. Then we have $A(S \cap K) \in \mathfrak{X}^*(AK)$. From Lemma 3.1(i), $1 \neq S \cap K \subseteq A(S \cap K)$ whence $1 \neq S \cap K \cap \Omega_1(Z(R)) \subseteq A \cap K$ by the maximality of $A$.

(ii) By Lemma 3.3(ii) and part (i), $A \cap K \neq 1$. The argument of [2, (2.37)(iii)] suffices to establish the other assertion of (ii) (the case $K/Z(K) \cong Sz(8)$ and $Z(K) \neq 1$, which is not difficult, is not covered there).

**Lemma 8.3.** Let $H$ be an $L(S)$-group with $H = \langle S^H \rangle$ and $O_2(H) = 1$. Suppose $A$ is a maximal elementary abelian subgroup of $S$ which normalizes $K$, where $K$ is a component of $H$ with $K/Z(K) \in \mathfrak{L}$.

(i) Either $A \cap K \in \mathfrak{A}_4(S)$ or one of the following holds:

(a) $K \cong L_2(q)$ $(q \equiv 3, 5(8))$ with $|A \cap K| = 2$; or
(b) \( K \cong L_2(2^{2^n}) \text{ with } |A \cap K| = 2^n. \)

(If \( A \in \mathfrak{S}_p(S) \), then possibility (b) can be omitted.)

(ii) If, moreover, \( K = F^*(H) \), then either

(a) \( A \leq K; \) or

(b) \( K \cong L_2(q^2) \quad (q \equiv 3, 5(8)) \) or \( A \) and \( R \cong Z_2 \times D_8 \) where \( R \in F^*(AK) \); or

(c) \( K \cong L_2(2^{2^n}) \) and \( |A| = 2^{n+1}; \) or

(d) \( K \cong L_2(q) \quad (q \equiv 3, 5(8)), \) \( |A \cap K| = 2 \) and \( |A| = 4. \)

**Proof.** Omitted.

**Lemma 8.4.** Suppose \( H \) is an \( \mathcal{L}(S) \)-group, \( K \) is a component of \( H^* \) with \( K \not\cong SL(2, q) \) for any odd \( q \) and \( A \) is a non-trivial elementary abelian subgroup of \( S \) which normalizes \( K \). If \( m(A) > m(J) \) where \( J = \langle C_K(a) \mid a \in A^* \rangle \), then \( K = J. \)

**Proof.** Since \( A \) normalizes \( K \) we may assume that \( H = KA \). Put \( \overline{H} = H/C_H(K) \). If \( C_A(K) \neq 1 \), then the result clearly holds, so we may suppose that \( C_A(K) = 1 \). By Lemma 3.1(i) we have that \( \overline{K} = K/Z(K) \in \mathcal{L} \). Also note that \( m(A) > m(J) \geq m(C_K(a)) \geq 1 \) (where \( a \in A^* \)), and so we may assume \( O_{2^4}(H) = 1 \).

Suppose every element of \( A \) induces inner automorphisms upon \( K \). Then \( A \leq C_H(K)K \) and hence, as \( C_A(K) = 1, m(A) \leq m(\overline{K}). \) Hence (for \( a \in A^* \)) \( m(C_K(a)) \leq m(J) < m(A) \leq m(\overline{K}). \) Therefore, \( K \neq \overline{K} \) because otherwise (2.18(i)) would yield \( m(C_K(a)) = m(K) \) for all \( a \in A^* \). Consequently \( \overline{K} \) is isomorphic to either \( Sz(8) \) or \( L_3(4) \). For \( a \in A^*, \) \( \alpha = \beta y \) where \( \beta \in C_H(K) \) and \( y \in S \cap K \). Note that \( y \) must have order 4 as \( m(A) \leq m(\overline{K}). \) Let \( B \in \mathfrak{S}_p(S) \). Then \( B_0 = B \cap K \in \mathfrak{S}_p(S \cap K) \) by Lemma 8.2(ii), and \( y \) induces an involutory automorphism on \( B_0. \) If \( \overline{K} \cong Sz(8), \) then \( m(C_K(a)) \geq m(C_{B_0}(y)) \geq m(\overline{K}) + 1. \) So \( 3 = m(\overline{K}) \geq m(A) \geq m(\overline{K}) + 1, \) which forces \( m(A) = m(\overline{K}). \) \( K \cong L_3(4) \) similarly yields \( 4 = m(\overline{K}) \geq m(A) > m(J) \geq 3, \) whence we also have \( m(A) = m(\overline{K}). \) So \( A \in \mathfrak{S}_p(S \cap K). \) From \( m(J) < m(A) \) we also see that \( m(J) = m(C_K(\alpha)) = 2 \) (if \( \overline{K} \cong Sz(8)) \) and \( 3 \) (if \( \overline{K} \cong L_3(4) \)) for all \( a \in A^* \). The fact that \( A^* \) fuses in \( N_K(A) \) shows that such a situation cannot occur. Thus we conclude that \( A \leq C_H(K)K \) and therefore, by (2.16)(ii), \( K \) is isomorphic to either \( L_2(q) \) or \( A \). Inspection of (2.17) yields that \( J = K, \) as required.

**Lemma 8.5.** Suppose \( K \cong PSp_4(2^n) \) for some \( n \geq 2, \) and let \( P \in \text{Syl}_2(K \) and \( \sigma \in \mathcal{N}(P). \) Then \( \langle C_K(\beta) \mid \beta \in \sigma^6 \cap P \rangle. \)

**Proof.** This can be verified by either appealing to (2.20) or examining the subgroup structure of \( K. \)
Lemma 8.6. Let $p^a$ and $q^b$ be two prime powers whose difference is 1. Then one of the primes is 2 and the other is a Fermat or Mersenne prime. Further, if (say) $p = 2$, then $b = 1$ unless $p^a = 8$ and $q^b = 9$.

Proof. By hypothesis one of $p$ and $q$ must be even, so we may suppose $p = 2$. Suppose $q^b = 2^a - 1$. Clearly $a > 1$ and thus $q^b = -1(4)$. Therefore, $b$ is odd and so $(q^b + 1)/(q + 1) = q^{b-1} - q^{b-2} + \ldots + 1$ is an odd divisor of $q^b + 1 = 2^a$, which forces $b = 1$. Now consider the case $q^b = 2^a + 1$. If $b = 1(2)$, then $(q^b - 1)/(q - 1)$ will be an odd divisor of $q^b - 1 = 2^a$ whence $b = 1$. So we may suppose $b = 2c$, and then $2^a = (q^c - 1)(q^c + 1)$. Thus $q^c = 3$ and so either $b = 1$ or $p^a = 8$ and $q^b = 9$, as required.

9. THE $m(S) = 3$ CASE

The purpose of this section is to show that $m(S) > 3$. A consequence of this is a strengthening of Lemma 7.3(ii) which will be heavily used in subsequent sections.

Lemma 9.1. Suppose $m(S) = 3$. Then

(i) $m(Z(S)) = 2$.

(ii) If $S_0 \subseteq \Sigma$, then $[S : S_0] \leq 2$.

Proof. Part (i) is clear from Lemmas 4.1(i), (iii).

Let $S_0 \subseteq \Sigma$ and set $N = N_{G}(S_0)$, $\bar{N} = N/S_0O_2(N)$ and $Z = \Omega_1(Z(S_0))$. Suppose $[S : S_0] > 2$. Since $K = \langle S^0 \rangle$ acts faithfully upon $Z$ by [13, Lemma 5.1] and $m(Z) \leq 3$, (2.7)(ii) and [1] imply that $K \cong L_2(4)$. But then $m(Z) \geq 4$ by [2, (2.66)], a contradiction. Therefore, $[S : S_0] \leq 2$.

Lemma 9.2. Suppose that $m(S) = 3$ and that $\sigma \in E(S)$. Set $C = C_{\sigma}(\sigma)$, $E = E_\sigma(C)$, $\bar{E} = E/O_2(E)$ and $R = C_{\bar{E}}(\bar{E})$ where $S^* \in S^*(C)$. Then

(i) $\bar{E} \cong L_2(q)$ (q odd), $A_7$ or $U_3(4)$;

(ii) $R \cong \mathbb{Z}_2$, $\mathbb{Z}_4$ or $Q_8$; and

(iii) $S^* \cap E \cong D_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ or a Sylow 2-subgroup of $U_3(4)$.

Proof. Since $C$ is an $S^*$-group, Lemmas 3.1 and 4.3 and the supposition $m(S) = 3$ yield (i) and (ii). Part (iii) follows from (i).

Lemma 9.3. Suppose $m(S) = 3$ and $S = S_1 \times S_2$ where $S_1 \neq 1 \neq S_2$. Then

(i) $m(S_i) \neq 2$; and

(ii) If $S_2$ does not contain a subgroup isomorphic to $Q_8$, then $S_1 \neq Q_8$. 


Proof: Suppose the lemma is false. Since \( S'_1 \leq S' \leq Z(S) \neq S \) by Lemma 4.1(i) there exists \( \mu \in S'^*_1 \) such that \( \mu^G \cap (S \setminus S'_1) \neq \emptyset \). By (2.7)(i) we may assume that this fusion takes place in \( N = N_\mu(S_0) \) for some \( S_0 \in \Sigma \). Set \( \bar{N} = N/S_0O_2(N) \). If \( S_1 \triangleright S_0 \), then \( S_0 = S_1 \times (S_2 \cap S_0) \). By the Krull–Remak–Schmidt theorem [9, Satz 12.3, p. 661, since \( S'_1 \triangleleft N \), \( S_2 \cap S_0 \) contains a subgroup isomorphic to \( S' \), so (i) must be false. But then \( m(S) \geq 4 \), and so \( S_1 \nleq S_0 \). Thus \( S_1 \neq 1 \). From [1] and (2.7)(ii), \( \langle S'^*_1 \rangle = \langle S'^*_0 \rangle \). If \( m(S'_1) = 2 \), then \( m(S) = 3 \) implies that \( \Omega_1(S_1) = S'_1 = Z(S_1) \). If (ii) is false, then we also have \( \Omega_1(S_1) = Z(S_1) \). Thus \( S_1 \) centralizes \( \Omega_1(S) \) and hence centralizes \( \Omega_1(Z(S_0)) \). Therefore, \( \langle S'^*_0 \rangle \) centralizes \( \Omega_1(Z(S_0)) \). Because \( \mu \in Z(S) \) and \( \bar{N} = \langle S'^*_0 \rangle N_\mu(S) \), the fusion may be assumed to take place in \( N_\mu(S_0) \). So we may suppose \( S_0 = S \). But this contradicts \( S_1 \nleq S_0 \), and completes the proof of the lemma.

**Lemma 9.4.** \( m(S) \geq 3 \).

**Proof.** We suppose \( m(S) < 3 \), and argue for a contradiction. So \( m(S) = 3 \) by Lemma 4.1(iii). Let \( \sigma \in \mathscr{C}(S) \). We set \( C = C_\sigma(\sigma) \), \( E = E_2(C) \), \( \bar{C} = C/O_2(C) \) and \( R = C_\sigma(E) \), where \( S^* \in \mathscr{Z}^*(C) \). Without loss of generality we may suppose \( S^* = C_\sigma(\sigma) \). Note that \( R(S^* \cap E) = R \times (S^* \cap E) \) by Lemma 9.2.

We divide the proof into two parts beginning with

**Case 1.** \( S = S^* \).

(9.1) \( S \neq R(S \cap E) \).

Suppose \( S = R(S \cap E) = R \times (S \cap E) \). By Lemmas 9.2, 9.3 and 4.1(ii), \( R \cong \mathbb{Z}_2 \), while \( S \cap E \cong D_8 \) or \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) by Lemmas 9.2 and 9.3. Hence \( S \cong \mathbb{Z}_2 \times D_8 \) and then, by (2.5), \( S \in \text{Syl}_2 G \) which contradicts the simplicity of \( G \). Hence \( S \neq R(S \cap E) \).

Assume, for the moment, the following situation: \( \bar{E} \cong A_7 \) or \( L_2(q^2) \), \( q = 3, S(8) \) and \( \mathcal{T}(S \setminus R(S \cap E)) = \emptyset \). Then every involution in \( S \) is conjugate to an element of \( Z(S) \). Let \( \langle \mu \rangle = Z(S \cap E) \) and suppose that \( \mu \) and \( \sigma \) fuse in \( G \). Thus \( \mu^f = \sigma \) for some \( g \in N_\mu(S) = N \) by (2.1)(v). Since \( Z((S \cap E)^f \setminus (S \cap E) = 1 \), we have \( (S \cap E) \cap (S \cap E)^f = 1 \) whence \( [S \cap E, (S \cap E)^f] = 1 \), contrary to \( m(S) \leq 3 \). Therefore, \( \mu \) and \( \sigma \) do not fuse in \( G \), and so, since \( R(S \cap E) \setminus (S \cap E) \) must fuse in \( G \) (by Glauberman's \( Z^* \)-theorem), \( S \cap E \) is a strongly involution closed dihedral 2-subgroup of \( G \). By [5, Corollary B4], \( G = \langle S \cap E \rangle \) has dihedral or quasidihedral Sylow 2-subgroup, contrary to \( m(S) = 3 \). Therefore, if \( \bar{E} \cong A_7 \) or \( L_2(q^2) \), \( q = 3, 5(8) \), then \( \mathcal{T}(S \setminus R(S \cap E)) = \emptyset \).

From (9.1) and Lemma 9.2 we deduce (note that \( m(S) = 3 \) and the above exclude the possibilities (2.17)(v) and (vii))
(9.2) (i) \(|S : R(S \cap E)| = 2\) with \(S \cap E \cong \mathbb{Z}_2 \times \mathbb{Z}_2\) and \(E \cong L_2(q)\), \(q \equiv 3, 5(8)\).

(ii) \(S/R \cong D_8\).

(9.3) (i) \(\mathcal{N}(S) \not\subseteq R(S \cap E)\),

(ii) \(R \not\subseteq \mathbb{Z}_2\),

(iii) \(For \mu \in \mathcal{N}(S) \setminus R(S \cap E), C_S(\mu)\) is abelian and \(|S : C_S(\mu)| \geq 4\).

Since \(\Omega_1(R(S \cap E))\) is elementary abelian, Lemma 4.1(i) implies (i). Let \(\mu \in \mathcal{N}(S) \setminus R(S \cap E)\). Since \(\langle \mu \rangle(S \cap E) \cong D_8\), \(R \cong \mathbb{Z}_2\) would yield \(S \cong \mathbb{Z}_2 \times D_8\), which is impossible. Thus \(R \not\subseteq \mathbb{Z}_2\).

Set \(P = R(S \cap E)\). Since \(S/R \cong D_8\), \([C_P(\mu) : C_R(\mu)] = 2\). If \(C_R(\mu) = R\), then either \(S \cong Q_8 \times D_8\) or \(S \cong \mathbb{Z}_4 \times D_8\). The former is impossible by Lemma 9.3(ii) and the later contradicts Lemma 4.1(ii). So \(C_R(\mu) \neq R\). Hence, since \(C_P(\mu) = C_R(\mu) S_{S \cap E}(\mu)\), we have \(C_P(\mu)\) is abelian and \(|P : C_P(\mu)| \geq 4\), whence (iii) holds.

Since \((S \cap E)^*\) are conjugate, \(\Omega_1(R(S \cap E))\) are conjugate and \(Z(S) = (Z(S) \cap R) \times (Z(S) \cap E)\), if \(\sigma\) is conjugate to an involution in \(R(S \cap E)\), then it must be conjugate to an involution in \(Z(S)\). By (2.1)(v) we may assume this fusion takes place in \(N_G(S)\). If \(R \cong Q_8\), then \(\sigma\) is the only involution of \(Z(S)\) contained in more than one subgroup of \(S\) of order 4, while, if \(R \cong \mathbb{Z}_2\), then \(S\) cannot possess any non-trivial automorphisms of odd order. Therefore, \(\sigma^G \cap R(S \cap E) = \{\sigma\}\).

Hence by Glauberman's \(Z^*\)-theorem, \(\sigma\) has a conjugate \(\mu\) with \(\mu \in S \setminus R(S \cap E)\). Let \(S_0 \in \Sigma\) be such that this conjugation occurs in \(N = N_G(S_0)\). Since \(\sigma \in Z(S)\) and \(\mu \notin Z(S)\), we may suppose the conjugation takes place in \(\langle S^\mathcal{N} \rangle\). By [13, Lemma 5.1], \(\langle S^\mathcal{N} \rangle, S_0 \rangle \leq S_0 \leq Z(S_0)\) and so \(\mu = \sigma^\tau = \sigma \tau\) for some \(\tau \in Z(S_0)\). Clearly \(\tau \in \mathcal{N}(S) \setminus R(S \cap E)\). So \(C_S(\tau)\) is abelian by (9.3)(iii). Because \(C_S(S_0) \leq S_0\), we see that \(S_0 = C_S(\tau)\). But then \(|S : S_0| \geq 4\) by (9.3)(ii), contrary to Lemma 9.1(ii).

This eliminate case 1.

Case 2. \(S \neq S^*\). From Lemmas 4.1(ii) and 9.2(ii) we note that \(R \cong \mathbb{Z}_2\).

(9.4) \(S^* = R(S^* \cap E)\).

If \(S^* \neq R(S^* \cap E)\), then \(S^* \cap E \cong \mathbb{Z}_2 \times \mathbb{Z}_2\) with \(S^*/R \cong D_8\). Hence \(Z(S) \leq R(S^* \cap E)\) and then, since \(S^* \neq S\), \(R(S^* \cap E) = RZ(S)\). But then \(S^*\) is abelian which is impossible. Thus (9.4) holds.

(9.5) \(\mathcal{N}(S^* \cap E)\) are conjugate and \(\mathcal{N}(R(S^* \cap E)) \setminus (S^* \cap E)\) are conjugate.

From (9.4) and the possibilities for \(\mathcal{E}\) we see that \(\mathcal{N}(S^* \cap E)\) fuses and
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Theorem 7.3 yields that \( Z(S \cap (S^* \cap E)) \subseteq (R \cap (S^* \cap E)) \) fuses. Now Glauberman's \( Z^* \)-theorem and the fact that \( Z(S) \cap (S^* \cap E) \neq 1 \) yields (9.5).

In view of (9.5) we have that \( Z(S) = \Omega_1(S^* \cap E) \), and so \( S^* \cap E \cong D_8 \) cannot occur.

By Lemma 4.1(i) there exists \( \mu \in Z(S) \) and \( \tau \in \gamma(S) \) with \( \mu \) and \( \tau \) conjugate. From (9.5) we have \( \tau \notin R(S^* \cap E) = S^* \). Let \( S_0 \in \Sigma \) be such that \( \mu \) and \( \tau \) are conjugate in \( N = N_6(S_0) \). Clearly we may assume that the conjugation happens in \( \langle S^* \rangle \) and that \( S \neq S_0 \). Thus \( |S : S_0| = 2 \) by Lemma 9.1(ii). Since \( \mu \in Z(S_0) \), \( \tau \in Z(S_0) \), and so \( Z(S_0) = Z(S(\langle \tau \rangle) = \Omega_1(S_0) \).

Also note, that as \( \tau \notin S^* \), \( R \cap S_0 = 1 \).

Suppose \( S^* \cap E \) is isomorphic to a Sylow 2-subgroup of \( U_3(4) \). Because \( |S^* : S_0| \leq 2 \) and \( R \cap S_0 = 1 \), we see that \( S_0 \) covers \( S^*/R \) and hence \( Z(S) \leq S_0 \) by the structure of \( S^* \cap E \). But \( S^* \) acts trivially upon \( S_0 \) by [13, Lemma 5.1(vi)], contradicting the choice of \( S_0 \). Therefore, we conclude that \( S^* \cap E \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \).

Since \( N_6(S^* \cap E) = S \leq N_6(S^*) = M \), we see that \( S^*/(S^* \cap E) \) fuses in \( M \) and so \( |S : S^*| = 4 \). Thus \( |S| = 2^5 \) and \( |S_0| = 2^4 \), whence \( S_0 \) is abelian. By Lemma 4.1(vi), \( |S'| = 2 \) and hence, by (2.1)(v), \( Z(S)^* \) does not fuse, against (9.5).

So case 2 also leads to the desired contradiction, and hence we have verified Lemma 9.4.

10. SOME CONSEQUENCES OF HYPOTHESIS 10.3

Many of our subsequent results are proved under the assumption that Hypothesis 10.3 holds. In this section Hypothesis 10.3 is described, and certain consequences of such a hypothesis pertaining are then established.

**Lemma 10.1.** Suppose \( \sigma \in \gamma(S) \) and \( H \in \mathcal{H}(\sigma) \) with \( m(O_2,(F(H))) \leq 2 \). Set \( E = E_2(H) \).

(i) If \( K \) is a quasisimple subgroup of \( H^* \) with \( K \neq SL(2, q) \) for any \( q \), then \( |K, O_2,(H)| = 1 \).

(ii) If \( K \) is a quasisimple group of \( H \) with \( K \neq SL(2, q) \) for any \( q \), then \( \{H\} \subseteq \mathcal{H}(\sigma, K) \).

(iii) \( E = E(H) O_2(E) \).

**Proof.** Set \( H_0 = H^* O_2(H) \).

(i) Let \( p \) be an odd prime. From Lemma 3.5(i) we have \( K \leq C_{H_0}(O_2(H)) \) since \( K \neq SL(2, q) \). So \( |K, F(O_2(H))| = 1 \) whence (i) holds.

(ii) This follows from (i) and the definition of \( \mathcal{H}(\sigma, K) \).
(iii) By Lemma 3.1, \( E \leq H_0 \). Since by Lemma 6.1, \( H \) does not have any 2-components isomorphic to \( SL(2, q) \) and \( E \) is perfect, Lemma 3.5(i) yields \([E, O_2(H)] = 1\) and thus \( E = E(H) O_2(E)\).

**Lemma 10.2.** Suppose \( A \) is an elementary abelian subgroup of \( S \) such that either \( m(A) \geq 5 \) or \( A \in \mathcal{U}_e(S) \). Then for \( \sigma \in A^* \)

(i) \( \mathcal{H}(\sigma) = \mathcal{H}(\sigma, K) \) for any quasisimple group \( K \not\cong SL(2, q) \); and 

(ii) \( E_2(H) = E(H) O_2(E_2(H)) \) for all \( H \in \mathcal{H}(\sigma) \).

**Proof.** Combining Lemmas 7.3 and 9.4 with Lemma 10.1 gives the lemma.

We now introduce the following hypothesis.

**Hypothesis 10.3.** is satisfied by the quadruple \((\sigma, A, H, K)\) if \( A \) is an elementary abelian subgroup of \( S \) with \( |A| \) maximal subject to there existing \( \sigma \in A^* \) and \( H \in \mathcal{H}(\sigma) \) such that \( E(H) \neq 1 \). Among all such triples \((\sigma, A, H)\), \( K \) is chosen to be a component of \( H \) with, if possible, \( K \not\cong L_2(q), q = 3, 5(8) \) or \( A_8 \). Also, if it is the case that \( A \not\in \mathcal{U}_e(S) \) and \( K \cong L_2(2^n), n \geq 2 \), then \( K \) is chosen so that \( n \) is as large as possible. Furthermore, in the situation when \( A \not\in \mathcal{U}_e(S) \) we suppose \( m(A) \geq 5 \).

When, in the future, Hypothesis 10.3 is stated to hold, we shall always use the above notation.

**Remarks.** (i) A consequence of Hypothesis 10.3 is that \( A \) is a maximal elementary abelian subgroup of \( S \). So \( Z(S) \leq \Omega_1(C_S(A)) = A \) and \( A \leq S \).

(ii) Hypothesis 10.3 is so phrased as to smooth our path past the exceptional embeddings in Lemma 3.4(ii) and, in particular, to allow us to use Lemmas 5.2 and 10.2.

**Lemma 10.4.** Assume Hypothesis 10.3 holds.

(i) Let \( \mu \in A^* \) and \( H_\mu \in \mathcal{H}(\mu) \). Then \( A \) normalizes each component of \( H_\mu \). In particular, \( A \) normalizes each component of \( H \).

(ii) Let \( L \) be a component of \( H \). Then every \( L \)-conjugacy class of involutions has a representative contained in \( A \).

(iii) \( \langle H \rangle = \mathcal{H}(\sigma_0) \) for all \( \sigma_0 \in C_A(K)^* \).

**Proof.** (i) Let \( L \) be a component of \( H_\mu \), and put \( A_1 = N_A(L) \) (note that \( A \leq H_\mu \)). By Lemma 3.1(ii) \( L \leq H_\mu^* \). Suppose \( A \neq A_1 \). By Lemma 3.3(i), \([A : A_1] = 2\) and \( L \) has abelian Sylow 2-subgroups. So \( A \) interchanges the components \( L \) and \( L_0 \). Hence, as may be checked, \( A_1 \) centralizes \( L \) and \( L_0 \). Thus \( A_1 \times L_0 \) centralizes \( L \). Let \( R \in \mathcal{E}^*(H_\mu) \) be such that \( A_1 \leq R \). Since \( m(L) \geq 2 \), \( R \cap (A_1 \times L_0) \) contains an elementary abelian subgroup \( B \) with \( B \geq A_1 \) and \( m(B) > m(A) \). Therefore, \( A \not\in \mathcal{U}_e(S) \) and hence \( m(B) > m(A) \geq 5 \).
by Hypothesis 10.3. Let $\tau \in A^*_1$ and $H_\tau \in \mathcal{H}(\tau)$. Since $[\tau, \mu] = 1$ and $L \leq E(H_\mu \cap H_\tau)$, Lemmas 3.4(ii) and 10.2(i) force $L \leq E(H_\tau)$. This is contrary to the maximal choice of $|A|$, and so conclude that (i) holds.

(ii) This follows from part (i) and Lemma 8.3(i) with $\mu = \sigma$ and $H = H_\mu$.

(iii) By part (ii) and Lemma 8.2(i) there exists $\sigma_1 \in (A \cap H)_0$. Since $K \not\cong \text{SL}(2, q)$, $\mathcal{H}(\sigma_1, K) = \mathcal{H}(\sigma_1) \neq \emptyset$ by Lemma 10.2(i) and so, using Lemma 5.2, we have $\mathcal{H}(\sigma_0, K) \subseteq \{H\}$ for all $\sigma_0 \in C_A(K)^\#$. Therefore (iii) holds by Lemma 10.2(i).

**Lemma 10.5.** Assume Hypothesis 10.3 holds. Then for each component $L$ of $H$, $\{H\} = \mathcal{H}(\sigma_0)$ for all $\sigma_0 \in \mathcal{H}(C_A(E(H)) \cap C_G(L))$.

**Proof:** We set $B = C_A(E(H)) \cap C_G(L)$ and let $\beta \in B^\#$ and $H_\beta \in \mathcal{H}(\mu)$. In view of Lemma 10.4(ii) to prove the lemma it will be sufficient to show that $H = H_\beta$. By Lemma 10.4(iii)

\[(10.1) \quad \{H\} = \mathcal{H}(\sigma_0) \text{ for all } \sigma_0 \in C_A(K)^\#,\]

and so we may suppose $K \neq L$. Combining (10.1) with Lemma 10.4(ii) we obtain

\[(10.2) \quad \text{if } \tau \text{ is an involution in } C_A(E(H))(K_1 | K_1 \text{ is a component of } H, K_1 \neq K) \text{, then } \{H\} = \mathcal{H}(\tau).\]

We also may suppose that

\[(10.3) \quad L \cong L_2(q) \ (q \equiv 3, 5(8)), \ L_2(2^n) \text{ or } A_6.\]

For if $L$ is not isomorphic to either of $L_2(q) \ (q \equiv 3, 5(8)), \ L_2(2^n)$ and $A_6$, then we have Hypothesis 10.3 holding with $L$ in place of $K$ whence Lemma 10.5 follows from Lemma 10.4.

\[(10.4) \quad \text{If } F \text{ is a product of components of } H \text{ each of which is isomorphic to } K, \text{ then } N_G(F) \leq H.\]

If $F \leq H$, then (10.4) holds. So we may assume $F \nleq H$. Thus there exists a component $K_1$ of $H$ with $K_1 \cong K$ and $|F, K_1| = 1$. Hence (10.1) holds with $K_1$ in place of $K$. Therefore, as $A \cap F \neq 1$, we conclude that $C_G(F) \leq H$. So $F^*(H) \leq FC_G(F) \leq H$ from which we infer that $E(H) = E(FC_G(F))$. Thus $N_G(F) \leq N_G(E(FC_G(F))) = H$, as required.

\[(10.5) \quad \text{(i) } F^*(H_\mu) \leq H.\]

\[(10.5) \quad \text{(ii) Each component of } H_\mu \text{ is contained in a component of } H.\]

\[(10.5) \quad \text{(iii) } L \text{ is a component of } H_\mu.\]
Clearly $A, L \leq C_G(\mu) \leq H_\mu$. So $L \leq C(E(H) \cap H_\mu)$ which is $C_{H_\mu}(\sigma)$-invariant. Hence, by Lemmas (3.4)(ii), 10.2(i) and 10.4(i), $L \leq J$ where $J$ is a component of $H_\mu$, and if $K \neq J$ then by (10.3) $J$ must be isomorphic to one of $L_2(q^2)$ ($q = 3, 5(8)$), $L_2(2^n), A$, or a group of type $JR$. The embedding of $L$ in $J$ and (10.2) then yield $F^*(H_\mu) \leq H$. Now $E(H_\mu)$ is $C_{H_\mu}(\mu)$-invariant whence (ii) holds by Lemmas 3.4(ii) and 10.2(i). From (ii) we see that $L = J$, and so we have verified (10.5).

(10.6) $K \leq H_\mu$.

Suppose that $K \leq H_\mu$ holds. The choice of $\sigma, K$ and $H$ and Lemmas 3.4(ii) and 10.2(i) forces $K$ to be a component of $H_\mu$ (the possibility $K \cong L_2(2^n)$ when $A \in \mathcal{U}(S)$ is eliminated as in Lemma 5.2). Let $J$ be a component of $H_\mu$ with $J \cong K$. Since $J \leq E(H)$ by (10.5)(ii), similar considerations show that $J$ is a component of $H$. Therefore, $1 \neq F = \langle J \mid J$ a component of $H_\mu$ with $J \cong K \rangle$ is a product of components of $H$. Hence $H_\mu = N_G(F) \leq H$ by (10.4). So we may assume that $K \leq H_\mu$.

(10.7) There exists $h \in H_\mu$ such that $L^h \nleq K$.

Set $F = \langle L^h \mid h \in H_\mu \rangle$. If (10.7) were false, then $K \leq N_G(F)$ by (10.5)(ii). Since $L$ is a component of $H_\mu$ by (10.5)(iii), $F \nleq H_\mu$ and so $K \leq H_\mu$, contrary to (10.6). Thus (10.7) holds.

(10.8) $K$ has one conjugacy class of involutions.

Because $L^h$ is a component of $H_\mu$ properly contained in $K$, (10.3) and Lemma 3.4(ii) show that $K$ is isomorphic to one of $L_2(q^2)$ ($q = 3, 5(8)$), $L_2(2^n), A$, or a group of type $JR$. So (10.8) holds.

(10.9) If $\zeta \in \mathcal{J}(K)$, then $\mathcal{M}(\zeta) = \{H\}$.

Suppose for the moment that $\zeta \in B$ and let $H_\zeta \in \mathcal{M}(\zeta)$. Also suppose that $H \neq H_\zeta$. Thus, since (10.1)–(10.8) have been proved under the hypothesis $H \neq H_\mu$, statements (10.1)–(10.8) hold with $\mu$ and $H_\mu$ replaced by $\zeta$ and $H_\zeta$. In particular $L^h \nleq K$ for some $h \in H_\zeta$ by (10.7) and so, since $K$ has one conjugacy class of involutions, $\zeta$ is conjugate to an element of $L$. Further, (10.5)(ii) and (10.7) give that $|E(H_\zeta)| < |E(H)|$. Now (10.2) yields that $H_\zeta$ is conjugate to $H$ which is impossible. Therefore, $H = H_\zeta$ must hold when $\zeta \in B$.

From Lemma 10.4(ii), $\zeta$ is conjugate in $K$ to an element of $B$ and thus we conclude that $\mathcal{M}(\zeta) = \{H\}$ for all $\zeta \in \mathcal{J}(K)$.

Let $h \in H_\mu$ be as in (10.7) and let $\eta \in L^h$. By (10.2) and (10.9), $\mathcal{M}(\eta) = \{H\} = \mathcal{M}(\eta^{-1})$ and so $h \in H$. But then, as $L$ is a component of $H$, $L^h$ is a component of $H$, contrary to (10.7).

With this contradiction the proof of Lemma 10.5 is complete.
Lemma 10.6. Assume Hypothesis 10.3 holds and let \( R \in \mathcal{R}^*(H) \). If \( H \) has exactly two components, then \( \mathcal{M}(\alpha) = \{ H \} \) for all \( \alpha \in \mathcal{Y}(C_A(E(H))) \).

Proof. Put \( K_1 K_2 = E(H) \). Let \( a \in \mathcal{Y}(C_A(E(H))) \) and \( M \in \mathcal{M}(\alpha) \). By Lemma 10.5, \( \mathcal{M}(\beta) = \{ H \} \) for all \( \beta \in \mathcal{Y}(K_i) \), \( i = 1, 2 \). Clearly \( E(H) \leq C_A(\alpha) \leq M \). Since \( m(C_A(\alpha)) \geq 5 \), by Lemmas 3.4(ii) and 10.2(i), \( K_i \leq J_i^1 \) or \( J_i^1 J_i^2 \) (\( i = 1, 2 \)), where \( J_i^j \) are components of \( M \) (and, in the latter case \( J_i^1 J_i^2 \)). In the first possibility the embedding of \( K_i \) in \( J_i^1 \) and in the second the fact that \( K_i \) covers \( (J_i^1 J_i^2)/Z(J_i^1 J_i^2) \) force \( K_i \) and \( K_2 \) to be contained in separate components of \( M \). Then, since \( C_A(\xi) \leq H \) for each \( \xi \in \mathcal{Y}(K_i) \), we obtain \( E(H) \leq E(M) \leq H \). Applying Lemma 3.4(ii) to the \( C_A(\alpha) \)-invariant subgroup \( E(M) \) yields \( E(H) = E(M) \) whence \( M = H \) and Lemma 10.6 follows.

Before proceeding further we introduce the following notation.

Suppose Hypothesis 10.3 holds for the quadruple \((A, a, H, K)\) and let \( K_1, \ldots, K_r \) be the components of \( H \). Let \( p \in C_A(E(H)) E(H) \). Then \( p = \rho_0 \rho_1 \cdots \rho_r \), where \( \rho_0 \in C_A(E(H)) \) and \( \rho_i \in K_i \). Now we define

\[
\text{supp}_H p = \{|i \mid \rho_i \in Z(K_i), 1 \leq i \leq r\}.
\]

Note that \( \text{supp}_H p \) is well defined and that \( \text{supp}_H p = 0 \) implies \( p \in C_A(E(H)) Z(E(H)) \langle C_A(E(H)) \rangle O_2(E(H)) \) by Lemma 8.3. We have suppressed mention of \( A \) in the notation \( \text{supp}_H p \) since we shall always be working with \( A \) fixed. When no confusion can occur we will write \( \text{supp}_H p \) as just \( \text{supp} p \).

Also, \( r(H) \) will denote the number of components possessed by \( H \).

Lemma 10.7. Suppose Hypothesis 10.3 holds, and set \( r = r(H) \). Suppose \( \rho, \mu \in \mathcal{Y}(C_A(E(H)) E(H)) \) are such that \( \text{supp} \mu \leq r \). If \( \rho^g = \mu \) for some \( g \in G \), then \( g \in H \).

Proof. By Lemma 10.5, \( \mathcal{M}(\rho) = \{ H \} = \mathcal{M}(\mu) = \mathcal{M}(\rho^g) \) and hence \( g \in H \).

Lemma 10.8. Assume Hypothesis 10.3 holds. Suppose \( \mu \in \mathcal{Y}(C_A(E(H)) E(H)) \) with \( \text{supp} \mu \leq r \), where \( r = r(H) \). If \( \mu^g \in H \) for some \( g \in G \) and \( r \geq 3 \), then \( \mu^g \) normalizes each component of \( H \).

Proof. Put \( \rho = \mu^g \) and \( F = C_A(E(H)) E(H) \). Let \( K_1, \ldots, K_r \) denote the components of \( H \). So \( \mu = \mu_0 \mu_1 \cdots \mu_r \), where \( \mu_0 \in C_A(E(H)) \) and \( \mu_i \in K_i \) for \( i = 1, \ldots, r \). Since, by hypothesis, \( \text{supp} \mu \leq r \), we may assume \( \mu_j \in Z(K_j) \) for some \( j, 1 \leq j \leq r \).

Suppose the lemma is false, and argue for a contradiction. So \( \rho \) interchanges components \( K_1 \) and \( K_2 \) of \( H \), and \( r \geq 3 \). Because \( \text{supp} \mu \leq r \) and Hypothesis 10.3 holds, Lemma 10.5 gives \( \{ H \} = \mathcal{M}(\mu) \). Clearly \( \{ H^g \} = \mathcal{M}(\rho) \). Put \( X = \{ kk^g \mid k \in K_1 \} \). Then \( X \leq C_A(\rho) \leq H^g \). Also, without loss of generality, we may assume \( \rho \in R \), where \( R \in \mathcal{R}^*(H) \) and \( A \leq R \).
It is claimed that

(10.10) $F^s \cap H$ contains a fours subgroup $U$ such that $\rho \in U$ and $\text{supp}_\rho, \xi < r$ for all $\xi \in U^w$.

Since $r > 3$, $\text{supp}_\rho x < r$ for all $x \in X$. Suppose there exists $\eta \in \mathcal{Y}(R \cap X)$ such that $\eta$ interchanges the components $L_1$ and $L_2$ of $H$. Then, since $m(R \cap X) > 1$, $\eta$ is contained in a fours subgroup $V$ of $R \cap X$ and so, by (2.15)(iv) and Lemma 10.5,

$$L_1, L_2 \leq \langle C_\mathfrak{g}(\zeta) \mid \zeta \in V^w \rangle \leq H.$$

Therefore, $L_1, L_2 \leq E(E(H^s) \cap H)$. Lemmas 3.4(ii) and 10.2(i) then force $L_1, L_2 \leq E(H)$. Now $L_1 = J^s$ for some component $J$ of $H$. Thus $\lambda, \lambda^s \in E(H)$, where $\lambda \in \mathcal{Y}(J)$ and $\text{supp}_\lambda, \lambda, \text{supp}_\lambda, \lambda^s \leq 1$. Hence $g \in H$ by Lemma 10.6 which gives $\rho \in E(H) C_\mathfrak{g}(E(H))$, contrary to $K_0^s \neq K_1$. Consequently each element of $\mathcal{Y}(R \cap X)$ normalizes each component of $E(H^s)$. Let $L_i = K_i^s$ for $i = 1, \ldots, r$. Choose $i \neq j$ and let $\eta \in \mathcal{Y}(R \cap X)$. Since $\langle \rho, \eta \rangle$ is a 2-group (recall that $[X, \rho] = 1$) and $\langle \rho, \eta \rangle$ normalizes $L_i$, $\langle \rho, \eta \rangle$ must centralize some involution $\zeta_i$ of $L_i$. If $\rho \neq \zeta_i$, then $U = \langle \rho, \zeta_i \rangle$ will satisfy the requirements of (10.10). So we must deal with the situation when $\rho = \zeta_i$ for all $i \neq j$. Since $r > 3$, we see that $L_i/Z(L_i)$ (if $i \neq j$) must be isomorphic to either $Sz(8)$ or $L_3(4)$ with $\rho \in Z(L_i)$. But then $\eta$ will induce inner automorphisms on $L_i$ and so $m(C_{L_i}(\eta)) > 1$. If we take $U \leq C_{L_i}(\eta) \leq E(H)$ with $m(U) = 2$ and $\rho \in U$ (which is possible by Lemma 8.1), then we have established (10.10).

Since $K_i^s = K_i$, (2.15)(iv) and Lemma 10.5 (applied to $H^s$) together with (10.10) yield $K_1 K_2 \leq E(H^s)$. Hence we obtain $K_1 K_2 \leq E(H^s)$ using Lemma 3.4(ii) on an element of $\mathcal{Y}(R \cap X)$. Therefore, there exists a component $L_i$ of $H^s$ with $\lambda \in L_i$ such that $\lambda, \lambda^s \in E(H^s)$ with $\text{supp}_\lambda, \lambda, \text{supp}_\lambda, \lambda^s \leq 1 < r$. Applying Lemma 10.7 to $H^s$ gives $g^{-1} \in H^s$, contrary to $K_i^s \neq K_i$. With this contradiction the proof of Lemma 10.8 is complete.

**Lemma 10.9.** Suppose Hypothesis 10.3 holds, $\mu \in \mathcal{Y}(C_\mathfrak{g}(E(H)) \cap E(H))$ with $\text{supp}_\mu < r(H) = r$ and $r > 2$. If $r = 2$, we further suppose that $C_\mathfrak{g}(E(H)) \neq 1$. If $\mu^s \in H$ ($g \in G$) is such that $\mu^s$ normalizes each component of $H$, then $g \in H$.

**Proof.** Let $K_1, \ldots, K_r$ denote the components of $H$. Put $L_i = K_i^s$, $U_i = K_i \cap H^s$, $V_i = L_i \cap H$ for $i = 1, \ldots, r$, $U_0 = C_\mathfrak{g}(E(H)) \cap H^s$ and $V_0 = C_\mathfrak{g}(E(H))^s \cap H$. Also, we set $U = U_0 U_1 \cdots U_r$ and $V = V_0 V_1 \cdots V_r$. Without loss of generality we may suppose $\mu^s \in R$ where $R \supset A$ and $R \in \mathcal{F}^s(H)$. By Lemma 10.5. $\mathcal{M}(\mu) = \{H\}$ and so $\mathcal{M}(\mu^s) = \{H^s\}$.

We may suppose

(10.11) for $i = 1, \ldots, r$ $L_i \leq H$ and $K_i \leq H^s$. 


Suppose $L_i \leq H$ holds for some $i$. Then since $\mu^g \in H$, $C_G(\mu^g) \leq H^g$ and $\mu^g$ normalizes each component of $H$, Lemmas 3.4(ii) and 10.1(i) yield that $L_i \leq K_j$ for some $j$. Because $r \geq 2$ by hypothesis, using Lemma 10.7 we obtain $g \in H$, as required. So we may assume $L_i \not\leq H$. Now consider the situation when $K_i \leq H^g$ for some $i$. Since $\mu^g$ normalizes $K_i$, there exists $\tau \in \mathcal{N}(K_i)$ such that $\tau \in C_G(\mu^g)$. So $\tau \in H^g$ and $\mathcal{N}(\tau) = \{H\}$. Then using Lemmas 3.4(ii) and 10.1(i) forces $K_i \leq L_j$ for some $j$. Again Lemma 10.7 gives $g \in H$, and so we have (10.11).

(10.12) For $i = 1, \ldots, r$, $U_i$ and $V_i$ have even order and, in the case $r(H) = 2$, $U_0 \neq 1 \neq V_0$.

Since $\mu^g$ normalizes $K_i$, $C_K(\mu^g)$ has even order, and hence $U_i = K_i \cap H^g$ has even order, for $i = 1, \ldots, r$. If $r = 2$, then, since $A \leq R$, $\mu^g$ normalizes $C_A(E(H))$ ($\neq 1$) and so $C_A(E(H))(\mu^g) \neq 1$. Thus $U_0 \neq 1$ when $r = 2$. If for some $j$ $Z(K_i) \neq 1$, then $K_j/Z(K_j) \cong Sz(8)$ or $L_j(4)$ whence $\mu^g$ induces an inner automorphism upon $K_j$, and so $m(U_j) \geq 2$. Therefore, since either $r \geq 3$ or $r = 2$ and $C_A(E(H)) \neq 1$, we see that $U$ contains a subgroup $W$ with $m(W) \geq 2$ and $\text{supp}_H \omega < r$ for all $\omega \in \mathcal{I}(W)$. Combining (2.15)(iv), (10.11) and Lemma 10.5 gives that $W$ normalizes each component of $H^g$. Employing Lemma 10.5 now yields that $V_i$ has even order for $i = 1, \ldots, r$, and, if $r = 2$, that $V_0 \neq 1$. This establishes (10.12).

Let us suppose that $V_1$ is such that $m(V_1) \leq m(V_i)$ and $m(V_1) \leq m(U_i)$ for all $i = 1, \ldots, r$.

Suppose $r \geq 3$. Then $\text{supp}_H \omega < r$ for all $\omega \in W$ where $W = U_1 U_2$. Let $B$ be an elementary abelian 2-subgroup of $W$ with $m(B) = m(W)$. From Lemma 10.6, $J = \langle C_{\beta}(\beta) \mid \beta \in B^* \rangle \leq V_1$. Now, by (10.12) and properties of quasisimple $\mathcal{L}$-groups, $m(W) > m(U_1) \geq 1$, and so (2.15)(iv) and (10.11) imply that $W$ normalizes $L_1$. Since $m(B) = m(W) > m(U_1) \geq m(V_1) \geq m(J)$, employing Lemma 8.4 yields $L_1 = J \leq V_1$, against (10.11). Noting that since the situation is symmetric in the $U_i$ and $V_i$, it only remains to deal with the case $r = 2$.

If it is the case that $U_0 \leq U_i$ for $i = 1$ or 2, then $m(U_0 U_i) > m(U_i) \geq m(V_i)$ with $\text{supp}_H \omega < 2$ for all $\omega \in U_0 U_i$. Then we may argue as in the case $r \geq 3$.

So we may assume that $U_0 \leq U_1 \cap U_2$. Therefore, by (10.12), $Z(K_i) \neq 1$ and so $K_i/Z(K_i) \cong Sz(8)$ or $L_j(4)$ for $i = 1, 2$. Thus $K_i \leq H^*$ and $L_i \leq (H^g)^*$ for $i = 1, 2$ by Lemma 3.3(i). If $m(U_i) > m(V_i)$, $i = 1$ or 2, then Lemmas 8.4 and 10.5 yield $L_i \leq H$. So, by (10.11), we have $m(U_i) = m(U_2) = m(V_i)$. Let $A_1$ and $B_1$ be (respectively) elementary abelian 2-subgroups of $U_1$ and $V_1$ with (respectively) $m(A_1) = m(U_1)$ and $m(B_1) = m(V_1)$. We claim that $m(V_1) < m(L_1)$. Suppose $m(V_1) = m(L_1) (= m(K_1))$ since $L_1 = K_1^g$ were to hold. Since $B_1$ must induce inner automorphisms upon $K_1$ and, by Lemma 8.1, $m(K_1) > m(K_1/Z(K_1))$ we infer that $B_1 \cap C_G(K_1) \neq 1$, whence
$K_1 \leq H^g$. Thus $m(V_1) < m(L_1)$. In particular, for each $\beta \in B_1^g$, $\beta = \delta \gamma$ where $\delta \in C_H(K_1)$ and $\gamma$ is an element in $R \cap K_1$ of order 4. Thus we must have $K_1/Z(K_1) \cong L_1(4)$. Since $A_1$ must induce inner automorphisms upon $L_1/Z(L_1) \cong L_1(4)$, $m(B_1) \geq 3$. But for $C \in \mathcal{U}_r(R \cap K_1)$, $|R \cap K_1 : C| = 2^2$ and so not all elements $\beta \in B_1^g$ can be expressed in the desired form.

This exhausts all the possibilities for the situation $r = 2$ and completes the proof of the lemma.

11. The $r(H) \geq 3$ Case

**Theorem 11.1.** Assume Hypothesis 10.3 holds. Then $r(H) \leq 2$.

**Proof.** Supposing that Hypothesis 10.3 holds and that $r(H) \geq 3$ we shall derive a contradiction. Put $r = r(H)$.

Since $r \geq 3$, Lemmas 10.8 and 10.9 yield the following:

\[(11.1) \text{If } \mu \in \mathcal{Y}(E(H)) \text{ is such that } \text{supp } \mu < r(H) \text{ and } \mu^\sigma \in H \text{ for some } \sigma \in H, \text{ then } \sigma \in H.\]

Now let $g$ be some fixed element of $G$ which does not belong to $H$. Then $M = H^g \neq H$. Let $E_1, ..., E_r$ be the components of $H$ and put $F_i = E_i^g$ for $i = 1, ..., r$. Choose $\tau \in \mathcal{Y}(E_1)$ and $\theta \in \mathcal{Y}(E_2 E_3 \cdots E_r)$ such that $\text{supp } \theta = r - 1$. Because $E_i \not\cong SL(2, q)$ where $q$ is odd, such a choice of $\theta$ is possible. Since $\tau$ and $\theta$ are not conjugate in $H$, by Lemma 10.7 they cannot be conjugate in $G$. Put $\sigma = \theta^g$. Since $g \in H$, $(11.1)$ gives

\[(11.2) \tau \not\in M \text{ and } \sigma \neq \sigma \not\in H.\]

Noting that $\tau$ and $\sigma$ are not conjugate we deduce that $\lambda = \tau \sigma$ has order $2m$. Set $\rho = \lambda^m$. Recall that if $m$ is odd, then $\tau$ and $\sigma \rho$ are conjugate (in $\langle \tau, \sigma \rangle$) and if $m$ is even then $\tau$ and $\tau \rho$ are conjugate (in $\langle \tau, \sigma \rangle$). We consider these two possibilities separately.

**Case 1:** $2 \mid m$. Now $\rho \in Z(\langle \sigma, \tau \rangle)$ and so $\rho \in C_G(\tau)$. From Lemma 10.5, since $\text{supp}_H \tau \leq 1 < r$, we have $C_G(\tau) \leq H$. So $\rho \in H$ and hence $\tau \rho \in H$. Therefore, by $(11.1)$, $\tau$ and $\tau \rho$ are conjugate in $H$ whence $\tau \rho \in E(H)$ and $\rho \in E(H)$. Since $\text{supp}_H \tau \leq 1$, we have $\text{supp}_H \tau \rho \leq 1$ and hence $\text{supp}_H \rho = \text{supp}_H \tau(\tau \rho) \leq 2 < r$. Then Lemma 10.5 forces $C_G(\rho) \leq H$ whence $\sigma \in H$ which contradicts $(11.2)$. Thus case 1 cannot occur.

**Case 2:** $2 \nmid m$. Since $\text{supp}_H \tau \leq 1$ and $\text{supp}_H \sigma = \text{supp}_H \theta = r - 1$, Lemma 10.5 gives $C_G(\tau) \leq H$ and $C_G(\sigma) \leq N$. Therefore, $\rho \in C_G(\tau) \cap C_G(\sigma) \leq H \cap M$, and hence $\tau \rho \in H$ and $\sigma \rho \in M$. Because $2 \nmid m$, $\tau \rho$ is conjugate to $\sigma$ and so $\tau \rho$ and $\sigma$ are conjugate. From $(11.1)$ we have that $\tau \rho$
and \( \theta \) are conjugate in \( H \). In particular, \( \tau \rho \in E(H) \) and \( \text{supp}_H \tau \rho = \text{supp}_H \theta = r - 1 \). Since \( \sigma \rho \) is conjugate (in \( \langle \sigma, \tau \rangle \)) to \( \tau \), a similar argument applied to \( \sigma \rho \) and \( \tau^* \) yields that \( \sigma \rho \in E(M) \) and \( \text{supp}_M \sigma \rho \leq 1 \). From \( \tau, \tau \rho \in E(H) \) and \( \sigma, \sigma \rho \in E(M) \) we infer that

\[
(11.3) \quad \rho \in E(H) \cap E(M).
\]

By (11.2) and the fact that \( \sigma, \tau \in C_G(\rho) \) we see that \( C_G(\rho) \leq H \) and \( C_G(\rho) \leq M \). Thus, by (11.1) and (11.3), \( \text{supp}_H \rho = \text{supp}_M \rho = r \). Set \( \tau \rho = \alpha \) and \( \sigma \rho = \beta \). Then (see above) \( \text{supp}_H \alpha = r - 1 \) and \( \text{supp}_M \beta \leq 1 \). Combining the facts \( \text{supp}_H \rho = r, \text{supp}_H \alpha = r - 1 \) and \( \tau \in F_1 \) yields that \( \alpha \in E_2 \cdots E_r Z(E(H)) \). Similar consideration give that \( \beta \in F_1 Z(E(M)) \). Thus we have shown

\[
(11.4) \quad \tau = \alpha \rho = \rho \alpha \quad \text{where} \quad \alpha \in E_2 \cdots E_r Z(E(H)), \quad \text{and} \quad \sigma = \beta \rho = \rho \beta, \quad \text{where} \quad \beta \in F_1 Z(E(M)).
\]

Now let \( \theta^* \in \mathcal{J}(E_2 \cdots E_r) \) be such that \( \text{supp}_H \theta^* = r - 1 \). Repeating the above arguments with \( \theta^* \) in place of \( \theta \) we obtain involutions \( \sigma^* (= \theta^* \tau) \), \( \rho^* \), \( \alpha^* \), \( \beta^* \) satisfying

\[
(11.5) \quad \begin{array}{l}
(i) \quad \sigma^* \in F_2 \cdots F_r, \\
(ii) \quad \rho^* \in C_G(\tau) \cap C_G(\sigma^*) \cap E(H) \cap E(M), \\
(iii) \quad \tau = \alpha^* \rho^* = \rho^* \alpha^*, \quad \text{where} \quad \alpha^* \in E, \cdots E, Z(E(H)) \text{ and } \sigma^* = \beta^* \rho^* = \rho^* \beta^* \quad \text{where} \quad \beta^* \in F_1 Z(E(M)).
\end{array}
\]

\[
(11.6) \quad \begin{array}{l}
(i) \quad \rho \rho^* \in E(H) \cap E(M), \\
(ii) \quad \text{supp}_M \rho \rho^* = \text{supp}_M \beta \beta^* + \text{supp}_M \sigma \sigma^*.
\end{array}
\]

Part (i) follows from (11.3) and (11.5)(ii). From (11.4) and (11.5)(iii)
\[
\rho \rho^* = \beta \alpha^* \sigma^* = \beta \beta^* \sigma \sigma^* \quad \text{(since} \quad \beta, \beta^* \in F_1 Z(E(M)) \text{ and } \sigma, \sigma^* \in F_2 \cdots F_r Z(E(M)), \text{ whence (ii) holds.}
\]

Because \( r \geq 3 \) we can choose \( \theta^* \) such that \( 1 \leq \text{supp}_M \sigma \sigma^* < r - 1 \). In addition we may choose \( \theta^* \) so that \( \overline{\sigma \sigma^*} \) has even order in \( F_2 \cdots F_r Z(E(M)) = F_2 \cdots F_r Z(E(M))/Z(E(M)) \). Thus \( \rho \rho^* \) has even order and, since \( \text{supp}_M \beta \beta^* \leq 1, \text{supp}_M \rho \rho^* \leq r - 1 \) by (11.6)(ii). Let \( \lambda \in \mathcal{J}(\rho \rho^*) \). Then \( \text{supp}_M \lambda \leq r - 1 \) and, by (11.6)(i), \( \lambda \in E(H) \cap E(M) \). Hence there exists \( \xi \in \mathcal{J}(E(H)) \) such that \( \text{supp}_H \xi \leq r - 1 \) and \( \xi^* \in H \). But then (11.1) forces \( g \in H \), which is contrary to the initial choice of \( g \). Therefore, case 2 also leads to a contradiction, and so Theorem 11.1 is proven.
12. The Case $C_A(K) = 1$

**Theorem 12.1** Assume Hypothesis 10.3 holds. Then $C_A(K) \neq 1$.

**Proof.** Supposing $C_A(K) = 1$ we derive a contradiction in a series of statements. Since the arguments to follow are independent of whether $O_2^+(K)$ is trivial or not, to simplify notation we shall assume $O_2^+(K) = 1$.

(12.1) (i) $K$ is simple; and

(ii) $K = E(H)$.

If $Z(K) \neq 1$, then $K/Z(K)$ is isomorphic to either $Sz(8)$ or $L_3(4)$, and then $A \cap K \geq Z(K)$, contrary to $C_A(K) = 1$. So $Z(K) = 1$ and (i) holds. Lemma 10.4(ii) implies that $A \cap K_i \neq 1$ for all components $K_i$ of $H$. Thus, as $C_A(K) = 1$, $K = E(H)$.

(12.2) $A \subseteq K$.

Since $A$ normalizes $K$ and $C_A(K) = 1$, $AK$ is a subgroup of $\text{Aut} K$. Then, by Lemma 8.3(ii), either $A \subseteq K$, $R \cong \mathbb{Z}_2 \times D_8$ ($R \in \mathcal{F}^*(AK)$), $|A| = 4$ or $|A| = 2^{n+1}$ and $K \cong L_2(2^n)$. Since, by Hypothesis 10.3 and Lemma 9.4, $m(A) > 3$ we have only to show that $|A| = 2^{n+1}$ and $K \cong L_2(2^n)$ ($n > 2$) is impossible. Let $R \in \mathcal{F}^*(H)$ with $R \supseteq A$. We may suppose $S \cap H = R$. Then $C_R(K) \cap A = 1$ and hence, as $A = O_1(C_R(A))$, $C_R(K) = 1$. Thus $R = (R \cap K)A$. Because $m(R \cap K) > m(A)$, Hypothesis 10.3 dictates that $\sigma \in A \cap K$ and that $\sigma$ is not conjugate to any element of $R \cap K$. Since all involutions of $R \cap K$ are $AK$-conjugate and $C_G(\sigma) \leq H$, $R = S$. But, as $|S'| > 2$, this is impossible by Lemma 4.1(vi). Therefore, $A \subseteq K$ must hold.

(12.3) $K \cong PSp_4(2^n)$ for some $n \geq 2$.

Suppose $K \cong PSp_4(2^n)$. Then $K$ has one conjugacy class of involutions by (12.2)(i) and (2.18)(ii). Since, using (12.2), $\sigma \in K$, we obtain $C_G(\mu) \leq H$ for all $\mu \in \mathcal{F}(K)$. Now $Z(S) \leq A$, and thus $S \leq H$. If $S \leq K$, then (2.4) yields a contradiction to the choice of $G$. Therefore, $S \not\subseteq K$, and so $K$ is isomorphic to either $L_2(q)$ ($q$ odd), $A_1$, or $L_2(2^n)$. Either of the first two possibilities imply, since $A \subseteq K$, that $m(A) = 2$, and thus we have $K \cong L_2(2^n)$. Hence $S = \langle \rho \rangle (S \cap K)$, where $\rho$ induces a field automorphism upon $K$. Now $\mathcal{F}(S) \setminus (S \cap K)$ are conjugate in $K(\rho)$ and so, since $S \cap K$ is not a strongly closed 2-subgroup, $S$ has one conjugacy class of involutions. Hence $\sigma \in \mathcal{F}(S)$. But then $E_2^+(C_G(\sigma)) \leq E_2^+(H) = KO_2^+(H)$, contrary to $C_K(\sigma)$ being 2-closed. Thus we conclude that $K \cong PSp_4(2^n)$ for some $n \geq 2$.

(12.4) $S \subseteq K$. 
Let \( R \in \mathcal{X}^{*}(H) \). Then, by (12.3), \( R \cap K \in \text{Syl}_{2} K \), and hence \( Z(R \cap K) = (R \cap K)' \leq Z(S) \). By (12.2) and (2.18)(i), \( \sigma \) is \( K \)-conjugate to an element of \( Z(R \cap K) \) whence \( S \leq H \), and then, since \( K \leq H \), \( S = C_{S}(K) \times (S \cap K) \). Hence \( S \leq K \), as required.

From (12.3) and (12.4) we have that \( S \in \text{Syl}_{2} K \), from which we deduce

\[
(12.5) \quad \begin{align*}
& \text{(i) } \mathfrak{A}_{e}(S) = \{A, B\}, \ Z(S) = S' \text{ and } S = AB; \\
& \text{(ii) } \mathfrak{A}(S) \text{ consists of three } K\text{-conjugacy classes each of which has a} \\
& \text{representative in } Z(S). \text{ Also } Z(S) = F_{1} \times F_{2}, \text{ where } |F_{i}| = 2^{n} \text{ and the } K\text{-conjugacy classes of } Z(S) \text{ are } F_{1}^{*}, F_{2}^{*} \text{ and } Z(S) \setminus (F_{1} \cup F_{2}).
\end{align*}
\]

Since \( C_{G}(\sigma) \leq H \), \( E_{2}(C_{G}(\sigma)) \leq E_{2}(H) = KO_{2}(H) \) and \( K \simeq \text{PSp}_{4}(2^{n}) \), we see that \( \sigma \notin \mathfrak{C}(S) \). Let \( \mu \in \mathfrak{C}(S) \cap Z(S) \) and let \( H_{\mu} \in \mathfrak{M}(\mu) \). So \( S \leq H_{\mu} \). From Lemma 10.2 and \( E_{2}(C_{\mu}(\mu)) \leq E_{2}(H_{\mu}) \) we have that \( E(H_{\mu}) \neq 1 \). Let \( L \) be a component of \( H_{\mu} \).

\[
(12.6) \quad C_{S}(L) \geq C_{A}(L) + 1 \text{ and } S \leq N_{G}(L).
\]

If it were the case that \( C_{A}(L) = 1 \), then steps (12.1)-(12.3) (since by (2.18)(iii), we know that \( \mu \) is contained in either \( A \) or \( B \)) would yield \( E(H) = L \simeq \text{PSp}_{4}(2^{l}) \) for some \( l \), contradicting the choice of \( \mu \in \mathfrak{C}(S) \). Thus \( C_{A}(L) \neq 1 \). From Lemma 3.3(ii) and (12.5)(i), \( S \leq N_{G}(L) \).

We now consider two cases depending on whether \( L \) is simple or not.

**Case 1: \( L \) is simple.** Since \( S \leq N_{G}(L) \), \( S \) contains a subgroup \( R \) of index at most two, with \( R = C_{S}(L) \times (S \cap L) \). Put \( R_{1} = C_{L}(L) \) and \( R_{2} = S \cap L \), and note that \( R_{i} \leq S \).

Suppose \( S = R \) holds. Then, it is claimed, \( R_{i} (i = 1, 2) \) is nonabelian. For, if (say) \( R_{1} \) were abelian, then \( Z(S) = S' \leq R_{1} \) whenever \( R_{2} = 1 \), a contradiction. Similarly \( R_{2} \) being abelian would contradict (12.6). Therefore, \( L \notin L_{3}(q) \) \( (q \equiv 3, 5(8)) \) and so \( \{A \cap L, B \cap L\} \in \mathfrak{A}(R_{2}) \) by Lemma 8.2. As a consequence \( A = A_{1} \times A_{2} \), where \( A_{1} = A \cap R_{i} \), and \( B = B_{1} \times B_{2} \), where \( B_{1} = B \cap R_{i} \). Because \( AB = (A_{1} \times A_{2})(B_{1} \times B_{2}) = A_{1}B_{1} \times A_{2}B_{2} \) and \( S = AB \) by (12.5)(ii) we obtain \( R_{i} = A_{i}B_{i} \) for \( i = 1, 2 \). Therefore, because the \( R_{i} \) are non-abelian, \( A_{i} \not\leq B \) and \( B_{i} \not\leq A \). From \( m(A_{1}) = m(B_{1}) \) and \( m(A_{2}) = m(B_{2}) \) we see that \( m(A_{1}) = m(B_{1}) \). Hence \( A_{1}B_{2} \in \mathfrak{A}_{e}(S) \). But then \( A \neq A_{i}B_{i} \neq B \) contradicts (12.5)(i). Thus \( S \neq R \) and so \( |S : R| = 2 \).

Now \( Z(S) = S' \leq R, \) and so \( Z(S) \leq Z(R) = Z(R_{i}) \times Z(R_{j}) \). Since \( R_{i} \leq S \) for \( \tau \in S \setminus R \) we have \( Z(S) = C_{Z(R)}(\tau) = C_{Z(R_{i})}(\tau) \times C_{Z(R_{j})}(\tau) = (Z(S) \cap R_{i}) \times (Z(S) \cap R_{j}) \). If \( |Z(S) \cap R_{2}| = 2 \), then \( (Z(S) \cap R_{i}) \cap F_{i} = 1 \) for \( i = 1, 2 \), and hence each involution of \( S \) is \( K \)-conjugate to an element in \( C_{A}(L) \) (see (2.15)). In particular, for some \( k \in K \), \( L \leq C_{G}(\sigma^{k}) \leq H \). Appealing to Lemmas 3.4(iii) and 10.2 yields \( L = K \), which is impossible since \( C_{A}(L) \neq 1 \). Therefore, \( |Z(S) \cap R_{2}| > 2 \). Hence \( L \simeq L_{2}(2^{2m}) \), \( m \geq 2 \) is
the only possibility. But then $A \cap L, B \cap L \in \mathcal{H}_e(R_2)$ implies that $R_2 = A \cap L = B \cap L \leq Z(S)$, whereas $R_2 \notin Z(S)$.

Thus case 1 cannot occur.

**Case 2: L is not simple.** Thus $L/Z(L) \cong Sz(8)$ or $L_3(4)$. If $L/Z(L) \cong Sz(8)$, then $A \cap L = B \cap L = (S \cap L)'$, whence $AB \leq C_S(L)$ $(S \cap L)' \neq S$. Therefore, $L/Z(L) \cong L_3(4)$ by (12.5)(i). Since $L \neq K$, we must have $L \notin K$. Therefore, from (12.5) we conclude that $F_j \cap C_G(L) = 1$ for at least one of $j = 1$ and $j = 2$. Since $|Z(S \cap L)/Z(L)| = 4$, this forces $|F_j| \leq 4$ whence $n = 2$. So $K \cong PSU_3(4)$ and $|S| = 2^8$. Also, $|S \cap L| = 2^k |S \cap Z(L)| = 2^7$ or $2^6$, and clearly $L = E(H^*)$.

Let $\rho \in \mathcal{Y}(Z(S))$ with $\rho \notin \sigma^k \cup \mu^k$. Now we show that $\sigma^g \cap S \neq \sigma^H \cap S$. Suppose $\sigma^g \cap S = \sigma^H \cap S$ and argue for a contradiction. Now $|A \cap L, B \cap L| = \mathcal{H}_e(S \cap L)$ and $|A: A \cap L| \leq 2$ (since $|A \cap L| = 2^5$ or $2^6$). From (2.19) there exists $\sigma^g \in \sigma^\mathcal{Y}$ with $\sigma^A \in A \cap L$. Then our supposition and the fact that $C_G(\sigma^g) \leq H$ force $N_o(A \cap L) \leq H$. Similarly we also obtain $N_o(B \cap L) \leq H$. But then, by [2, (2.36)(iii)], $L = \langle N_o(A \cap L), N_o(B \cap L) \rangle \leq H$, a contradiction. Therefore $\sigma^g \cap S \neq \sigma^H \cap S$. Consequently, since $\mu \in \mathcal{H}(S)$ and $\sigma \in \mathcal{H}(S)$, $S_o = \sigma^k \cap S$ and $S_{\rho} = \rho^k \cap S$ must be fused in $G$ but not in $H$. Since $\sigma, \rho \in \mathcal{Y}(S)$, by (2.1)(v) there exists $g \in N_o(S) \setminus H$ such that $\sigma^g = \rho$. Note that $S_{\rho} = S_o \neq \emptyset$ would force $g \in H$. Hence $S_o \cap S_{\rho} = \emptyset$ and so $S_o \neq S_{\rho}$.

If $S_o = S_{\rho}$, then using Lemma 8.5 gives $K = \langle C_k(\rho) \mid \rho \in S_{\rho} \rangle \leq H^g$, which yields $K \leq E(H^g) = K^g$, contrary to $g \notin H$. Thus $S_o \neq S_{\rho}$ and so $|S_o| < |S_{\rho}|$. Therefore, $\mu^g \cap Z(S) = \mu^k \cap Z(S) = F_1^*$ or $F_2^*$. Because $H_{\mu}, L$ and $A$ satisfy Hypothesis 10.3, Lemma 10.2 yields $\{H_\mu\} = \mathcal{H}(\mu^g)$ for all $\mu^g \in C_A(L)^*$. Since $C_A(L) \leq Z(S)$ and $|C_A(L)| = 4$, we conclude that $C_A(L)^* = \mu^g \cap Z(S)$. Moreover, we also have that $N_o(S) \leq H_{\mu}$. Hence, because $L = E(H^g_{\mu}) \leq H_{\mu}$, (2.1)(v) forces $C_A(L) \leq L$ and so $C_A(L) = Z(L) \cap S \leq H$. Since $C_A(L) < Z(S)$ and $L_3(4)$ has one conjugacy class of involutions, we see that every involution of $L$ is $L$-conjugate to an element of $Z(S)$. This observation together with $N_o(S) \leq H_{\mu}$ and (2.1)(v) then forces $\mu^g \cap S = C_A(L)^*$, whereas $|\mu^k \cap S| > 3$. With this contradiction case 2 is also eliminated.

The proof of Theorem 12.1 is complete.

### 13. The Case $r(H) = 2$ and $C_A(E(H)) \neq 1$

**Theorem 13.1.** If Hypothesis 10.3 holds with $r(H) = 2$, then $C_A(E(H)) = 1$.

**Proof.** Supposing $C_A(E(H)) \neq 1$ we argue for a contradiction. Put $K_1K_2 = E(H)$. 


Since $A \cap K_1 \neq 1$, Lemmas 10.4(i) and 10.9 imply $N_G(A) \leq H$, and hence, in particular, $S \leq H$. Now, since $A \leq S$, $Z(S) \cap C_A(E(H)) \neq 1$ and so Lemmas 10.4(i) and 10.9 also yield $N_G(Z(S)) \leq H$. Therefore, $T \leq H$ where $T \in Syl_1 G$ is such that $S \leq T$. Noting that $1 \neq C_S(E(H)) \leq T$ we have $Z(T) \cap C_S(E(H)) \neq 1$. Let $\alpha \in \mathcal{Y}(Z(T) \cap C_S(E(H)))$. Clearly $\alpha \in Z(S)$ and so $\alpha \in S_\alpha(H))$.

Let $g \in G$ be such that $\alpha^g \in S$. We will show that $g \in H$. If $\alpha^g$ normalizes both $K_1$ and $K_2$, then $g \in H$ by Lemma 10.9. So we must examine the situation when $\alpha^g$ interchanges $K_1$ and $K_2$. Put $R = C_S(E(H))(S \cap E(H))$ and $Q = C_S(E(H))$. So $Z(S) \cap K_i = 1$, $i = 1, 2$ and hence $U_1(S) \cap K_i = 1$, $i = 1, 2$. Therefore, $\mathcal{Y}(N_S(K_1))R = \emptyset$.

From Lemma 10.6, $\mathcal{M}(\rho) = \{H\}$ for all $\rho \in \mathcal{Y}(Q)$. Put $P = C_G(\alpha^g)$.

\[ (13.1) \quad m(P) = 1. \]

Suppose $m(P) \geq 2$. Recalling that $\alpha \in Z(S)$, we see that $P$ contains a noncyclic abelian group $V$ with $\alpha \in V$. Now $V \leq P \leq C_G(\alpha^g) \leq H^g$. If $\alpha$ interchanges $K_1^g$ and $K_2^g$, then $E(H^g) = K_1^gK_2^g \leq H$ by (2.15)(iv). Since $\alpha^g \in H$ and $E(H^g)$ is $C_%G(\alpha^g)$-invariant, this gives $E(H^g) \leq E(H)$, and thus $g \in H$. Therefore, either $g \in H$ or $\alpha$ normalizes $K_1^g$ and $K_2^g$. If the latter possibility holds, then $\alpha^{-1}$ normalizes $K_1$ and $K_2$. Because $\alpha \in H^g$, $\alpha^{-1} \in H$ and then applying Lemma 10.7 gives $g^{-1} \in H$. Therefore, $m(P) \geq 2$ implies $g \in H$, and so we may suppose $m(P) = 1$.

\[ (13.2) \quad |P| = 2. \]

From (13.1), $\langle \alpha \rangle = \Omega_1(P)$. Suppose $|P| > 2$. Then there exists $\zeta \in P$ such that $\zeta^2 = \alpha$. Since $\alpha^g \in Z(S^g) \leq C_G(\alpha^g)$, there exists $c \in C_G(\mu^g)$ such that $P^c \leq S^g$, and thus $\alpha^g \in Z(S^g)$. By (2.15)(v), $\alpha^g = \alpha^k$ for some $k \in N_G(S^g) \leq H^g$ (since $N_G(S) \leq N_G(Z(S)) \leq H$). Hence $g^{-1} \in C_G(\alpha^g) \leq H^g$ which gives $g^2 \in H^g$. Therefore, we may suppose that (13.2) holds.

Setting $Q\langle \alpha^g \rangle = Q_0$ we have that $C_{Q_0}(\alpha^g)$ is elementary abelian of order 4. By a well-known result of Suzuki $Q_0$ must be dihedral or semidihedral, and so $Q_0 \cong D_8$ or $Z_2 \times Z_2$. In particular, $Q$ is abelian.

Using Lemma 3.3(i) gives $R = Q \times (S \cap E(H))$ with $S \cap E(H)$ elementary abelian. The fact that $\mathcal{Y}(N_\alpha(K_1))R = \emptyset$ and Lemmas 3.3(i), (ii) imply that $\mathcal{Y}(\Omega_1(R))$ and thus $\Omega_1(R)$ is a weakly closed elementary abelian subgroup of $T$. Hence Hypothesis 9.2 of [4] is satisfied (with $\Omega_1(R) = W$). Since $\max(m(B/C_G(\Omega_1(R)))) \leq S$, $B \leq \Omega_1(R)$ and $B$ is $G$-conjugate to a subgroup of $\Omega_1(R) \leq 1$, [4, Corollary 4] implies that $m([\Omega_1(R), \alpha^g]) \leq 1$, whereas $m([S \cap E(H), \alpha^g]) \geq 2$. From this contradiction we infer that $g \in H$, as desired.

Since $C_G(\alpha) \leq H$ by Lemma 10.6(ii), (2.20) may be applied to show that $G$ is not a minimal counterexample and so we conclude that $C_A(E(H)) = 1$. 

14. The Case $r(H) = 2$ and $C_A(E(H)) = 1$

**Hypothesis 14.1.** This hypothesis holds if Hypothesis 10.3 is satisfied with $r(H) = 2$ and $C_A(E(H)) = 1$ (and we put $E(H) = K_1 K_2 = K_1 \times K_2$).

Recall that Lemma 10.5 gives

\[(14.1) \quad \text{if Hypothesis 14.1 holds, then } \mathcal{M}(\eta) = \{H\} \text{ for all } \eta \in \mathcal{Y}(K_i) \text{ for } i = 1, 2.\]

The purpose of this section is to show that Hypothesis 14.1 is untenable in our minimum counterexample (Theorem 14.8). First we establish the following result, which is similar in spirit to Lemma 10.9.

**Theorem 14.2.** Assume Hypothesis 14.1 holds, and let $R \in \mathcal{E}^*(H)$ with $R \geq A$ and $S \cap H = R$. Suppose $\mu \in \mathcal{Y}(R \cap K_i)$, $i = 1$ or 2, and $\mu^g \in R$, where $g \in G$. Then $g \in H$.

Before commencing the proof of Theorem 14.2 we note in Lemmas 14.3–14.7 some straightforward consequences of Hypothesis 14.1.

**Lemma 14.3.** Assume Hypothesis 14.1 holds. Then there exists $g \in N_{\mathcal{E}}(A)$ such that $S^g \leq H$. Also we have $A \in \mathcal{U}_e(S)$.

**Proof.** Let $R \in \mathcal{E}^*(H)$ be such that $A \leq R$. If $Z(S) \cap K_i \neq 1$ for $i = 1$ or 2, then $S \leq H$ by (14.1). So we may suppose $Z(S) \cap K_i = 1$ for $i = 1, 2$. Thus $K_i \cap U^1(R) = 1$ for $i = 1, 2$, and hence $K_i \cap R$ must be abelian for $i = 1, 2$. In particular the $N_{E(H)}(R \cap E(H))$ conjugacy classes of $\mathcal{Y}(R \cap E(H))$ are $(R \cap K_1)^{\mathcal{Y}}$, $(R \cap K_2)^{\mathcal{Y}}$ and $\Delta = (R \cap E(H)) \setminus ((R \cap K_1) \cup (R \cap K_2))$. Moreover, if $\eta \in A^{\mathcal{Y}}$, then $\eta \in R \cap E(H)$. For, if $n \notin E(H)$ we must have by Lemma 10.4(i), since $C_A(E(H)) = 1$, $K_i \leq K_i(\eta) \leq \text{Aut } K_i$ for $i = 1$ or 2, whence $K_i \cap U(R) \neq 1$. Thus $A = R \cap E(H)$. Consequently $Z(S)^{\mathcal{Y}} \leq \Delta$. Let $n \in N_{\mathcal{E}}(A)$ and $\xi \in (A \cap K_i)^{\mathcal{Y}}$. So $\xi^n \in A$. If $\xi^n \in R \cap K_1$ or $R \cap K_2$, then $g \in H$ by Lemma 10.7, while $\xi \in \Delta$ implies $\xi^{nk} \in Z(S)$ for some $k \in N_{E(H)}(A)$ and then, using (14.1), $S^{k^{-1}n^{-1}} \leq H$. So either $S \leq N_{\mathcal{E}}(A) \leq H$ or $S^g \leq H$ for some $g \in N_{\mathcal{E}}(A)$, are required.

Now we show that $A \in \mathcal{U}_e(S)$. If $Z(S) \cap K_i \neq 1$, then (14.1) and Hypothesis 10.3 forces $A \in \mathcal{U}_e(S)$. So we may suppose that $K_i \leq H^{*} \text{ and that (see above) } R \cap E(H) = A$. Let $g \in N_{\mathcal{E}}(A)$ be such that $S^g \leq H$, and let $B \in \mathcal{U}_e(S^g)$. Suppose $A \notin \mathcal{U}_e(S)$. Then $m(A) \geq 5$ and, hence, since $K_1 \cong K_2$, $K_i \leq L_2(q)$ ($q = 3, 5(8)$). Thus, by Lemma 8.3(i), $B > A$, contradicting the maximality of $|A|$. Therefore, $A \in \mathcal{U}_e(S)$.

In view of Lemma 14.3, when Hypothesis 14.1 is satisfied, we may (and shall), by a change of notation, suppose that $S \leq H$. 


(i) Suppose that neither of $K_1$ and $K_2$ is isomorphic to $L_2(q^2)$, $q = 3, 5(8)$ or $A_7$. Then $S \cap E(H)$ contains an element of $V_e(S)$; and

(ii) $C_S(E(H)) = 1$.

Proof. If $A \cap K_i \in V_e(S \cap K_i)$ for $i = 1, 2$, then $A \cap E(H) \in V_e(S \cap E(H))$. Since $K_i \not\cong L_2(q^2)$ ($q \equiv 3, 5(8)$) or $A_7$ and $C_A(E(H)) = 1$, we have $A = A \cap E(H)$, as required. Otherwise, by Lemma 8.2(ii), we have either (say) $K_1 \cong L_2(q_1)$, $K_2 \cong L_2(q_2)$; or $K_i \cong L_2(q_i)$ for $i = 1, 2$ (where $q_i \equiv 3, 5(8)$ for $i = 1, 2$). In the first case (using $C_A(E(H)) = 1$), $m(A) = 2 + m(K_2) = m(S \cap E(H))$ and thus $S \cap E(H)$ contains an element of $V_e(S)$. The second possibility gives $m(A) = 3$ or $4$ but $m(S \cap E(H)) = 4$, and hence the desired conclusion holds in this case. This proves (i).

Because of Lemma 14.3 we may suppose $S \leq H$. Since $C_S(E(H)) \leq S$, $C_S(E(H)) \neq 1$ would give $1 \neq C_{2(S)}(E(H)) \leq C_A(E(H))$. Thus $C_S(E(H)) = 1$.

LEMMA 14.5. Assume Hypothesis 14.1, and let $T \in \text{Syl}_2 G$ be such that $T \geq S$.

(i) If $K_i^\delta = K_2$ for some $\delta \in S$, then $S \cap E(H)$ is abelian and is weakly closed in $T$.

(ii) If $K_i \cong L_2(2^n)$ for $i = 1, 2$, then either $S \cap E(H)$ is weakly closed in $T$ or $K_i \cong L_2(2^2)$, $i = 1, 2$.

Proof. Since $K_i^\delta = K_2$ for some $\delta \in S$, $Z(S) \cap K_i = 1$ and so $V(S) \cap K_i = 1$ ($i = 1, 2$). Thus, $K_i \not\cong L_2(q^2)$ ($q \equiv 3, 5(8)$) or $A_7$ and therefore $C_S(E(H)) = 1$ by Lemma 14.14(ii). Consequently $\mathcal{N}(S \cap E(H)) = \emptyset$. Put $B = S \cap E(H) = (S \cap K_1) \times (S \cap K_2)$. By Lemma 3.3(i), $B$ is elementary abelian. If $B^k \leq S$ for some $g \in G$, then $B^k \leq S$, whence $[B : B_1] < 2$ where $B_1 = B \cap B^k$. Since $m(S \cap K_i) \geq 2$ ($i = 1, 2$), we have (say) $B_1 \cap (S \cap K_1) \neq 1$ and $B_1^{-1} \cap (S \cap K_2) \neq 1$, where $j \in \{1, 2\}$. Appealing to Lemma 10.7 gives $g \in H$, and so $B^k \leq S \cap E(H) = B$. This verifies (i).

In view of (i) we may suppose $N_S(K_i) = S$. Since $S \leq H$ and $C_S(E(H)) = 1$ by Lemma 14.4(ii), $[S : S \cap E(H)] \leq 4$. Suppose $K_i \not\cong L_2(2^2)$ and put $B = S \cap E(H)$. If $B^k \leq S$ for some $g \in G$, then $[B : B_1] < 4$ where $B_1 = B \cap B^k$. Now $m(B) \geq 3 + m(S \cap K_i)$ and so $m(B_1) \geq 1 + m(S \cap K_i)$. Hence $B_1 \cap (S \cap K_i) \neq 1 \neq B_1^{-1} \cap (S \cap K_i)$. Arguing as in part (i) yields that $B = B^k$. Therefore, either $S \cap E(H)$ is weakly closed in $T$ or $K_i \cong L_2(2^2)$, $i = 1, 2$.

LEMMA 14.6. If Hypothesis 14.1 holds, then $N_S(K_i) = S$, $i = 1, 2$.

Proof. Suppose the lemma is false, and argue for a contradiction.
From Lemma 14.5(i), \( S \cap E(H) \) is abelian and weakly closed in some Sylow 2-subgroup of \( G \). As in Lemma 14.5(i) we also have 
\[ N_5(K_i) \cap (S \cap E(H)) = \emptyset. \]
Therefore, \( s \leq 1 \), where \( s = \max \{ m(D/C_2(S \cap E(H))) \mid D \leq S, D \leq S \cap E(H) \) and \( D \) is \( G \)-conjugate to a subgroup of \( S \cap E(H) \). So Hypothesis 9.4 of [4] holds with \( S \cap E(H) = W \).

Since \( S \cap E(H) \) is not a strongly closed 2-subgroup, there exists \( \zeta \in S \backslash (S \cap E(H)) \) with \( \zeta \) \( G \)-conjugate to an element of \( S \cap E(H) \). Thus \( K_i^j = K_i \) and so \( m([S \cap E(H), \zeta]) = m(S \cap K_i) > 1 \). But then, by [4, Corollary 4], \( S \cap E(H) \) is a strongly closed 2-subgroup, a contradiction. Therefore, \( N_5(K_i) = S \), \( i = 1, 2 \) must hold.

**Lemma 14.7.** Assume Hypothesis 14.1 holds with \( K_i \cong L_2(2^n) \), \( i = 1, 2 \). Then \( K_i \cong L_2(2^2) \), \( i = 1, 2 \).

**Proof:** Put \( R_i = S \cap K_i \), \( i = 1, 2 \), and suppose at least one of \( K_i \) is not isomorphic to \( L_2(2^2) \). By Lemma 14.5(ii), \( S \cap E(H) = R_1 \times R_2 \) is weakly closed in some Sylow 2-subgroup of \( G \).

Let \( \xi \in Z(S) \) (\( \leq S \cap E(H) \)) and \( \eta \in \mathcal{Y}(S) \). Assume \( \xi \) and \( \eta \) are conjugate and let \( f \in G \) be chosen so that \( \xi^f = \eta \) and \( C_2(\eta) \leq S^f \). Now \( C_{K_i}(\eta) \) is non-cyclic and hence \( C_{K_i}(\eta) \cap C_{S^f}(K_i^j) \neq \emptyset \), \( i = 1, 2 \). Therefore, (14.1), \( R_i^j \triangleleft H \) and so \( (S \cap E(H))^f \leq S \cap (S \cap E(H))^f \leq H \). Thus \( (S \cap E(H))^f \leq S \) for some \( h \in H \) whence \( (S \cap E(H))^f = S \cap E(H) \). Consequently \( \eta \in S \cap E(H) \) which implies, since by Lemma 14.6 all involutions of \( E(H) \) are \( E(H) \)-conjugate to an element of \( Z(S) \), that \( S \cap E(H) \) is a strongly closed abelian 2-subgroup of \( G \). Therefore, we must have \( K_i \cong L_2(2^2), i = 1, 2 \), as desired.

**Proof of Theorem 14.2.** We suppose \( g \notin H \), and show this leads to a contradiction. From Lemma 14.3 we may suppose \( S \leq H \).

\( (14.2) \ K_j^j \triangleleft H \) for \( j = 1, 2 \).

Suppose \( K_j^j \triangleleft H \) were to hold. Then, since \( \mu^t \) normalizes \( K_1 \) and \( K_2 \) by Lemma 14.6, \( K_j^j \triangleleft K_j^j, j' = 1 \) or 2 by Lemmas 3.4(ii) and 10.2. Then \( g \in H \) by Lemma 10.7, a contradiction. Thus (14.2) holds.

We now subdivide the proof into the following two cases:

1. \( m(C_{K_i}(\mu^i)) \geq 2 \) for some \( i \in \{1, 2\} \); and
2. \( m(C_{K_i}(\mu^i)) \leq 1 \) for \( i = 1, 2 \).

Since \( N_5(K_i) = S \) by Lemma 14.6, (1) and (2) cover all the possibilities.

**Case 1:** \( m(C_{K_i}(\mu^i)) \geq 2 \) for some \( i \in \{1, 2\} \)

Combining (2.18)(v), (14.1) and (14.2) we immediately obtain

\( (14.3) \ K_1 \) and \( K_2 \) are isomorphic to one of the following: \( L_2(2^n), Sz(2^n), U_3(2^n), L_3(2^n) \) and \( PSp_4(2^n) \) \( n \geq 2 \).
We also note that

\[(14.4) \text{ all involutions in } E(H) \text{ are } E(H)\text{-conjugate to an element in } Z(S).\]

Since \( S \cap K_i \leq S \) for \( i = 1, 2 \), \( Z(S) \cap K_i \neq 1 \), \( i = 1, 2 \), and so, if both \( K_1 \) and \( K_2 \) have one conjugacy class of involutions, then \((14.4) \) holds. Otherwise at least one of \( K_1 \) and \( K_2 \) is isomorphic to \( PSp_4(2^n) \). If, say, \( K_1 \cong PSp_4(2^n) \), then \( S = (S \cap K_1) \times C_S(K_1) \) and hence \( Z(S \cap K_1) \leq Z(S) \). So, by \((2.18)(i)\), \((14.4) \) will also hold in this case.

We now further restrict our attention to

**Subcase 1(a):** \( \mu^s \in E(H) \). Since \( C_g(\mu) \leq H \), to obtain a contradiction to \( g \notin H \) it will suffice to show that \( \mu \) and \( \mu^s \) fuse in \( H \). Hence, by \((2.1)(v)\) and \((14.4)\) we may assume that \( g \in N_G(S) \).

The possibilities for the \( K_i \) as given in \((14.3) \) will be examined in the following three cases:

(a.1) \( K_i \neq L_2(2^n) \) for \( i = 1, 2 \);

(a.2) exactly one of \( K_i \) is isomorphic to \( L_2(2^n) \); and

(a.3) \( K_i \cong L_2(2^n) \) for \( i = 1, 2 \).

**Case (a.1).** By Lemma 14.4(ii) and \((14.3) \) \( C_G(E(H)) = 1 \) and thus \( S = S_1 \times S_2 \), where \( S_i = S \cap K_i \). Since \( g \notin H \), \( S_i^g \cap S_i = 1 \) for \( i = 1, 2 \). Because \( g \in N_G(S) \), \( S_i^g \leq S \) and thus \( [S_i^g, S_i] = 1 \) for \( i = 1, 2 \). That is, \( S_i^g \leq Z(S) \). Similarly \( S_i^g \leq Z(S) \) and so \( S \) is abelian, contrary to \( \text{cl } S = 2 \). This deals with case (a.1).

**Case (a.2).** To fix notation we suppose \( K_1 \cong L_2(2^n) \) some \( n \). So \( S = C_S(K_1) \times (S \cap K_2) \) and, since \( C_s(E(H)) = 1 \), \( [S : S \cap E(H)] \leq 2 \). We next establish

\[(14.5) \text{ if } \xi \in \mathcal{F}(S \cap K_2), \text{ then either } \xi \text{ is not conjugate to an element of } C_S(K_2) \text{ or } K_2 \cong U_3(4) \text{ and } K_1 \cong L_2(4).\]

Suppose \( \xi \) is conjugate to \( \eta \in C_S(K_2) \). Then \( K_2 \leq C_G(\eta) \leq H_n \in \mathcal{H}(\eta) \). If it is the case that \( m(C_G(\eta)) \geq 5 \), then Lemmas 3.4(iii) and 10.2 force \( K_2 \leq E(H_n) \). However, by \((14.3) \), as \( E_2(C_G(\xi)) \leq E_2(H), \) \( C_G(\xi) \) has only one component and that is isomorphic to \( L_2(2^n) \), and so we must have \( m(C_G(\eta)) \leq 4 \). Since \( m(C_{C_S(K_2)}(\eta)) \geq 2 \) (because \( \eta \) normalizes \( S \cap K_1 \), \((14.3) \) implies that \( K_2 \cong U_3(4) \). Also we must have \( m(S \cap K_1) = 2 \), and thus \( K_1 \cong L_2(4) \). This proves \((14.5) \).

Considering the first possibility of \((14.5) \) we obtain, since \( g \in N_G(S) \), \((S \cap K_2)^g \cap C_S(K_2) = 1 \). From Lemma 10.7 we also have \((S \cap K_2)^g \cap (S \cap K_2) = 1 \) and hence \( [(S \cap K_2)^g, C_S(K_2)] = [(S \cap K_2)^g, S \cap K_2] = 1 \). Thus \((S \cap K_2)^g \leq Z(S) \), and so \( S \cap K_2 \leq Z(S) \). Then \( S \cap E(H) \) is abelian.
and so $|S'| = 2$ by (2.12), whence $K_i \cong L_2(2^2)$. Therefore, $|Z(S) : S \cap K_2| = 2$ and hence, as $g \in N_G(S)$, $(S \cap K_2) \cap (S \cap K_2)^G \neq 1$, which gives $g \in H$ by Lemma 10.7. So it remains to deal with the situation $K_i \cong U_4(4)$ and $K_i \cong L_2(4)$. Here we have $|Z(S) : Z(S \cap K_2)| = 2$ and thus, since $m(S \cap K_2) = 2$ and $g \in N_G(S)$, $Z(S \cap K_2)^G \cap Z(S \cap K_2) \neq 1$, which forces $g \in H$. This disposes of case (a.2).

**Case (a.3).** By Lemma 14.7, $K_i \cong L_2(2^2)$ for $i = 1, 2$. So $|Z(S)| = 4$ and $|Z(S) \cap K_i| = 2$ for $i = 1, 2$. Since $g \in N_G(S)$, Lemma 10.7 then forces $g \in H$.

This contradiction eliminates case (a.3) and completes consideration of subcase 1(a). So Theorem 14.2 holds when $m(C_{\psi}(\mu^G)) \geq 2$, $i \in \{1, 2\}$ and $\mu^G \in E(H)$.

**Subcase 1(b):** $\mu^G \not\in E(H)$. We first state some consequences of our earlier work.

(14.6) (i) If $R_0 \leq S \cap E(H)$ and $R_0 \cap K_i \neq 1$ for $i = 1$ or $2$, then $N_G(R_0) \leq H$.

(ii) $N_G(Z(S)) \leq H$.

(iii) If $R_0 \leq S$ and $|R_0 \cap K_i| > [R_0 : R_0 \cap E(H)]$ for $i = 1$ or $2$, then $N_G(R_0) \leq H$.

Observe that for $\zeta \in S \cap E(H)$, $m(C_{\psi}(\zeta)) \geq 2$, $i = 1, 2$. Therefore, combining subcase 1(a) with (14.1) yields (i). Part (ii) follows from (i) and Lemmas 14.4(i) and 14.6.

If, say, $|R_0 \cap K_1| > [R_0 : R_0 \cap E(H)]$, then for $n \in N_G(R_0)$, $(R_0 \cap K_1)^n \cap E(H) \neq 1$ and then $n \in H$ by subcase 1(a). Thus (iii) holds.

Let $S_0 \in \Sigma$ be such that $\mu, \mu^G \in S_0$ with $g \in N_G(S_0) = N$. So $N_G(S_0) \nleq H$. By (14.6)(ii), $S_0 \neq S$ and we may assume $g \in N^*$. Since (without loss of generality, by (14.4)) $\mu \in Z(S) \leq Z(S_0), \mu^G \in Z(S_0)$ also. Hence $Z(S) \leq S_0 \leq C_3(\mu^G)$. Further, from (14.7)(iii), $|S_0 \cap K_i| \leq |S : S \cap E(H)| \leq 4$ for $i = 1, 2$.

If, say $K_2 \not\cong L_2(2^2)$, then, by (14.3), $S = C_5(K_2) \times (S \cap K_2)$ and $|S : S \cap E(H)| \leq 2$, which gives $4 \leq |Z(S \cap K_2)| \leq |Z(S) \cap K_2| \leq 2$, a contradiction. Therefore, $K_i \cong L_2(2^2), i = 1, 2$. Hence, by Lemma 14.7, $K_i \cong L_2(2^2), i = 1, 2$.

To fix notation we suppose $|C_{S_0}(\mu)| = 2$. If $|S : S_0| > 2$, then $|S_0| = 2^4$ and $N^*/S_0 \cong L_2(2^2)$. But, for $\zeta \in (S \cap K_2) \setminus C_{S_0}(\mu^G)$, $m(C_{S_0}(\zeta)) \geq 3$ and so $S_0$ is not a standard module for $N^*/S_0$. Therefore, $|S : S_0| = 2$. Since $|N^*/S_0 : O_2(N^*/S_0)| = 2$, we may suppose that $g$ has odd order. By (2.12) and (14.6)(iii), $[S : S \cap E(H)] = 4$. Now $S_0 \leq Z(S)$ and so, since $|Z(S)| = 4$, $|S_0| = 2$ by (14.6)(ii). Since $E(H) \cap S_0 \leq C_{E(H)}(\mu) = C_{S \cap K_2}(\mu^G) \times C_{S \cap K_2}(\mu^G)$ and $|S_0| = 2^4$, $E(H) \cap S_0 = C_{S \cap K_2}(\mu^G) \times (S \cap K_2)$. From [13, Lemma 5.1(vi)], $[N^*, \Omega(S_0)] = 1$ and so $\Omega(S_0) \cap K_i = 1, i = 1, 2$. Conse-
quently \( \gamma(S_0 \langle \mu^\ell \rangle (S \cap E(H))) = \emptyset \), and so \( \Omega_1(S_0) \neq S_0 \). Since \( m(S) = 4 \) by Lemma 14.4(i) and \( S \cap K_1 \leqslant Z(S_0) \), we see that \( Z(S_0) = \langle \mu^\ell \rangle Z(S) \). By [13, Lemma 5.1(vi)], \( |S_0, \langle g \rangle| \leqslant Z(S_0) \), whence \( S_0 = [S_0, \langle g \rangle] \times C_{S_0}(g) \) with \( |S_0, \langle g \rangle| = |Z(S_0), \langle g \rangle| \) elementary abelian of order 4. Because \( m(S_0) = 4 \), \( C_{S_0}(g) \cong D_8 \) and hence, in particular, \( \Omega_1(S_0) = S_0 \), contrary to the previously obtained \( \Omega_1(S_0) \neq S_0 \).

With this contradiction subcase 1(b) is finished, and so is case 1.

Case 2: \( m(C_{k_i}(\mu^\ell)) \leqslant 1 \) for \( i = 1, 2 \)

Thus \( K_i \cong L_2(q_i), \ q_i = 3, 5(8) \), for \( i = 1, 2 \). Let \( S_0 \in \Sigma \) be such that \( \mu, \mu^\ell \in S_0 \) and \( g \in N_{S_0}(S_0) = N \). As previously we may suppose \( \mu \in Z(S) \), whence \( \mu^\ell \in Z(S_0) \). Hence \( |S_0 \cap E(H)| = 4 \) and so \( |S : S_0| > 2 \). Then \( S_0 \) must be a standard module for \( N^*/S_0 \), whereas \( m(C_{S_0}(\xi)) = 3 \) for some \( \xi \in \gamma(S) \). Therefore, case 2 cannot occur.

The proof of Theorem 14.2 is complete.

**Theorem 14.8.** Hypothesis 14.1 cannot hold.

**Proof.** Assuming Hypothesis 14.1 holds we seek a contradiction.

Let \( T \in \text{Syl}_2 G \) be such that \( S \leqslant T \). From Theorem 14.2 and Lemma 14.6, \( N_G(Z(S)) \leqslant H \). So, in particular, \( T \leqslant H \). Clearly \( Z(T) \cap E(H) \neq 1 \).

\[
(14.7) \text{There exists } t \in T \text{ such that } K_1 = K_2 \text{ (and so } K_1 \cong K_2 \text{ and } \gamma(Z(T) \cap E(H)) \cong \text{diag}(S \cap K_1) \times (S \cap K_2)).
\]

If (14.7) were false, then we would have \( K_1 \leqslant K_1 T \) and hence \( Z(T) \cap K_1 \neq 1 \). Let \( \mu \in \gamma(Z(T) \cap K_1) \). By (14.1) and Theorem 14.2 we may apply (2.20) to \( H \) and \( \mu \) and hence \( G \) is not a counterexample to Theorem A. Thus (14.7) holds.

Let \( \alpha \in \gamma(Z(T) \cap E(H)) \) and \( M \in \mathcal{M}(\alpha) \). So \( \alpha \in Z(S) \) and hence \( \alpha \in A \).

By Lemma 10.2(ii), \( E_2(M) = E(M) O_2(M) \).

\[
(14.8) \ E(M) \neq 1.
\]

Suppose \( E(M) = 1 \). Then \( M \) is 2-constrained and so, by a Frattini argument, \( M = O_2(M) N_M(S^*) \), where \( Z(S) \leqslant S^* \leqslant S \). Using (14.1) gives \( O_2(M) \leqslant H \). Further, \( Z(S) \cap K_i \neq 1 \) by Lemma 14.6 and thus Theorem 14.2 yields \( N_G(S^*) \leqslant H \). Therefore, \( M = H \) which is impossible since \( E(H) \neq 1 \). Thus \( E(M) \neq 1 \).

\[
(14.9) \text{Either of the following conditions imply } H = M.
\]

(a) \( J \leqslant H \) where \( J \) is a component of \( M \).
(b) \( K_i \leqslant M \) for \( i = 1 \) or 2.
First we prove (a) implies $H = M$. Since $\alpha \in Z(S)$, Lemmas 3.4(ii) and 10.2(i) give (say) $J \leq K_1$. If $J < K_1$, then, as $\alpha \in K_1 C_6(K_1)$, we must have $K_1$ of type $JR$ whence $S$ is abelian by Lemmas 14.3 and 14.6. Thus $J = K_1$ must hold. Employing Theorem 14.2 yields $E(M) \leq H$ whence by the above arguments we obtain $E(H) = E(M)$. Hence $M = H$ and we have (b).

Suppose $K_i \leq M$. By (14.1) and Lemma 14.6, $\{H\} = \mathcal{H}(\beta)$ for some $\beta \in Z(S)$. Lemmas 3.4(ii) and 10.2 imply $K_i \leq J_1$ for some component $J_1$ of $M$ and then $J_1 \leq H$ by (14.1) and the embedding of $K_i$ in $J_1$. By (a), $H = M$, and we have proved (14.9).

(14.10) For some component $J$ of $M$, $(\alpha, A, M, J)$ satisfies Hypothesis 10.3.

If $M = H$, then clearly (14.10) holds. So we may assume $M \neq H$. Suppose $(\alpha, A, M, J)$ does not satisfy Hypothesis 10.3 for any component $J$ of $M$. Then since $A \in \mathfrak{A}_e(S)$, each component of $M$ is isomorphic to either $A_6$ or $L_2(q)$, $q = 3, 5(8)$. Since $m(S \cap K_i) \geq 2$, by (2.15)(iv), (2.18)(v), (14.1) and (14.9) we see that each component of $M$ must be isomorphic to $L_2(5)$. By (14.9) we then have, as $m(\text{Aut } L_2(5)) = 2$, $m(S \cap K_i) = 2$. Hence $K_i$ is isomorphic either to $L_2(q')$, $q' = 3, 5(8)$ or $U_3(4)$. If $K_i \cong U_3(4)$ (recall $K_1 = K_2$), then, as $C_5(E(H)) = 1$ and $S \cap K_i \leq S$, $S \leq E(H)$ which gives $\Omega_1(S)$ abelian, a contradiction. So $K_i \cong L_2(q')$, $q' = 3, 5(8)$ and then $(\alpha, A, M, J)$ must satisfy Hypothesis 10.3 for any component $J$ of $M$. This establishes (14.10).

In view of (14.10) Theorem 11.1 gives

(14.11) $r(M) \leq 2$.

Our attention is now directed to showing that

(14.12) $M = H$.

Assuming $M \neq H$ we derive a contradiction.

Suppose, for the moment, that $r(M) = 2$. Because of Theorem 13.1, $C_4(E(M)) = 1$. Put $E(M) = J_1 J_2$. Let $B \in \mathfrak{A}_e(S)$ be such that $B \leq S \cap E(H)$ (by (14.1) applied to $M$, (2.18)(v) and (14.9), this is possible by Lemma 14.4(i)). If $K_1$ has one $K_1$-conjugacy class of involutions, then, since $K_1 = K_2$, $B$ has two $H$-conjugacy classes, namely, $(B \cap K_1)^* \cup (B \cap K_2)^* = C_1$ and $B \setminus (B \cap K_1) \cup (B \cap K_2) = C_2$. Let $\xi \in (J_1 \cap B)^*$. By (14.1) applied to $M$, $\mathcal{H}(\xi) = \{M\}$. If $\xi \in C_1$, then $\{H\} = \mathcal{H}(\xi) = \{M\}$, whereas $\xi \in C_2$ implies that $\xi$ and $\alpha$ are conjugate and so $C_6(\xi)$ contains a Sylow 2-subgroup of $G$. Since Theorem 14.2 holds for $M$, (2.20) shows $G$ is not a counterexample to Theorem A. So we must have $K_1 \cong PSp_q(2^n)$. By (14.1) and (14.9), $(S \cap K_i) \cap (S \cap J_i) = 1$, $i, j \in \{1, 2\}$, and therefore, by
Lemma 14.6, \([S \cap K_i, S \cap J_i] = 1, \ i, j \in \{1, 2\}\). Hence, since \(S = (S \cap K_1)(S \cap K_2), S \cap E(M) \leq Z(S)\), which is not possible. This untenable situation arose from the assumption \(r(M) = 2\). So by (14.8) and (14.11), \(r(M) = 1\).

Put \(J = E(M)\). Note that (2.18)(v), (14.1) and (14.9) yield the following possibilities for \(J/Z(J): L_2(2^n), U_3(2^n), Sz(2^n), L_3(2^n)\) or \(PSp_4(2^n) (n \geq 2)\). We first consider the situation when \(m(C_\alpha(J)) = 1\). Then \(m(S/S \cap J) \leq 2\) (recall that \(S \leq M\)). If \(S \cap K_1 \cap J = 1, \) say, then, since \(m(S \cap K_i) \geq 2\), we must have \([S: C_\alpha(J)(S \cap J)] = 2, J \cong L_2(2^n)\) some \(n\) and \(K_1 \cong L_2(q), q = 3, 5(8)\) or \(U_3(4)\). As we have seen earlier \(K_1 \cong U_3(4)\) implies \(\Omega_4(S)\) is abelian, and so \(K_1 \cong L_2(q), q = 3, 5(8)\). Since \(K_1 = K_2\) and \(H\) satisfies Hypothesis 10.3, this forces \(J \cong L_2(q'), q' = 3, 5(8)\) or \(A_6\). Hence \(J \cong L_2(2^n)\).

But then \(m(S) = 3\), whereas (looking in \(H\)) \(m(S) = 4\). Therefore, \(S \cap K_1 \cap J \neq 1\) must hold. Let \(\xi \in \mathcal{Y}(S \cap K_1 \cap J)\). By Theorem 14.2 the elements of \(\xi \cap S\) are \(H\)-conjugate and thus, in particular, \(C_\xi(\eta) \leq H\) for all \(\eta \in \xi \cap S\). Consequently, either \(J \leq \langle C_\xi(\eta) \mid \eta \in \xi \cap S \rangle \leq H\) or \(J/Z(J) \cong L_3(4)\) with \(|Z(J) \cap S| = 2\). In the latter case (looking in \(M\)) \(m(S) = 5\) and \(|\mathfrak{S}_!(S)| = 2\). However, either \(m(S) = 2m(K_i)\) or \(K_i \cong L_2(q), q \in 3, 5(8)\) or \(A_7\), by Lemma 14.4, so the case \(J/Z(J) \cong L_3(4)\) is impossible.

Thus we infer that \(m(C_\alpha(J)) \geq 2\). Now Hypothesis 10.3 holds for \((\alpha, B, M, J)\), where \(B \in A_3(S)\) is such that \(m(B \cap C_\alpha(S)) \geq 2\). Then Lemma 10.4, (2.18)(v) and (14.9) imply \(K_i\) is isomorphic to one of \(L_2(2^n), U_3(2^n), L_3(2^n)\) and \(PSp_4(2^n)\).

Clearly \(C_\alpha(J) \cap K_i = 1\) and so \([C_\alpha(J), (K_i \cap S)] = 1, i = 1, 2\). If \(K_i \cong L_3(2^n)\) or \(PSp_4(2^n)\), then \(S = (S \times K_i) \times (S \cap K_2)\) whence \(C_\alpha(J) \leq Z(S)\) and thus (looking in \(M\)) \(|\mathfrak{S}_!(S)| = 2\), contrary to \(|\mathfrak{S}_!(S)| = 4\). Therefore, since \(\Omega_4(S)\) is not abelian, \(K_i \cong L_2(2^n)\) is the only possibility by Lemma 14.7. By Lemma 14.4, \(m(S) = 4\) and \(C_\alpha(J) \leq Z(S) \subseteq S \cap E(H)\). Hence \(C_\alpha(J) \cong Z_2 \times Z_2\) as \(C_\alpha(J) \cap K_i = 1\). Also, \(J \cong L_2(2^n)\) (\(U_3(4)\) is not possible by order considerations). Therefore, \(|S'| = 2\) since \(C_\alpha(J)(S \cap J)\) is an abelian subgroup of \(S\) of index 2. However, there exists \(\xi \in \mathcal{Y}(S)(S \cap E(H))\) such that \(\langle \xi \rangle(S \cap K_i) \cong D_8\), and hence \(\langle \xi \rangle(S \cap K_{i-1}) \cong D_8\), which implies \(|S'| > 2\).

This is the desired contradiction, and so we have established (14.12).

We are now in a position to conclude the proof of Theorem 14.8.

(14.13) \(a^G \cap S = a^H \cap S\).

Recall that \(N_G(S) \leq H\). So, if all involutions of \(S\) are \(H\)-conjugate to one in \(Z(S)\), then \(a^G \cap S = a^H \cap S\) by (2.1)(v). Therefore, we may suppose \(K_1 \cong L_2(q^2), L_2(q)q = 3, 5(8)\) or \(A_7, (K_1 \cong L_2(2^{2n}), n > 1\) is ruled out by Lemma 14.7) and \(S > S \cap E(H)\). From Lemmas 14.4(i) and 14.6, \(|S: S \cap E(H)| \leq 4\) with \(\Omega^1(S) \leq E(H)\).
If (14.13) were false, then there would exist \( S_0 \subseteq \Sigma \) with \( \alpha \in S_0 \) and \( N^* \leq H \), where \( N = N_G(S_0) \). Note that, for all \( \gamma \in \mathcal{Y}(S \cap E(H)) \), \( C_G(\gamma) \leq H \) (by (14.1) and (14.12)), and \( \gamma^* \in S \cap E(H) \) implies \( g \in H \). Therefore, since \( |\mathcal{U}(S_0)| = 1 \), \( S_0 \) is elementary abelian, \( m(S_0) = 4 \) and \( N^*/S_0 \cong L_2(2^2) \). So \( |S| = 2^6 \), and hence \( K_i \cong L_2(q) \), \( q \equiv 3 \), 5(8). But then \( m(C_{S_0}(\zeta)) = 3 \) for some \( \zeta \in \mathcal{Y}(S \cap E(H))^* \), contrary to \( S_0 \) being a standard module for \( N^*/S_0 \).

Thus (14.13) holds.

Now combining (2.20), (14.12) and (14.13) gives the final contradiction, and the proof of Theorem 14.8 is complete.

15. THE CASE \( r(H) = 1 \)

Throughout this section we suppose that Hypothesis 10.3 holds with \( r = r(H) = 1 \). As usual \( R \in \mathcal{X}(H) \) with \( R \not\supseteq A \), and we may suppose \( R = S \cap H \). Also we set \( A_0 = C_A(K) \). Recall that \( A_0 \neq 1 \) by Theorem 12.1. Our ultimate aim is to show that this situation cannot hold in a minimal counterexample. We begin by restricting the possibilities for \( K \).

**Lemma 15.1.** \( K/Z(K) \in \mathcal{L}_2 \).

**Proof.** Suppose the lemma is false. Then \( K/Z(K) = \overline{K} \in \mathcal{L}_1 \).

(15.1) Suppose \( \alpha \in A_0^* \) and \( \alpha^k \in R \), \( g \in G \). If either \( K \not\cong L_2(2^2) \) and \( m(C_{A_0}(\alpha^k)) \geq 2 \) or \( m(C_{A_0}(\alpha^k)) \geq 3 \), then \( g \in H \).

Assume either \( K \not\cong L_2(2^2) \) and \( m(C_{A_0}(\alpha^k)) \geq 2 \) or \( m(C_{A_0}(\alpha^k)) \geq 3 \). Then (2.18)(v) or the fact that \( m(\text{Aut} L_2(2^2)) = 2 \) and Lemma 10.5 imply \( K^x \leq H \), whence \( K = K^x \) by Lemma 3.4(ii). So \( g \in H \), as required.

(15.2) \( m(A_0) \leq 2 \).

Suppose \( m(A_0) \geq 3 \) and argue for a contradiction. Thus \( m(C_{A_0}(\beta)) \geq 2 \) for all \( \beta \in \mathcal{Y}(R) \). Therefore, if \( K \not\cong L_2(2^2) \), then \( S \leq N_G(A) \leq H \) by (15.1). Hence, as now \( Z(S) \cap A_0 \neq 1 \), we then have \( N_G(Z(S)) \leq H \). Selecting \( \mu \in \mathcal{Y}(Z(T) \cap A_0) \), (15.1) and (2.20) yield that \( G \) is not a counterexample. So it only remains to deal with the case \( K \cong L_2(2^2) \).

Since \( m(A_0) \geq 3 \), (15.1) implies that \( g \in H \) when \( \alpha \in A_0^* \) and \( \alpha^k \in A \). Thus \( S \leq N_G(A) \leq H \), and then \( N_G(Z(S)) \leq H \). Also note that \( A \in \mathcal{U}_e(S) \) by Hypothesis 10.3. Let \( \alpha \in \mathcal{Y}(Z(T) \cap A_0) \) and \( \alpha^k \in S \), \( g \in G \). We aim to show that \( g \in H \), which using (2.20) will give the desired contradiction. Therefore, by (15.1), we may suppose \( m(C_{A_0}(\alpha^k)) \leq 2 \), and consequently \( m(A_0) \leq 4 \). Hence \( m(S) \leq 6 \).
Let $B$ be an elementary abelian subgroup of $S$ with $m(B) \geq 5$. Put $B_0 = C_B(K)$. Clearly $m(B_0) \geq 3$. It is claimed that $\mathcal{M}(B) = \{H\}$ for all $B \in B^*_0$. Let $B \in B^*_0$ and $M \in \mathcal{M}(B)$. Put $C = C_A(B)$. Then $m(C) \geq 2$ and $KC \leq M$. Using Lemmas 10.2 and 3.4(ii) we obtain, for some component $J$ of $M$, either $K \leq J$ or $K \leq J^J$ ($J \neq J^J$ and $\xi \in C^*$). We now show that $J \leq H$. If $J \neq J^J$ for some $\eta \in C^*$, then (2.15)(iv) forces $J \leq H$. So we may suppose that $C$ normalizes $J$, and that $K < J$. Hence $J$ is isomorphic to one of $L_2(16)$, $J_1$, and $L_3(5^2)$ by Lemma 3.4(ii). Then, appealing to (2.18)(v), we obtain $J \leq H$, except when $J \cong L_2(16)$, in which case $\xi$ acts upon $J$ as a field automorphism, for some $\xi \in C^*$. But then there exists $\eta \in C \setminus \langle \xi \rangle$ inducing an inner automorphism on $J$ and so $J = \langle C_\eta(\eta), C_\eta(\xi) \rangle \leq H$. Thus $J \leq H$ and so $J = K$ by Lemma 3.4(ii) whence $E(M) \leq H$ which then gives $E(M) = H$. Therefore, $M = H$, so establishing the above claim.

Now let $S_0 \in \Sigma$ and put $N = N_G(S_0)$. We need to show that $N \leq H$. Suppose, for the moment, that $m(S_0) > 5$ and let $B \in \mathfrak{A}_e(S_0)$. Setting $B_0 = C_B(K)$ we have $m(B_0) \geq 3$ and (by the above) $\mathcal{M}(B) = \{H\}$ for all $B \in B^*_0$. Clearly, for $n \in N$, $B_0^n \in \mathfrak{A}_e(S_0)$, and so $m(C(B_0^n)) \geq 3$ and $\mathcal{M}(\gamma) = \{H\}$ for all $\gamma \in C_B(K)^*$. Since $B_0^n \cap C_B(K) \neq 1$, this gives $H^n = H$ whence $N \leq H$. Therefore, we may assume $m(S_0) \leq 4$. If $m(S_0) = 3$, then we must have $|S : S_0| = 2$ and then $m(S) \leq 4$ whereas $m(S) \geq 5$. Thus $m(S_0) = 4$. Because $m(S) \geq 5$, if $|S : S_0| = 2$, then $A \cap S_0 \in \mathfrak{A}_s(S_0)$ with $S = S_0 A$. Then, since $Z(S) \leq \Omega_1(Z(S)) \leq A \cap S_0$, $\Omega_1(Z(S)) = Z(S)$ which yields $N \leq N_G(Z(S)) \leq H$. Thus we may suppose $|S : S_0| > 2$, which then forces $|S_0| = 2^4$ and $|S : S_0| = 4$. So $|S| = 2^8$. But then $m(S) \geq 5$ and Lemma 4.1(vi) imply that $A_0 \leq Z(S)$, contrary to an earlier supposition that $m(C_{A_0}(\alpha^i)) \leq 2$.

This concludes the proof of (15.2).

(15.3) $m(A_0) = 1$.

Suppose (15.3) is false. By (15.2), $m(A_0) = 2$ and hence $m(A) \geq 4$. First we consider the case when $K \not\cong L_2(2^2)$. Thus we can still make use of (15.1) to infer that $S \leq N_G(A) \leq H$, $N_G(Z(S)) \leq H$ and $A \in \mathfrak{A}_s(S)$.

As in (15.2) we shall show that $H$ control fusion in $S$. Then (2.20) will yield a contradiction. So let $S_0 \in \Sigma$ and suppose that $N = N_G(S_0) \leq H$. If $m(S_0) = 3$, then $|S : S_0| = 2$ and, since $m(S) \geq 4$, $S = AS_0$ with $A \cap S_0 \in \mathfrak{A}_s(S_0)$. Therefore, $\Omega_1(Z(S_0)) = Z(S)$, contrary to $N \leq H$. Thus $m(S_0) > 3$.

Suppose $K \cong L_2(q)$, $q = 3$, $5(8)$, $q > 5$ or $JR$. Then $m(C_B(K)) = 2$ for all $B \in \mathfrak{A}_c(S)$. If $\alpha \in Z(S) \cap A_0$ ($\leq S_0$) and $n \in N$ with $\alpha^n \in B$ for some $B \in \mathfrak{A}_c(S)$, then arguing as in (15.1) gives $n \in H$. Consequently $m(S_0) < m(S)$. Since in the case $K \cong L_2(q)$, $q = 3$, $5(8)$, $q > 5$, $m(S) = 4$ (and $m(S_0) > 3$), we must have $K \cong JR$. Then $m(S) = 5$ with $S \cap K \leq Z(S)$
and hence, since $C_5(K) \cap Z(S) \neq 1$, $m(Z(S)) \geq 4$. But then $Z(S) = \Omega_1(Z(S_0))$, against $N \nsubseteq H$.

We now turn to the case when $K \cong L_2(q^2)$, $q \equiv 3 \pmod{5}$ or $A_7$. Here we have $m(S) = 4$ or 5. Consider the case $m(S) = 5$ first. Then an element of $A^*$ must induce a non-inner automorphism upon $K$ and so $m(C_5(K)) \geq 2$ for each $B \in \mathfrak{A}_c(S)$. As in the previous paragraph this implies $m(S_0) < m(S) = 5$. So $m(S_0) = 4$. We cannot have $[S : S_0] = 2$ since this would give $\Omega_1(Z(S_0)) = Z(S)$. Therefore, $[S : S_0] = 4$ and $|S_0| = 2^4$.

However $|A \cap S_0| \geq 2^3$ contradicts the fact that $S_0$ is a standard module for $(N/S_0)^*$. So we must have $m(S) = 4$. Since $m(S_0) > 3$, $m(S_0) = 4$. Suppose $|S/S_0| > 2$. Then $Z = \Omega_1(Z(S_0)) \in \mathfrak{A}_c(S_0)$ ($\subseteq \mathfrak{A}_c(S)$). By Lemma 8.3(i), $Z \cap K \in \mathfrak{A}_c(S \cap K)$. Now (15.1) holds with $A$ replaced by $A^+ = A_0(Z \cap K)$ ($\subseteq \mathfrak{A}_c(S)$) and therefore, as $N \nsubseteq H$, $A^+ \nsubseteq S_0$. However, since $1 \neq Z(S) \cap C_5(K) \subseteq Z \cap A^+$, $m(Z \cap A^+) > 3$ which means that $Z$ is not a standard module for $(N/S_0)^*$. Thus $|S/S_0| = 2$ must hold.

Choose $a \in \mathfrak{Y}(A_0 \cap Z(T))$ ($S \leq T \in \text{Syl}_2 \text{G}$) and suppose $a^g \in S$ with $g \notin H$. By the above we may suppose, without loss of generality, that $g \in N = N_G(S_0)$ for some $S_0 \in \Sigma$ with $[S : S_0] = 2$. Since $a \in Z(S)$, $a^g \in Z(S_0)$, and $C_5(a^g) = S_0$ since $a^g \notin Z(S)$. From (15.1) we see that $m(C_5(K)(a^g)) = 1$. Because $[N^*, \mathfrak{U}^*(S_0)] = 1$ and $C_G(a)$, $N_G(S) \leq H$, we cannot have $a$ being the square of an element of order 4 in $S_0$, and so $|C_5(K)(a^g)| = 2$. Hence, since $[S : C_5(a^g)] = 2$, we have $C_5(K) = A_0$. Because $|C_5(K)(a^g)| = 2$, $S \neq C_5(K)(S \cap K)$, from which we conclude that $S \cong D_8 \times D_8$. Hence $S \in \text{Syl}_2 \text{G}$ by (2.5) which then contradicts the simplicity of $G$. This completes our considerations of the case $K \cong L_2(q^2)$, $q \equiv 3 \pmod{5}$ or $A_7$.

To complete the proof of (15.3) we must deal with the Jekyll and Hyde group $L_2(2^2)$. Suppose $K \cong L_2(2^2)$. Note that, since $m(C_5(K)(\beta)) = 2$ for each $\beta \in \mathfrak{Y}(C_5(K))$, we have $\mathfrak{Y}(\beta) = \{H\}$ for all $\beta \in \mathfrak{Y}(C_5(K))$ by (15.1).

First we deal with the situation when $S \leq H$. Again we manoeuvre into a position to apply (2.20). We begin by observing that if $C_5(K)$ is abelian, then $C_5(K) \leq Z(S)$ and no element of $C_5(K)^*$ is conjugate to an element of $S \cap K$. For $C_5(K)$ being abelian implies that $S$ contains an abelian subgroup of index 2 and so $|S'| = 2$ by Lemma 4.1(vi). Clearly $S' \leq S \cap K$ and hence $[S, C_5(K)] \leq (S \cap K) \cap C_5(K) = 1$, and so $C_5(K) \leq Z(S)$. By (2.1)(v), $(S')^*$ is not $G$-conjugate to an element of $Z(S) \setminus S'$. Since $(S \cap K)^*$ fuses in $K$, we have verified the second part of the above observation. If $N_G(S) \nsubseteq H$, then, for $n \in N_G(S) \cap H$, $C_5(K) \cap C_5(K)^n = 1$ (recall that $\mathfrak{Y}(\beta) = \{H\}$ for all $\beta \in \mathfrak{Y}(C_5(K)))$. Hence $C_5(K)$ is isomorphic to either $Z_2 \times Z_2$ or $D_8$. Suppose the former possibility occurs. Then by the above observation $C_5(K) \leq Z(S)$ and thus $m(Z(S)) \geq 4$ whence $S = Z(S)$, a contradiction, while $C_5(K) \cong D_8$ implies that $S \cong D_8 \times D_8$, which is also untenable. Therefore, $N_G(S) \leq H$ holds.
Let $S_0 \in \Sigma$ with $S \neq S_0$. Suppose $|S:S_0| > 2$. By standard module considerations and the fact that $m(S) = 4$, $|S| = 2^6$ and $|S:S_0| = 4$. Hence either $|C_5(S)| = 2^3$ or $S = C_5(K) \times (S \cap K)$. If $|C_5(S)| = 2^3$, then either $C_5(S)$ is abelian or $C_5(S) \cong \mathbb{Z}_8$. The latter possibility yields the impossible situation $S \cong D_8 \times D_8$. On the other hand, if $C_5(S)$ is abelian, then $C_5(S) \leq Z(S)$. Since $Z(S)$ is elementary abelian, this then gives $m(C_5(K)) = 3$, contrary to $m(A_5) = 2$. Now we examine the possibility $S = C_5(K) \times (S \cap K)$. Here we have $S \cap K \leq Z(S) \leq O_1(S_0) = S_0$ and so, because $S_0$ is a standard module, every involution of $C_5(K)$ must lie in $S_0$. Thus $O_1(S) = S_0$, a contradiction. Therefore, we conclude that $|S:S_0| = 2$. Put $W = S_0 \cap C_5(K)$. If there exists $n \in N_{C_5(S)}(W)$, then $W \times W' \leq S_0$ which, by order consideration, yields that $|W| \leq 4$. Thus $|C_5(S)| \leq 2^3$. Suppose, for the moment, that $C_5(S)$ is abelian. Then $C_5(S) \leq Z(S)$ and so $C_5(S) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with $C_5(S) \times C_5(S)' = S_0$. Clearly we have $\Pi_{C_5(S)} = \{A, B\}$ with $B \neq A$. If $A$ and $B$ are conjugate, then they must be conjugate in $N_{C_5(S)}(\leq H)$. However $A = C_5(K)(S \cap K)$ and so we see that $A$ and $B$ are not conjugate. If $N_{C_5(S)}(\Pi_{C_5(S)})$ is a 2-group, then $S \in Syl_2 G$ and then standard fusion arguments can be used to eliminate this case. So let $x \in N_{C_5(S)}C_5(S)$ with $xC_5(S)$ of odd order, then $x$ must normalize both $A$ and $B$ and hence centralizes $A/Z(S)$ and $B/Z(S)$. Therefore, $S = D \times [S, x]$ with $xC_5(S)$ of order 3, $D \cong D_8$, $[D: A \cap K] = 2$ and $|S, x| = C_5(K)$. Let $S_0 \in \Sigma$ with $S \neq S_0$. Then $S_0 \leq \Pi_{C_5(S)}$ and $S_0$ is weakly closed in $S$. Put $N = N_{C_5(S)}(S_0)$ and $\overline{N} = N/C_5(S_0)$. Since $[S, x] \leq Z(S) \leq S_0$, $1 \neq x \in \overline{N}$. Also we note that fusion in $S_0^c$ is determined by $N$, that $\overline{N} = C_5(S)O_2(\overline{N})$ and that $N_{C_5(S)}(\Pi_{C_5(S)}) = SC_5(K)$. Set $\overline{Q} = \langle \overline{x} \rangle O_2(\overline{N})$. Suppose $S_0 = A$. Then, since $(A \cap K)^*$ fuses in $K$ but are not fused by $(x)$, we have $O_2(\overline{N}) \neq 1$. Hence, using the fact that $\overline{Q}$ is a subgroup of $Aut A \cong GL(4, 2)$, $|\overline{Q}| = 9$, 15 or 21. If $|\overline{Q}| = 15$, then $A^*$ fuses, but this is easily seen to be impossible. While $|\overline{Q}| = 21$ (since then $\overline{Q}$ must be non-abelian) implies that the orbit lengths of $\overline{Q}$ on $A^*$ are 1, 7 and 7 (as $Z(S)$ possesses at least three conjugacy classes of involutions). Hence $|C_5(\overline{Q})| = 2$. Now $S$ normalizes $\overline{Q}$ and therefore normalizes $C_5(\overline{Q})$. Hence $S$ normalizes $C_5(\overline{Q})$ which implies $C_5(\overline{Q}) \leq C_{Z(S)}(\langle x \rangle) = Z(S) \cap K$, and so $C_5(\overline{Q}) = Z(S) \cap K$. This contradicts the fact that $(A \cap K)^*$ fuses. Thus $|\overline{Q}| \neq 21$, and so $|\overline{Q}| = 9$. Therefore, $\overline{Q}$ is abelian and hence $\overline{Q}$ normalizes $[A, \overline{x}] = [S, x] = C_5(K)$. Moreover, since $\langle \overline{x} \rangle$ centralizes $Z(S) \cap K$, $(Z(S) \cap K)^*$ has only three $\overline{Q}$-conjugates, which must be the elements of $(A \cap K)^*$. Thus we have shown that $\overline{Q}$ normalizes $C_5(K)$ and $A \cap K (= S \cap K)$, and hence $N$ normalizes $C_5(K)$ and $A \cap K$ when $S_0 = A$. For the case $S_0 = B$ we may argue as above to obtain the conclusion that either $|\overline{Q}| = 3$ or 9 or $\overline{Q}$ centralizes $Z(S) \cap K$. Hence, for $S_0 = B$, either $\overline{N}$ normalizes $C_5(K)$ or $N$ centralizes $Z(S) \cap K$. Recalling that $\Sigma = \{S, A, B\}$ and that $(S \cap K)^*$ fuses, we see we have proved that at least one of $C_5(K)$ and $S \cap K$ is a strongly closed 2-subgroup of $G$. With this contradiction we
have eliminated the possibility that \( C_S(K) \) is abelian. Consequently \( C_S(K) \cong D_8 \) must hold. Therefore, either \( S = C_S(K) \times (S \cap K) \) with \( S \cap K = \mathbb{Z}_2 \times \mathbb{Z}_2 \) or \( S/C_S(K) \cong D_8 \). Consider the former case, and let \( S_0 \in \Sigma \) with \( S \neq S_0 \). Then \( S_0 = (S \cap K)B \) for some \( B \in \mathfrak{M}_4(C_S(K)) \). Note that \( S \cap K \leq N_G(S_0) \). Suppose there exists \( n \in N_G(S_0) \setminus H \). If \( B^n \cap (S \cap K) \neq 1 \), then, since \( S \cap K \leq Z(S) \), \( S^n_{-1} \leq C_S(\beta) \leq H \) for some \( \beta \in B^n \), whence \( n \in H \) because \( N_G(S) \leq H \). Hence all elements of \( S_0 \setminus (S \cap K) \) are \( N_G(S_0) \)-conjugate to some element of \( B \). From this we conclude that \( S \cap K \leq N_G(S_0) \) when \( N_G(S_0) \not\leq H \). Therefore, \( S \cap K \leq N_G(S_0) \) for all \( S_0 \in \Sigma \), which cannot occur. Thus \( S/C_S(K) \cong D_8 \) must hold. This possibility yields \( S \cong D_8 \times D_8 \), but we shall omit the details. This completes our considerations of the case \( K \leq S_0 \) with \( S \leq H \).

Now we consider the situation \( K \cong L_2(2^3) \) when \( S \geq H \). So \( R < S \). Then we have \( C_R(K) = \Omega_4(C_R(K)) \leq Z(R) \) and no element of \( C_R(K) \) is conjugate to an element of \( Z(S) \). Thus \( C_R(K) = A_0 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). Also \( C_R(K) \cap C_R(K)^t = 1 \) for all \( \xi \in S \setminus R \). Suppose \( (R \cap K) \cap C_R(K)^t \neq 1 \) for some \( \xi \in S \setminus R \), and argue for a contradiction. Hence, since \( (R \cap K)^t \) are conjugate in \( K \), we have \( Z(S) \cap (R \cap K) = 1 \) and so \( S' \cap (R \cap K) = 1 \). Thus \( R = C_R(K) \times (R \cap K) \). Because \( Z(S) \cap (R \cap K) = 1 = Z(S) \cap C_R(K) \) we note that \( |Z(S)| = 4 \).

If \( (R \cap K) \cap (R \cap K)^t \neq 1 \) for some \( \xi \in S \setminus R \), then there exists \( \rho, \eta \in R \cap K \) such that \( p^t = \eta \), and then \( [\rho, \xi] \in (R \cap K) \cap S' \leq (R \cap K) \cap Z(S) = 1 \) whence \( |R| < |C_S(\rho)| \). This contradicts the fact that \( \rho \) is conjugate to an element of \( C_R(K) \) and we thus must have \( (R \cap K) \cap (R \cap K)^t = 1 \) for all \( \xi \in S \setminus R \). Since \( K \) contains an element of order 3 which centralizes \( C_R(K) \) and acts transitively on \( (R \cap K)^t \), we see that \( R \) contains at least nine involutions conjugate to some element of \( Z(S) \). Thus \( \forall (R \cap K \cup C_R(K)) \) contains all the involutions of \( R \) not conjugate to an element of \( Z(S) \). Let \( \xi \in S \setminus R \). Then, since \( C_R(K) \cap C_R(K)^t = 1 = (R \cap K) \cap (R \cap K)^t \), \( \xi \) must interchange \( C_R(K) \) and \( R \cap K \). Thus \( |S' : R| = 4 \) with \( |S'| = 4 \), contradicting Lemma 4.1(vi). Thus we have verified that \( (R \cap K) \cap C_R(K)^t = 1 \) for all \( \xi \in S \setminus R \).

From the fact that \( (R \cap K) \cap C_R(K)^t = 1 = C_R(K) \cap C_R(K)^t \) and fusion in \( K \) we deduce that every element of \( (R \cap K) \cap C_R(K)^t \cap (R \cap K) \) is conjugate to an element of \( C_R(K) \). Since we must have \( Z(S) \leq (R \cap K) \cap C_R(K)^t \) it follows that \( Z(S) = R \cap K \). Hence \( R = (R \cap K) \times C_R(K) \). Since \( Z(S)^t \) fuses, \( |S'| > 2 \) and then it is not difficult to show that \( |S : R| = 4 \) (see [2, Lemma 5.6]). We may now proceed as in [2, Lemma 5.8] to obtain \( S = RQ \) where \( R \cap Q = Z(S) \) and either \( Q \) is homocyclic abelian of exponent 4 or elementary abelian and finally to obtain a contradiction. (In considering the case when \( Q \) is elementary abelian one needs the fact that \( R \) and \( Q \) are normal in \( N_G(S) \). If this were not the case then it follows that \( R^n = Q \) some
n \in N_0(S). Since \( \gamma(S) = \gamma(R \cup Q) \) and no element of \( Z(S) \neq 1 \) fuses to \( R \setminus Z(S) \), this yields that \( Z(S) \) is a strongly closed 2-subgroups, a contradiction.)

This completes the case \( K \cong L_2(2^2) \), and the proof of (15.3).

To complete the proof of Lemma 15.1 we must eliminate the remaining possibility, \( m(A_0) = 1 \). In view of the fact that \( m(S) > 3 \) and \( (\sigma, A, H, K) \) satisfies Hypothesis 10.3, \( K \) must be isomorphic to one of \( J_2, L_2(q^2), q \equiv 3, 5(8) \) and \( A_7 \), (with \( K \) inducing a non-trivial outer automorphism in the last two cases).

If \( S \leq H \), then \( K \cong J_2 \) implies \( \Omega_1(S) \) is abelian, while \( K \cong L_2(q^2), q \equiv 3, 5(8) \) or \( A_7 \) gives either \( S \cong Q_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) or \( S \cong D_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \). Suppose \( S \cong Q_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) and let \( \alpha \in Z(C_S(K))^* \) (\( C_S(K) \cong Q_8 \) here). Observe that \( \alpha \in Z(N_0(S)) \) since \( C_S(C_S(K)) \) does not contain any subgroups isomorphic to \( Q_8 \). Let \( S_0 \in \Sigma \) with \( S_0 \neq S \). Then it is claimed \( \alpha \in Z(N_0(S_0)) \). If this is false, then \( \alpha \) cannot be a square of an element in \( S_0 \), and so \( S_0 \cap C_S(K) = \langle \alpha \rangle \). Thus \( |S : S_0| > 4 \) whence, as \( m(S) = 4 \), standard module considerations imply \( |S| = 2^6 \), a contradiction. Thus \( \alpha \in Z(N_0(S_0)) \) for every \( S_0 \in \Sigma \), which is impossible. So we must have \( S \cong D_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \).

Since this situation may be eliminated by arguments similar in nature to those used in the \( L_2(2^2) \) case, we omit the details.

Now we assume \( S \not\leq H \). If \( K \cong L_2(q^2), q \equiv 3, 5(8) \) or \( A_7 \), then \( R = R_1 \times (R \cap K) \), where \( R_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \), \( R_1 \supseteq C_K(K) \) and \( R_1 \) induces a nontrivial outer automorphism on \( K \). Following [2, Lemma 5.10] we also deduce \( R_1 \cap Z(S) \neq 1 \) and then as in [2, Lemmas 5.11 and 5.12] we obtain \( R_1 \leq S \) and \( |S| = 2^6 \). (In showing that \( R_1 \leq S \) we note the following: using the notation of [2, Lemma 5.11], when showing that \( S : C_S(\langle a \rangle) \) = 4 we have, since \( N/C_S(B) \) is a subgroup of \( GL(3, 2) \) and \( \tilde{S}C_S(B)/C_S(B) \neq 1 \), that \( \tilde{S}C_S(B)/C_S(B) \in \text{Syl}_2(C_S(B)). \) Next the arguments of [2, p. 44] yield that \( R \cap K \leq S \) and that \( \gamma(S \setminus R) \neq \emptyset \). Consequently, as \( R \leq Z(S) \), we have \( S \cong D_8 \times D_8 \), a contradiction. The remaining case of \( K \cong J_2 \) may be handled by the same arguments as for the case \( K \cong L_2(2^2) \) and \( S \not\leq H \).

The proof of Lemma 15.1 is complete.

The balance of this section deals with the case when \( K/Z(K) \in \Sigma_2 \). Theorem 15.2 gives us our first foothold in this situation.

**Theorem 15.2.** There exists an elementary abelian subgroup \( B \) of \( R \) with \( m(B) = m(A) \), \( Z(S) A_0 \leq B \) and \( O^2(N_0(B)) \leq H \). In the case when \( K/Z(K) \cong L_2(2^n) \), \( Sz(2^n) \) or \( U_3(2^n) \), we also have \( S \leq H \).

**Remark.** If \( K \cong PSp_4(2^n) \), then we may take \( B = A \) in the above theorem.

Before commencing the proof of Theorem 15.2 we establish some preliminary lemmas.
LEMMA 15.3. Either $N_G(A) \leq H$ or $m(A_0) \leq m(K/Z(K))$.

Proof. Suppose $m(A_0) > m(K/Z(K))$. Let $\alpha \in A_0^*$ and $g \in N_G(A)$. So $\alpha^g \in A$. Since $\mathcal{M}(\rho) = \{H\}$ for all $\rho \in A_0^*$ by Lemma 10.5, Lemma 8.4 forces $K^g \leq H$ and then $K = K^g$ by Lemma 3.4(ii). Thus $g \in H$, and the lemma holds.

LEMMA 15.4. (i) If $A_0^g \cap A_0 \neq 1$, then $g \in H$.

(ii) If $A_0 \leq A^* \leq A$, then $N_{N_G(A^*)}(A_0) = N_G(A^*) \cap H$.

Proof. (i) is just a restatement of Lemma 10.7. From (i), $N_G(A_0) \leq H$, and so $N_{N_G(A^*)}(A_0) \leq N_G(A^*) \cap H$. Now $N_G(A^*) \cap H$ normalizes $K$ and hence normalizes $C_A(K) = A_0$. Therefore, $N_G(A^*) \cap H \leq N_{N_G(A^*)}(A_0)$ and (ii) holds.

LEMMA 15.5. Suppose $A_1 \leq A \cap K$ and that $A_0 \cap A_1 = 1$. Put $B = A_0 A_1$. If there exists $g \in N_G(B) \setminus H$ such that $A_0^g \cap A_1 = 1$, then $[C_B(A_1), A_0] = 1$. Further, if $K$ is isomorphic to one of $L_2(2^n)$ ($n > 2$), $Sz(2^n)$ and $U_3(2^n)$ and $A_1 = \Omega_1(R \cap K)$, then $S \leq H$.

Proof. Note that since $N_G(B) \leq H$, $A_1 \neq 1$ by Lemma 15.4. First we establish that $[C_R(A_1), A_0] = 1$. Set $V = C_R(A_1)$. Since $R$ normalizes $A_0$, $C_R(V) > A_1$ and hence $V$ centralizes an element of $A_0^n$ for any $n \in N_G(B)$. So $V \leq H^n \cap N_G(B) = N_{N_G(B)}(A_0^n)$ by Lemmas 15.4(i), (ii) (with (ii) applied to $A_0^n \leq B \leq A^n$).

Because $g \notin H$ by hypothesis, Lemma 15.4(i) gives $A_0 \cap A_0^g = 1 = A_1 \cap A_0^g$. Therefore as $V$ normalizes $A_0$ and $A_0^g$ and centralizes $A_1$ we obtain $[A_0, V] \leq [B, V] \leq [A_0^g, V] \leq A_0^g$ whence $[A_0, V] = 1$, as required.

Now we suppose, additionally, that $K$ is isomorphic to one of $L_2(2^n)$ ($n > 2$), $Sz(2^n)$, $U_3(2^n)$ and $A_1 = \Omega_1(R \cap K)$ and show that $S \leq H$ yields a contradiction. Clearly $C_S(K) \leq C_S(A_1)$ and so $A = A_0 A_1 \leq Z((S \cap K) C_S(K))$ by the first part of the lemma. Hence $A = \Omega_1((S \cap K) C_S(K))$ since $A = \Omega_1(C_S(A_1))$. Consequently $\mathcal{Y}(S \setminus (S \cap K) C_S(K)) \neq \emptyset$ and $K \cong L_2(2^m)$ for some $m$. We claim that $A$ is weakly closed in some Sylow 2-subgroup of $G$. If this were false, we would have $A^k \leq S$, $k \in G$ and $A \neq A^k$. Then $A \cap A^k \leq Z(S)$ and, since $[A : A \cap A^k] = 2$, $[A_1 : A_1 \cap Z(S)] \leq 2$. But then, as $K \not\leq L_2(4)$, this contradicts the fact that $S/C_S(K)(S \cap K)$ induces a field automorphism upon $K$, and verifies the claim. Now appealing to [4, Corollary 4] gives $||A, \zeta|| = 2$ for some $\zeta \in \mathcal{Y}(S \setminus (S \cap K) C_S(K))$ which forces $K \cong L_2(4)$, a contradiction.

This completes the proof of the lemma.

In proving Theorem 15.2 we are forced to pursue different approaches depending on whether $K/Z(K)$ has Lie rank 1 or 2. Accordingly the proof is presented in two parts (Lemmas 15.6 and 15.7).
LEMMA 15.6. If $K/Z(K)$ is isomorphic to one of $L_2(2^n)$ ($n > 2$), $Sz(2^n)$ and $U_3(2^n)$, then $O^2(N_o(A))S \leq H$.

Proof. Set $N = N_o(A)$ and $A_1 = A \cap K$. So $A = A_0A_1$ and $A_1 \leq \mathfrak{A}_t(R \cap K)$. Define $\hat{A}_1 = [\xi, A_1]$ if $K/Z(K) \cong Sz(8)$ and $Z(K) \neq 1$, where $\langle \xi \rangle \in \text{Syl}_2(N_k(A_1))$, and $\hat{A}_1 - A_1$ otherwise. In the former case $|\hat{A}_1| = 2^3$ and $\langle \xi \rangle$ acts regularly upon $\hat{A}_1^*$. Note that $A = A_0 \times \hat{A}_1$. Recall that $N_o(A_0) \leq H$.

We suppose $O^2(N)S \leq H$ and seek a contradiction. Thus $m(A_0) \leq m(\hat{A}_1)$ by Lemma 15.3.

(15.4) If $A_0^n \cap \hat{A}_1 \neq 1$ where $n \in N$, then $A_0^n \leq \hat{A}_1$.

Put $m_0 = m(A)$, $\hat{m}_1 = m(\hat{A}_1)$ and $m = m(A_0^n \cap \hat{A}_1)$. So $m \leq m_0 \leq \hat{m}_1$. Using counting arguments we show that $m = m_0$, which will prove (15.4).

Observe from Lemma 15.4 that distinct $N$-conjugates of $A_0$ intersect trivially. Putting $J = N \cap K (= N_k(\hat{A}_1))$ we have that $J/C_j(A_1)$ is cyclic of order $2^{\hat{m}_1} - 1$ and acts regularly on $\hat{A}_1^*$. Now $J \cap H^n$ and $C_j(\hat{A}_1)$ both normalize $A_0^n \cap \hat{A}_1$ (since $N \cap H = H_0(A_0)$ by Lemma 15.4(ii)). Hence $((J \cap H^n)/C_j(\hat{A}_1))/C_j(A_1)$ has order at most $2^{m_1} - 1$ and so $|J : J \cap H^n| \geq (2^{m_1} - 1)/(2^m - 1)$. Now if $j$ and $k$ are representatives of distinct cosets of $J \cap H^n$ in $J$, then $(A_0^n \cap \hat{A}_1)^* \cap (A_0^n \cap \hat{A}_1)^* = (2^{m_1} - 1)/(2^m - 1)$ and thus $|J : J \cap H^n| - (2^{m_1} - 1)/(2^m - 1)$.

Because $J$ centralizes $A/\hat{A}_1$, clearly it normalizes $A_1A_0^n$ and hence, since $N \cap H^n$ normalizes $A_0^n$, we see that $A_0^n$ has $(2^{m_1} - 1)/(2^m - 1)$ distinct $J$-conjugates in $A_1A_0^n$. Since such distinct conjugates intersect trivially, counting the elements of $(A_1A_0^n)^*$ yields

$$(2^{m_0} - 1)(2^{\hat{m}_1} - 1)/(2^m - 1) \leq 2^{m_0 + \hat{m}_1 - m},$$

which then gives

$$2^{m_0} + 2^{\hat{m}_1} > 2^m + 2^{m_0 + \hat{m}_1 - m}.$$ 

Then $m \leq m_0 \leq \hat{m}_1$, forces $m = m_0$, as required.

(15.5) There exists $n \in N \cap H$ such that $A_0^n \cap \hat{A}_1 = 1$.

Suppose (15.5) is false. Then, by (15.4), all $N$-conjugates of $A_0$ distinct from $A_0$ are contained in $\hat{A}_1$. Therefore, the $N$-conjugacy classes of $A$ are $A_0^n \cup A_0^*$ and $A_1(A_0 \cup \hat{A}_1)$ since $N \cap K$ acts transitively upon $\hat{A}_1^*$ and trivially on $A_0$. The square of the classes $A_0^n \cup A_0^*$ in the group ring $\mathbb{Z}(A)$ is (setting $m_0 = m(A_0)$ and $\hat{m}_1 = m(\hat{A}_1)$)

$$(2^{\hat{m}_1} + 2^{m_0} - 2)1 + (2^{m_1} - 2)A_0^* + (2^{m_0} - 2)A_0^* + 1(A \setminus (A_0 \cup \hat{A}_1)).$$
and thus, since this must be a linear combinations of $N$-conjugacy classes of $A$, $\hat{m}_1 = m_0$. Consequently $\hat{A}_1$ is the only other $N$-conjugate of $A_0$ apart from $A_0$ and so $[N: N \cap H] \leq 2$ by Lemma 15.4. Thus $O^2(N) \leq H$. Hence $S \leq H$ with $[S: S \cap H] = 2$. Let $\eta \in S \setminus S \cap H$. Then $\eta$ interchanges $A_0$ and $\hat{A}_1$ and so $A_0 \cap Z(S) = 1 = \hat{A}_1 \cap Z(S)$. So $\Omega_1(R) = A$. Because $\eta$ interchanges $A_0$ and $\hat{A}_1$, $A$ is weakly closed and then $[4, Corollary]$ yields a contradiction. Therefore, (15.5) holds.

Since $A \leq S$, we have $S \leq N$. Let $T_1 \in \text{Syl}_2 N$ with $S \leq T_1$.

Suppose $K/Z(K) \cong \text{Sz}(8)$ with $Z(K) \neq 1$. Then $R \cap K = A_1$. Let $V \in \text{Syl}_2 K$ with $V \supseteq A_1$. By Sylow's theorem $V \leq T_1$ for some $f \in N$. Since $Z(S) \leq A$ and $S \leq T_1$, $1 \neq Z(S) \cap Z(T_1) \leq Z(AV^f)$. By [12, Lemma 2.6(v)], $Z(S) \cap Z(T_1) \leq C_A(K^f)$. Hence $S \leq H$ by Lemma 15.4(i). By a change of notation we may suppose $S \leq H$. Combining (15.5) with Lemma 15.5 (taking $B = A_0 \hat{A}_1$) gives that $\Omega_1(S)$ is abelian. Therefore, $K/Z(K) \cong \text{Sz}(8)$ and $Z(K) \neq 1$ is impossible. So $\hat{A}_1 = A_1$ from now on.

Also from (15.5) and Lemma 15.5.

(15.6) $S \leq H$.

(15.7) (i) Each $N$-conjugacy class of $A \setminus A_1$ contains an element of $A_0^\#$.

(ii) $A_1 \leq N$.

Again put $m_0 = m(A_0)$ and $m_1 = m(A_1)$. Because $N \cap K$ acts transitively on $A_0^\#$ and trivially on $A_0$, the $N \cap K$ conjugacy classes of $A \setminus A_0$ each have $2^{m_1} - 1$ elements. Conjugating $A_0$ by the element $n$ in (15.5) adds one further element to these classes with the exception of $A_1$. Thus the $N$-conjugacy classes of $A \setminus A_1$ are unions of sets each containing $2^{m_1}$ elements and each containing an element of $A_0$. So (i) holds.

Now we prove (ii). Let $C_1$ be the $N$-conjugacy class of $A$ containing $A_1^\#$. So we must show $C_1 = A_1^\#$. Suppose $A_1^\# \subset C_1$. By the remarks above there exists $a \in C_1 \cap A_0^\#$ and $|C_1| = 1 \pmod{2^{m_1}}$. Hence $C_N(a)$ contains a Sylow 2 subgroup of $N$ and so $R = S$ by Lemma 10.5, contrary to (15.6) and so $C_1 = A_1^\#$ must hold.

Put $\tilde{N} = N/A_1$. Then $1 \neq \tilde{A}_0 = \tilde{A} \leq \tilde{T}_1$ and so there exists $a \in A_0^\#$ such that $\tilde{a} \in Z(\tilde{T}_1)$. Set $\Omega = aA_1$ and $M = C_N(\tilde{a})$. Then $M$ leaves $\Omega$ invariant. Let $k \in K$ be an element of order $|A_1| - 1$ which acts transitively on $A_1^\#$. So $k \in M$ and hence $M_\alpha$ (the stabilizer of $\alpha$) is transitive on $\Omega \setminus \{\alpha\}$. Since $T_1 \leq M$ and $T_1 \leq C_\alpha(\alpha)$ (because $T_1 \leq H$ by (15.6)), we observe that $M$ is 2-transitive on $\Omega$. Putting $\bar{M} = M/W$ where $W$ is the kernel of the action of $M$ on $\Omega$ we see, from $W \leq C_\alpha(\alpha) \leq H$ and the structure of $K$, that $\bar{M}$ is a 2-transitive permutations group on $\Omega$, whose 2 point stabilizer is cyclic. Thus the structure of $\bar{M}$ is given in [10].

If $\bar{M} \cong U_3(q)$, $PGU_3(q)$ or $R(q)$ (a group of Ree type), then $|\Omega| =
$A_1 = 2^{m_1} = q^3 + 1$, while if $\tilde{M} \cong Sz(q)$, then $2^{m_1} = q^2 + 1$. Neither of these situations is possible by Lemma 8.6. Suppose $\tilde{M} \cong L_2(q)$ or $\text{PGL}_2(q)$ ($q > 3$). Then $2^{m_1} = q + 1$ and so $q$ is a Mersenne prime. Also $2^{m_1}(2^{m_1} - 1)c = |M| = q(q^2 - 1)/d = 2^{m_1 + 1}(2^{m_1 - 1} - 1)(2^{m_1} - 1)/d$, where $c$ is the order of the stabilizer of two points and $d = 1$ or 2. Inspection of the above equalities yields $m_1 = 3$, $d = 2$, $c = 3$, and $q = 7$ as the only possibility, in which case $\tilde{M} \cong L_2(7)$. Thus we conclude that either $\tilde{M}$ has a regular normal subgroup or $\tilde{M} \cong L_2(7)$.

Before ruling out the possibility $\tilde{M} \cong L_2(7)$ we pause to make some general observations.

(15.8) (i) $Z(S) \leq A_1$.

(ii) $C_K(K) \leq \Omega_t(Z(R))$.

Because $R \neq S$ by (15.6) and $Z(S) \leq A_1$, (15.7)(i) forces $Z(S) \leq A_1$. Since $[C_K(K), R]/C_K(K) \leq Z(S) \cap C_K(K) \leq A_0$, (15.6) and Lemma 10.5 imply that (ii) also holds.

Now suppose $\tilde{M} \cong L_2(7)$ holds. Since $1 \neq \bar{S} \in \tilde{M}$, $\tilde{M}$ acts faithfully on $A_1$ (with its natural action). Consequently, as $Z(S) (\leq A_1)$ is non-cyclic and $S/S \cap W$ is elementary abelian, we must have $|S : S \cap W| = 2$. From $|A_1| = 2^3$ it follows that $R = C_K(K)(R \cap K)$, and hence, by (15.8)(ii), $A = \Omega_t(R) = Z(R)$ and $S \cap W = R$. Now $A_1$ is not strongly closed and so there exists $\delta \in A_1$ and $\beta \in S \setminus A_1$ with $\delta$ and $\beta$ conjugate. Since the elements of $A_1^*$ are all conjugate, we may choose $\delta \in Z(S)$. From (15.7)(i) and $A = \Omega_t(R)$ we see that $\beta \in S \setminus R$. By the action of $\tilde{M}$ on $A_1$, $|C_{A_1} (\beta)| \leq 2^2$ and so $C_{A_1}(\beta) = Z(S)$. If $\beta$ centralizes $\gamma \in A_1 \setminus A_1$, then $|R| < |C_5(\gamma)|$, contradicting (15.7)(i). Then $C_{A_1}(\gamma) = Z(S)$. Let $S_0 \in \Sigma$ be such that $\delta$ and $\beta$ fuse in $N_{S_0}(S_0)$. Since $\delta \in Z(S) \leq Z(S_0)$, $\beta \in Z(S_0)$, and so $A \cap S_0 = Z(S_0)$. Therefore, $|S/S_0| > 4$. However, $m(N_1(Z(S_0))) = 3$ and so $\Omega_t(Z(S_0))$ is not a standard module for $N(S_0)^*/S_0$, a contradiction. Therefore, we conclude that $\tilde{M} \cong L_2(7)$ and so $\tilde{M}$ possesses a regular normal subgroup. From this fact we deduce

(15.9) $N = S(N \cap H)$, $[S : R] = |A_1|$, $Z(S) = A_1$ and $R = C_K(K) \times (R \cap K)$.

By the t.i. property of $A_0$ and counting the possible $N$-conjugates of $A_0$ we obtain $[N : N \cap H] \leq |A_1|$. Now let $W \cap T_1 < T_2 \leq T_1$ be such that $T_2$ is the regular normal subgroup of $\tilde{M}$. Let $1 \neq \beta \in Z(S) \cap Z(T_1)$ ($\leq A_1$) and $x \in T_1 \setminus (W \cap T_1)$. So $a\beta \in \Omega_t$ with $(a\beta)^x = a\beta_1$ for some $\beta_1 \in A_1$, and $(a\beta_1)^x = a\beta$. Hence $a^x\beta = a\beta_1$ and substituting for $a^x$ in the other equation yields $\beta_1^x = \beta_1$. Since $x$ was arbitrary and we already have $C_K(K)(R \cap K) = W \cap T_1$ centralizing $A_1$, $\beta_1 \in Z(T_2)$. Hence $A_1 = Z(T_2)$ by the transitivity of
T_2 on \Omega. Thus no element of T_2 can induce a field automorphism on K (when K \cong L_2(2^{2n})) and so T_2 \cap R = C_\kappa(K)(R \cap K) = W \cap S.

Now \bar{M}/T_2 is soluble because it is the stabilizer of a point in \Omega and so \bar{M} is a soluble 2-transitive permutation group. Clearly \bar{S} is a strongly closed 2-subgroup of \bar{M} and \bar{S} \neq 1 because S \leq H and S \cap W \leq H. Suppose \bar{S} \cap \bar{T}_2 = 1 were to hold. Then [\bar{S}, \bar{T}_2] = 1, contradicting the structure predicted for \bar{M} by [8]. Therefore, \bar{S} \cap \bar{T}_2 \neq 1 and thus, as \bar{M} acts irreducibly on \bar{T}_2, we obtain \bar{T}_2 \leq \bar{S}. So S \cap T_2 covers \bar{T}_2. Thus

$$\bar{T}_2 = S \cap T_2 \cong (S \cap T_2)/(S \cap T_2 \cap W).$$

By considering

$$(S \cap T_2)R/R \cong (S \cap T_2)/(S \cap T_2 \cap R) = (S \cap T_2)/(S \cap T_2 \cap W)$$

(using T_2 \cap R = S \cap W) we obtain |A_1| \leq |S/R|. Hence, since |N: N \cap H| \leq |A_1|, this gives N = S(N \cap H) and |S/R| = |A_1|.

For \gamma \in A_\kappa (\text{recall that } C_\kappa(\gamma) = R) we have |A_1| = |S : R| = |S : C_\kappa(\gamma)| = |S, \gamma| \leq |Z(S)| \leq |A_1| whence A_1 = Z(S). From A_1 = Z(S) we see that R = C_\kappa(K) \times (R \cap K). This proves (15.9).

One of the main reasons for establishing A_1 = Z(S) is that it helps us to prove.

(15.10) \quad O_2.(N^*)S \leq N.

First we show that N^* is 2-constrained. Put E = E_2.(N^*). Clearly [A_1, E] = 1 and so E \leq C_\kappa(A_1) \leq H. Then using [7, Theorem 3.1] gives E = KO_2.(H) whence E \leq C_{K_0^2.(H)}(A_1) and so E is of odd order. Thus N^* is 2-constrained.

Since Z(S) = A_1 \leq N by (15.7)(ii), A_1 \leq Z(N^*) and so N^*/A_1 is also 2-constrained. Hence O_2(N^*)S = N^*, so giving (15.10).

(15.11) \quad R = A.

In view of (15.9), of (15.11) is false, then K is isomorphic to either S_2(2^m) or U_3(2^m) (m_1 = m(A_1)) and R \cap K \in \text{Syl}_2 K. Since A_1 is not a strongly closed 2-subgroup, there exists \beta \in A_\kappa and \gamma \in S_A with \beta and \gamma conjugate. By (15.6) and (15.7)(i), \gamma \in S_A. Let S_0 \in \Sigma be such that \beta and \gamma are conjugate in N_\kappa(S_0) = N_0. Put \bar{N}_0 = N_0/S_0. Because \beta \in A_1 = Z(S) we may suppose \beta and \gamma are conjugate in N_\kappa^8. From (15.7)(i), since \gamma \in Z(S_0), we have S_0 \cap A = A_1. We may now argue as in [2, Lemma 8.19] that |S : S_0| \geq 4 and that \bar{N}_0^8 is isomorphic to one of L_2(2^m), Sz(2^m), U_3(2^m) and SU_2(2^m) (for some m).

Putting E_0 = U^1(S_0) \leq Z(S) we may further follow [2, Lemma 8.19] to obtain |S_0 \cap R| \leq |R \cap K| \leq |E_0|. If \bar{N}_0^8 \cong L_2(2^m), then |S : S_0| \leq |A_1 : E_0|
by [2, (2.66)(iv)]. If \( \bar{N}^*_0 \cong U_3(2^m) \) or \( SU_3(2^m) \), then, as \( \bar{N}^*_0 \) will then contain a subgroup isomorphic to \( L_2(2^m) \), we see that \( [S : S_0] \leq |A_1 : E_0| \) also holds in this case. For the case \( \bar{N}^*_0 \cong Sz(2^m) \) we consider \( \bar{N}^*_0 \) acting upon \( Z(S_0)/E_0 \). Since \( E_0 \leq C_{Z(S_0)}(\bar{N}^*_0) \) and \( (S/E_0)' \leq Z(S)/E_0 \), [3, Theorem A] yields \( [S : S_0] \leq |A_1 : E_0| \). Thus \( [S : S_0] \leq |A_1 : E_0| \) holds. Using (15.9) we can now obtain a contradiction as in [2, Lemma 8.19], and so we have (15.11).

Working in \( N/O_2(N^*) \), with the aid of (15.10) and (15.11), the same proof as [2, Lemma 8.21] yields

\[
(15.12) \text{There exists } D \leq S \text{ such that } S = A_0D \text{ with } D \cap A_0 = 1, A_1 \leq D, |D| = 2^{2m_1} (m_1 = m(A_1)) \text{ and } D \text{ is either an elementary or homocyclic abelian group.}
\]

A final contradiction may now be reached as in Lemmas 8.22 and 8.23 of [2] (note for the case \( D \) is elementary abelian: if \( D \) and \( A \) are not normal in \( N_G(S) \), then since \( A_1 = Z(S) \), \( (D \setminus A_1)_{q^*} = (A \setminus A_1) \) for some \( \omega \in N_G(S) \). Since \( \gamma(S) = D^{*} \cup A^{*} \), every element of \( \gamma(S) \setminus A_1 \) is conjugate to an element of \( A_1^{*} \) by (15.7)(i). Because of (15.6) no element of \( Z(S)^{*} \) fuses to an element of \( A_1^{*} \) and hence \( Z(S) \) is a strongly closed 2-subgroup, a contradiction).

This finishes the proof of Lemma 15.6.

**Lemma 15.7.** If \( K \) is isomorphic to one of \( L_3(2^n) \) and \( PSp_4(2^n) \) or \( K/Z(K) \cong L_2(4) \), then \( O^{2}(N_G(B)) \leq H \) for some \( B \) having the properties stated in Theorem 15.2.

**Proof.** Again we put \( N = N_G(A) \). Suppose the lemma is false and argue for a contradiction. Set \( \bar{N} = N/C_G(A) \), \( F = N \cap K \) and \( R \cap K = Q \). So \( \bar{F} = Z(\bar{F}) \times \bar{J} \), where \( Z(\bar{F}) \) is cyclic of order dividing \( 2^n - 1 \) and \( \bar{J} \cong L_2(2^n) \). As in Lemma 15.6 we have \( S \leq \bar{N} \).

\[
(15.13) \bar{J} \text{ is a component of } \bar{N}.
\]

Since \( S' \leq Z(S) \leq A \), \( \bar{S} \) is a strongly closed abelian 2-subgroup of \( \bar{N} \). Clearly \( \bar{J} \leq \bar{N}^* \) and hence \( \bar{J} \leq E_2(\bar{N}^*) \) by (2.9). Put \( \bar{X} = O_2(\bar{N}^*) \). From the structure of \( K \). \( C_A(\bar{\rho}) = Q' \leq Z(S) \) for all \( \bar{\rho} \in \bar{Q}^* \) and so, since \( [K, A_0] = 1 \), \( C_A(\bar{\rho}) = Q'A_0 = C_A(\bar{Q}) \). Consequently \( \bar{X} = \langle C_X(\bar{\rho}) \mid \bar{\rho} \in \bar{Q}^* \rangle \) normalizes \( C_A(\bar{Q}) \). Since \( C_A(\bar{Q}) = Q'A_0 \geq Z(S) \), we have \( \bar{Q} \) centralizing \( A/C_A(\bar{Q}) \) and \( C_A(\bar{Q}) \) and hence \( [\bar{X}, \bar{Q}] \) centralizes both \( A/C_A(\bar{Q}) \) and \( C_A(\bar{Q}) \). Therefore, \( [\bar{X}, \bar{Q}] \) centralizes \( A \) and hence \( [\bar{X}, \bar{Q}] = 1 \). Thus \( [\bar{J}, \bar{X}] = 1 \) and so \( \bar{J} \leq E(\bar{N}^*) \).

By [13, Lemma 5.2] if \( \bar{L}_1 \) and \( \bar{L}_2 \) are distinct components of \( \bar{N}^* \), then \( \bar{L}_1 \) and \( \bar{L}_2 \) act non-trivially on distinct composition factors in an \( E(\bar{N}^*) \) composition series of \( A \). However, by the structure of \( K \), \( \bar{J} \) has just one composition factor on \( A \) and so \( \bar{J} \leq \bar{L} \) for some component \( \bar{L} \) of \( \bar{N}^* \). We
now show that $\bar{L} \cong L_2(2^m)$ for some $m$. Suppose this is false. Then $\bar{L}$ is of type $L_2(q)$, $q \equiv 3, 5(8)$, $q > 5$, JR, $S_z(2^m)$ of $U_3(2^m)$. The above argument shows that for any non-cyclic 2-subgroup $Q_1$ of $Q \cap \bar{L}$, $Q_1$ centralizes any $Q_1$-invariant subgroup of $\bar{L}$ of odd order. Hence this rules out $\bar{L} \cong L_2(q)$, $q = 3, 5(8)$, $q > 5$ or of Ree type. The remaining possibilities for $L$ are impossible by [13, Lemma 2.10]. Therefore, $L \cong L_2(2^m)$.

From the structure of $K$, for $\bar{p} \in \bar{O}^*[A : C_4(\bar{p})] = 2^n$, whereas [13, Lemma 2.12] gives $[V : C_4(\bar{p})] = 2^m$ for any non-trivial $\bar{L}$-composition factor of $A$ whence $[A : C_4(\bar{p})] \geq 2^m$ by [3, Lemma 2.5, Chapter 2]. Thus $n = m$ and $\bar{L} = L_2$, as desired.

(15.14) $\bar{J} \leq O^2(\bar{N})S$.

By Lemma 15.3, as $O^1(N) \leq H$, $m(A) \leq 2m(A_0)$. From the structure of $\bar{N}^*$ we have that $\bar{S}$ normalizes $\bar{J}$. So we only need to establish that $\bar{J} \leq O^2(\bar{N})$.

If $K/Z(K) \cong L_3(2^n)$, then $m(A) \leq 4n$. Then [13, Lemmas 2.12 and 5.2] imply that $\bar{N}^*$ has at most two components isomorphic to $L_2(2^n)$ and therefore $\bar{J} \leq O^2(\bar{N})$.

Now suppose $K \cong PSp_4(2^n)$. Then $m(A) \leq 6n$ and so $\bar{N}^*$ can have at most three components isomorphic to $L_2(2^n)$. Suppose $J = L_1, L_2, L_3$ are distinct and are conjugate in $\bar{N}$. Then, by considering a $L_1L_2L_3$ composition series of $A$, we obtain $[A, L_1] = 2^n$. Clearly $[A, L_1] \leq A_1$ as $[L_1, A_0] = 1$. But then $[A_1, L_1] \neq A_1$, a contradiction. Thus we conclude that $\bar{J} \leq O^2(\bar{N})$ in this case also.

By (15.14) we have that $O^2(\bar{N})S$ normalizes $C_4(\bar{J})$. Hence $O^2(N)$ normalizes $C_4(\bar{J})$.

If $K/Z(K) \cong L_3(2^n)$, then $C_4(\bar{J}) = A_0$ and so $A_0$ is normalized by $O^2(N)$. But then $O^2(N) \leq H$ by Lemma 15.4(i), contrary to our supposition. Therefore, $K \cong PSp_4(2^n)$. In this case $C_4(\bar{J}) = A_0 \times C_4(\bar{J})$ with $m(C_4(\bar{J})) = n$. Also we have $Z(\bar{F})$ acts transitively and regularly upon $C_4(\bar{J})^*$ and, of course, centralizes $A_0$. Thus, since $O^2(N) \leq H$, we are in a position to mimic the arguments of (15.4) and (15.5) and show there exists $n \in O^2(N) \setminus H$ such that $A_0^\sigma \cap C_4(\bar{J}) = 1$. Hence $[C_4(C_4(\bar{J})), A_0] = 1$ by Lemma 15.5.

Let $A_2 \in \mathfrak{V}_e(Q)$ with $A_2 \neq A_1$, and put $B = A_0A_2$. Then $(\mu, B, H, K)$, $\mu \in A_0$ satisfies Hypothesis 10.3. Since we are also assuming $O^2(N_0(B)) \leq H$, all the previous conclusions obtained for $A$ are also valid for $B$. In particular we have $S$ normalizing $C_4(\bar{J}) = A_0C_4(\bar{J})$, where $\bar{J} = E(N_2(A_2)/C_2(A_2))$ (so $\bar{J} \cong L_2(2^n)$). Therefore, $S$ normalizes $A_0C_4(\bar{J}) \cap A_0C_4(\bar{J})$. Since $A_0 \cap K = 1$, $A_0C_4(\bar{J}) \cap A_0C_4(\bar{J}) = A_0C_4(\bar{J}) \cap C_4(\bar{J})$. Now $K = \langle N_2(A_4), N_2(\bar{A}) \rangle$ normalizes $C_4(\bar{J}) \cap C_4(\bar{J})$ and hence this group must be trivial. Thus $S$ normalizes $A_0$ and so $S \leq H$ by Lemma 15.4(i). Hence $S = C_4(K) \times (S \cap K)$. Since $C_4(K) \leq C_4(C_4(\bar{J})), A_0 \leq Z(C_4(K))$ and conse-
quently \( \mathfrak{U}_e(S) = \{A, B\} \). From the structure of \( S \) we see that \( \bar{J} = E(\bar{N}^*) \) and \( \bar{I} = E((N_e(B)/C_e(B))^*) \) and so \( \bar{J} \subseteq \bar{N} \) and \( \bar{I} \subseteq N_e(B)/C_e(B) \). Hence arguing as above we can obtain that \( N \cap N_e(B) \) normalizes \( A_0 \) and thus \( N \cap N_e(B) \leq H \).

Now let \( M \) denote the inverse image in \( N \) of \( \bar{J} \). Then \( M \subseteq N \) and \( S \leq M \). Hence \( N = MN_\delta(S) \) by (2.1)(iv). But then, as \( M \leq H \) and \( N_\delta(S) \leq N \cap N_e(B) \leq H \), we have a contradiction.

The proof of Lemma 15.7 is complete.

Now combining Lemmas 15.1, 15.6 and 15.7 yields Theorem 15.2. We now seek to strengthen the conclusion of Theorem 15.2 in the next result. By a possible change of notation we shall suppose that \( A \) satisfies the conclusion of Theorem 15.2.

**Theorem 15.8.** Suppose \((\sigma, A, H, K)\) satisfies Hypothesis 10.3. If \( \mu \in A^\# \) and \((\mu, A, H_\mu, K_\mu)\) satisfies Hypothesis 10.3 with \( K \cong K_\mu \), then \( K \cong H_\mu \).

**Proof.** Combining Theorem 11.1, 13.1 and 14.8 gives that \( K_\mu = E(H_\mu) \). We seek to show that \( K = K_\mu \), from which the theorem follows.

Put \( A_\mu = C_A(K_\mu) \). Then \( A = A_\mu(A \cap K_\mu) \) and \( A = A_0(A \cap K) \). Appealing to Theorem 15.2 yields

\[
N_{K_\mu}(A \cap K_\mu) = O^2(N_{K_\mu}(A \cap K_\mu)) \leq O^2(N_e(A)) \leq H.
\]

Consequently \( N_{K_\mu}(A \cap K_\mu) \) normalizes \( K \), and hence normalizes both \( A_0 \) and \( A \cap K \). Note that Lemmas 3.4(ii) and 10.2 imply \( K = K_\mu \) if either of \( K \leq H_\mu \) or \( K_\mu \leq H \) holds, so we may assume \( K \nsubseteq H_\mu \) and \( K_\mu \nsubseteq H \).

Suppose \( K/Z(K) \cong L_3(2^n) \) or \( PSp_4(2^n) \). Then the arguments of [2, Lemma 8.29], which depend only on properties of \( K \) and the above observation, produce the desired conclusion \( K = K_\mu \).

So now we consider the possibility when \( K/Z(K) \cong L_2(2^n), Sz(2^n) \) or \( U_3(2^n) \). Again we put \( \hat{A}_1 = [\xi, A \cap K] \) if \( K/Z(K) \cong Sz(8) \) and \( Z(K) \neq 1 \) (where \( \langle \xi \rangle \in Syl_1(N_\delta(A \cap K)) \)) and \( \hat{A}_1 = A \cap K \) otherwise. We use \( \hat{A}_\mu \) to denote the analogue of \( \hat{A}_1 \) for \( K_\mu \). The same consideration as precede [2, Lemma 8.31] give

\[
(15.15) \quad \hat{A}_1 = A_\mu \text{ and } A_0 = \hat{A}_\mu.
\]

Let \( F \) denote a (cyclic) Hall 2'-subgroup of \( N_{K_\mu}(A \cap K_\mu) \).

\[
(15.16) \quad [F, K] = 1.
\]

The same proof as [2, Lemma 8.32] will establish (15.16).

\[
(15.17) \quad \text{Let } S_1 \text{ be any } H \cap H_\mu\text{-conjugate of } S \text{ (recall } S \leq H \cap H_\mu). \text{ Then } C_{S_1}(K) \cap C_{S_1}(K_\mu) = 1.
\]
Applying Theorem 15.2 to $H$ and $H_\mu$ gives that $S_1$ normalizes both $K$ and $K_\mu$. Consequently $C_{S_1}(K) \cap C_{S_1}(K_\mu) \trianglelefteq S_1$. Suppose $C_{S_1}(K) \cap C_{S_1}(K_\mu) \neq 1$. Then $1 \neq Z(S_1) \cap C_{S_1}(K) \cap C_{S_1}(K_\mu) \trianglelefteq A_0^h$ for some $h \in H \cap H_\mu$ whence $K_\mu \leq H$ by Lemma 15.4(i) whereas we are assuming $K_\mu \nleq H$. Therefore, $C_{S_1}(K) \cap C_{S_1}(K_\mu) = 1$.

Now set $C = C_G(F)$. Then $K \subseteq C \neq G$ and $K \subseteq C^*$. Recall that, by Theorem 15.2, $F \leq H$. Since we also have $S \leq H$, there exists $h \in H$ such that $S^h \cap C \subseteq X^*(C)$ with $S^h \cap C \geq S^h \cap K = S \cap K$ (since $K \leq H$). So $V = S^h \cap C$ normalizes $K$ and $S \cap K$. Hence $V$ normalizes $Z(S \cap K)$. Suppose $Z(K) = 1$. Then, by (15.15), $Z(S \cap K) \leq A \cap K = A_1 = A_\mu$, and then $V$ centralizes an element of $A_\mu^w$. If, on the other hand, $Z(K) \neq 1$, then $K/Z(K) \cong Z(S)(8)$ whence $V = C_{A_1}(K)(V \cap K)$ and so $V$ centralizes $A_1 = A_\mu$. Consequently $V \leq H_\mu$ by applying Lemma 15.4(i) to $H_\mu$, and so $V$ normalizes $K_\mu$. However $V$ centralizes $F$ and hence $V$ must centralize $K_\mu$. Since $S$, $V \subseteq H \cap H_\mu$, $V \subseteq S_1$, where $S_1$ is some $H \cap H_\mu$-conjugate of $S$, and then $C_{A_1}(K) = 1$ by (15.7). Therefore, we must have $|V: S \cap K| \leq 2$.

Now arguing as in [2, Lemma 8.3] and using Lemma 3.1(i) gives

\[(15.18) \quad K = E(C^*).\]

It is claimed that $K \leq N_G(F)$. Suppose $K \nsubseteq N_G(F)$ and argue for a contradiction. So there exists $n \in N_G(F)$ such that $K^n \leq C$ and $K^n \neq K$. By (15.8), $[K, K^n] = 1$. Since $A_\mu = A_1 \leq K$, $K^n \leq H_\mu$ by Lemma 15.4(i) applied to $H_\mu$. Thus $K^n$ normalizes $K_\mu$ and centralizes $F$, and therefore, since $O^2(K^n) = K^n$, $[K^n, K_\mu] = 1$. Let $\lambda \in A_\mu^w$. Then $\lambda^n \in K^n$. Because $\mathcal{A}(\mu_0) = \{H_\mu\}$ for all $\mu_0 \in A_\mu^w$ by Lemma 10.5, $(\lambda^n, A_\mu^w, H_\mu^w, K_\mu^w)$ satisfies Hypothesis 10.3. Since $\lambda^n$ centralizes both $K$ and $K_\mu$, $K, K_\mu \leq H$ whence $K = K_\mu = K_\mu$ by Lemma 3.4(ii) and 10.2, contrary to $K_\mu \nleq H$. This establishes the above claim. Hence $N_G(F) \leq H$ and then $K_\mu = (A \cap K_\mu, N_K(F)) \leq H$, contrary to the supposition $K_\mu \nleq H$.

This concludes the proof of Theorem 15.8.

An immediate corollary of Theorem 15.8 is

**Lemma 15.9.** $S \leq N_G(A) \leq H$ and $N_G(Z(S)) \leq H$.

Let $T \in \text{Syl}_1 G$ with $S \leq T$. Since $1 \neq C_T(K) \cap S \leq T$, we can choose $a \in \mathcal{A}(Z(T) \cap A_\mu)$. So $C_G(a) \leq H$. Let $g \in G$ be such that $\beta = a^g \in S$. We now show that $g \in H$.

Suppose it is the case that for some $k \in K$, $\gamma = \beta^k$ centralizes $A \cap K$ and $\gamma \in S$. Since there exists $1 \neq \delta \in Z(S) \cap A_\mu$, $B = (A \cap K)\langle \delta \rangle \leq A$ is an elementary abelian subgroup of $C_G(\gamma)$ with $m(B) > m(K/Z(K))$. Now $H^k \in \mathcal{A}(\gamma)$ and $K^k \leq H$ and so there exists $\lambda \in B^w$ with $[K^k, \lambda] = 1$. Then $K^k$ is $C_M(\gamma)$-invariant where $M \in \mathcal{A}(\lambda)$ and hence $K^k \leq E(M)$ by Lemmas 3.4(ii) and 10.2. Consequently $(\lambda, A, M, K^k)$ satisfies...
Hypothesis 10.3 and thus $H^{rs} = H$ by Theorem 15.8. Hence $g \in H$ in this case.

We now show that if $\beta$ induces an inner automorphism on $K$, then there exist some $k \in K$ such that $\gamma = \beta^k$ centralizes $A \cap K$ and $\gamma \in S$. Put $X = K\langle \beta \rangle$ and $Y = C_A(K)$, and set $\bar{X} = X/Y$. Then $S \cap X \in \mathcal{E}^e(X)$. Also, it is claimed, $Y \leq S$. If $S \in \text{Syl}_2 X$, then it is clear that $Y \leq S$. Otherwise we have $K/Z(K) \cong L_2(2^n)$, $Sz(2^n)$ or $U_3(2^n)$. Since $Z(K) \leq S$ when $K/Z(K) \cong Sz(3)$, there is no loss in supposing here that $Z(K) \cap S = 1$. If $U \in \text{Syl}_2 K$ is such that $U \geq S \cap K$ and $W$ is a complement to $U$ in $N_K(U)$, then considering the action of $W$ upon $S \cap X$ and using the fact that $C_1(W) = 1$ it follows that $Y \leq S$. Since $A \in \mathcal{H}_e(S)$ and $K \not\leq L_2(2^n)$ by Lemma 15.1, $A \cap K \in \mathcal{H}_e(S \cap K)$ and hence $A \cap K \in \mathcal{H}_e(S \cap \bar{K})$ by Lemma 8.8. Therefore, since $\bar{X} = \bar{K}$, $\bar{x}^k \in A \cap \bar{K}$ by some $k \in \bar{K}$ by (2.18)(i). But then $\gamma = \beta^k \in (A \cap K)Y$ for some $k \in \bar{K}$. Hence $\gamma \in S$ and $[\gamma, A \cap K] = 1$, as required.

Combining the above observations with Lemma 15.1 we see that we now have only to verify that $g \in H$ in the case when $K \cong L_2(2^n)$ $(n > 1)$ and $\beta$ induces a field automorphism upon $K$. As usual, $K \leq H^s$ or $K^s \leq H$ implies $g \in H$; so we may suppose $K \leq H^s$ and $K^s \leq H$. Put $L = C_K(\beta) \cong L_2(2^n)$. Then $H^s \cap K = L$ and so $L \leq K^s$ by Lemmas 3.4(ii) and 10.2. Since $K^s \leq H$, $C_K(\alpha) = L$ and so $\alpha$ induces a field automorphism on $K^s$. Let $\delta (= \alpha)$ be a $K^s$-conjugate of $\alpha$ such that $\delta \in \langle K \cap A \rangle \leq A$ (this is possible since all involutions of $K^s \langle \alpha \rangle$ are $K^s$-conjugate). Using Theorem 15.8 we see that $[H] = m(\delta)$. Consequently $K^s = \langle C_K(\alpha), C_K(\delta) \rangle \leq H$, contrary to $K^s \not\leq H$. Thus we conclude that $g \in H$.

Appealing to (2.20) now yields a contradiction to the minimal choice of $G$ and so we have established

**Theorem 15.10.** If Hypothesis 10.3 holds for $(\sigma, H, K)$, then $r(H) \neq 1$.

**16. Conclusion of the Proof**

Here we tidy up some loose ends which have escaped our attention in the previous sections. Combining Theorem 11.1, 12.1, 13.1, 14.8 and 15.10 yields

**Lemma 16.1.** If $\sigma \in \mathcal{H}(S)$, then $m(C_\sigma(\sigma)) \leq 4$ and $\sigma \not\in B$ for any $B \in \mathcal{H}_e(S)$.

For the remainder of this section $\sigma \in \mathcal{H}(S)$, $C = C_\sigma(\sigma)$, $R \in \mathcal{E}^e(C)$ and $K$ is a 2-component of $C$. Putting $\bar{K} = K/O_2.(K)$ we have the following consequence of Lemmas 4.3 and 16.1.
LEMMA 16.2. Either $\overline{K}$ is isomorphic to one of $L_2(q), q$ odd; $A_7$; $U_3(8)$; $L_2(8)$; $Sz(8)$; a group of type $JR$ or $|Z(\overline{K})| = 2$ and $\overline{K}/Z(\overline{K}) \cong Sz(8)$.

LEMMA 16.3. Either (i) $m(S) \geq 5$ and $\overline{K}$ is isomorphic to one of $A_6$; $I_2(4)$; $I_2(8)$; $U_3(4)$; $U_3(8)$; $Sz(8)$ or $|Z(\overline{K})| = 2$ and $\overline{K}/Z(\overline{K}) \cong Sz(8)$; or

(ii) $m(S) = 4$, $m(C) = 3$ with $\overline{K}$ isomorphic to one of $L_2(q), q$ odd; $A_7$; and $U_3(4)$.

Proof. Suppose $m(S) \geq 5$ and let $B \in \mathfrak{U}_e(S)$. Then $m(C_B(\sigma)) \geq 3$. Since $\mathfrak{C}(S) \cap C_B(\sigma) = \phi$, [7, Theorem 3.1] and Lemma 10.2(ii) together with Lemma 16.2 yield the possibilities for $\overline{K}$ as listed in (i) (see, for example, [2, (2.49)]), whereas, by Lemma 9.4, $m(S) \leq 4$ implies $m(S) = 4$ and hence $m(C) = 3$ by Lemma 16.1. Then $m(K) = 2$ and so (ii) holds. This proves the lemma.

LEMMA 16.4. $m(S) = 4$.

Proof. Suppose the lemma is false. Thus $m(S) \geq 5$ and Lemma 16.3(i) applies. If $B \in \mathfrak{U}_e(S)$, then $m(C_B(\sigma)) \geq 3$. Put $W = C_B(\overline{K})$. If $W \cap B \neq 1$, then Lemma 10.1 and [7, Theorem 3.1] would contradict Lemma 16.1. So $W \cap B = 1$. In particular, $Z(S) \cap W = 1$ and so, by [12, Lemma 2.6(v)], $\overline{K}/Z(\overline{K}) \not\cong Sz(8)$ with $Z(\overline{K}) \cap S \neq 1$. Since $m(C_B(\sigma)) \geq 3$ and $m(U_3(4)) = 2$, we also see that $K \not\cong U_3(4)$. Similarly, $K \not\cong L_2(4)$, since $m(Aut L_2(4)) = 2$. Thus $\overline{K}$ is isomorphic to either $A_6$, $L_2(8)$, $Sz(8)$ or $U_3(8)$.

We consider the situation when $\overline{K} \not\cong A_6$. Then $R = W \times (R \cap K)$ and so, because $m(C_B(\sigma)) \leq 4$ and $W \cap Z(S) = 1$, $W = \langle \sigma \rangle$. Set $N = N_G(D)$, where $D = W \Omega_1(R \cap K)$. By the structure of $\overline{K}$, $N$ contains an element which acts transitively on $\Omega_1(R \cap K)^*$ and centralizes $W$. Moreover, since $Z(S) \leq D$, we have $S \leq N$. Because $R \not= S$, we see that $\sigma$ must be $N$-conjugate to one of the sets $\Omega_1(R \cap K)^*$ and $D \setminus (W \cup \Omega_1(R \cap K))$. However $C_B(\sigma) \cap (R \cap K) \neq 1$ and hence, by Lemma 16.1, $C_B(\sigma) = \Omega_1(R \cap K)$ and $\sigma^N \cap D = D \setminus \Omega_1(R \cap K)$. So $2^3 | [N : C_B(\sigma)]$ and hence $2^3 | |S : R|$. Since $R \leq S \leq N$, we also have $|S : R| \leq 2^3$ and so $|S : R| = 2^3$. Hence $|[S, \sigma]| = 2^3$. Now $|S, \sigma| \leq D \cap Z(S) \leq \Omega_1(R \cap K)$ and thus $\Omega_1(R \cap K) = Z(S)$.

Since $Z(S)$ is not a strongly closed 2-subgroup of $G$, there exists $\alpha \in Z(S)$ and $\beta \in S \setminus Z(S)$ with $\alpha$ and $\beta$ conjugate. In fact, because $D \setminus Z(S) \leq \mathfrak{U}_e(S)$ and $\Omega_1(R) = D$ we have $\beta \in S \setminus R$. Let $S_0 \in \Sigma$ be such that $\alpha, \beta \in S_0$ with $\alpha$ and $\beta$ $M$-conjugate, where $M = N_G(S_0)$. Since $N_M(S)$ leaves $Z(S)$ invariant and $M = M^* N_M(S)$, we may suppose that $\alpha$ and $\beta$ are conjugate in $M^*$. Further $\beta \in Z(S_0)$, as $\alpha \in Z(S) \leq Z(S_0)$. For $\gamma \in D \setminus Z(S)$, $\gamma = \sigma^\delta$ for some $\delta \in Z(S)$ and hence $C_B(\gamma) = R$. Therefore, since $\beta \in S \setminus R$, $S_0 \cap D = Z(S)$.

We now show that $R \cap K$ is elementary abelian. If this were not the case, then $R \cap K$ would be isomorphic to a Sylow 2-subgroup of either $U_3(8)$ or
Sz(8). We derive a contradiction to this situation as follows. First, it is asserted that \([S : S_0] > 2\). If this were false, then \(S_0 \cap \langle \sigma \rangle = 1\) and \([D : D \cap S_0] \leq 2\) would force \(S_0\) to cover \(D/\langle \sigma \rangle\) and then the structure of \(R \cap K\) gives \(Z(S) \leq S_0\). However \([M^*, S_0'] = 1\), contrary to \(\alpha\) and \(\beta\) being conjugate in \(M^*\). Thus \([S : S_0] > 2\). Because \(m(C_6(\sigma)) \leq 4\), we note that \(m(S) \leq 6\). Hence, by \([2, (2.71)(i)]\), \(M^*/S_0\) is isomorphic to either \(L_2(4)\) or \(L_2(8)\) with, respectively, \(m(L_2(4)) = 4\) or \(6\). Since \(\sigma \not\in S_0\) and \(m(Z(S)) = 3\), we see that \(M^*/S_0 \cong L_2(8)\) must hold. By \([2, (2.66)(iv)]\), \(\langle Z(S) : \Omega^*(S_0) \rangle \geq 8\) and so \(S_0\) is elementary abelian. But then \([S_0] = 2^6\) and \([S : S_0] = 2^3\), whereas \(|S| < 2^8\), a contradiction. Thus we have shown that \(R \cap K\) is elementary abelian (and so \(R = \langle \sigma \rangle Z(S)\)).

From the structure of \(K\) there exists a cyclic group \(Y_0\) of odd order with \(Y_0\) centralizing \(\sigma\) and acting transitively on \((R \cap K)^* = Z(S)^*\). Clearly \(Y_0\) normalizes \(F = C_6(Z(S)) \cap N_6(R)\). Now \(F/C_6(R)\) is a 2-group and hence \(SC_6(R)/C_6(R) \leq O_2(FY_0/C_6(R))\). By a Frattini argument \(N_6(S)\) contains a cyclic group \(Y\) of odd order which centralizes \(\sigma\) and acts transitively on \(Z(S)^*\). Since \([S : R] = 2^3\) and \(\phi(S) = Z(S)\) (because \(Z(S) = [S, \sigma] \leq S'\)), we see that \(Y\) is transitive on \(S/R\). Hence \(S/Z = R/Z(S) \times V/Z(S)\) by \([6, \text{Theorem } 5.2.3]\).

It is claimed that \(V\) is abelian. If \(V \setminus Z(S)\) contains an involution, then as \(Y\) is transitive on \((V/Z(S))^*\) every coset of \(Z(S)\) in \(V\) contains an involution and so \(V\) is elementary abelian. So we may suppose \(\gamma(V \setminus Z(S)) = \emptyset\), and thus \(\Omega_1(V) = Z(S)\). Thus \(\beta = \sigma \gamma\) for some \(\gamma \in V\) (recall that \(\beta \in S \setminus R\)). Since \(\beta \in \gamma(S)\), \(\sigma\) must invert \(\gamma\) and therefore, conjugating by elements of \(Y\), we deduce that \(\sigma\) inverts \(V\), whence \(V\) is abelian as claimed. Appealing to Lemma 4.1(vi) yields \(|S'| = 2\), whereas \(S' = Z(S)\) with \(|Z(S)| = 2^3\). This contradiction arose from the supposition \(K \cong A_6\). Therefore, we must have \(K \cong A_6\).

Since \(C_6(\sigma) \leq R\) and \(\sigma \in Z(S)\), \(R = R_1 \times (R \cap K)\) with \(R_1 = \langle \sigma \rangle \times \langle \tau \rangle\) for some \(\tau \in Z(S)\). Put \(D = C_6(\sigma)/\langle \sigma \rangle\). Note that \(D = R_1 \times R_2\) for some \(R_2 \in \mathcal{H}_6(R \cap K)\). From the structure of \(K\), \(K\) contains a cyclic 3-subgroup \(Y\) which centralizes \(R_1\) and is transitive on \(R_3^*\). Set \(F = \langle Y, S \rangle\) and \(\bar{F} = F/\langle \tau \rangle\). Since \(R \neq S\), \(\sigma\) is conjugate to an element of \(Z(R)/\langle \sigma \rangle\) and thus all elements of \(\bar{D} \setminus C_6(\sigma)\) are \(\bar{F}\)-conjugate to \(\bar{\sigma}\). Hence, since \(\bar{D} \leq \bar{F}\), \(\bar{\sigma}\) has four conjugates in \(\bar{F}\). Consequently \(4 = [\bar{S} : C_{\bar{F}}(\bar{\sigma})] = [\bar{S}, \bar{\sigma}] = |Z(S)| = 2\), and so \(K \cong A_6\) is untenable.

This completes the proof of Lemma 16.4.

By Lemmas 16.3 and 16.4, \(m(C) = 3\) and \(\bar{K}\) is isomorphic to one of \(L_2(q)\), \(q\) odd; \(A_7\); and \(U_3(4)\). Since \(\sigma \in Z(S)\) and \(Z(S)\) is noncyclic, we must have \(R = \langle \sigma \rangle \times (R \cap K)\) with \(R \cap K \cong \mathbb{Z}_2 \times \mathbb{Z}_2\) or a Sylow 2-subgroup of \(U_3(4)\) and \(\bar{K} \cong L_2(q)\), \(q = 3, 5(8)\) or \(U_3(4)\). Now \(Z(S) \cap (R \cap K) \neq 1\) and \(\sigma\) is \((S^*\text{-})\)conjugate to an element of \(\Omega_1(R)/\langle \sigma \rangle\) implies that \(\Omega_1(R \cap K) = Z(S)\).
So the elements of $Z(S)^*$ are conjugate and hence $S' = Z(S)$ by (2.1)(v). Also $\sigma^{G_{O_2}(R)} = R'(R \cap K)$.

Suppose $R \cap K$ is isomorphic to a Sylow 2-subgroup of $U_3(4)$. Let $\alpha \in Z(S)^*$ and $\beta \in S\setminus Z(S)$ with $\alpha$ and $\beta$ conjugate. Then $\beta \in S' \cap R$. Choose $S_0 \in \Sigma$ such that $\alpha, \beta \in S_0$ with $\alpha$ and $\beta$ conjugate in $M = N_G(S_0)$. Arguing as in Lemma 16.4 we conclude that $|S : S_0| \geq 4$. Then, since $m(S) = 4$, $|\sigma^{G_{O_2}(R)}| = 4$ and $|\sigma^{G_{O_2}(R)}| = 4$ give $M^*/S_0 \cong L_2(4)$ and $|Z(S) : G'(S_0)| \geq 4$ whence $|S| = 2^6$, which is not the case. Thus $R \cap K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. From $|\sigma^{G_{O_2}(R)}| = 4$ we deduce that $|S : R| = 2^2$ and hence $|S| = 2^5$. Consequently $S$ contains an abelian subgroup of index 2 whence $|S'| = 2$ by Lemma 4.1(vi), a contradiction.

This contradiction together with Lemma 16.1 implies that $\mathcal{G}(S) = \emptyset$, which is contrary to Lemma 4.2(i). Thus we conclude that there exist no counterexamples to Theorem A, so proving the theorem.

17. Corollaries

The purpose of this section is to give proofs for the corollaries stated in Section 1.

**Corollary 1.** Suppose that $G$ is a finite group, that $O_{2'}(G) = 1$ and that $S$ is strongly closed 2-subgroup of $G$ of nilpotent class at most two. Set $K = \langle S^G \rangle$. Then $K = HE(K)$ where each of the components of $E(K)$ is of type $L$ for some $L \in \mathcal{L}$ and $H/O_{2',2',2'}(H)$ is a central product of an abelian 2-group and quasisimple Goldschmidt groups.

**Proof.** Let $R \in \text{Syl}_2 O_2(K) E(K)$, and put $N = N_k(R)$. By a Frattini argument $K = NE(K)$. Let $L$ be a component of $K$. Since $L \leq \mathcal{L} \leq K$, $L \cap S$ is a strongly closed 2-subgroup of $L$. From Lemma 3.1(i), $L = \langle (S \cap L)^L \rangle$, and so $S \cap L$ is a non-trivial strongly closed 2-subgroup of $\overline{L} = L/Z(K)$. Appealing to Theorem A yields that $\overline{L} \in \mathcal{L}$.

By Lemma 2.1(iv), $\langle S^K \rangle = K$, and thus $K = N^*E(K)$ (recall that $N^* = \langle \mathcal{Z}^*(N) \rangle$). Since $N$ is 2-constrained by [7, Proposition 3.3], $N^*$ is also 2-constrained and hence, using [13, Lemma 5.1(iii)], $N^*/O_{2',2',2'}(N^*)$ is the central product of an abelian 2-subgroup and quasisimple Goldschmidt groups. Taking $H = N^*$ completes the proof of Corollary 1.

**Corollary 2.** Let $G$ be a non-abelian finite simple group containing a non-trivial strongly closed 2-subgroup $S$. Then either

1. $G \in \mathcal{L}$; or
2. for each $1 \neq x \in N_G(S)$ with $x^3 = 1$, $C_S(x) \neq 1$. 

(i) $G \in \mathcal{L}$; or

(ii) for each $1 \neq x \in N_G(S)$ with $x^3 = 1$, $C_S(x) \neq 1$. 

Proof. If $C_S(x) = 1$ for some $1 \neq x \in N_S(x)$ with $x^3 = 1$, then $S$ admits a fixed-point-free automorphism of order 3 and so $\text{cl} \ S \leq 2$. Hence $G \in \mathcal{L}$ by Theorem A, so proving the corollary.

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