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Maximal reflexive cacti with four cycles: The approach via Smith graphs

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ABSTRACT

Cacti, or treelike graphs, are graphs whose all cycles are mutually edge-disjoint. Graphs with the property $\lambda_2 \leqslant 2$ are called reflexive graphs, where λ_2 is the second largest eigenvalue of the corresponding $(0, 1)$-adjacency matrix. The property $\lambda_2 \leqslant 2$ is a hereditary one, i.e. all induced subgraphs of a reflexive graph are also reflexive. This is why we represent reflexive graphs through the maximal graphs within a given class (such as connected cacti with a fixed number of cycles). In previous work we have determined all maximal reflexive cacti with four cycles whose cycles do not form a bundle and pointed out the role of so-called pouring of Smith graphs in their construction. In this paper, besides pouring, we show several other patterns of the appearance of Smith trees in those constructions. These include splitting of a Smith tree, adding an edge to a Smith tree and then splitting of the resulting graph, identifying two vertices of a Smith tree and then splitting the resulting graph. Our results show that the presence of Smith trees is evident in all such maximal reflexive cacti with four cycles and that in most of them Smith graphs appear in the described way.

1. Introduction

Consider a simple connected graph G (undirected, with no loops or multiple edges). A *cactus* or a *treelike graph* is a graph in which any two cycles have at most one common vertex. The *characteristic polynomial* of the graph G is $P_G(\lambda) = \det(\lambda I - A)$ where A is its $(0, 1)$-adjacency matrix. *Eigenvalues* of G are the roots of its characteristic polynomial. The family of eigenvalues is called the *spectrum* of G. Since A is a real and symmetric matrix, the eigenvalues are real and can be ordered from the largest to the smallest one: $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$. For connected graphs $\lambda_1(G) > \lambda_2(G)$ holds.

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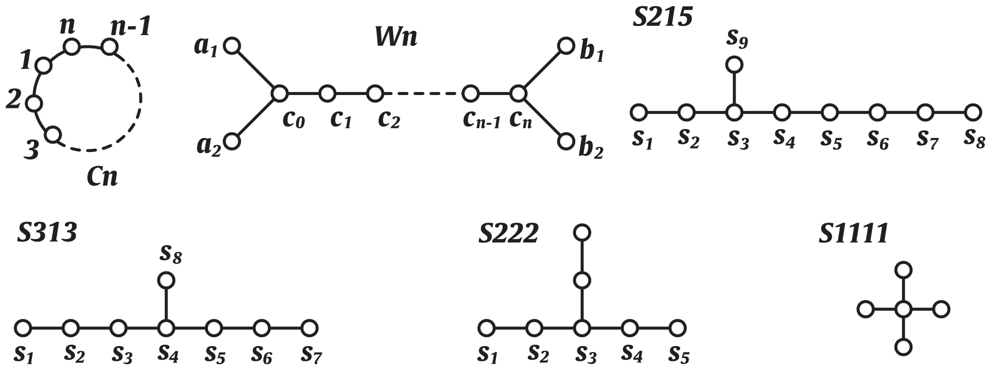

Fig. 1. Smith graphs.

The *interlacing theorem* shows the interrelation between the spectrum of a graph *G* and the spectrum of its induced subgraph *H*.

The Interlacing Theorem. *Let* $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ *be the eigenvalues of a simple graph G and* $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m$ *the eigenvalues of its induced subgraph H. Then the inequalities* $\lambda_{n-m+i} \leqslant \mu_i \leqslant \lambda_i (i = 1, 2, \ldots, m)$ *hold.*

If *G* is connected and $m = n - 1$, we have $\lambda_1 > \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \cdots$.

Graphs shown in Fig. 1 are called *Smith graphs* [21]. They are the only connected graphs for which $\lambda_1 = 2$ holds.

Proper induced subgraphs of Smith graphs all have the property $\lambda_1 < 2$ and are called Coxeter–Dynkin graphs.

We study *reflexive graphs*, i.e. graphs with the property $\lambda_2 \leqslant 2$. Graphs with the property $\lambda_2 \leqslant 2 \leqslant \lambda_1$ are called *hyperbolic graphs*. The property $\lambda_2 \leqslant 2$ is a hereditary one, i.e. every induced subgraph of a reflexive graph preserves this property. This is why we represent reflexive graphs through the maximal graphs within a given class, such as connected cacti with a fixed number of cycles.

Graphs whose second largest eigenvalue is bounded by a constant $c \in R$ have been investigated by various authors. Some of the results for the bounds considered so far include: $c = 1/3$ [1], $c = \sqrt{2} - 1$ [11], $c = (\sqrt{5} - 1)/2$ [5,20], $c = 1$ [2,9].

Most of the results on reflexive graphs are related to graphs having a cut-vertex. Reflexive trees have been investigated in [7,9], bicyclic reflexive graphs with a bridge between the cycles in [17], see also [6,12]. Various classes of multicyclic reflexive cacti whose cycles do not form a bundle have been studied in [8,13–16,18]. Particularly, the maximal reflexive cacti with four cycles are shown in [15,16].

Reflexive graphs correspond to sets of vectors in the Lorentz space $R^{p,1}$ with Gram matrix *2I-A* and consequently norm 2 and mutual angles 90° and 120°. They are Lorentzian counterparts of the spherical and Euclidean graphs, which occur in the theory of reflection groups. They have direct application to the construction and the classification of such groups [10].

In former investigations aimed at finding maximal reflexive graphs the central role of Smith graphs was obvious and some large classes of these graphs were constructed and described by means of Smith graphs and their pouring [13]. Is it possible to apply some more complex operations and modifications of Smith graphs in constructions of various classes of maximal reflexive graphs?

In this paper, we examine some other roles of Smith graphs in maximal reflexive cacti with four cycles. We consider extensions of certain types of cacti without bridges by some modified Smith trees. The modifications of Smith trees are denoted by σ_i, $i = 1, 2, \ldots, 7$, and the corresponding extensions are called σ_i-extensions. These modifications are defined as follows:

- σ_1 is a whole Smith tree (Fig. 2(a)).
- σ_2 is a Smith tree split into two parts.

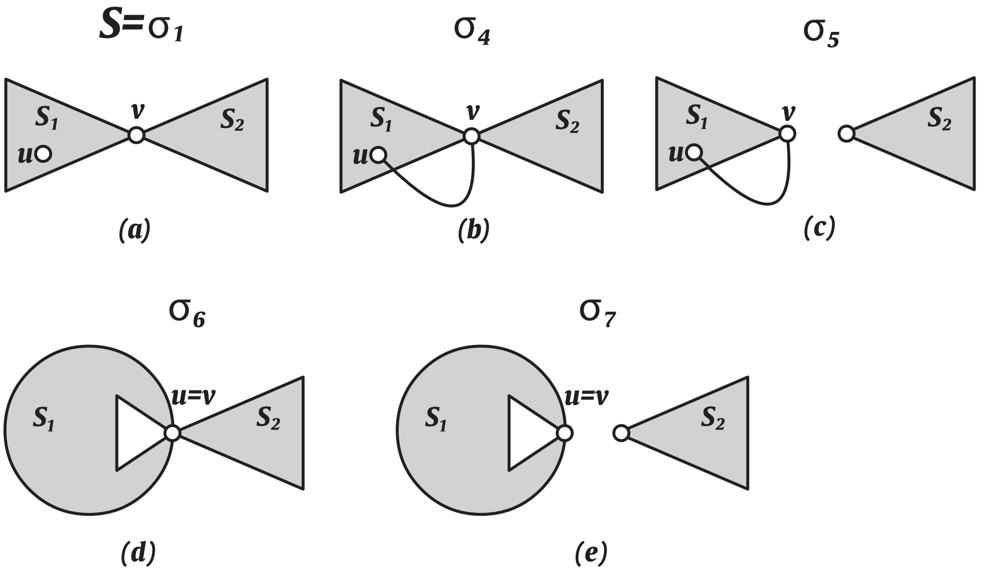

Fig. 2. The modifications of Smith trees (S is a Smith tree, S1 and S2 are its parts connected at v).

- σ_3 is a Smith tree split into three parts (Fig. 12).
- σ_4 is a Smith tree with an added edge (Fig. 2(b)).
- σ_5 is a Smith tree with an added edge (Fig. 2(c)) and then split at one of the vertices of this new edge.
- σ_6 is a Smith tree with two vertices u and v identified (Fig. 2(d)).
- σ_7 is a Smith tree with two vertices u and v identified (Fig. 2(e)) and then split at that vertex into two parts.

Note that by adding an edge to a Smith tree or by identifying two vertices we obtain a cycle; thus corresponding cacti that we extend should be tricyclic.

In Appendix A the tables of values $P_G(2)$ are given for some subgraphs G of Smith graphs. In [4], in Theorem 3, values $P_G(-2)$ are given for the graphs G whose least eigenvalue is greater than -2 which is applicable to the Smith trees, since for trees $P_G(2) = P_G(-2)$ holds.

This paper is structured in the following way. In Section 2 we give some previous and auxiliary results that we use in the rest of the paper. In Section 3, maximal reflexive graphs with four cycles of certain type from [15] are investigated for presence of some types of modifications of Smith trees. In Section 4, splitting of Smith trees into three parts within a class of maximal reflexive graphs with four cycles from [16] is studied. In Section 5 modified Smith trees are detected within another class of graphs from [15]. In Section 6 we give some conclusions.

We use the terminology of the theory of graph spectra as in [3]. Also, in this paper *subgraph* means *induced subgraph*.

2. Auxiliary and previous results

In our search for maximal reflexive cacti following lemmas and theorems are important tools.

Lemma 1 ([21], see also [3], pp. 78–79)**.** *Let $\lambda_1(G)$ be the index (the largest eigenvalue) of a graph G. Then $\lambda_1(G) \leqslant 2$ ($\lambda_1(G) < 2$) if and only if each component of G is a subgraph (resp. proper subgraph) of one of the graphs shown in Fig. 1, all of which have index equal to 2.*

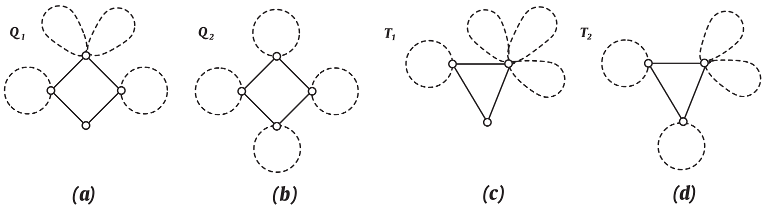

Fig. 3. Maximal reflexive cacti with the maximal number of cycles (5).

Lemma 2 (Schwenk [19]). *Given a graph G, let $C(v)$ and $C(uv)$ denote the set of all cycles containing a vertex v and an edge uv of G, respectively. Then*

$$1^\circ\ P_G(\lambda) = \lambda P_{G-v}(\lambda) - \sum_{u \in Adj(v)} P_{G-v-u}(\lambda) - 2\sum_{C \in C(v)} P_{G-V(C)}(\lambda),$$

$$2^\circ\ P_G(\lambda) = P_{G-uv}(\lambda) - P_{G-v-u}(\lambda) - 2\sum_{C \in C(uv)} P_{G-V(C)}(\lambda),$$

where $Adj(v)$ denotes the set of neighbours of v, while $G - V(C)$ is the graph obtained from G by removing the vertices belonging to the cycle C.

Whether a cactus is reflexive or not can often be tested in a simple way by using the following theorem.

Theorem RS [17]. *Let G be a graph with a cut-vertex u.*

1° *If at least two components of $G - u$ are supergraphs of Smith graphs, and if at least one of them is a proper supergraph, then $\lambda_2(G) > 2$.*
2° *If at least two components of $G - u$ are Smith graphs, and the rest are subgraphs of Smith graphs, then $\lambda_2(G) = 2$.*
3° *If at most one component of $G - u$ is a Smith graph, and the rest are proper subgraphs of Smith graphs, then $\lambda_2(G) < 2$.*

If after removing vertex *u* we get one proper supergraph, while the rest are proper subgraphs of Smith graphs, the theorem does not answer the question whether the graph is reflexive or not (we call such a graph RS-undecidable) and such cases are interesting for further investigations.

If all cycles of a cactus have a common vertex (form a bundle), it is clear that the number of cycles of such reflexive graph is not limited. On the other hand, it was shown in [16] that an RS-undecidable reflexive cactus whose cycles do not form a bundle has at most five cycles. The only such graphs, which are all maximal, are the four families of graphs of Fig. 3. In Fig. 3 and all other figures in this paper cycles depicted with closed dotted lines denote cycles of arbitrary length.

That is why, when looking for maximal reflexive cacti, we restrict ourselves to those of them that are RS-undecidable and whose cycles do not form a bundle.

Note that graphs of Fig. 3 consist only of Smith graphs (cycles). By removing one of the outer cycles of these graphs we get the starting graphs with four cycles shown in Fig. 4.

In [15,16] we have constructed all maximal reflexive RS-undecidable cacti with four cycles whose cycles do not form a bundle. There are about 200 infinite families of such graphs and most of them (170) are classified into groups denoted by H, I, J, K, L, M, N. For the convenience of the reader we reproduce some of them in Appendix B. For the complete list and diagrams of these graphs we direct the reader to [15,16].

As shown in [16], from graphs depicted in Fig. 4(a) and (b), where the central cycle is quadrangle, we obtain maximal reflexive cacti shown in Fig. 5 (S is a Smith tree and S_1 and S_2 are parts of a Smith tree as in Fig. 2(a)). The presence and the role of Smith trees here is obvious.

Note that graphs in Fig. 5(b) and (c) can be interpreted as special cases of (d).

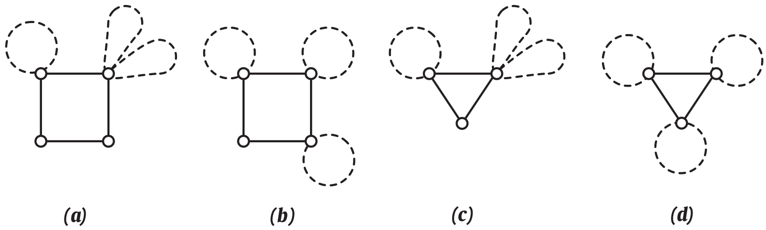

Fig. 4. The cyclic structure of the studied graphs.

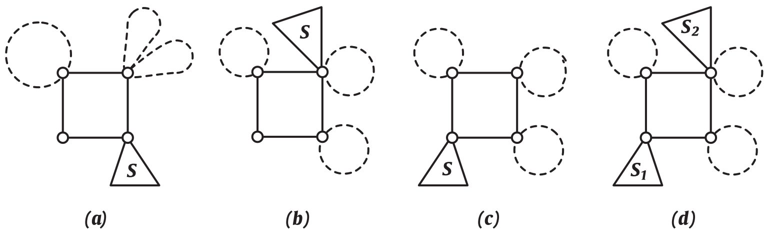

Fig. 5. Maximal reflexive graphs generated by the graphs of Fig. 4(a) and 4(b).

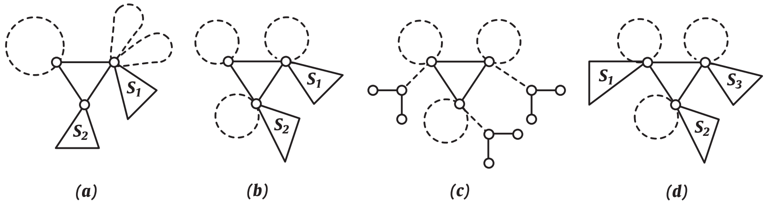

Fig. 6. Maximal reflexive graphs generated by the graphs of Fig. 4(c) and 4(d).

Also, we get graphs of type H, found in [15], and we will discuss the presence of Smith trees in them in Section 5.

For the cacti with four cycles whose central cycle is a triangle it was shown in [16] that graphs in Fig. 6(a), (b), (c) are maximal reflexive cacti. The case of splitting of a Smith tree into three parts (Fig. 6(d)) will be discussed in Section 4.

Here, the presence of the Smith trees is clear, too, and one can also notice how the Smith tree W_n is involved in the exception (Fig. 6(c)). The remaining cases are graphs of type I, J, K, L, M, N from [15,16], and we will discuss them in Section 3.

3. The graphs of type J, K, M and N

In this section, we consider σ_4 and σ_5-extensions of the tricyclic graph in Fig. 7. As explained in Section 1, graph σ_4 is obtained from the Smith tree by adding an edge connecting the vertices u and v (Fig. 2(b)), and graph σ_5 is obtained by splitting the graph σ_4 at vertex $u = v$ (Fig. 2(c)).

Attaching the graph σ_4 at its vertex v to one of the vertices d_1 or d_2 of the tricyclic graph in Fig. 7 we get graphs G_2 and G_3 in Fig. 8. Attaching the components of graph σ_5 to vertices d_1 and d_2 of the graph in Fig. 7 we obtain graphs G_1 and G_4 in Fig. 8.

G_1, G_2, G_3 and G_4 are graphs with four cycles. The fourth cycle stems from the graph σ_4 (and σ_5). Graphs G_1 and G_2 have cyclic structure as in Fig. 4(c) and graphs G_3 and G_4 have cyclic structure as in Fig. 4(d).

For a Smith graph S (Fig. 2(a)) $P_S(2) = 0$ holds. Let us introduce the notation. $P_{S_1}(2) = A_1$, $P_{S_2}(2) = B_1$, $P_{S_1-v}(2) = A$, $P_{S_2-v}(2) = B$, $\sum_{x\in Adj(v)} P_{S_1-v-x}(2) = \Sigma_A$, $\sum_{x\in Adj(v)} P_{S_2-v-x}(2) = \Sigma_B$. (Since $P_S(2) = 2AB - \Sigma_A B - \Sigma_B A = 0$, we have $AB_1 - B\Sigma_A = 0$.)

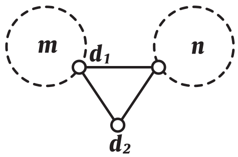

Fig. 7. Starting graphs for σ_4 and σ_5 extensions.

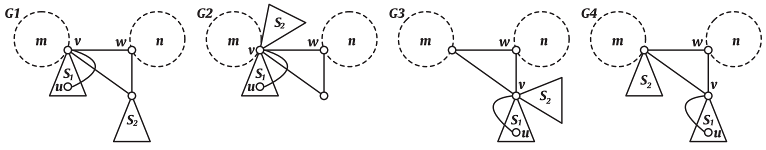

Fig. 8. Graphs G1, G2, G3, G4.

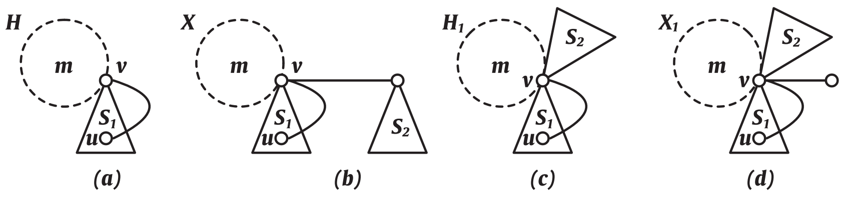

Fig. 9. Auxiliary graphs in the proof of Proposition 1.

$U_S = P_{S-u}(2)$, $U = P_{S_1-v-u}(2)$, $V_S = P_{S-v}(2)$, $P_{S-u-v}(2) = W$ and $P_{S-p}(2) = C$, where p is the unique path connecting vertices u and v (within the Smith tree).

Proposition 1. *For graphs G_i in Fig. 8 we have $P_{G_i}(2) = mn(-2V_S + W + 2C)$ for $i = 1, 2, 3, 4$.*

Proof. Let us consider the graph G_1. We will use auxiliary graphs H and X (Fig. 9(a) and (b)). Applying Lemma 2.1 at the vertex v we get $P_H(2) = m(-\Sigma_A - U - 2P_{S_1-p}(2))$.

Now, applying Lemma 2.1 at vertex w we compute $P_{G_1}(2)$.

$$\begin{aligned} P_{G_1}(2) &= 2nP_X(2) - nmAB_1 - nBP_H(2) - 2(n-1)P_X(2) - 2P_X(2) - 2mnAB \\ &= -nmAB_1 - nBP_H(2) - 2mnAB = mn(-AB_1 + B\Sigma_A + UB + 2B \cdot P_{S_1-p}(2) - 2AB) \\ &= mn(-2V_S + W + 2C). \end{aligned}$$

Note that $AB = V_S$, $UB = W$, $B \cdot P_{S_1-p}(2) = C$.

Now, let us consider the graph G_2. Using auxiliary graphs H_1 and X_1 (Fig. 9(c) and (d)) and applying Lemma 2.1 at vertex w, we can compute $P_{G_2}(2)$. First, for graph H_1 we find $P_{H_1}(2) = -m(\Sigma_A B + \Sigma_B A + UB + 2C)$. Now,

$$\begin{aligned} P_{G_2}(2) &= 2nP_{X_1}(2) - 2(n-1)P_{X_1}(2) - 2nmAB - nP_{H_1}(2) - 2P_{X_1}(2) - 2mnAB \\ &= -2nmAB - nP_{H_1}(2) - 2mnAB = mn(-2AB + B\Sigma_A + \Sigma_B A - 2AB + UB + 2C) \\ &= mn(-2AB + UB + 2C) = mn(-2V_S + W + 2C). \end{aligned}$$

Similarly, $P_{G_3}(2) = mn(W + 2C) - 2mnAB = mn(-2V_S + W + 2C)$ and

$$\begin{aligned} P_{G_4}(2) &= -nmB(A_1 - U - 2P_{S_1-p}(2)) + mn\Sigma_B A - 2mnAB \\ &= mn(-A_1 B + BU + \Sigma_B A + 2C - 2V_S) = mn(-2V_S + W + 2C). \end{aligned}$$

Thus, we proved that $P_{G_i}(2) = mn(-2V_S + W + 2C)$, $i \in \{1, 2, 3, 4\}$. $\square$

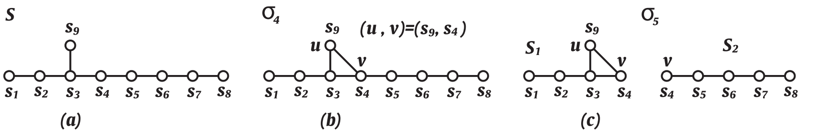

Fig. 10. An example for σ_4 and σ_5.

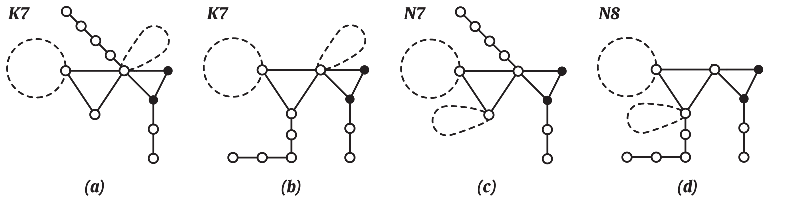

Fig. 11. The resulting graphs of type G_1, G_2, G_3 and G_4 from the example.

This value depends on the choice of the pair of vertices u, v. We will go through all Smith trees and find all the cases for which $-2V_S + W + 2C = 0$ holds. Let us show an example first.

Example. Let S be the Smith tree S215 (Fig. 10(a)), with an added edge connecting u and v; $(u, v) = (s_9, s_4)$. Then: $V_S = 25$, $W = 20$, $C = 15$ (see the tables in Appendix A) and, therefore, $-2V_S + W + 2C = 0$.

The resulting graphs of type G_1, G_2, G_3, G_4 are shown in Fig. 11(a), (b), (c) and (d) respectively. These are maximal reflexive graphs of type K and N [15].

Now, let us examine all Smith trees S and the corresponding values $P_S(2)$ for all ordered pairs (u, v). For S=S215 $-2V_S+W+2C = 0$ holds if and only if $(u, v) \in \{(s_1, s_5), (s_8, s_1), (s_7, s_3), (s_7, s_4), (s_9, s_4), (s_7, s_5)\}$. In the following table we show all the graphs of type J, K, M and N that correspond to each pair (s_i, s_j).

(s_1, s_5)	J7, J8, M11, M4
(s_8, s_1)	J2, M7
(s_7, s_3)	K20, K21, K33, K34, N20, N21, N33, N34
(s_7, s_4)	K27, K28, N27, N28
(s_9, s_4)	K7, K8, N7, N8
(s_7, s_5)	K25, K26, N25, N26

Now, consider the graph S313. For this graph $-2V_S + W + 2C = 0$ holds if and only if $(u, v) \in \{(s_1, s_6), (s_2, s_4)\}$. The following table shows all the graphs of type J, K, M and N that correspond to each pair (s_i, s_j).

(s_1, s_6)	J9, J10, M12
(s_2, s_4)	K16, K17, K29, K30, N16, N17, N29, N30

Also, for graph K3 we notice the appearance of S313 with $(u, v) = (s_1, s_8)$, but $-2V_S + W + 2C \neq 0$. This is one of the cases where a graph is maximal reflexive within the given class, but $\lambda_2 < 2$. These graphs are marked with $(*)$ in corresponding figures.

For the graphs S222 and S1111 there are no pairs (u, v) such that $-2V_S + W + 2C = 0$ holds.

For graph W_n the relation$-2V_S + W + 2C = 0$ holds for $(u, v) = (a_1, c_2)$. For $n = 2$ we get graphs J4, M9, and for $n > 2$ J5, J6, M10 and a subgraph of the maximal graph M2.

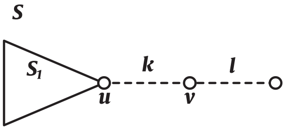

Fig. 12. Splitting Smith trees into three parts.

The following theorem summarizes what we have just proved.

Theorem 1. *σ_4 and σ_5-extensions of the tricyclic graph in Fig. 7 lead to maximal reflexive cacti with four cycles.*

Remark. Maximal reflexive cacti with four cycles of type J, K, M, N from [15] that are at the same time graphs of type G_1, G_2, G_3, G_4 are J2, J4–J10; K3, K7–K8, K16–K17, K20–K21, K25–K30, K33–K34; M4, M7, M9–M12; N7–N8, N16–N17, N20–N21, N25–N30, N33–N34.

As for graphs of type J, K, M, N that do not follow previously shown patterns, in many of them Smith trees appear as in Fig. 2(c): two vertices are identified, and thus we get a cycle within this graph. There are, of course, some exceptions.

In graphs of type I and J from [15] the presence of Smith trees is noticeable, but there are too few of these graphs to make any kind of generalization.

4. Splitting of a Smith tree into three parts

Let us now focus on another interesting class of graphs, which involves splitting of Smith trees into three parts as in Fig. 6(d). Let us consider Smith trees S215, S313 and S222 with cut-vertices u and v, as in Fig. 12.

Theorem 2. *σ_3-extensions of graphs in Fig. 4(d) are maximal reflexive cacti with four cycles, for suitable choices of cut-vertices u and v of Smith trees S215, S313 and S222.*

Remark. A cactus with four cycles and the cyclic structure as in Fig. 4(d), and with three parts of a Smith tree S215, S313 or S222 attached to its three vertices of the central triangle (Fig. 6(d)) is a maximal reflexive cactus (within the given class) if and only if it is one of the ten graphs described in [16] (Fig. 14). (See Fig. B2 in Appendix B.)

Proof. If we obtain three parts of a Smith tree (S215, S313 or S222) by cutting it at two vertices (and keeping copies of them in each part), it is clear that two of these parts have to be paths. Consider a Smith tree shown in Fig. 12. S is split into three graphs: S_1, a path of length k, and a path of length l.

Applying Lemma 2.1 at vertex u, and knowing that $P_S(2) = 0$, we get:

$$\begin{aligned} P_S(2) &= 2P_{S_1-u}(2)(k+l+1) - \sum_{x \in Adj(x)_{S_1}} P_{S_1-u-x}(2)(k+l+1) - P_{S_1-u}(2)(k+l) \\ &= (k+l+1)\left(P_{S_1-u}(2) - \sum_{x \in Adj(x)_{S_1}} P_{S_1-u-x}(2)\right) + P_{S_1-u}(2) = 0. \end{aligned}$$

Let us attach these three parts to the graph in Fig. 4(d). We obtain the graph G in Fig. 13.

Let us compute the value $P_G(2)$ for this graph ($k, l \geqslant 1$). Applying Lemma 2.1 at vertex u, we get:

$$\begin{aligned} P_G(2) &= -2mnpP_{S_1-u}(2)(k+l+1) - m(-pl)nP_{S_1-u}(2)(k+1) - p(l+1)(-mk)nP_{S_1-u}(2) \\ &\quad -2(n-1)P_{S_1-u}(2)(-mp(k+l+1)) - n(-mp(k+l+1)) \sum_{x \in Adj(x)_{S_1}} P_{S_1-u-x}(2) \end{aligned}$$

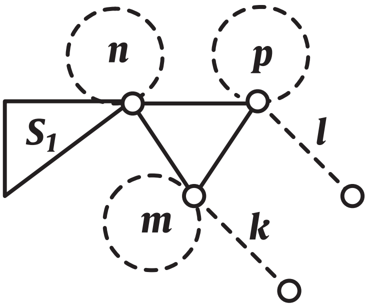

Fig. 13. Graphs of Theorem 2.

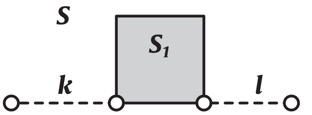

Fig. 14. Another way of splitting of Smith trees.

$$-2(-mp(k+l+1))P_{S_1-u}(2) - 2mnp(k+1)(l+1)P_{S_1-u}(2)$$
$$= -mnp\left((k+l+1)\left(P_{S_1-u}(2) - \sum_{x\in Adj(x)_{S_1}} P_{S_1-u-x}(2)\right) + P_{S_1-u}(2)\right) =$$
$$= -mnpP_S(2) = 0.$$

Now let us find all possible splittings of the Smith graphs S215, S313 and S222 from Fig. 1.

S215 gives: for $u = s_3$ we have $(k, l) \in \{(1, 1), (1, 4), (2, 3), (3, 2), (4, 1)\}$; $u = s_4$ implies $(k, l) \in \{(1, 3), (2, 2), (3, 1)\}$, $u = s_5$ gives $(k, l) \in \{(1, 2), (2, 1)\}$ and for $u = s_6$, $(k, l) = (1, 1)$.

For graph S222 we get $u = s_3$ and $(k, l) = (1, 1)$ and for graph S313 we get: for $u = s_4$ we have $(k, l) \in \{(1, 2), (2, 1)\}$ and for $u = s_5$, $(k, l) = (1, 1)$.

In this way we found all 10 graphs from [16] (Fig. 14). (Five of them are shown in Fig. B2, Appendix B.)

Thus, we characterized the whole set of maximal reflexive cacti of this cyclic structure whose all vertices of the central triangle are loaded by some additional trees, with one exception (the graph in Fig. 6(c)), by means of splitting of Smith trees into three parts in the described way.

Let us now examine all other possibilities of splitting of a Smith tree into three parts. If we split a Smith tree (S215, S313 or S222) at the vertex of degree three and get three paths, corresponding cacti are subgraphs of the graph in Fig. 6(c). The splitting of the form as in Fig. 14, where S_1 is between the two paths results in cacti that are either subgraphs of the graph in Fig. 6(c), or some of the ten graphs we have just described. □

5. H graphs

In this section, we will discuss graphs of type H from [15]. A few of them are shown in Appendix B, Fig. B1. We will examine σ_4, σ_5, σ_6, and σ_7-extensions of the tricyclic graph in Fig. 15(a) to obtain maximal reflexive cacti with four cycles of type H.

Example. We start with the tricyclic graph shown in Fig. 15(a). In the Smith tree S215 we identify vertices s_6 and s_9 to obtain the graph in Fig. 15(c). Now we split that graph at the vertex $s_6 = s_9$, as in Fig. 15(d) and attach these two graphs to get the graph in Fig. 15(e). This is the graph H16 from [15], a maximal reflexive graph of type H.

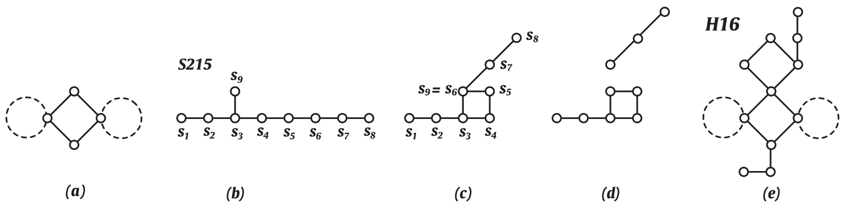

Fig. 15. An example for σ_4 and σ_5.

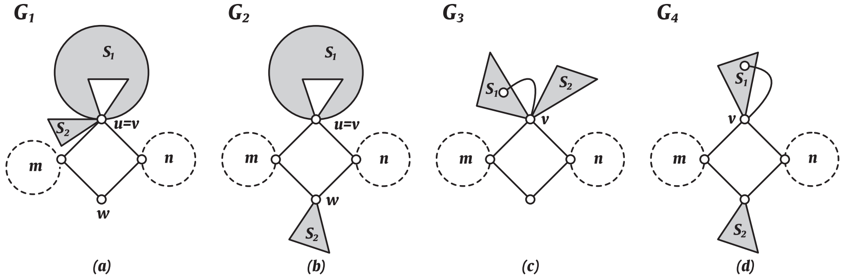

Fig. 16. Graphs of Proposition 2.

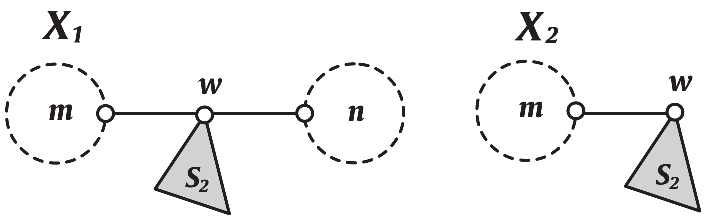

Fig. 17. Auxiliary graphs in the proof of Proposition 2.

The tricyclic graph in Fig. 15(a) is the starting graph in our investigations. Although it is a proper subgraph of the graphs we are studying, for this graph we already have $\lambda_2 = 2$. Extending this graph we get maximal reflexive cacti within the given class and for many of them $\lambda_2 = \lambda_3 = 2$ holds.

Proposition 2. *For all graphs of type G_i, shown in Fig. 16, $P_{G_i}(2) = 0$ holds, $i \in \{1, 2, 3, 4\}$.*

Proof. Consider the graph G_2 in Fig. 16(b). Applying Lemma 2.1 at the vertex $u = v$, we compute the value $P_{G_2}(2)$. We use auxiliary graphs X_1 and X_2 in Fig. 17, where $P_{X_1}(2) = 0$ and $P_{X_2}(2) = -mP_{S_2-w}(2)$ hold.

$$P_{G_2}(2) = 2 \cdot 0 - n(-m)P_{S_1-u}(2)P_{S_2-w}(2) - (-n)mP_{S_1-u}(2)P_{S_2-w}(2) - 2mnP_{S_1-u}(2)P_{S_2-w}(2) = 0.$$

Analogously, we get $P_{G_i}(2) = 0$ for $i \in \{1, 3, 4\}$ in Fig. 16(a), (c) and (d). □

Examining all cases of graphs of type H, we come to the following conclusions.

Theorem 3. *σ_4, σ_5, σ_6, and σ_7-extensions of the tricyclic graph in Fig. 15(a) lead to maximal reflexive cacti with four cycles.*

Remark. Maximal reflexive cacti H1, H5, H7, H9, H11, H30, H32, H33, H35, H36 and H38 are graphs of type G_1, Fig. 16(a). Maximal reflexive cacti H3, H10, H14, H16, H18, H23, H42, H43, H44, H45 and H48 are graphs of type G_2, Fig. 16(b). Maximal reflexive cacti H2, H4, H12, H19, H31, H35, H37, H39 and H41 are graphs of type G_3, Fig. 16(c). Maximal reflexive cacti H6, H8, H13, H24, H25, H26, H29, H46 and H47 are graphs of type G_4, Fig. 16(d).

Maximal reflexive graphs H15, H20, H21, H22, H34 are supergraphs of some of the graphs of type (a), obtained by adding a pendent edge. Graphs H17, H40 and H27, H28 are subgraphs of the graphs of types (c) and (d), respectively, obtained by removing a pendent edge.

For corresponding graphs of type (c) and (d), mentioned in the previous theorem, $P_{G_i}(2) = 0$ holds, but $\lambda_2 > 2$ and $\lambda_3 = 2$.

6. Conclusions

We have studied the role of Smith trees in maximal reflexive cacti with four cycles. We have established and proved presence of several types of modifications of Smith trees within these graphs. We discussed whole Smith trees and following modifications of Smith trees: a Smith tree split into two or three parts, a Smith tree with two identified vertices, with or without splitting afterwards, a Smith tree with an added edge, with or without splitting afterwards (modifications σ_i, $i = 1, 2, \ldots, 7$).

Many of the maximal reflexive cacti with four cycles follow these patterns. Those that do not are exceptions in which we clearly see some forms of the presence of Smith trees, but so far it has not led to any generalization within these classes.

These exceptions, and the cases of the maximal reflexive cacti with $\lambda_2 < 2$, as well as other maximal reflexive cacti with less than four cycles, are interesting subjects for future investigations of this type.

Acknowledgements

We used the expert system *newGRAPH* [22] for computations of eigenvalues of graphs.

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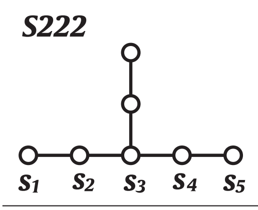
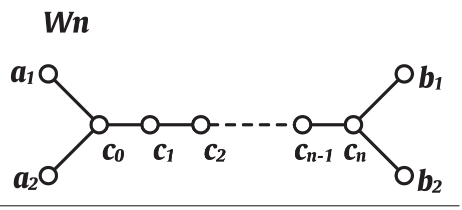
Appendix A. Tables

For a Smith tree S let us introduce the notation: $U_S = P_{S-u}(2), V_S = P_{S-v}(2), P_{S-u-v}(2) = W, P_{S-p}(2) = C$, where p is the unique path connecting vertices u and v, within the Smith tree.

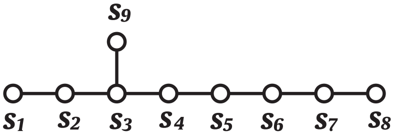




	Us	Vs	W	C		Us	Vs	W	C
(s_1, s_2)	3	12	6	6	a_1a_2	4	4	4	4
(s_1, s_3)	3	27	18	9	a_1c_k	4	16	$4(k+2)$	8
(s_1, s_4)	3	12	10	6	a_1b_1	4	4	$n+4$	4
(s_1, s_5)	3	3	4	3	c_kc_{k+l}	16	16	$16l$	16
(s_2, s_3)	12	27	18	18					
(s_2, s_4)	12	12	16	12					

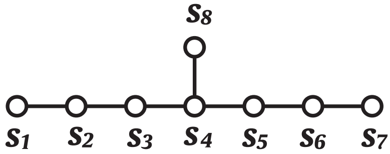
$k, n \in N_0, l \in N$

S215



	Us	Vs	W	C
(s_1, s_2)	4	16	8	8
(s_1, s_3)	4	36	24	12
(s_1, s_4)	4	25	20	10
(s_1, s_5)	4	16	16	8
(s_1, s_6)	4	9	12	6
(s_1, s_7)	4	4	8	4
(s_1, s_8)	4	1	4	2
(s_1, s_9)	4	9	8	6
(s_2, s_3)	16	36	24	24
(s_2, s_4)	16	25	30	20
(s_2, s_5)	16	16	32	16
(s_2, s_6)	16	9	30	12
(s_2, s_7)	16	4	24	8
(s_2, s_8)	16	1	14	4
(s_2, s_9)	16	9	16	12
(s_3, s_4)	36	25	30	30
(s_3, s_5)	36	16	48	24
(s_3, s_6)	36	9	54	18
(s_3, s_7)	36	4	48	12
(s_3, s_8)	36	1	30	6
(s_3, s_9)	36	9	18	18
(s_4, s_5)	25	16	20	20
(s_4, s_6)	25	9	30	15
(s_4, s_7)	25	4	30	10
(s_4, s_8)	25	1	20	5
(s_4, s_9)	25	9	20	15
(s_5, s_6)	16	9	12	12
(s_5, s_7)	16	4	16	8
(s_5, s_8)	16	1	12	4
(s_5, s_9)	16	9	20	12
(s_6, s_7)	9	4	6	6
(s_6, s_8)	9	1	6	3
(s_6, s_9)	9	9	18	9
(s_7, s_8)	4	1	2	2
(s_7, s_9)	4	9	14	6
(s_8, s_9)	1	9	8	3

S313



	Us	Vs	W	C
(s_1, s_2)	2	8	4	4
(s_1, s_3)	2	18	12	6
(s_1, s_4)	2	32	24	8
(s_1, s_5)	2	18	15	6
(s_1, s_6)	2	8	8	4
(s_1, s_7)	2	2	3	2
(s_1, s_8)	2	8	7	4
(s_2, s_3)	8	18	12	12
(s_2, s_4)	8	32	32	16
(s_2, s_5)	8	18	24	12
(s_2, s_6)	8	8	16	8
(s_2, s_8)	8	8	12	8
(s_3, s_4)	18	32	24	24
(s_3, s_5)	18	18	27	18
(s_3, s_8)	18	8	15	12
(s_4, s_8)	32	8	16	16

Appendix B

A selection of maximal reflexive RS-undecidable cacti with four cycles (whose cycles do not form a bundle) from [15,16].

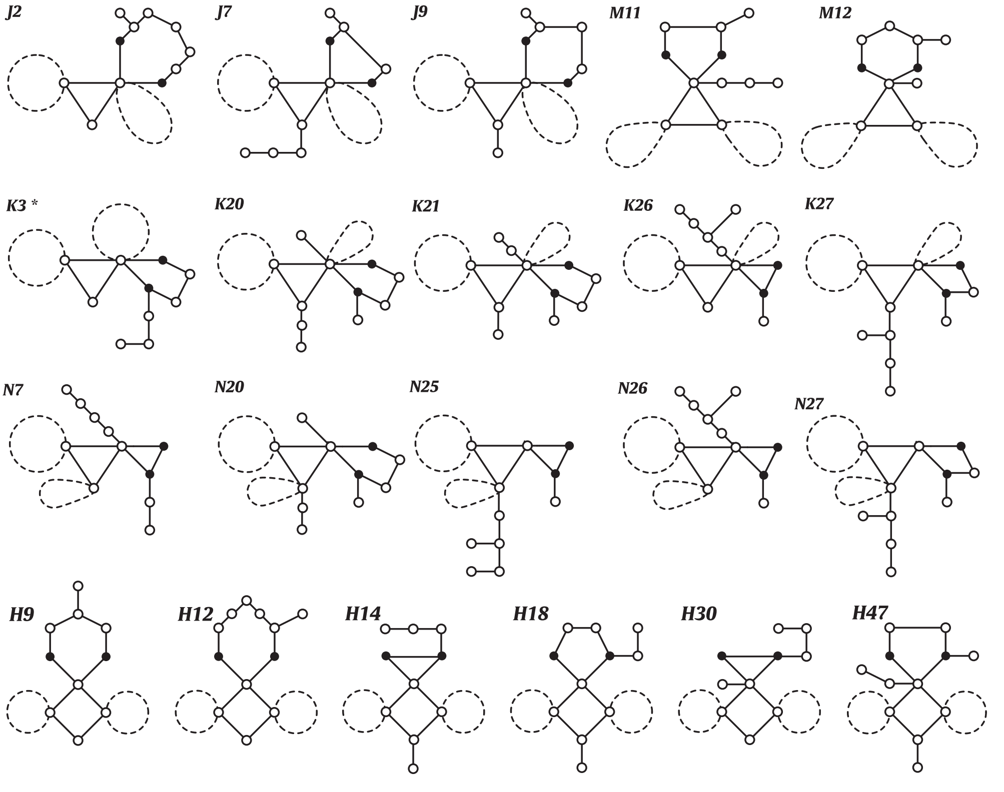


Fig. B1. A selection of graphs of type J, K, M, N and H to accompany Sections 3 and 5. (We show 21 out of 170 families of these graphs.)

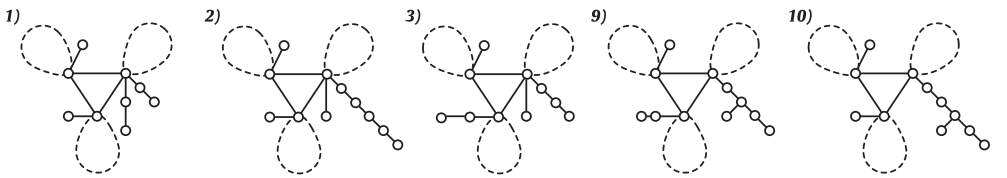


Fig. B2. A selection of graphs studied in Section 4. (We show 5 out of 10 of these families.)

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