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The change of feedback invariants under one row perturbation ☆

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Abstract

In this paper we completely characterize possible feedback invariants of a rectangular matrix under small additive perturbations on one of its rows.

Keywords: Row completions; Perturbations; Feedback invariants

1. Introduction

In [3], Beitia et al. have described the possible similarity invariants of a square matrix under small additive perturbations on one of its rows. Their result combines the problems of describing the possible Jordan canonical forms of matrices obtained by addition of a complex matrix E with sufficiently small entries to a complex square matrix M, see [12,4], and on the other hand, the problem of completion of a rectangular matrix to a square one, see [13].

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Also, Gracia et al. [8] have described the possible feedback invariants of matrices obtained by addition of a complex matrix E with sufficiently small entries to an arbitrary rectangular complex matrix M.

This paper is a natural prolongation of those results.

Let $A \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{n\times m}$, $b^{\mathrm{T}} \in \mathbb{C}^{n\times 1}$, $a \in \mathbb{C}^{1\times n}$, $c \in \mathbb{C}^{1\times m}$ and $x \in \mathbb{C}^{1\times 1}$. Observe the following rectangular matrix:

$$M = \left[\begin{array}{c|c|c} A & b^{\mathrm{T}} & C \\ \hline a & x & c \end{array}\right] \in \mathbb{C}^{(n+1)\times(n+1+m)}. \tag{1}$$

Recently, in [9], is given a generalization of the result from [13]. This together with the result from [8] allows us to study the feedback invariants of M under small perturbations on one of its rows, i.e. on

$$\begin{bmatrix} a & x & c \end{bmatrix} \in \mathbb{C}^{1\times(n+1+m)}.$$

Throughout the paper $\mathbb{F}$ denotes an arbitrary field and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If f is a polynomial, $d(f)$ denotes its degree. If $f(\lambda) = \lambda^k - a_{k-1}\lambda^{k-1} - \cdots - a_1\lambda - a_0 \in \mathbb{F}[\lambda]$, $k > 0$, then $C(f)$ denotes the companion matrix

$$C(f) = \left[e_2^{(k)} \cdots e_k^{(k)} a\right]^{\mathrm{t}},$$

where $e_i^{(k)}$ is ith column of the identity matrix I_k and

$$a = [a_0 \cdots a_{k-1}]^{\mathrm{t}}.$$

For the polynomials $\alpha_1|\cdots|\alpha_n$ by $\sum d(\alpha_i)$ we denote $\sum_{i=1}^n d(\alpha_i)$ and by $\prod \alpha_i$ we denote $\prod_{i=1}^n \alpha_i$.

In this paper we consider partitions as sequences of nonincreasing integers. If a and b are partitions $a \cup b$ is defined as the partition whose components are those of a or b reordered in nonincreasing order. For any partition $a = (a_1, \ldots, a_m)$ we can define its length $l(a)$ as the number of nonnegative elements in a, and its weight $|a|$ as their sum. Also by $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_m)$, $\bar{a}_i = \sharp\{j | a_j \geqslant i\}$, we denote the conjugate (dual) partition of a.

Also, we will use majorization in Hardy–Littlewood–Polya sense, see [10]: for any two partitions $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_m)$,

$$a \prec b$$

means

$$\sum_{i=1}^k a_i \leqslant \sum_{i=1}^k b_i, \quad k = 1, \ldots, m-1$$

and

$$\sum_{i=1}^m a_i = \sum_{i=1}^m b_i.$$

If

$$\sum_{i=1}^k a_i \leqslant \sum_{i=1}^k b_i, \quad k = 1, \ldots, m,$$

then we write

$$a \prec\prec b.$$

2. Previous results

Let $X + \lambda Y \in \mathbb{F}[\lambda]^{q\times p}$ be an arbitrary singular pencil of rectangular matrices. Consider the equation

$$(X + \lambda Y)x = 0,$$

where x is a polynomial column vector. Among all its solutions we choose a nonzero solution $x_1(\lambda)$ of least degree ϵ_1. Among all the solutions of the same equation that are linearly independent of $x_1(\lambda)$ we take a solution $x_2(\lambda)$ of least degree ϵ_2. Obviously $\epsilon_1 \leqslant \epsilon_2$. Continuing this finite process, we obtain a fundamental series of solutions of our equation

$$x_1(\lambda), x_2(\lambda), \ldots, x_{p-r}(\lambda)$$

having the degrees

$$\epsilon_1 \leqslant \epsilon_2 \leqslant \cdots \leqslant \epsilon_{p-r},$$

where $r = \operatorname{rank}(X + \lambda Y)$. In general a fundamental series of solutions is not uniquely determined by the pencil $X + \lambda Y$. However, the set of degrees is. If denote by $c_1 \geqslant \cdots \geqslant c_{p-r}$ the numbers $\epsilon_1, \ldots, \epsilon_{p-r}$ in nonincreasing order, then we call $c_1 \geqslant \cdots \geqslant c_{p-r}$ the column minimal indices of the pencil $A + \lambda B$, for details see [7].

Furthermore, we shall deal with the pencils of the form

$$\begin{bmatrix}\lambda I - A & -B\end{bmatrix} \in \mathbb{F}[\lambda]^{n\times(n+m)} \quad \text{with } A \in \mathbb{F}[\lambda]^{n\times n}. \tag{2}$$

It is easy to see that the number of column minimal indices of (2) coincides with the number of columns of the matrix B (denote them by $c_1 \geqslant \cdots \geqslant c_m$), and the number of nonzero among them is equal to the rank of the matrix B. Hence, in this case by abuse of notation, we shall also call the numbers $c_1 \geqslant \cdots \geqslant c_{\operatorname{rank} B}$ the column minimal indices of (2).

Also, we can introduce Brunovsky indices as $r_i = \sharp\{j | c_j \geqslant i\}$, i.e. partition $r = (r_1, \ldots, r_t)$ is the conjugate (dual) partition of the partition of column minimal indices $(c_1, \ldots, c_m)$.

Definition 1. Let $A, A' \in \mathbb{F}^{n\times n}$, $B, B' \in \mathbb{F}^{n\times l}$. Two rectangular matrices

$$S = \begin{bmatrix}A & B\end{bmatrix}, \qquad S' = \begin{bmatrix}A' & B'\end{bmatrix} \tag{3}$$

are feedback equivalent if there exists a nonsingular matrix

$$P = \begin{bmatrix}N & 0\\ V & T\end{bmatrix},$$

where $N \in \mathbb{F}^{n\times n}$, $V \in \mathbb{F}^{l\times n}$, $T \in \mathbb{F}^{l\times l}$, such that $S' = N^{-1}SP$.

Two matrices of the form (3) are feedback equivalent if and only if the matrix pencils

$$R = \begin{bmatrix}\lambda I - A & -B\end{bmatrix} \quad \text{and} \quad R' = \begin{bmatrix}\lambda I - A' & -B'\end{bmatrix} \tag{4}$$

are strictly equivalent. Therefore, the matrices (3) are feedback equivalent if and only if the pencils (4) have the same invariant factors and the same column minimal indices (frequently we shall call this set of invariants the feedback invariants of the pencil R). The feedback invariants of the matrix S we define as the feedback invariants of the corresponding pencil R. The column minimal indices of the matrix S (and of the corresponding pencil R), coincide (unordered) with the controllability indices of the pair (A, B) (see e.g. [7,13]), and the nonzero among them we shall call the controllability indices of the matrix S.

Definition 2. Let $A, A' \in \mathbb{F}^{n\times n}$, $B, B' \in \mathbb{F}^{n\times l}$, $C, C' \in \mathbb{F}^{n\times m}$. Two matrices

$$L = \begin{bmatrix} A & B & C \end{bmatrix} \quad \text{and} \quad L' = \begin{bmatrix} A' & B' & C' \end{bmatrix}$$

are (n, l)-feedback equivalent if there exists a nonsingular matrix

$$P = \begin{bmatrix} Q & 0 & 0 \\ T & U & 0 \\ V & G & H \end{bmatrix} \in \mathbb{F}^{(n+l+m)\times(n+l+m)},$$

where $Q \in \mathbb{F}^{n\times n}$, $U \in \mathbb{F}^{l\times l}$, such that $L' = Q^{-1}LP$.

Using the previous notation and Lemma 4 from [6], it is easy to obtain the following result (see also [9]):

Lemma 1. *The matrix* $L = \begin{bmatrix} A & B & C \end{bmatrix}$ *is* (n, l)*-feedback equivalent to a unique matrix* $L' = \begin{bmatrix} A' & B' & C' \end{bmatrix}$, *where*

$$\begin{aligned} A' &= C(\alpha_1) \oplus \cdots \oplus C(\alpha_n) C(\lambda^{\nu_1}) \oplus \cdots \oplus C(\lambda^{\nu_\rho}) \\ &\quad \oplus C(\lambda^{\mu_1}) \oplus \cdots \oplus C(\lambda^{\mu_l}) \in \mathbb{F}^{n\times n}, \\ B' &= \left[e^{(n)}_{\nu_1+\cdots+\nu_\rho+\mu_1+p} \cdots e^{(n)}_{\nu_1+\cdots+\nu_\rho+\mu_1+\cdots+\mu_l+p} 0 \right] \in \mathbb{F}^{n\times l}, \\ C' &= \left[e^{(n)}_{\nu_1+p} \quad e^{(n)}_{\nu_1+\nu_2+p} \cdots e^{(n)}_{\nu_1+\cdots+\nu_\rho+p} 0 \right] \in \mathbb{F}^{n\times m}, \end{aligned}$$

$p = \sum d(\alpha_i)$, *for some numbers* $\mu_1 \geqslant \cdots \geqslant \mu_l \geqslant 0$, $\nu_1 \geqslant \cdots \geqslant \nu_\rho > 0$ *and polynomials* $\alpha_1(\lambda) | \cdots | \alpha_n(\lambda)$, $l, \rho, n \geqslant 0$.

Definition 3. The matrix L' is called the canonical form for (n, l)-feedback equivalence of the matrix L.

Remark 1. Note that the union of the nonzero numbers among μ_i and ν_i coincide with the nonzero column minimal indices (unordered) of

$$\begin{bmatrix} \lambda I - A & -B & -C \end{bmatrix}. \tag{5}$$

We shall call μ_i (respectively, ν_i) the minimal indices of the first (respectively, second) kind of the pencil (5) (and of the corresponding matrix L).

Polynomials $\alpha_1(\lambda) | \cdots | \alpha_n(\lambda)$ are the invariant factors of the pencil (5).

Let X be an $m \times n$ complex matrix with $m \leqslant n$. By invariant polynomials of X we assume the invariant factors of the polynomial matrix $\begin{bmatrix} \lambda I_m & 0 \end{bmatrix} - X$.

Definition 4. Let $X = \begin{bmatrix} A & B \end{bmatrix} \in \mathbb{K}^{n\times(n+m)}$ with $A \in \mathbb{K}^{n\times n}$. λ_0 is called an eigenvalue of X if there exists a nonzero vector $x \in \operatorname{Ker} B^{\mathrm{T}}$ such that

$$A^{\mathrm{T}} x = \lambda_0 x.$$

The eigenvalues of the pair $(A, B) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}$ we define as the eigenvalues of the corresponding matrix $\begin{bmatrix} A & B \end{bmatrix}$.

Let $\lambda_1, \dots, \lambda_r$ be distinct eigenvalues of (A, B). The set of eigenvalues (zeros of the $D_n(\lambda)$ – nth determinantal divisor) of (A, B) denote by $\sigma(A, B)$; $m(\lambda_i, (A, B))$ is the algebraic multiplicity of λ_i as an eigenvalue of (A, B); and $s(\lambda_i, (A, B))$, $w(\lambda_i, (A, B))$ are the partitions of λ_i in the Segre, respectively, the Weyr characteristic of (A, B), see [8]. By abuse of notation, we shall adopt all the previous notation for the corresponding matrix $X = \begin{bmatrix} A & B \end{bmatrix}$.

Norm of the matrix X and of the polynomial vector space will be l_1 norm, i.e.

$$\|X\| = \sum_{i,j} |x_{ij}| \quad \text{for } X = [x_{ij}],$$

$$\|b(\lambda)\| = \sum_{i=0}^{n} |b_i| \quad \text{for } b(\lambda) = b_n\lambda^n + \cdots + b_1\lambda + b_0.$$

For polynomial matrices we define $\|M(\lambda)\| = \sum_{i,j} \|m_{ij}(\lambda)\|$, where $M(\lambda) = [m_{ij}(\lambda)]$.

Let $\eta > 0$ be a real number. By $B(\lambda_i, \eta)$ denote the open ball with center at λ_i and radius η. The η neighbourhood of the spectrum of X is the set $\mathscr{V}_\eta(X) = \bigcup_{i=1}^{r} B(\lambda_i, \eta)$ whenever the balls are pairwise disjoint.

Lemma 2 [3,2]. *Let $b(\lambda) \in \mathbb{C}[\lambda]$ be a polynomial of degree n, $b(\lambda) = b_n\lambda^n + \cdots + b_1\lambda + b_0 = b_n(\lambda - \mu_1)\cdots(\lambda - \mu_n)$.*

1. *Given $\epsilon > 0$ there exists $\delta > 0$ such that if $b'(\lambda)$ is a polynomial of degree at most n satisfying $\|b(\lambda) - b'(\lambda)\| < \delta$, then the roots of $b'(\lambda)$ are in $\bigcup_{i=1}^{n} B(\mu_i, \epsilon)$.*
2. *Reciprocally, given $\epsilon > 0$ there exists $\delta > 0$ such that if $\mu_i' \in B(\mu_i, \delta)$, $i = 1, \dots, n$, and $b'(\lambda) = b_n(\lambda - \mu_1')\cdots(\lambda - \mu_n')$ then $\|b(\lambda) - b'(\lambda)\| < \epsilon$.*

Theorem 1 [9]. *Let $A \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{n\times l}$ and $C \in \mathbb{F}^{n\times m}$. Let $\mu_1 \geqslant \cdots \geqslant \mu_l \geqslant 0$ and $\nu_1 \geqslant \cdots \geqslant \nu_\rho > 0$ be the minimal indices of the first and of the second kind, respectively, and let $\alpha_1 | \cdots | \alpha_n$ be the invariant factors of*

$$\begin{bmatrix} \lambda I - A & -B & -C \end{bmatrix}.$$

Let $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_{\bar{\rho}} > 0$ and $\gamma_1 | \cdots | \gamma_{n+l}$ be positive integers and monic polynomials, respectively. There exist matrices $D \in \mathbb{F}^{l\times n}$, $E \in \mathbb{F}^{l\times l}$ and $F \in \mathbb{F}^{l\times m}$ such that the pencil

$$\lambda \begin{bmatrix} I_{n+l} & 0 \end{bmatrix} - G = \begin{bmatrix} \lambda I - A & -B & -C \\ -D & \lambda I - E & -F \end{bmatrix} \tag{6}$$

has $\gamma_1 | \cdots | \gamma_{n+l}$ as invariant factors and $d_1 \geqslant \cdots \geqslant d_{\bar{\rho}}$ as nonzero column minimal indices if and only if the following conditions are valid:

(i) $d_i \geqslant s_i,\ i = 1, \dots, \bar{\rho}$,
(ii) $\rho \leqslant \bar{\rho} \leqslant \min(l + \rho, m)$,
(iii) $\gamma_i | \alpha_{i+\rho-\bar{\rho}} | \gamma_{i+l+\rho-\bar{\rho}},\ i = 1, \dots, n + \bar{\rho} - \rho$,
(iv) $\sum f_i + \sum d(\alpha_i) = \sum d_i + \sum d(\gamma_i)$,

(v) $\sum_{i=1}^{h_q} f_i - \sum_{i=1}^{h_q-q} d_i \leqslant d(\pi_{\rho+l-\bar{\rho}}) - d(\pi_{\rho+l-\bar{\rho}-q})$,
$h_q = \min\{i | d_{i-q+1} < f_i\},\ q = 1, \ldots, \rho + l - \bar{\rho}$,
$\pi_j = \prod_{i=1}^{n+j} \operatorname{lcm}(\alpha_{i-j}, \gamma_{i+\bar{\rho}-\rho}),\ j = 0, \ldots, \rho + l - \bar{\rho}$,

where $f_1 \geqslant \cdots \geqslant f_{\rho+l}$ *is nonincreasing ordering of numbers* $\mu_1 + 1, \ldots, \mu_l + 1,\ \nu_1, \ldots, \nu_\rho$ *and* $s_1 \geqslant \cdots \geqslant s_{\bar{\rho}}$ *is nonincreasing ordering of numbers* $\nu_1, \ldots, \nu_\rho, \mu_l + 1, \ldots, \mu_{l+\rho+1-\bar{\rho}} + 1$.

Theorem 2 [3,1,11]. *Let* $\gamma_1' | \cdots | \gamma_n'$ *be monic polynomials. Let* $M \in \mathbb{C}^{n\times n}$ *be a matrix with* $\gamma_1 | \cdots | \gamma_n$ *as invariant polynomials. In every neighbourhood of M there exists a matrix* M' *such that* $\gamma_1' | \cdots | \gamma_n'$ *are its invariant polynomials if and only if*

$$\gamma_1' \cdots \gamma_i' | \gamma_1 \cdots \gamma_i, \quad i = 1, \ldots, n-1,$$

and

$$\gamma_1' \cdots \gamma_n' = \gamma_1 \cdots \gamma_n.$$

Theorem 3 [8]. *Let* $A \in \mathbb{K}^{n\times n}$ *and* $B \in \mathbb{K}^{n\times m}$. *Let* $\alpha_1 | \cdots | \alpha_n$ *be the invariant factors, and let* $k_1 \geqslant \cdots \geqslant k_m$ *be the column minimal indices of* $\begin{bmatrix}\lambda I - A & -B\end{bmatrix}$, $\sum d(\alpha_i) = p$. *Then there exists a neighbourhood* $\mathscr{V}$ *of* $\begin{bmatrix}A & B\end{bmatrix}$ *such that* $\begin{bmatrix}A' & B'\end{bmatrix} \in \mathscr{V}$ *implies*

$$(k_1', \ldots, k_m', 0, \ldots) \prec (k_1 + t, k_2, \ldots, k_m, 0, \ldots),$$
$$(d(\alpha_n), \ldots, d(\alpha_1), 0, \ldots) \prec (d(\alpha_n') + t, d(\alpha_{n-1}'), \ldots, d(\alpha_1'), 0, \ldots),$$

where $\alpha_1' | \cdots | \alpha_n'$ *are the invariant factors and* $k_1' \geqslant \cdots \geqslant k_m'$ *are the column minimal indices of* $\begin{bmatrix}\lambda I - A' & -B'\end{bmatrix}$, *and* $t = \sum_{i=1}^n d(\alpha_i) - \sum_{i=1}^n d(\alpha_i') \geqslant 0$.

Theorem 4 [8]. *Let* $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times m}$, $\eta > 0$. *Let* $r = (r_1, r_2, \ldots)$ *be the partition of the Brunovsky indices of* (A, B). *Let* a_i *be the partition corresponding to* $\lambda_i \in \sigma(A, B)$ *in the Weyr characteristic of* (A, B), $i = 1, \ldots, p$.

Then there exists a neighbourhood $\mathscr{V}$ *of* $\begin{bmatrix}A & B\end{bmatrix}$ *such that* $\begin{bmatrix}A' & B'\end{bmatrix} \in \mathscr{V}$ *implies*

(i) $\sigma(A', B') \subset \mathscr{V}_\eta(A, B)$,
(ii) *if* $\mu_{i1}, \ldots, \mu_{it_i}$ *are the eigenvalues of* (A', B') *in* $B(\lambda_i, \eta)$ *and* b_{ij} *is the partition corresponding to* μ_{ij} *in the Weyr characteristic of* (A', B'), $j = 1, \ldots, p$, *then*

$$\bigcup_{j=1}^{t_i} b_{ij} \prec\prec a_i, \quad i = 1, \ldots, p,$$

(iii) *if* $r' = (r_1', r_2', \ldots)$ *is the partition of the Brunovsky indices of* (A', B') *then* $r \prec\prec r'$ *and* $r_1' \leqslant m$.

Note that the condition (iii) is equivalent to the condition

$$(k_1', \ldots, k_m', 0, \ldots) \prec (k_1 + t, k_2, \ldots, k_m, 0, \ldots),$$

where r and $(k_1, \ldots, k_m)$, and also, r' and $(k_1', \ldots, k_m')$, are conjugate partitions.

Theorem 5 [8]. *Let* $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $\eta > 0$. *Let* $r = (r_1, r_2, \ldots)$ *be the partition of the Brunovsky indices of* (A, B). *Let* a_i *be the partition corresponding to* $\lambda_i \in \sigma(A, B)$ *in the Weyr* *characteristic of* (A, B), $i = 1, \dots, p$. *Let* $b_{i1}, \dots, b_{it_i}$, $i = 1, \dots, p$, *and* $r' = (r'_1, r'_2, \dots)$ *be given partitions.*

There exists in any neighbourhood of $\begin{bmatrix} A & B \end{bmatrix}$ *a matrix* $\begin{bmatrix} A' & B' \end{bmatrix}$ *such that*

(i) $\sigma(A', B') \subset \mathscr{V}_\eta(A, B)$,
(ii) (A', B') *has* t_i *eigenvalues* $\mu_{i1}, \dots, \mu_{it_i}$ *in* $B(\lambda_i, \eta)$, *and* b_{ij} *is the partition corresponding to* μ_{ij} *in the Weyr characteristic of* (A', B'), $j = 1, \dots, t_i$, $i = 1, \dots, p$,
(iii) r' *is the partition of the Brunovsky indices of* (A', B'),

if and only if the following conditions are satisfied:

$$\bigcup_{j=1}^{t_i} b_{ij} \prec\prec a_i, \quad i = 1, \dots, p,$$
$$r \prec\prec r' \quad \textit{and} \quad r'_1 \leqslant m.$$

3. Technical results

Lemma 3. *Let* P, Q *and* R *be nonzero polynomials in* $\mathbb{F}[\lambda]$, *such that* $\gcd(P, Q) = 1$. *Then there exist solutions* $x, y \in \mathbb{F}[\lambda]$ *of the equation* $Px + Qy = R$, *such that*

$$d(x) \leqslant \max\{d(R) - d(P), d(Q)\} \quad \textit{and} \quad d(y) \leqslant \max\{d(R) - d(Q), d(P)\}.$$

Proof. Suppose that $d(P) \geqslant d(Q)$. By applying the Euclide's algorithm we obtain the sequence of polynomials $q_1, \dots, q_{n+1}$ and $r_0 = Q, r_1, \dots, r_n$, such that $d(r_0) > d(r_1) > \dots > d(r_n) = 0$ and

$$P = Qq_1 + r_1, \quad Q = r_1 q_2 + r_2, \dots, r_{n-2} = r_{n-1} q_n + r_n, \quad r_{n-1} = r_n q_{n+1}. \tag{7}$$

Obviously, $d(q_1) = d(P) - d(Q)$ and $d(q_k) = d(r_{k-2}) - d(r_{k-1})$, $k = 2, \dots, n + 1$. Further, for each $k = 1, \dots, n$ there exist polynomials a_k and b_k such that $r_k = Pa_k + Qb_k$ and

$$d(a_k) = \sum_{i=2}^{k} d(q_i) = d(Q) - d(r_{k-1}), \qquad d(b_k) = \sum_{i=1}^{k} d(q_i) = d(P) - d(r_{k-1}). \tag{8}$$

Indeed, from (7) we have that $r_1 = P - Qq_1$ and $r_2 = Q - r_1 q_2 = -Pq_2 + Q(1 + q_1 q_2)$. Now, by induction, we have that for each $k \geqslant 3$, $r_k = r_{k-2} - r_{k-1} q_k = P(a_{k-2} - a_{k-1} q_k) + Q(b_{k-2} - b_{k-1} q_k)$, which gives (8).

Divide the polynomial R by r_0 with the quotient l_0 and the remainder R_1: $R = r_0 l_0 + R_1$. Obviously, $d(l_0) = d(R) - d(r_0) = d(R) - d(Q)$, and $d(R_1) < d(r_0)$. Now, divide R_1 by r_1: $R_1 = r_1 l_1 + R_2$, and continue the process: $R_k = r_k l_k + R_{k+1}$, $k = 0, \dots, n$, $R_0 = R$, $R_{n+1} = 0$ (since $r_n = 1$). We have that $d(l_k) = d(R_k) - d(r_k)$ and $d(R_k) < d(r_{k-1})$ for $k = 1, \dots, n$, and hence $d(l_k) < d(r_{k-1}) - d(r_k)$. So,

$$R = \sum_{i=0}^{n} r_i l_i = r_0 l_0 + \sum_{i=1}^{n} r_i l_i = Ql_0 + \sum_{i=1}^{n} (Pa_i + Qb_i) l_i = Px + Qy,$$

where $x = \sum_{i=1}^{n} a_i l_i$ and $y = l_0 + \sum_{i=1}^{n} b_i l_i$.

Finally, we have that $d(x) \leqslant \max_{i=1,\ldots,n}\{d(a_i l_i)\} = \max_{i=1,\ldots,n}\{d(a_i) + d(l_i)\} \leqslant \max_{i=1,\ldots,n} \times \{d(Q) - d(r_{i-1}) + d(r_{i-1}) - d(r_i)\} = \max_{i=1,\ldots,n}\{d(Q) - d(r_i)\} = d(Q)$, and also $d(y) \leqslant \max\{d(l_0), \max_{i=1,\ldots,n}\{d(b_i) + d(l_i)\}\} \leqslant \max\{d(R) - d(Q), d(P)\}$.

Completely analogously, in the case $d(P) < d(Q)$, we obtain the existence of polynomials x and y such that $d(x) \leqslant \max\{d(R) - d(P), d(Q)\}$ and $d(y) \leqslant d(P)$, which concludes our proof. □

Let $P_1, \ldots, P_n$ be polynomials in $\mathbb{F}[\lambda]$. Consider the polynomial equation

$$P_1 x_1(\lambda) + \cdots + P_n x_n(\lambda) = 0. \tag{9}$$

Then the set of all n-tuples $\begin{pmatrix} x_1(\lambda) & \cdots & x_n(\lambda)\end{pmatrix}^{\mathrm{T}}$ of the solutions of (9), with the natural addition and multiplication by an arbitrary polynomial, forms a $\mathbb{F}[\lambda]$-module (a submodule of $\mathbb{F}[\lambda]^n$). The following, well-known, Quillen–Suslin theorem is valid over the rings of polynomials in arbitary number of variables.

Theorem 6 [5, Chapter 5, Theorem 1.8]. *Let $Q_1, \ldots, Q_n \in \mathbb{F}[\lambda_1, \ldots, \lambda_k]$ such that $1 \in \langle Q_1, \ldots, Q_n\rangle$. Then the module of the solutions of the equation*

$$Q_1 y_1 + \cdots + Q_n y_n = 0$$

is free.

Corollary 7. *Let $P_1, \ldots, P_n \in \mathbb{F}[\lambda] \setminus \{0\}$. Then the module of the solutions of Eq. (9) is free.*

Proof. Let $P = \gcd(P_1, \ldots, P_n)$ and $P_i' = \frac{P_i}{P}, i = 1, \ldots, n$. Then after cancelling Eq. (9) by the polynomial P, we obtain $P_1' x_1 + \cdots + P_n' x_n = 0$. Finally, since $\gcd(P_1', \ldots, P_n') = 1$, we have that $1 \in \langle P_1', \ldots, P_n'\rangle$, which finishes our proof. □

Lemma 4. *Let $P_1, \ldots, P_n \in \mathbb{F}[\lambda] \setminus \{0\}$. Then there are at most $n - 1$ linearly independent solutions of (9).*

Proof. By Corollary 7, we have that the module of the solutions of (9), M, is free and its rank is at most n. Suppose that there are n linearly independent solutions $e_1, \ldots, e_n$. Then for each $k = 1, \ldots, n$ we have that there exist polynomials $z_k \neq 0, \alpha_k^1, \ldots, \alpha_k^n$, such that

$$\alpha_k^1 e_1 + \cdots + \alpha_k^n e_n = (\underbrace{0, \ldots, 0}_{k-1}, z_k, 0, \ldots, 0)^{\mathrm{T}} = f_k.$$

However, this means that f_k belongs to M for every $k = 1, \ldots, n$, and hence that it is the solutions of (9). Thus, $P_k z_k = 0$ which implies $P_k = 0$, which is a contradiction. □

Lemma 5. *Let $n > 0$. Let $P' | P_1 | P_2 | \cdots | P_n$ be polynomials from $\mathbb{F}[\lambda]$. Let $m_{n-1} \geqslant \cdots \geqslant m_1 = d(P') \geqslant 0$ be integers, $m_i \leqslant d(P_i), i = 1, \ldots, n-1$. There exist polynomials $X_i, d(X_i) \leqslant d(P_i), i = 2, \ldots, n$, with arbitrary small coefficients, such that*

$$d(\gcd(P_{n-1} + X_{n-1}, P_n + X_n)) = m_{n-1}$$
$$d(\gcd(P_{n-2} + X_{n-2}, P_{n-1} + X_{n-1}, P_n + X_n)) = m_{n-2}$$
$$\cdots$$

$$d(\gcd(P_2 + X_2, \ldots, P_{n-2} + X_{n-2}, P_{n-1} + X_{n-1}, P_n + X_n)) = m_2$$
$$\gcd(P_1, P_2 + X_2, \ldots, P_{n-2} + X_{n-2}, P_{n-1} + X_{n-1}, P_n + X_n) = P'.$$

Proof. Write the polynomials P_i, $i = 1, \ldots, n$ as

$$P_i = P' \frac{P_1}{P'} \frac{P_2}{P_1} \frac{P_3}{P_2} \frac{P_4}{P_3} \cdots \frac{P_i}{P_{i-1}}.$$

Let $a^i_1, \ldots, a^i_{(d(P_i)-d(P_{i-1}))}$ be the zeros of the polynomial $\frac{P_i}{P_{i-1}}$, $i = 1, \ldots, n$, $P_0 = P'$. Then

$$S = \left\{a^1_1, \ldots, a^1_{d(P_1)-d(P')}, a^2_1, \ldots, a^2_{d(P_2)-d(P_1)}, \ldots, a^n_1, \ldots, a^n_{d(P_n)-d(P_{n-1})}\right\}$$

is the set of all zeros of the polynomial $\frac{P_n}{P'}$.

Denote the elements of S in this order by b_i, $i = 1, \ldots, d(P_n) - d(P')$. Let $n_i = m_i - d(P')$, $i = 2, \ldots, n-1$. Let ϵ_i, $i = 1, \ldots, d(P_n) - d(P')$, be arbitrary small positive numbers. Then from Lemma 2 we conclude that there exist polynomials X_i with arbitrary small coefficients, such that $d(X_i) \leqslant d(P_i)$ and

$$P_i + X_i = P'(\lambda - b_1 - \epsilon_1) \cdots (\lambda - b_{n_i} - \epsilon_{n_i})(\lambda - b_{n_i+1}) \cdots (\lambda - b_{d(P_i)-d(P')})$$

for $i = 2, \ldots, n-1$, and

$$P_n + X_n = P'(\lambda - b_1 - \epsilon_1) \cdots (\lambda - b_{d(P_n)-d(P')} - \epsilon_{d(P_n)-d(P')}).$$

Obviously, they satisfy

$$d(\gcd(P_i + X_i, \ldots, P_{n-1} + X_{n-1}, P_n + X_n)) = m_i$$

for $i = 2, \ldots, n-1$, and

$$\gcd(P_1, P_2 + X_2, \ldots, P_{n-2} + X_{n-2}, P_{n-1} + X_{n-1}, P_n + X_n) = P',$$

as wanted. □

Lemma 6. *Let* $A \in \mathbb{F}^{n\times n}$, $F \in \mathbb{F}^{l\times l}$ *and*

$$M = \begin{bmatrix} A & B & C \\ E & F & G \end{bmatrix} \in \mathbb{F}^{(n+l)\times(n+l+m)}.$$

Let $P \in \mathbb{F}^{n\times n}$, $L \in \mathbb{F}^{l\times l}$ *and* $N \in \mathbb{F}^{m\times m}$ *be invertible matrices, and* $Q \in \mathbb{F}^{l\times n}$, $R \in \mathbb{F}^{m\times n}$ *and* $K \in \mathbb{F}^{m\times l}$ *be such that*

$$M = \begin{bmatrix} P^{-1} & 0 \\ -L^{-1}QP^{-1} & L^{-1} \end{bmatrix} \begin{bmatrix} \frac{A_c}{E} & \frac{B_c}{F} & \frac{C_c}{G} \end{bmatrix} \begin{bmatrix} P & 0 & 0 \\ Q & L & 0 \\ R & K & N \end{bmatrix} = P_1 \begin{bmatrix} \frac{A_c}{E} & \frac{B_c}{F} & \frac{C_c}{G} \end{bmatrix} P_2,$$

where $\begin{bmatrix} A_c & B_c & C_c \end{bmatrix}$ *is the* (n, l)*-feedback canonical form of the matrix* $\begin{bmatrix} A & B & C \end{bmatrix}$. *Let*

$$\overline{M} = \begin{bmatrix} \frac{A_c}{E} & \frac{B_c}{F} & \frac{C_c}{G} \end{bmatrix}.$$

Then in every neighbourhood of M *there exists a matrix*

$$M' = \begin{bmatrix} A & B & C \\ E' & F' & G' \end{bmatrix}$$

such that $\gamma_1'|\cdots|\gamma_n'$ and $d_1' \geqslant \cdots \geqslant d_m'$ are its feedback invariants if and only if in every neighbourhood of $\overline{M}$ there exist a matrix

$$\overline{M}' = \begin{bmatrix} A_c & B_c & C_c \\ \overline{E}' & \overline{F}' & \overline{G}' \end{bmatrix}$$

such that $\gamma_1'|\cdots|\gamma_n'$ and $d_1' \geqslant \cdots \geqslant d_m'$ are its feedback invariants.

Proof. Let $\epsilon > 0$ and define $\epsilon' > 0$ as a positive real number satisfying

$$\epsilon' < \frac{\epsilon}{\|P_1\|\|P_2\|}.$$

If $X \in \mathbb{F}^{l\times n}$, $Y \in \mathbb{F}^{l\times l}$ and $Z \in \mathbb{F}^{l\times m}$ verify

$$\begin{bmatrix} A_c & B_c & C_c \\ \overline{E}+X & \overline{F}+Y & \overline{G}+Z \end{bmatrix} = \overline{M}'$$

with $\|X+Y+Z\| < \epsilon'$, and $\overline{M}'$ has prescribed feedback invariants, then

$$M' = P_1\overline{M}'P_2$$

will have the same feedback invariants as $\overline{M}'$ and we have that

$$\|M - M'\| \leqslant \|P_1\|\|\overline{M} - \overline{M}'\|\|P_2\| \leqslant \|P_1\|\|\epsilon'\|\|P_2\| \leqslant \epsilon,$$

as wanted. The converse is proved analogously. □

The following result is not hard to prove:

Lemma 7. *Let $A \in \mathbb{F}^{n\times n}, B \in \mathbb{F}^{m\times m}, C \in \mathbb{F}^{m\times s}, X \in \mathbb{F}^{m\times n}$ be such that the pencil $\begin{bmatrix}\lambda I - B & -C\end{bmatrix} \in \mathbb{F}[\lambda]^{m\times(m+s)}$ has all invariant factors equal to 1. Then the following two matrices are feedback equivalent*:

$$\left[\begin{array}{c|cc} A & 0 & 0 \\ \hline X & B & C \end{array}\right] \in \mathbb{F}^{(n+m)\times(n+m+s)}$$

and

$$\left[\begin{array}{c|cc} A & 0 & 0 \\ \hline 0 & B & C \end{array}\right] \in \mathbb{F}^{(n+m)\times(n+m+s)}.$$

4. Main result

Theorem 8. *Let $A \in \mathbb{C}^{n\times n}, C \in \mathbb{C}^{n\times m}, b^{\mathrm{T}} \in \mathbb{C}^{n\times 1}, a, a' \in \mathbb{C}^{1\times n}, c, c' \in \mathbb{C}^{1\times m}$ and $x, x' \in \mathbb{C}^{1\times 1}$. Let μ_1 be the minimal index of the first kind and $\nu_1 \geqslant \cdots \geqslant \nu_\rho > 0$ be the minimal indices of the second kind, and let $\alpha_1|\cdots|\alpha_n$ be the invariant factors of*

$$\begin{bmatrix}\lambda I - A & -b^{\mathrm{T}} & -C\end{bmatrix}.$$

Let a_i be the partition corresponding to $\lambda_i \in \sigma(M)$, $i = 1, \ldots, p$, in the Weyr characteristic of M, where

$$M = \left[\begin{array}{c|c|c} A & b^{\mathrm{T}} & C \\ \hline a & x & c \end{array}\right] \quad \left(\textit{respectively},\ M' = \left[\begin{array}{c|c|c} A & b^{\mathrm{T}} & C \\ \hline a' & x' & c' \end{array}\right]\right).$$

Let $\gamma_1|\cdots|\gamma_{n+1}$ and $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_{\bar\rho} > 0$ (respectively, $\gamma'_1|\cdots|\gamma'_{n+1}$ and $d'_1 \geqslant d'_2 \geqslant \cdots \geqslant d'_{\bar\rho'} > 0$) be the feedback invariants of the matrix M (respectively, M').

Then for arbitrary small $\eta > 0$, there exists $\epsilon > 0$ such that if

$$\|M - M'\| = \left\|\left[\begin{array}{c|c|c} A & b^{\mathrm{T}} & C \\ \hline a & x & c \end{array}\right] - \left[\begin{array}{c|c|c} A & b^{\mathrm{T}} & C \\ \hline a' & x' & c' \end{array}\right]\right\| < \epsilon,$$

then

(i) $d'_i \geqslant s_i$, $i = 1, \ldots, \bar\rho'$,
(ii) $\rho \leqslant \bar\rho' \leqslant \min(1 + \rho, m)$,
(iii) $\gamma'_i|\alpha_{i-\bar\rho'+\rho}|\gamma'_{i+1+\rho-\bar\rho'}$, $i = 1, \ldots, n$,
(iv) $\sum f_i + \sum d(\alpha_i) = \sum d'_i + \sum d(\gamma'_i)$,
(v) $\sum_{i=1}^{h'_q} f_j - \sum_{i=1}^{h'_q-q} d'_j \leqslant d(\pi'_{\rho+1-\bar\rho'}) - d(\pi'_{\rho+1-\bar\rho'-q})$,
$h'_q = \min\{i | d_{i-q+1} < f_i\}$, $q = 1, \ldots, \rho + 1 - \bar\rho'$,
$\pi'_j = \prod_{i=1}^{n+j} \mathrm{lcm}(\alpha_{i-j}, \gamma'_{i+\bar\rho'-\rho})$, $j = 0, \ldots, \rho + 1 - \bar\rho'$,
(vi) $\sigma(M') \subset \mathscr{V}_\eta(M)$,
(vii) *if $\mu_{i1}, \ldots, \mu_{it_i}$ are the eigenvalues of M' in $B(\lambda_i, \eta)$ and b_{ij} is the partition corresponding to μ_{ij} in the Weyr characteristic of M', $j = 1, \ldots, p$, then*

$$\bigcup_{j=1}^{t_i} b_{ij} \prec\prec a_i, \quad i = 1, \ldots, p,$$

(viii) *if $r' = (r'_1, r'_2, \ldots)$ is the partition of the Brunovsky indices of M', then $r \prec\prec r'$ and $r'_1 \leqslant m$*

where $f_1 \geqslant \cdots \geqslant f_{\rho+1}$ is nonincreasing ordering of numbers $\mu_1 + 1$, $\nu_1, \ldots, \nu_\rho$ and $s_i = \nu_i$, $i = 1, \ldots, \bar\rho'$ if $\bar\rho' = \rho$, or $s_i = f_i$, $i = 1, \ldots, \bar\rho'$ if $\bar\rho' = \rho + 1$ and r is the dual partition of the partition $(d_1, \ldots, d_{\bar\rho})$.

Proof. This theorem is a direct consequence of the previous ones (see Theorems 1 and 4). □

Lemma 8. *Under the same notation as in the previous theorem we can obtain the necessity of the following conditions as well*:

$$(d'_1, \ldots, d'_{\bar\rho'}, 0, \ldots) \prec (d_1 + t, d_2, \ldots, d_{\bar\rho}, 0, \ldots)$$
$$(d(\gamma_n), \ldots, d(\gamma_1), 0, \ldots) \prec (d(\gamma'_n) + t, d(\gamma'_{n-1}), \ldots, d(\gamma'_1), 0, \ldots),$$

where $t = \sum d(\gamma_i) - \sum d(\gamma'_i) \geqslant 0$.

Proof. Trivially follows from Theorem 3. □

Let $\rho, \nu_1 \geqslant \cdots \geqslant \nu_\rho \geqslant \mu_1 + 1$ and $d_1 \geqslant \cdots \geqslant d_\rho$ be positive integers and let $\alpha_1|\cdots|\alpha_n$ and $\gamma_1|\cdots|\gamma_{n+1}$ be monic polynomials such that they satisfy the following conditions:

(i) $d_i \geqslant \nu_i$, $i = 1, \ldots, \rho$,
(ii) $\gamma_i|\alpha_i|\gamma_{i+1}$, $i = 1, \ldots, n$,
(iii) $\sum f_i + \sum d(\alpha_i) = \sum d_i + \sum d(\gamma_i)$.

Let $P = \frac{\prod \gamma_i}{\prod \alpha_i}$ and let

$$k_1 = \sum \nu_i + \mu_1 + 1 - \sum d_i (= d(P)), \tag{10}$$

$$k_i = \sum_{j=1}^{i-1} (d_j - \nu_j) + k_1, \quad i = 2, \ldots, \rho. \tag{11}$$

Then (like in [9]) the following matrix

$$L = \left[\begin{array}{c|c|c|c|c|c}
C(\lambda^{\nu_1}) & & & & & e_1 \\ \hline
 & C(\lambda^{\nu_2}) & & & & e_2 \\ \hline
 & & \ddots & & & \ddots \\ \hline
 & & & C(\lambda^{\nu_\rho}) & & e_\rho \\ \hline
 & & & & C(\lambda^{\mu_1}) \quad e_0 & \\ \hline
t_1 & t_2 & \cdots & t_\rho & t_{\rho+1} &
\end{array}\right] \tag{12}$$

has P as the only nontrivial invariant factor and $d_1, \ldots, d_\rho$ as controllability indices. Here (and further on) $e_i = e_{\nu_i}^{(\nu_i)}$, $i = 1, \ldots, \rho$, $e_0 = e_{\mu_1}^{(\mu_1)}$,

$$t_i = \left[\underbrace{0 \quad (-1)^{\rho-i+1}x_1 \quad \cdots \quad (-1)^{\rho-i+1}x_{k_1}}_{k_i} \;\middle|\; \underbrace{(-1)^{\rho-i} \quad 0}_{\nu_i - k_i}\right] \in \mathbb{F}^{1\times\nu_i},$$

$i = 1, \ldots, \rho$, and

$$t_{\rho+1} = \begin{bmatrix} 0 & x_1 & \cdots & x_{k_1} \end{bmatrix} \in \mathbb{F}^{1\times(\mu_1+1)},$$

where $x_1, \ldots, x_{k_1}$ are such that

$$P = \lambda^{k_1} - x_{k_1}\lambda^{k_1-1} + \cdots - x_2\lambda - x_1.$$

By using the result from Lemma 6, we can consider the matrix (1) in the following feedback equivalent form:

$$M = \left[\begin{array}{c|c|c|c|c|c|c}
N & & & & & & \\ \hline
 & C(\lambda^{\nu_1}) & & & & & e_1 \\ \hline
 & & C(\lambda^{\nu_2}) & & & & e_2 \\ \hline
 & & & \ddots & & & \ddots \\ \hline
 & & & & C(\lambda^{\nu_\rho}) & & e_\rho \\ \hline
 & & & & & C(\lambda^{\mu_1}) \quad e_0 & \\ \hline
w_0 & w_1 & w_2 & \cdots & w_\rho & w_{\rho+1} &
\end{array}\right]. \tag{13}$$

Here $N = \operatorname{diag}(C(\alpha_1), \ldots, C(\alpha_n))$ and $w_i = \begin{bmatrix} w_i^1 & \cdots & w_i^{j_i} \end{bmatrix}$, for some scalars $w_i^1, \ldots, w_i^{j_i}$, $i = 0, \ldots, \rho+1$, $j_0 = p$, $j_k = \nu_k$, $k = 1, \ldots, \rho$, $j_{\rho+1} = \mu_1 + 1$.

Let S be the submatrix of M formed by its last $n + \rho + 1 - p$ columns and its last $n + 1 - p$ rows, $p = \sum d(\alpha_i)$. Using the previous notation, it is not hard to conclude that the matrices L and S have the same controllability indices ($d_1 \geqslant \cdots \geqslant d_\rho$) and the same polynomial $P \left(= \frac{\prod \gamma_i}{\prod \alpha_i}\right)$ as the only nontrivial invariant polynomial. Thus, they are feedback equivalent, i.e. they are both feedback equivalent to the following matrix:

$$\left[\begin{array}{c|ccc|ccc} C(P) & & & & & & \\ \hline & C(\lambda^{d_1}) & & & e^{(d_1)}_{d_1} & & \\ & & \ddots & & & \ddots & \\ & & & C(\lambda^{d_\rho}) & & & e^{(d_\rho)}_{d_\rho} \end{array}\right],$$

and there exist invertible matrices $T \in \mathbb{F}^{(n+1-p)\times(n+1-p)}$ and $Q \in \mathbb{F}^{\rho\times\rho}$ and a matrix $R_3 \in \mathbb{F}^{\rho\times(n+1-p)}$ such that

$$TS\begin{bmatrix} T^{-1} & 0 \\ R_3 & Q \end{bmatrix} = L.$$

Furthermore, from the form of the matrices M and S, by applying Lemma 7, we have that there exist matrices $R_1 \in \mathbb{F}^{(n+1-p)\times p}$ and $R_2 \in \mathbb{F}^{\rho\times p}$ such that

$$\begin{bmatrix} I & 0 \\ -TR_1 & T \end{bmatrix} M \begin{bmatrix} I & 0 & 0 \\ R_1 & T^{-1} & 0 \\ R_2 & R_3 & Q \end{bmatrix} = \begin{bmatrix} N & 0 \\ X & L \end{bmatrix}. \tag{14}$$

Here the matrix $X \in \mathbb{F}^{(n+1-p)\times p}$ is of the form $\begin{bmatrix} 0 \\ t_0 \end{bmatrix}$, where $t_0 = \begin{bmatrix} y_1 & \cdots & y_p \end{bmatrix}$, for some scalars $y_1, \ldots, y_p$.

Define the matrix M' as

$$M' = \left[\begin{array}{c|c|c|c|c|c|ccc} N & & & & & & & & \\ \hline & C(\lambda^{\nu_1}) & & & & & e_1 & & \\ \hline & & C(\lambda^{\nu_2}) & & & & & e_2 & \\ \hline & & & \ddots & & & & & \ddots \\ \hline & & & & C(\lambda^{\nu_\rho}) & & & & e_\rho \\ \hline & & & & & C(\lambda^{\mu_1}) \quad e_0 & & & \\ \hline t_0 & t_1 & t_2 & \cdots & t_\rho & t_{\rho+1} & & & \end{array}\right], \tag{15}$$

i.e., as the right-hand side of Eq. (14).

Now, using the previous notation, we can give the following lemma:

Lemma 9. *Let $\gamma_1|\cdots|\gamma_{n+1}$ and $\gamma'_1|\cdots|\gamma'_{n+1}$ be monic polynomials. Let $d_1 \geqslant \cdots \geqslant d_\rho$ and $d'_1 \geqslant \cdots \geqslant d'_\rho$ be positive integers. Let*

$$M = \left[\begin{array}{c|c|c|c|c|c|ccc} N & & & & & & & & \\ \hline & C(\lambda^{\nu_1}) & & & & & e_1 & & \\ \hline & & C(\lambda^{\nu_2}) & & & & & e_2 & \\ \hline & & & \ddots & & & & & \ddots \\ \hline & & & & C(\lambda^{\nu_\rho}) & & & & e_\rho \\ \hline & & & & & C(\lambda^{\mu_1}) \quad e_0 & & & \\ \hline w_0 & w_1 & w_2 & \cdots & w_\rho & w_{\rho+1} & & & \end{array}\right]$$

and

$$M' = \left[\begin{array}{c|c|c|c|c|c|c}
N & & & & & & \\ \hline
 & C(\lambda^{\nu_1}) & & & & & e_1 \\ \hline
 & & C(\lambda^{\nu_2}) & & & & e_2 \\ \hline
 & & & \ddots & & & \ddots \\ \hline
 & & & & C(\lambda^{\nu_\rho}) & & e_\rho \\ \hline
 & & & & & C(\lambda^{\mu_1}) \quad e_0 & \\ \hline
t_0 & t_1 & t_2 & \cdots & t_\rho & t_{\rho+1} &
\end{array}\right]$$

be the matrices (13) *and* (15), *respectively, both with* $\gamma_1 | \cdots | \gamma_{n+1}$ *and* $d_1 \geqslant \cdots \geqslant d_\rho$ *as feedback invariants. Then in every neighbourhood of* M *there exists a matrix*

$$\overline{M} = \left[\begin{array}{c|c|c|c|c|c|c}
N & & & & & & \\ \hline
 & C(\lambda^{\nu_1}) & & & & & e_1 \\ \hline
 & & C(\lambda^{\nu_2}) & & & & e_2 \\ \hline
 & & & \ddots & & & \ddots \\ \hline
 & & & & C(\lambda^{\nu_\rho}) & & e_\rho \\ \hline
 & & & & & C(\lambda^{\mu_1}) \quad e_0 & \\ \hline
w_0 + \eta_0 & w_1 + \eta_1 & w_2 + \eta_2 & \cdots & w_\rho + \eta_\rho & w_{\rho+1} + \eta_{\rho+1} &
\end{array}\right]$$

such that $\gamma'_1 | \cdots | \gamma'_{n+1}$ *and* $d'_1 \geqslant \cdots \geqslant d'_\rho$ *are its feedback invariants if and only if in every neighbourhood of* M' *there exists a matrix*

$$\overline{M}' = \left[\begin{array}{c|c|c|c|c|c|c}
N & & & & & & \\ \hline
 & C(\lambda^{\nu_1}) & & & & & e_1 \\ \hline
 & & C(\lambda^{\nu_2}) & & & & e_2 \\ \hline
 & & & \ddots & & & \ddots \\ \hline
 & & & & C(\lambda^{\nu_\rho}) & & e_\rho \\ \hline
 & & & & & C(\lambda^{\mu_1}) \quad e_0 & \\ \hline
t_0 + \epsilon_0 & t_1 + \epsilon_1 & t_2 + \epsilon_2 & \cdots & t_\rho + \epsilon_\rho & t_{\rho+1} + \epsilon_{\rho+1} &
\end{array}\right]$$

such that $\gamma'_1 | \cdots | \gamma'_{n+1}$ *and* $d'_1 \geqslant \cdots \geqslant d'_\rho$ *are its feedback invariants.*

Proof. Suppose that there exist arbitrary small scalars η_i, $i = 0, \ldots, \rho + 1$, such that the matrix $\overline{M}$ has $\gamma'_1 | \cdots | \gamma'_{n+1}$ and $d'_1 \geqslant \cdots \geqslant d'_\rho$ as feedback invariants. Using the previous notation, define the matrix Y by

$$Y = \begin{bmatrix} I & 0 \\ -TR_1 & T \end{bmatrix} \overline{M} \begin{bmatrix} I & 0 & 0 \\ R_1 & T^{-1} & 0 \\ R_2 & R_3 & Q \end{bmatrix}.$$

The matrix Y has $\gamma'_1 | \cdots | \gamma'_{n+1}$ and $d'_1 \geqslant \cdots \geqslant d'_\rho$ as feedback invariants, and it is in the small neighbourhood of M'. However, the matrix Y is obtained from the matrix M' by small perturbations of the last $n + 1 - p$ rows. By applying Lemma 7, we can transform the matrix Y in the form $\overline{M}'$, i.e. there exist invertible matrices $\widetilde{P} \in \mathbb{F}^{(n+1)\times(n+1)}$ and $\widetilde{Q} \in \mathbb{F}^{(n+1+\rho)\times(n+1+\rho)}$, such that $\overline{M}' = \widetilde{P} Y \widetilde{Q}$, and since Y is obtained by small perturbations of M', we have that $\widetilde{P} = I + \widetilde{P}_\epsilon$, $\widetilde{Q} = I + \widetilde{Q}_\epsilon$, where $\|\widetilde{P}_\epsilon\|$ and $\|\widetilde{Q}_\epsilon\|$ are small. Hence, the matrix $\overline{M}'$ is feedback equivalent to Y and hence to the matrix $\overline{M}$, and it is in the small neighbourhood of the matrix M', as wanted.

The converse is proved analogously. □

Our aim is to make small perturbations on the last row of M', in order to obtain $\gamma_1'|\cdots|\gamma_{n+1}'$ and $d_1' \geqslant \cdots \geqslant d_\rho'$ as its feedback invariants.

Let

$$\overline{L}' = \left[\begin{array}{c|c|c|c|c|c} C(\lambda^{\nu_1}) & & & & & e_1 \\ \hline & C(\lambda^{\nu_2}) & & & & e_2 \\ \hline & & \ddots & & & \ddots \\ \hline & & & C(\lambda^{\nu_\rho}) & & e_\rho \\ \hline & & & & C(\lambda^{\mu_1}) \quad e_0 & \\ \hline t_1+\epsilon_1 & t_2+\epsilon_2 & \cdots & t_\rho+\epsilon_\rho & t_{\rho+1}+\epsilon_{\rho+1} & \end{array}\right],$$

and

$$\overline{M}' = \begin{bmatrix} N & 0 \\ \overline{Y} & \overline{L} \end{bmatrix},$$

where $\overline{Y}$ is of the form $\begin{bmatrix} 0 \\ t_0+\epsilon_0 \end{bmatrix}$, where $\epsilon_0 = \begin{bmatrix}\epsilon_0^1 & \cdots & \epsilon_0^p\end{bmatrix}$ and $\epsilon_i = \begin{bmatrix}\epsilon_i^1 & \cdots & \epsilon_i^{k_i} & \epsilon_i^{k_i+1} & 0 & \cdots & 0\end{bmatrix}$, $i = 1, \ldots, \rho+1$, $k_{\rho+1} = \mu_1$ (for definition of $k_1, \ldots, k_\rho$ see (10) and (11)). Here ϵ_i^j, $i = 1, \ldots, \rho+1$, $j = 1, \ldots, k_i+1$ and ϵ_0^j, $j = 1, \ldots, p$ are some small numbers. Now, by using the definition of column minimal indices, we have that they are equal to the minimal degrees of linearly independent solutions of the following system:

$$\left(\lambda\begin{bmatrix} I_{n+1} & 0\end{bmatrix} - \overline{M}'\right)\begin{bmatrix} a_1 \\ \vdots \\ a_p \\ \bar{x}_1 \\ \vdots \\ \bar{x}_{n+\rho-p} \\ \bar{x}_{n+\rho-p+1}\end{bmatrix} = 0. \tag{16}$$

Because of the form of the matrix $\overline{M}'$, we have that Eq. (16) is equivalent to

$$a_i = 0, \quad i = 1, \ldots, p$$

and

$$\left(\lambda\begin{bmatrix} I_{n+1-p} & 0\end{bmatrix} - \overline{L}'\right)\begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_{n+\rho-p} \\ \bar{x}_{n+\rho-p+1}\end{bmatrix} = 0. \tag{17}$$

Finally, if $X_i = (-1)^{(\rho-i)}(\epsilon_i^{k_i+1}\lambda^{k_i} + \epsilon_i^{k_i}\lambda^{k_i-1} + \cdots + \epsilon_i^2\lambda + \epsilon_i^1)$, $i = 1, \ldots, \rho+1$, $k_{\rho+1} = \mu_1$, then Eq. (17) becomes

$$P_1'z_1 + P_2'z_2 + \cdots + P_\rho' z_\rho + P_{\rho+1}' z_{\rho+1} = 0. \tag{18}$$

Here $z_i = (-1)^{\rho+1-i}\bar{x}_{\sum_{j=1}^{i-1}\nu_j+1}$, $i = 1, \ldots, \rho+1$, and $P_i' = P\lambda^{k_i-k_1} + X_i$, $i = 1, \ldots, \rho$ and $P_{\rho+1}' = P\lambda^{\mu_1+1-k_1} + X_{\rho+1}$. The corresponding degree of the solution of (17) is equal to the

$$\max\{d(z_1)+\nu_1, d(z_2)+\nu_2, \ldots, d(z_\rho)+\nu_\rho, d(z_{\rho+1})+\mu_1+1\}. \tag{19}$$

In fact, each z_i, $i = 1, \ldots, \rho$, has “weight” ν_i, and $z_{\rho+1}$ has “weight” $\mu_1 + 1$, and we will call (19) the *weighted degree* of the solution of Eq. (18). Thus the weighted degree of the solution of Eq. (18) is equal to the degree of the corresponding solution of Eq. (17).

Now we can pass to the main results.

Theorem 9. *Let $\gamma_1'|\cdots|\gamma_{n+1}'$ be monic polynomials and let $d_1' \geqslant \cdots \geqslant d_{\bar\rho'}'$ be positive integers. Let M be the matrix (1) and let it has $\gamma_1|\cdots|\gamma_{n+1}$ and $d_1 \geqslant \cdots \geqslant d_{\bar\rho}$ as feedback invariants. Let $\alpha_1|\cdots|\alpha_n$, μ_1, and $\nu_1 \geqslant \cdots \geqslant \nu_\rho$ be the invariant polynomials, the minimal index of the first kind and the minimal indices of the second kind, respectively, of the matrix $\begin{bmatrix} A & b^{\mathrm T} & C\end{bmatrix}$. Then in every neighbourhood of M there exists a matrix*

$$M' = \begin{bmatrix} A & b^{\mathrm T} & C \\ a' & x' & c' \end{bmatrix}$$

with $\gamma_1'|\cdots|\gamma_{n+1}'$ and $d_1' \geqslant \cdots \geqslant d_{\bar\rho'}'$ as its feedback invariants, if and only if one of the following groups (I)–(III) *of conditions holds*:

(I)
- (o) $\bar\rho = \bar\rho' = \rho$,
- (i) $\gamma_i'|\alpha_i|\gamma_{i+1}'$, $i = 1, \ldots, n$,
- (ii) $\gamma_1'\cdots\gamma_i'|\gamma_1\cdots\gamma_i$, $i = 1, \ldots, n+1$,
- (iii) $(d_1', \ldots, d_\rho') \prec (d_1 + t, \ldots, d_\rho)$, $t = \sum d_i' - \sum d_i \geqslant 0$,
- (iv) $d_i' \geqslant \nu_i$, $i = 1, \ldots, \rho$,
- (v) $f_{i+1} = d_i'$, $h_1 \leqslant i \leqslant \rho$,
  $h_1 = \min\{i | d_i' < f_i\}$,
- (vi) $\sum_{i=1}^{\rho} d_i' + \sum_{i=1}^{n+1} d(\gamma_i') = n + 1$.

(II)
- (o) $\bar\rho = \bar\rho' = \rho + 1$,
- (i) $\gamma_i = \gamma_i' = \alpha_{i-1}$, $i = 1, \ldots, n+1$, $\alpha_0 = 1$,
- (ii) $d_i' = d_i = f_i$, $i = 1, \ldots, \rho + 1$.

(III)
- (o) $\bar\rho' = \rho + 1 > \bar\rho = \rho$,
- (i) $\gamma_i' = \alpha_{i-1}$, $i = 1, \ldots, n+1$, $\alpha_0 = 1$,
- (ii) $d_i' = f_i$, $i = 1, \ldots, \rho + 1$,
- (iii) $m \geqslant \rho + 1$,

where $f_1 \geqslant \cdots \geqslant f_{\rho+1}$ are the same as in Theorem 1.

Proof. *Necessity*: From Theorem 1 it is easy to conclude that in general we have the following three cases:

1. $\bar\rho = \bar\rho' = \rho$,
2. $\bar\rho = \bar\rho' = \rho + 1$,
3. $\bar\rho' = \rho + 1 > \bar\rho = \rho$.

The case 2 is possible only if our two matrices are feedback equivalent, thus

- (i) $\gamma_i = \gamma_i' = \alpha_{i-1}$, $i = 1, \ldots, n+1$, $\alpha_0 = 1$,
- (ii) $d_i' = d_i = f_i$, $i = 1, \ldots, \rho + 1$,

which are exactly the conditions (II).

The case 3 implies

(i) $\gamma_i' = \alpha_{i-1}, i = 1, \ldots, n+1,\ \alpha_0 = 1,$
(ii) $d_i' = f_i, i = 1, \ldots, \rho + 1.$

As we can easily conclude that $m \geqslant \rho + 1$, we obtain the condition (III).

So suppose that we have the case 1. Then, the necessity of the conditions (I) follows directly by unifying the results from Theorems 1 and 4. The only condition that we are left to prove is the condition (ii) from (I), i.e.

$$\gamma_1' \cdots \gamma_i' | \gamma_1 \cdots \gamma_i, \quad i = 1, \ldots, n+1. \tag{20}$$

From Theorem 4, we have that

$$\sigma(M') \subset \mathscr{V}_\eta(M)$$

for all η neighbourhoods of $\sigma(M)$. Thus we can conclude that $\sigma(M') \subset \sigma(M)$. So, from the condition (vii) in Theorem 8, we easily obtain the wanted result. $\square$

Remark 2. If a and b are partitions, then

(i) $\overline{a \cup b} = \bar{a} + \bar{b}$,
(ii) $a \prec b \Leftrightarrow \bar{b} \prec \bar{a}$.

Sufficiency: By Lemma 6 we can assume that $\begin{bmatrix} A & b^{\mathrm{T}} & C \end{bmatrix}$ is in its $(n, 1)$-feedback canonical form, i.e. we can assume that the matrix M is in the form (13).

The case (II) is trivial. In the case (III), let $\epsilon > 0$ be arbitrary small. Then adding ϵ to the zero at the position $(n, n + \rho + 1)$ of the matrix (13), we obtain the wanted result.

In the case (I), first suppose that $\nu_\rho \geqslant \mu_1 + 1$. By applying Lemmas 6 and 9 we can replace the matrix M by the matrix M' in the form (15). We will prove that we can make small changes in the last row of M' in order to obtain the matrix whose feedback invariants are $\gamma_1' | \cdots | \gamma_{n+1}'$ and $d_1' \geqslant \cdots \geqslant d_\rho'$.

From the definition of $k_i, i = 1, \ldots, \rho$, we have

$$\begin{aligned} d_1 &= \nu_1 - k_1 + k_2 \\ d_2 &= \nu_2 - k_2 + k_3 \\ &\cdots \\ d_\rho &= \nu_\rho - k_\rho + \mu_1 + 1, \end{aligned}$$

$k_1 = d\left(\frac{\prod \gamma_i}{\prod \alpha_i}\right)$, with $\frac{\prod \gamma_i}{\prod \alpha_i} = P$, and $\mu_1 + 1 \geqslant k_\rho \geqslant \cdots \geqslant k_1$ (since $d_i \geqslant \nu_i, i = 1, \ldots, \rho$).

Let $P' = \frac{\prod \gamma_i'}{\prod \alpha_i}$. Let $k_1' = d(P')$. From the condition (ii) we have $P' | P$, i.e. $k_1 \geqslant k_1'$. Further, define $k_2', \ldots, k_\rho'$ as follows:

$$k_i' = \sum_{j=1}^{i-1} (d_j' - \nu_j) + k_1', \quad i = 2, \ldots, \rho.$$

Thus, $d'_1, \ldots, d'_\rho$ satisfy

$$\begin{aligned} d'_1 &= \nu_1 - k'_1 + k'_2 \\ d'_2 &= \nu_2 - k'_2 + k'_3 \\ &\ldots \\ d'_\rho &= \nu_\rho - k'_\rho + \mu_1 + 1. \end{aligned}$$

From the conditions (iv) and (iii) of (I), we obtain

$$\mu_1 + 1 \geqslant k'_\rho \geqslant \cdots \geqslant k'_1,$$

and

$$k_i \geqslant k'_i, \quad i = 1, \ldots, \rho.$$

Let $P_1 = P$, $P_i = P\lambda^{k_i - k_1}$, $i = 2, \ldots, \rho$, and $P_{\rho+1} = P\lambda^{\mu_1+1-k_1}$. By applying Lemma 5, we obtain the existence of polynomials X_i, $d(X_i) \leqslant d(P_i)$, $i = 2, \ldots, n$, with arbitrary small coefficients, such that the following equations are valid:

$$\begin{aligned} &d(\gcd(P_\rho + X_\rho, P_{\rho+1} + X_{\rho+1})) = k'_\rho \\ &d(\gcd(P_{\rho-1} + X_{\rho-1}, P_\rho + X_\rho, P_{\rho+1} + X_{\rho+1})) = k'_{\rho-1} \\ &\ldots \\ &\gcd(P_1, P_2 + X_2, \ldots, P_{\rho-1} + X_{\rho-1}, P_\rho + X_\rho, P_{\rho+1} + X_{\rho+1}) = P'. \end{aligned}$$

Write $X_i = (-1)^{(\rho-i)}(\epsilon_i^{k_i+1}\lambda^{k_i} + \epsilon_i^{k_i}\lambda^{k_i-1} + \cdots + \epsilon_i^2\lambda + \epsilon_i^1)$, $i = 2, \ldots, \rho+1$, where ϵ_i^j, $i = 2, \ldots, \rho+1$, $j = 1, \ldots, k_i + 1$, $k_{\rho+1} = \mu_1$, are some small numbers. Let $\epsilon_i = \begin{bmatrix} \epsilon_i^1 & \cdots & \epsilon_i^{k_i} & \epsilon_i^{k_i+1} & 0 & \cdots & 0 \end{bmatrix}$, $i = 2, \ldots, \rho+1$.

By applying Lemma 7 we have that the controllability indices of the matrix

$$\overline{M}' = \left[\begin{array}{c|c|c|c|c|c|c} N & & & & & & \\ \hline & C(\lambda^{\nu_1}) & & & & & e_1 \\ \hline & & C(\lambda^{\nu_2}) & & & & e_2 \\ \hline & & & \ddots & & & \ddots \\ \hline & & & & C(\lambda^{\nu_\rho}) & & e_\rho \\ \hline & & & & & C(\lambda^{\mu_1}) \quad e_0 & \\ \hline t_0 & t_1 & t_2 + \epsilon_2 & \cdots & t_\rho + \epsilon_\rho & t_{\rho+1} + \epsilon_{\rho+1} & \end{array}\right]$$

are the same as the controllability indices of its submatrix $\overline{L}'$ formed by the last $n + \rho + 1 - p$ columns and the last $n + 1 - p$ rows.

Now, we shall prove that the matrix $\overline{L}'$, and hence the matrix $\overline{M}'$, has $d'_1 \geqslant \cdots \geqslant d'_\rho$ as the nonzero column minimal indices.

It is enough to observe Eq. (18), i.e.

$$P'_1 z_1 + P'_2 z_2 + \cdots + P'_\rho z_\rho + P'_{\rho+1} z_{\rho+1} = 0.$$

Let $i \in \{1, \ldots, \rho\}$. Observe the set of solutions of the equation

$$P'_i z_i + P'_{i+1} z_{i+1} + \cdots + P'_\rho z_\rho + P'_{\rho+1} z_{\rho+1} = 0 \tag{21}$$

such that $z_i \neq 0$. Let $D_j = \gcd(P'_j, \ldots, P'_{\rho+1})$, $\rho + 1 \geqslant j \geqslant i$. Let $\overline{P}_j = P'_j / D_j$, $Q_j = D_{j+1}/D_j$, $\rho \geqslant j \geqslant i$. Obviously, the polynomials $\overline{P}_j$ and Q_j are mutually prime for every $\rho \geqslant j \geqslant i$.

Eq. (21) is equivalent to the following one:

$$\overline{P}_i z_i = -Q_i(\overline{P}_{i+1}z_{i+1} + Q_{i+1}(\overline{P}_{i+2}z_{i+2} + Q_{i+2}(\cdots + Q_{\rho-1}(\overline{P}_\rho z_\rho + Q_\rho z_{\rho+1})\cdots))). \tag{22}$$

From the definition of P'_j, $\rho+1 \geqslant j \geqslant i$, we have that $d(D_j) = k'_j$ and hence $d(\overline{P}_j) = k_j - k'_j$, and $d(Q_j) = k'_{j+1} - k'_j$.

Since, $\gcd(\overline{P}_i, Q_i) = 1$ and $z_i \neq 0$, we have $Q_i | z_i$, and hence $d(z_i) \geqslant d(Q_i) = k'_{i+1} - k'_i$. From (19) we obtain that all the solutions of (21) have the weighted degree greater or equal than $\nu_i + k'_{i+1} - k'_i = d'_i$. We are left to prove that there exists a solution of (21) whose weighted degree is d'_i. Indeed, let $z_i = -Q_i$, and denote $y_j = \overline{P}_{j+1}z_{j+1} + Q_{j+1}(\cdots + Q_{\rho-1}(\overline{P}_\rho z_\rho + Q_\rho z_{\rho+1})\cdots))$, $\rho > j > i$, $y_\rho = z_{\rho+1}$. Then Eq. (21) becomes the following system of equations

$$\overline{P}_{i+1}z_{i+1} + Q_{i+1}y_{i+1} = \overline{P}_i \tag{23}$$

and

$$\overline{P}_{j+1}z_{j+1} + Q_{j+1}y_{j+1} = y_j, \quad \rho > j > i. \tag{24}$$

By Lemma 3 there exists a solution of Eq. (23) such that $d(z_{i+1}) \leqslant \max\{d(Q_{i+1}), d(\overline{P_i}) - d(\overline{P}_{i+1})\}$ and $d(y_{i+1}) \leqslant \max\{d(\overline{P}_{i+1}), d(\overline{P_i}) - d(Q_{i+1})\}$.

The weighted degree corresponding to z_{i+1} is less or equal than $\nu_{i+1} + \max\{d(Q_{i+1}), d(\overline{P_i}) - d(\overline{P}_{i+1})\} = \max\{\nu_{i+1} + k'_{i+2} - k'_{i+1}, \nu_{i+1} + k_i - k'_i - k_{i+1} + k'_{i+1}\} \leqslant \max\{d'_{i+1}, \nu_i - k'_i + k'_{i+1}\} = \max\{d'_{i+1}, d'_i\} = d'_i$. The last inequality follows from $\nu_{i+1} \leqslant \nu_i$ and $k_{i+1} \geqslant k_i$.

Analogously, by Lemma 3, there exist solutions of Eq. (24) such that $d(z_{j+1}) \leqslant \max\{d(Q_{j+1}), d(y_j) - d(\overline{P}_{j+1})\}$ and $d(y_{j+1}) \leqslant \max\{d(\overline{P}_{j+1}, d(y_j) - d(Q_{j+1})\}$, $i < j < \rho$. Hence, by induction we have

$$d(z_{j+1}) \leqslant \max\left\{ d(Q_{j+1}), d(\overline{P_j}) - d(\overline{P}_{j+1}), d(\overline{P}_{j-1}) - d(Q_j) - d(\overline{P}_{j+1}), \ldots, d(\overline{P_i}) - \sum_{l=i+1}^{j} d(Q_l) - d(\overline{P}_{j+1}) \right\}.$$

Further, for every $i \leqslant p \leqslant j$ we have that $d(\overline{P}_p) - \sum_{l=p+1}^{j} d(Q_l) - d(\overline{P}_{j+1}) = k_p - k'_p - (k'_{j+1} - k'_{p+1}) - k_{j+1} + k'_{j+1} \leqslant k'_{p+1} - k'_p = d(Q_p)$. Hence, the weighted degree corresponding to z_{j+1} is $\nu_{j+1} + d(z_{j+1}) \leqslant \nu_{j+1} + \max\{d(Q_{j+1}), d(Q_j), \ldots, d(Q_i)\} \leqslant \max\{\nu_{j+1} + k'_{j+2} - k'_{j+1}, \nu_j + k'_{j+1} - k'_j, \ldots, \nu_i + k'_{i+1} - k'_i\} = d'_i$.

We are left with proving that $d(z_{\rho+1}) = d(y_\rho) \leqslant d'_i$. Indeed, from (24) for $j = \rho - 1$, we have that $d(y_\rho) \leqslant \max\{d(\overline{P}_\rho), d(y_{\rho-1}) - d(Q_\rho)\}$. Further, by induction, we obtain $d(y_\rho) \leqslant \max_{p=i,\ldots,\rho}\{d(\overline{P}_p) - \sum_{l=p+1}^{\rho} d(Q_l)\} = \max_{p=i,\ldots,\rho}\{k_p - k'_p - k'_{\rho+1} + k'_{p+1}\}$, where $k'_{\rho+1} = \mu_1 + 1$. Finally, since $k_p \leqslant \nu_p$, we have that the weighted degree corresponding to $z_{\rho+1}$ is

$$\mu_1 + 1 + d(z_{\rho+1}) \leqslant \max_{p=i,\ldots,\rho}\{\nu_p - k'_p + k'_{p+1}\} = \max_{p=i,\ldots,\rho}\{d'_p\} = d'_i.$$

The module of all the solutions of Eq. (21), without the constraint $z_i \neq 0$, is free, by Corollary 7, and has at most $\rho + 1 - i$ linearly independent solutions, by Lemma 4. Since $d'_1 \geqslant \cdots \geqslant d'_\rho$ Eq. (18) has as the weighted degrees of its linearly independent ρ solutions, $d'_\rho \leqslant \cdots \leqslant d'_1$, and hence they are the nonzero column minimal indices of the matrix $\overline{L}'$ (and of $\overline{M}'$).

Observe now the matrix (15), i.e.

$$\begin{bmatrix} N & 0 \\ X & \overline{L} \end{bmatrix}.$$

We have that the above matrix is feedback equivalent to the following one:

$$\left[\begin{array}{c|ccc|ccc} N & & & & & & \\ \hline & C(P) & & & & & \\ & & C(\lambda^{d_1}) & & e_{d_1}^{d_1} & & \\ Z & & & \ddots & & \ddots & \\ & & & C(\lambda^{d_\rho}) & & & e_{d_\rho}^{d_\rho} \end{array}\right],$$

$Z \in \mathbb{F}^{(n+1-p)\times p}$, which is further, by applying Lemma 7, feedback equivalent to a matrix of the form

$$\left[\begin{array}{cc|ccc|ccc} N & & & & & & & \\ \widetilde{Y} & C(P) & & & & & & \\ \hline & & C(\lambda^{d_1}) & & & e_{d_1}^{d_1} & & \\ & & & \ddots & & & \ddots & \\ & & & & C(\lambda^{d_\rho}) & & & e_{d_\rho}^{d_\rho} \end{array}\right]. \tag{25}$$

Note that we can do feedback equivalent transformations in order that $\widetilde{Y}$ becomes of the form $\begin{bmatrix} 0 \\ s \end{bmatrix}$, where $s = \begin{bmatrix} s_1 & \cdots & s_p \end{bmatrix}$, for some scalars $s_1, \ldots, s_p$. By using the result from Theorem 2, i.e. making perturbations on the part s, in the same way as in [3], we finish our proof, since $\prod \gamma_i' = P \prod \alpha_i'$ and the matrix

$$\begin{bmatrix} N & 0 \\ \widetilde{Y} & C(P) \end{bmatrix}$$

has the same invariant polynomials as the whole matrix (25).

If $\mu_1 + 1$ is not the smallest among $f_1, \ldots, f_{\rho+1}$ then we will repeat the same procedure as in the proof of Theorem 1 in [9], and reduce to the case when $\mu_1 + 1$ is the smallest among $f_1, \ldots, f_{\rho+1}$.

Theorem 10. *Let $\eta > 0$ be arbitrary small. Let $t_i \geqslant 1, i = 1, \ldots, v$ be integers and $m'_{ij}, i = 1, \ldots, v,\ j = 1, \ldots, t_i$ be partitions. Let $d'_1 \geqslant \cdots \geqslant d'_{\bar\rho'}$ be positive integers. Let M be the matrix (1) and let it has $\gamma_1 | \cdots | \gamma_{n+1}$ and $d_1 \geqslant \cdots \geqslant d_{\bar\rho}$ as feedback invariants. Let $\alpha_1 | \cdots | \alpha_n$, μ_1, and $\nu_1 \geqslant \cdots \geqslant \nu_\rho$ be invariant polynomials, minimal index of the first kind and minimal indices of the second kind, respectively, of the matrix $\begin{bmatrix} A & b^{\mathrm{T}} & C \end{bmatrix}$. Then in every neighbourhood of M there exists a matrix*

$$M' = \begin{bmatrix} A & b^{\mathrm{T}} & C \\ a' & x' & c' \end{bmatrix}$$

such that

(a) $\sigma(M') \subset \mathscr{V}_\eta(M)$,
(b) *M' has $t_i - 1$ eigenvalues $\mu_{i2}, \ldots, \mu_{it_i}$ different from λ_i in $B(\lambda_i, \eta)$, $m'_{i1} = \omega(\lambda_i, M')$, and $m'_{ij} = \omega(\mu_{ij}, M')$, $i = 1, \ldots, v$, $j = 2, \ldots, t_i$,*
(c) *$d'_1, \ldots, d'_{\bar\rho'}$ are the controllability indices of M',*

if and only if:

(i) $0 \leqslant m'_{1k} - \omega(\lambda_i, [A, b^{\mathrm{T}}, C]) \leqslant 1,\ i = 1, \ldots, v,\ k = 1, \ldots, l(m'_{i1})$,
(ii) $0 \leqslant m'_{ijk} \leqslant 1,\ i = 1, \ldots, v,\ j = 2, \ldots, t_i,\ k = 1, \ldots, l(m'_{ij})$,
(iii) $\bigcup_{j=1}^{t_i} m'_{ij} \prec\prec \omega(\lambda_i, M),\ i = 1, \ldots, v$,
(iv) $(d'_1, \ldots, d'_{\bar{\rho}'}) \prec (d_1 + t, \ldots, d_{\bar{\rho}})$,
(v) $\sum_{i=1}^{\bar{\rho}'} d'_i + \sum_{i,j} m'_{ij} = n + 1$,
(vi) *one of the following conditions is satisfied*:
  (a) $\bar{\rho} = \bar{\rho}' = \rho$,
  $d'_i \geqslant \nu_i,\ i \in \{1, \ldots, \rho\}$,
  $f_{i+1} = d'_i,\ h_1 \leqslant i \leqslant \rho$,
  $h_1 = \min\{i | d'_i < f_i\}$,
  (b) $\bar{\rho} = \bar{\rho}' = \rho + 1$,
  $d'_i = d_i = f_i,\ i = 1, \ldots, \rho + 1$,
  (c) $\bar{\rho}' = \rho + 1 > \bar{\rho} = \rho$,
  $d'_i = f_i,\ i = 1, \ldots, \rho + 1$,
  $m \geqslant \rho + 1$,
  where $f_1, \ldots, f_{\rho+l}$ *are the same as in Theorem* 1.

Proof. The necessity is trivial to prove, calling the previous results.

For the sufficiency observe that if one of conditions (vi) (b) or (c) is satisfied then like in Theorem 9, we can trivially finish the proof.

If we have (vi) (a), then the proof goes similarly as the proof in [3]. Indeed, let $\tilde{m}_i = \bigcup_{j=1}^{t_i} m'_{ij}$, and $\tilde{n}_i = \overline{\tilde{m}_i}, i = 1, \ldots, v$. Define polynomials $\tilde{\gamma}_1 = \prod_{i=1}^{v} (\lambda - \lambda_i)^{\tilde{n}_{i,n+1}}, \ldots, \tilde{\gamma}_{n+1} = \prod_{i=1}^{v} (\lambda - \lambda_i)^{\tilde{n}_{i,1}}$.

It is not hard to verify that $\tilde{\gamma}_1 | \cdots | \tilde{\gamma}_{n+1}$ and $d'_1 \geqslant \cdots \geqslant d'_\rho$ satisfy the conditions from Theorem 9. Thus, for given $\epsilon > 0$, there exists $\widetilde{M} \in B(M, \frac{\epsilon}{2})$ such that $\tilde{\gamma}_1 | \cdots | \tilde{\gamma}_n$ and $d'_1 \geqslant \cdots d'_\rho$ are its feedback invariants.

Like in [3], the matrix $\lambda \begin{bmatrix} I & 0 \end{bmatrix} - \widetilde{M}$ is equivalent to the matrix

$$\begin{bmatrix} \mathrm{diag}(I, \widetilde{M}(\lambda)) & 0 \end{bmatrix},$$

where

$$\widetilde{M}(\lambda) = \left[\begin{array}{ccc|ccc} \alpha_1 & & & & & \\ & \ddots & & & & \\ & & \alpha_n & & & \\ y_1 & \cdots & y_n & a_1 & \cdots & a_{\rho+1} \end{array}\right].$$

Denote by $\widetilde{P}'$ the greatest common divisor of the polynomials $a_1, \ldots, a_{\rho+1}$. Obviously $\widetilde{P}' = \frac{\prod \tilde{\gamma}_i}{\prod \alpha_i}$. So, (as in [3]) $\widetilde{P}' = (\lambda - \lambda_1)^{c_1} \cdots (\lambda - \lambda_v)^{c_v}$ where $c_i = m(\lambda_i, M) - m(\lambda_i, [A \quad b \quad C]) \geqslant 0$.

If $c_i \neq 0$ and $t_i \geqslant 2$, for some $i \in \{1, \ldots, v\}$ we will change each of n_{ij} factors $(\lambda - \lambda_i)$, of the polynomial $\widetilde{P}'$, to $(\lambda - \lambda_i - \epsilon_{ij})$, $j = 2, \ldots, t_i$, where ϵ_{ij} are arbitrary small. Denote by P' the obtained polynomial. Obviously, it is close to $\widetilde{P}'$.

In fact, we have

$$
\begin{aligned}
&a_1 = P + X_1 = \widetilde{P}' A_1\\
&a_2 = \lambda^{k_2-k_1} P + X_2 = \widetilde{P}' A_2\\
&\ldots\\
&a_\rho = \lambda^{k_\rho-k_1} P + X_\rho = \widetilde{P}' A_\rho\\
&a_{\rho+1} = \lambda^{\mu_1+1-k_1} P + X_{\rho+1} = \widetilde{P}' A_{\rho+1},
\end{aligned}
$$

where $\gcd(A_1, \ldots, A_{\rho+1}) = 1$ and $P = \frac{\prod \gamma_i}{\prod \alpha_i}$. Since P' is close to $\widetilde{P}'$, there exist polynomials X'_i, $i = 1, \ldots, \rho + 1$ with small coefficients, such that

$$
\begin{aligned}
&P + X'_1 = a'_1 = P' A_1\\
&\lambda^{k_2-k_1} P + X'_2 = a'_2 = P' A_2\\
&\ldots\\
&\lambda^{k_\rho-k_1} P + X'_\rho = a'_\rho = P' A_\rho\\
&\lambda^{\mu_1+1-k_1} P + X'_{\rho+1} = a'_{\rho+1} = P' A_{\rho+1}.
\end{aligned}
$$

In this way we have obtained a matrix $M' \in B(M, \epsilon)$ by putting the appropriate coefficients of X_i as ϵ_i^j in M' ($M' \in B(\widetilde{M}, \frac{\epsilon}{2})$). Like in [3] we can conclude that the matrix M' has m'_{ij}, $i = 1, \ldots, v$, $j = 1, \ldots, t_i$ as Weyr characteristics. Finally, since our Eq. (18) is left unchanged, M' has d'_i, $i = 1, \ldots, \rho$, as controllability indices, as wanted. □

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