

**Abstract**

Let $H$ be a fixed graph. An $H$-covering of $G$ is a set $L = \{H_1, H_2, \ldots, H_k\}$ of subgraphs of $G$, where each subgraph $H_i$ is isomorphic to $H$ and every edge of $G$ appears in at least one member of $L$. If there exists an $H$-covering of $G$, $G$ is called $H$-coverable. An $H$-covering of $G$ with $k$ copies $H_1, H_2, \ldots, H_k$ of $H$ is called minimal if, for any $H_j, \bigcup_{i=1}^{k} H_i \neq H_j$ is not an $H$-covering of $G$. An $H$-covering of $G$ with $k$ copies $H_1, H_2, \ldots, H_k$ of $H$ is called minimum if there exists no $H$-covering with less than $k$ copies of $H$. A graph $G$ is called $H$-equicoverable if every minimal $H$-covering in $G$ is also a minimum $H$-covering in $G$. In this paper, we investigate the characterization of $P_3$-equicoverable graphs.

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1. Introduction and preliminaries

A vertex of degree 0 is called an isolated vertex. All graphs considered here are finite, simple and without any isolated vertices. A graph $G$ has order $|V(G)|$ and size $|E(G)|$. The edge-degree of an edge $e$ in a graph $G$, written $d_G^{(1)}(e)$ or $d^{(1)}(e)$, is the number of edges adjacent to $e$. We denote by $NG(e)$ the set of all the adjacent edges of $e$. The edge with edge-degree 0 is an isolated edge. The maximum edge-degree of $G$ is denoted by $d^{(1)}(G)$ and the minimum edge-degree of $G$ is denoted by $\delta^{(1)}(G)$. The path and circuit on $k$ vertices are denoted by $P_k$ and $C_k$, respectively. A star is a tree consisting of one vertex adjacent to all the others. The $(n+1)$-vertex star is the biclique $K_{1,n}$. A double-star is a tree containing two central vertices plus leaves. A graph $H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. Suppose that $E'$ is a nonempty subset of $E$. The subgraph of $G$ whose vertex set is the set of ends of edges in $E'$ and whose edge set is $E'$ is called the subgraph of $G$ induced by $E'$ and is denoted by $G[E']$. $G[E']$ is an edge-induced subgraph of $G$.

Let $H$ be a subgraph of $G$. By $G - H$, we denote the graph remaining after we delete from $G$ the edges of $H$ and any resulting isolated vertices. A collection of copies of $H$, say $H_1, H_2, \ldots, H_k$, is called an $H$-packing in $G$ if they are edge-disjoint. An $H$-packing in $G$ with $k$ copies $H_1, H_2, \ldots, H_k$ of $H$ is called maximal if $G - \bigcup_{i=1}^{k} E(H_i)$ contains no subgraph isomorphic to $H$. An $H$-packing in $G$ with $k$ copies $H_1, H_2, \ldots, H_k$ of $H$ is called maximum if no more than $k$ edge-disjoint copies of $H$ can be packed into $G$. A graph $G$ is called $H$-equipackable if every maximal $H$-packing in

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**P3-equicoverable graphs—Research on H-equicoverable graphs**

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\[ G \text{ is also a maximum } H\text{-packing in } G. \] Recently, Vestergaard et al. [2,5,3] characterized \( P_3 \)-equipackable graphs and the author and Fan [6] characterized \( M_2 \)-equipackable graphs.

A related idea of graph packing is graph covering. An \( H\text{-covering} \) of \( G \) is a set \( L = \{H_1, H_2, \ldots, H_k\} \) of subgraphs of \( G \), where each subgraph \( H_i \) is isomorphic to \( H \) and every edge of \( G \) appears in at least one member of \( L \). If \( G \) has an \( H\text{-covering} \), \( G \) is called \( H\text{-coverable} \). A graph \( G \) is called \( H\text{-decomposable} \) if it has an \( H\text{-packing} \) which is also an \( H\text{-covering} \). The following lemma is a well-known result for \( H = P_3 \):

**Lemma 1.1 (Caro, Ruiz [4])**. A connected graph is \( P_3\text{-decomposable} \) if and only if it has even size.

**Definition 1.2.** An \( H\text{-covering} \) of \( G \) with \( k \) copies \( H_1, H_2, \ldots, H_k \) of \( H \) is called minimal if, for any \( H_j, \bigcup_{i=1}^k H_i - H_j \) is not an \( H\text{-covering} \) of \( G \). An \( H\text{-covering} \) of \( G \) with \( k \) copies \( H_1, H_2, \ldots, H_k \) of \( H \) is called minimum if there exists no \( H\text{-covering} \) with less than \( k \) copies of \( H \). A graph \( G \) is called \( H\text{-equicoverable} \) if every minimal \( H\text{-covering} \) in \( G \) is also a minimum \( H\text{-covering} \) in \( G \).

In this paper, we characterize \( P_3\text{-equicoverable} \) graphs.

The following proposition is clearly true.

**Proposition 1.3.** A graph is \( P_3\text{-coverable} \) if and only if it has no isolated edges.

Note that when \( G \) is isomorphic to \( K_2 \) or \( M_2 \), it is not \( P_3\text{-coverable} \). When \( G \cong P_3 \), \( G \) is clearly \( P_3\text{-equicoverable} \). So we characterize \( P_3 \)-graphs with size at least 3 and without any isolated edges in the following.

**Proposition 1.4 (Ruiz [4]).** Every connected graph \( G \) with at least two edges has an edge \( f \) such that \( G - f \) contains exactly one nonempty component.

**Lemma 1.5.** Let \( G \) be a connected graph with size \( m \geq 3 \). The number of \( P_3 \) in a minimum \( P_3\text{-covering} \) of \( G \) is \( \lceil \frac{m}{2} \rceil \).

**Proof.** When \( m \) is even, \( G \) is \( P_3\text{-decomposable} \) by Lemma 1.1. So \( G \) has a \( P_3\text{-covering} \) with only \( \frac{m}{2} = \lceil \frac{m}{2} \rceil \) copies of \( P_3 \) which clearly is minimum.

When \( m \) is odd, by Proposition 1.4, there exists an edge \( f \) such that the subgraph induced by \( E(G - f) \) is connected and with size even. So \( G \) has a \( P_3\text{-covering} \) with \( \frac{m-1}{2} + 1 = \frac{m+1}{2} = \lceil \frac{m}{2} \rceil \) copies of \( P_3 \) which is minimum. \( \square \)

**Lemma 1.6.** Let \( G \) be a connected graph with size \( m \geq 3 \) and maximum edge-degree \( k \). If \( k > \lceil \frac{m}{2} \rceil \), then \( G \) is not \( P_3\text{-equicoverable} \).

**Proof.** Assume that \( e \) is an edge with edge-degree \( k \) and \( N_G(e) = \{e_1, e_2, \ldots, e_k\} \). Denote \( G[\{e, e_i\}] \) by \( H_i \). Then \( L = \{H_1, H_2, \ldots, H_k\} \) is a minimal \( P_3\text{-covering} \) of \( N_G(e) \cup e \). So the minimal \( P_3\text{-covering} \) of \( G \) that contains \( H_i(i = 1, 2, \ldots, k) \) has at least \( k > \lceil \frac{m}{2} \rceil \) copies of \( P_3 \). By Lemma 1.5, \( G \) is not \( P_3\text{-equicoverable} \). \( \square \)

By Lemma 1.6, we know that

**Remark 1.7.** All stars \( K_{1,t}(t \geq 4) \), the paw and the graph shown in Fig. 1 are not \( P_3\text{-equicoverable} \).

For convenience, we call a connected subgraph \( G_0 \) of \( G \) forbidden if \( G_0 \) is not \( P_3\text{-equicoverable} \) and \( G - G_0 \) contains no isolated edges. Then we have the following important lemma:

**Lemma 1.8.** Let \( G \) be a connected graph with size \( m > 2 \). If \( G \) contains a forbidden subgraph \( G_0 \), then \( G \) is not \( P_3\text{-equicoverable} \).

![Fig. 1. The paw and a graph which are not P3-equicoverable.](image)
Proof. Since $G_0$ is not $P_3$-equicoverable, by Lemma 1.5, it has a minimal $P_3$-covering with $k_0$ copies of $P_3$, where $k_0 > \lceil \frac{m_0}{2} \rceil$ ($m_0$ is the size of $G_0$). Suppose that $G - G_0$ has $s$ components $G_1, G_2, \ldots, G_s$. Since $G_0$ is forbidden, each $G_i$ ($i = 1, 2, \ldots, s$) has no isolated edges. Denote the size of $G_i$ by $m_i$. Then $G$ has a minimal $P_3$-covering with $k$ copies of $P_3$, where $k = k_0 + \sum_{i=1}^{s} \lceil \frac{m_i}{2} \rceil > \lceil \frac{m_2}{2} \rceil$. So $G$ is not $P_3$-equicoverable. □

2. Main results

We first characterize paths and cycles which are $P_3$-equicoverable.

Lemma 2.1. The path $P_n$ is $P_3$-equicoverable if and only if $n = 3, 4, 5, 6, 8$.

Proof. We can easily verify that $P_3, P_4, P_5, P_6, P_8$ are all $P_3$-equicoverable.

The path $P_7$ has a minimal $P_3$-covering with $4 > 3$ copies of $P_3$, so it is not $P_3$-equicoverable. When $n \geq 9$, $P_7$ is a forbidden subgraph of $P_n$. By Lemma 1.8, $P_n$ ($n \geq 9$) is not $P_3$-equicoverable. □

Lemma 2.2. The cycle $C_n$ is $P_3$-equicoverable if and only if $n = 3, 4, 5, 7$.

Proof. We can easily verify that $C_3, C_4, C_5, C_7$ are all $P_3$-equicoverable.

The cycle $C_6$ has a minimal $P_3$-covering with $4 \geq 3$ copies of $P_3$, so it is not $P_3$-equicoverable. When $n \geq 8$, $P_7$ is a forbidden subgraph of $C_n$. By Lemma 1.8, $C_n$ ($n \geq 8$) is not $P_3$-equicoverable. □

We introduce a useful definition.

Definition 2.3. A $k$-extendedstar is a tree obtained from a star $K_{1,k}$ by performing elementary subdivisions on each edge; that is, a $k$-extendedstar has one vertex of degree $k$ (called the center of the $k$-extendedstar), $k$ vertices of degree 2 and $k$ leaves. We denote it by $S^*_k$.

See Fig. 2 for a 7-extendedstar $S^*_7$.

The following lemma is clearly true:

Lemma 2.4. Each $k$-extendedstar is $P_3$-equicoverable.

Remark 2.5. Clearly, $P_3$ can be denoted by $S^*_1$ and $P_5$ can be denoted by $S^*_2$.

Then we consider graphs that contains a cycle.

For convenience, we denote by $C_3 \cdot S^*_k$ a graph obtained from a cycle $C_3$ and a $k$-extendedstar $S^*_k$ ($k \geq 1$) by identifying one vertex of the cycle $C_3$ with the center of $S^*_k$. See Fig. 3 for $k = 4$.

Lemma 2.6. Let $G$ be a connected graph that is not a cycle. If $G$ contains a 3-cycle, then $G$ is $P_3$-equicoverable if and only if $G$ is a graph of the form $C_3 \cdot S^*_k$.
Proof. Each graph \( C_3 \cdot S_k^+ \) is clearly \( P_3 \)-equicoverable.

Conversely, suppose that \( G \) is a \( P_3 \)-equicoverable graph that contains a 3-cycle. Let \( v_1, v_2, v_3 \) be the vertices of such a 3-cycle in \( G \). Since \( G \) is not a cycle and \( G \) is connected, there exists a vertex \( v_4 \) which is adjacent to some \( v_i (i = 1, 2, 3) \), say \( v_3 \).

Define \( S = \{v_1v_2, v_2v_3, v_1v_3, v_3v_4\} \). For the paw \( G[S] \), we first consider four cases.

Case 1: There exists no isolated edge in \( G - G[S] \). So \( G[S] \) is forbidden. By Lemma 1.8, \( G \) is not \( P_3 \)-equicoverable. This is a contradiction.

Case 2: There exist two isolated edges \( e_1, e_2 \) in \( G - G[S] \). Up to isomorphism, there are just six possibilities.

Subcase 1: One of the isolated edges is incident with \( v_1 \), the other is incident with \( v_2 \). We denote them by \( v_1u_1 \) and \( v_2u_2 \). Then the subgraph induced by the edges \( \{u_1v_1, v_1v_2, v_2u_2, v_2v_3\} \) is forbidden.

Subcase 2: One of the isolated edges is incident with \( v_1 \), the other is incident with \( v_3 \). We denote them by \( v_1u_1 \) and \( v_3u_2 \). Then the subgraph induced by the edges \( \{v_1v_3, v_2v_3, u_2v_3, v_4v_3\} \) is forbidden.

Subcase 3: One of the isolated edges is incident with \( v_1 \), the other is incident with \( v_4 \). We denote them by \( v_1u_1 \) and \( v_4u_2 \). Then the subgraph induced by the edges \( \{v_1v_3, v_2v_3, u_2v_4, v_4v_3\} \) is forbidden.

Subcase 4: One of the isolated edges is incident with \( v_3 \), the other is incident with \( v_4 \). We denote them by \( v_3u_1 \) and \( v_4u_2 \). Then the subgraph induced by the edges \( \{v_1v_3, v_2v_4, u_2v_4, u_4v_3\} \) is forbidden.

Subcase 5: One of the isolated edges is \( v_1v_4 \), the other is incident with \( v_3 \). We denote it by \( v_3u_1 \). Then the subgraph induced by the edges \( \{v_1v_3, v_2v_3, u_1v_3, v_4v_3\} \) is forbidden.

Subcase 6: One of the isolated edges is \( v_1v_4 \), the other is incident with \( v_2 \). We denote it by \( v_2u_1 \). Then the subgraph induced by the edges \( \{v_1v_2, v_2v_3, v_1v_3, v_2u_1\} \) is forbidden.

In all subcases, \( G \) is not \( P_3 \)-equicoverable; that is, \( G - G[S] \) cannot contain two isolated edges.

Case 3: There exist three isolated edges \( e_1, e_2, e_3 \) in \( G - G[S] \). Up to isomorphism, there are just three possibilities.

Subcase 1: \( e_1 = u_1v_1, e_2 = u_2v_2, e_3 = u_3v_3 \). Then the subgraph induced by the edges \( \{v_1v_3, v_2v_3, v_3u_4, v_3u_3\} \) is forbidden.

Subcase 2: \( e_1 = u_1v_1, e_2 = u_2v_3, e_3 = u_3v_4 \). Then the subgraph induced by the edges \( \{v_2v_3, u_2v_3, v_3v_4, v_4u_3\} \) is forbidden.

Subcase 3: \( e_1 = u_1v_1, e_2 = u_2v_2, e_3 = u_3v_4 \). Then the subgraph induced by the edges \( \{v_1v_3, v_2v_3, v_3u_4, v_4u_3\} \) is forbidden.

So \( G - G[S] \) cannot contain three isolated edges.
Case 4: There exist four isolated edges $e_1, e_2, e_3, e_4$ in $G - G[S]$. $e_1 = u_1v_1, e_2 = u_2v_2, e_3 = u_3v_3, e_4 = u_4v_4$. Then $G$ is the graph $G[S] \cup \{e_1, e_2, e_3, e_4\}$ which contains a forbidden subgraph induced by the edges $\{u_3v_3, v_2v_3, v_3v_4, v_4u_4\}$.

So $G - G[S]$ cannot contain four isolated edges.

Thus there remains only one possibility: the graph $G - G[S]$ has exactly one isolated edge $e$. Define $G_0 = G[S] \cup \{e\}$.

Then $G - G_0$ has no isolated edges. We can get the following statements:

1. The isolated edge $e$ must be only incident with $v_4$ (we denote $e$ by $v_4v_5$). Otherwise, if $e$ is incident with $v_1, v_2$ or $v_3$, then $A^{(1)}(G_0) > \frac{m(G_0)}{2}$. By Lemma 1.6, the subgraph $G_0$ is forbidden.

So $G$ is not $P_3$-equicoverable.

2. Only $v_3$ has neighbors in $G - G_0$.

   (i) By (1), $v_4$ and $v_5$ have no other neighbors in $G - G_0$.

   (ii) Vertices $v_1$ and $v_2$ have no other neighbors in $G - G_0$.

Otherwise, if $v_1$ and $v_2$ have other neighbors (edge $v_1v_2$ has adjacent edge) in $G - G_0$, let $G_1 = G_0 - v_1v_2$. Since there are no isolated edges at $v_3, v_4$ and $v_5$ in $G - G_0$, there are no isolated edges at $v_3, v_4$ and $v_5$ in $G - G_1$. Then $G - G_1$ has no isolated edges, $G_1$ is forbidden.

So $G$ is not $P_3$-equicoverable. This is a contradiction.

3. For any adjacent vertex $u$ of $v_3$, $d(u) = 2$.

   Otherwise, let $u$ be a neighbor of $v_3$. Whether $d(u) = 1$ or $d(u) \geq 3$, the subgraph induced by the edges $\{v_2v_3, uv_3, v_3v_4, v_4v_5\}$ is always forbidden.

4. In $G - G_0$, all the paths beginning with $v_3$ have length no more than 2.

   Otherwise, if there exists a $l$-path $v_3u_1u_2 \ldots u_l (l \geq 3)$ with one endpoint $v_3$ in $G - G_0$, we can find a forbidden subgraph induced by the edges $\{u_1v_3, u_2v_3, v_3v_4, v_4v_5\}$.

From above, $G$ is a graph of the form $C_3 \cdot S_k^*$. □

We denote by $C_4 \cdot P_2 \cdot S_k^*$ a graph obtained from a cycle $C_4$ and a $k$-extended star $S_k^* (k \geq 0)$ by adding an edge between a vertex of the cycle $C_4$ and the center of the $k$-extended star. See Fig. 4 for $k = 4$.

Lemma 2.7. Let $G$ be a connected graph that is not a cycle. If $G$ contains a 4-cycle, then $G$ is $P_3$-equicoverable if and only if $G$ is a graph of the form $C_4 \cdot P_2 \cdot S_k^*$.

Proof. Clearly, each graph $C_4 \cdot P_2 \cdot S_k^*$ is $P_3$-equicoverable.
Conversely, suppose that $G$ is a $P_3$-equicoverable graph that contains a 4-cycle. By Lemma 2.6, $G$ contains no 3-cycle. Let $v_1, v_2, v_3, v_4$ be the vertices of such a 4-cycle in $G$. Since $G$ is not a cycle and is connected, there exists a vertex $v_5$ which is adjacent to some $v_i (i = 1, 2, 3, 4)$, say $v_3$.

Let $G_0$ be the subgraph induced by the edges $\{v_1v_2, v_2v_3, v_3v_4, v_3v_5\}$. For $G - G_0$, we first consider four cases.

**Case 1:** The graph $G - G_0$ has four isolated edges. One must be $v_1v_4$. Denote the others by $u_2v_2, u_3v_3, u_4v_5$. Then $G$ is the graph $G_0 \cup \{v_1v_4, u_2v_2, u_3v_3, u_4v_5\}$ which contains a forbidden subgraph induced by the edges $\{u_2v_2, v_1v_2, v_2v_3, v_3u_3\}$.

So $G$ is not $P_3$-equicoverable. This is a contradiction; that is, $G - G_0$ cannot contain four isolated edges.

**Case 2:** The graph $G - G_0$ has three isolated edges $e_1, e_2, e_3$. Since $G$ has no 3-cycle, there are four possibilities.

**Subcase 1:** They are, respectively, incident with $v_2, v_3, v_5$. Denote them by $e_1 = u_2v_2, e_2 = u_3v_3, e_3 = u_4v_5$. Then the subgraph induced by the edges $\{u_2v_2, v_1v_2, v_2v_3, v_3u_3\}$ is forbidden.

**Subcase 2:** One of the isolated edges is $v_1v_4$, the others are, respectively, incident with $v_2$ and $v_3$ (denote them by $v_2u_2, v_3u_3$). Then the subgraph induced by the edges $\{v_2v_2, v_3u_4, v_3v_5, v_3u_3\}$ is forbidden.

**Subcase 3:** One of the isolated edges is $v_1v_4$, the others are, respectively, incident with $v_2$ and $v_5$ (denote them by $v_2u_2, v_5u_4$). Then the subgraph induced by the edges $\{v_3v_2, v_3u_4, v_3v_5, v_5u_4\}$ is forbidden.

**Subcase 4:** One of the isolated edges is $v_1v_4$, the others are, respectively, incident with $v_3$ and $v_5$ (denote them by $v_3u_3, v_5u_4$). Then the subgraph induced by the edges $\{v_3u_3, v_3v_4, v_3v_5, v_5u_4\}$ is forbidden.

So $G - G_0$ cannot contain three isolated edges.

**Case 3:** The graph $G - G_0$ has two isolated edges $e_1, e_2$. Since $G$ has no 3-cycle, there are six possibilities.

**Subcase 1:** One of the isolated edges is $v_1v_4$, the other is incident with $v_2$ (denote it by $v_2u_1$). Then the subgraph induced by the edges $\{v_4v_1, v_1v_2, v_2v_3, v_2u_1\}$ is forbidden.

**Subcase 2:** One of the isolated edges is $v_1v_4$, the other is incident with $v_3$ (denote it by $v_3u_1$). Then the subgraph induced by the edges $\{v_4v_3, v_2v_3, u_1v_3, v_3u_3\}$ is forbidden.

**Subcase 3:** One of the isolated edges is $v_1v_4$, the other is incident with $v_5$ (denote it by $v_5u_1$). Then the subgraph induced by the edges $\{v_4v_3, v_2v_3, v_3v_5, v_5u_1\}$ is forbidden.

**Subcase 4:** One of the isolated edges is incident with $v_2$, the other is incident with $v_3$. Denote them by $v_2u_1$ and $v_3u_2$. Then the subgraph induced by the edges $\{u_1v_2, v_2v_3, v_3u_2, v_3u_3\}$ is forbidden.

**Subcase 5:** One of the isolated edges is incident with $v_2$, the other is incident with $v_5$. Denote them by $v_2u_1$ and $v_5u_2$. Then the subgraph induced by the edges $\{v_4v_3, v_2v_3, v_3v_5, v_5u_2\}$ is forbidden.

**Subcase 6:** One of the isolated edges is incident with $v_3$, the other is incident with $v_5$. Denote them by $v_3u_1$ and $v_5u_2$. Then the subgraph induced by the edges $\{u_1v_3, v_2v_3, v_3v_5, v_5u_2\}$ is forbidden.
In all subcases, \( G \) is not \( P_3 \)-equicoverable; that is, \( G - G_0 \) cannot contain two isolated edges.

**Case 4:** The graph \( G - G_0 \) has no isolated edge. So \( G_0 \) is forbidden, and \( G \) is not \( P_3 \)-equicoverable.

Thus there remains one possibility: \( G - G_0 \) has only one isolated edge \( e \). The following statements are true.

1. The edge \( e \) must be \( v_1v_4 \).
   Otherwise, since \( G \) has no 3-cycles, there are three possibilities.
   (i) The isolated edge is incident with \( v_2 \). We denote it by \( uv_2 \). Then the subgraph induced by the edges \( \{uv_2, v_1v_2, v_2v_3, v_3v_5\} \) is forbidden.
   (ii) The isolated edge is incident with \( v_3 \). We denote it by \( uv_3 \). Then the subgraph induced by the edges \( \{uv_3, v_1v_2, v_2v_3, v_3v_5\} \) is forbidden.
   (iii) The isolated edge is incident with \( v_5 \). We denote it by \( uv_5 \). Then the subgraph induced by the edges \( \{uv_5, v_5v_3, v_3v_4, v_3v_2\} \) is forbidden.

So the isolated edge \( e \) must be \( v_1v_4 \).

2. None of the vertices \( v_1, v_2 \) and \( v_4 \) has a neighbor in \( G - G_0 - e \) since the edge \( v_1v_4 \) is isolated and \( v_2 \) is symmetric to \( v_4 \) in \( G_0 \cup v_1v_4 \) (we can let \( G_0 \) be the subgraph induced by the edges \( \{v_1v_4, v_2v_3, v_3v_4, v_3v_5\} \), then the edge \( v_1v_2 \) is isolated in \( G - G_0 \)).

3. The vertex \( v_3 \) has no neighbor in \( G - G_0 - e \).
   Otherwise, if \( v_3 \) has a neighbor \( u \) in \( G - G_0 - e \), we denote by \( G_2 \) the subgraph induced by the edges \( \{uv_3, v_1v_2, v_2v_3, v_3v_5\} \). For \( G - G_2 \), since \( G - G_0 - e \) has no isolated edge, there are two possibilities.
   (i) There exists no isolated edge in \( G - G_2 \). So \( G_2 \) is forbidden.
   (ii) There exists exactly one isolated edge \( e \) which must be incident with \( u \). Then the subgraph induced by the edges \( \{v_5v_3, v_4v_3, v_3u, e\} \) is forbidden.

We can get that \( G \) is not \( P_3 \)-equicoverable. So \( v_3 \) has no neighbor in \( G - G_0 - e \).

4. The vertex \( v_5 \) has no neighbor or has neighbors with degree 2 in \( G - G_0 - e \).
   (a) If \( v_5 \) has no neighbor, then \( G = G_0 \cup v_1v_4 \). And \( G \) is clearly \( P_3 \)-equicoverable.
   (b) If \( v_5 \) has neighbors, then for any neighbor \( u \) of \( v_5 \), \( d(u) = 2 \).
   Denote by \( G_3 \) the subgraph induced by the edges \( \{uv_5, v_5v_3, v_2v_3, v_3v_4\} \).
   If \( d(u) = 1 \), there are two possibilities for \( G - G_3 \).
   (i) There exists no isolated edge in \( G - G_3 \). So \( G_3 \) is forbidden.
   (ii) There exists exactly one isolated edge \( e \) which must be incident with \( v_5 \). Then the subgraph induced by the edges \( \{v_2v_3, v_3v_5, v_5u, e\} \) is forbidden.
If \( d(u) \geq 3 \), \( G_3 \) is forbidden.
So \( d(u) = 2 \).
Another neighbor \( u_1 \) of \( u \) is a leaf. Otherwise, \( G - G_3 \) contains no isolated edge and \( G_3 \) is forbidden.
From above, \( G \) is a graph of the form \( C_4 \cdot P_2 \cdot S_k \). \( \square \)

**Lemma 2.8.** Let \( G \) be a connected graph that is not a cycle. If there exists a cycle with length larger than 4 in \( G \), then \( G \) is not \( P_3 \)-equicoverable.

**Proof.** Assume that there exists a cycle \( C = v_1 v_2 \ldots v_k v_1 \) with \( k \geq 5 \) in \( G \). Let \( V_1 = \{ v_1, v_2, \ldots, v_k \} \). We consider two cases.

**Case 1:** \( V_1 = V(G) \); that is, \( G \) has no vertex outside \( V_1 \). Since \( G \) is not a cycle, there exist two vertices \( v_i \) and \( v_j \) in \( V_1 \) which are adjacent. We denote by \( G_0 \) the subgraph induced by the edges \( \{v_{i+2}v_{i+1}, v_{i+1}v_i, v_iv_j, v_jv_{i-1}\} \) (subscripts modulo \( k \)). Note that all the edges in \( G - G_0 \) are attached to the \( (k - 3) \)-path \( v_{i-1}v_{i-2} \ldots v_1v_kv_{k-1} \ldots v_{i+2} \) (\( k \geq 5 \)). So \( G - G_0 \) contains no isolated edges. And \( G_0 \) is forbidden. Consequently, \( G \) is not \( P_3 \)-equicoverable.

**Case 2:** \( V_1 \subset V(G) \); that is, \( G \) has other vertices outside \( V_1 \). Since \( G \) is connected, there exists a vertex \( v_l \) in \( V(G) - V_1 \) which is adjacent to some vertex \( v_i \) in \( V_1 \).

We denote by \( G'_0 \) the subgraph induced by the edges \( \{v_{i+2}v_{i+1}, v_{i+1}v_i, v_iv_j, v_jv_{i-1}\} \). Since \( v_{i-1} \) and \( v_{i+2} \) are endpoints of the \( (k - 3) \)-path \( v_{i-1}v_{i-2} \ldots v_1v_kv_{k-1} \ldots v_{i+2} \), there are four subcases for \( G - G'_0 \):

**Subcase 1:** The graph \( G - G'_0 \) contains only one isolated edge \( e \). Then there are three possibilities for \( e \).

(i) It is incident with \( v_i \). We denote it by \( v_iw \). Then \( w \) and \( v_i \) have no neighbors and there is no isolated edge at \( v_i \). The subgraph induced by the edges \( \{wv_i, v_iv_j, v_iv_{i+1}, v_{i+1}v_i\} \) is forbidden.

(ii) It is incident with \( v_l \). We denote it by \( v_iw \). Then \( w \) is not in \( V_1 \). The subgraph induced by the edges \( \{wv_i, v_iv_j, v_iv_{i+1}, v_{i+1}v_i, v_{i+1}v_{i+2}\} \) is forbidden.

(iii) It is incident with \( v_{i+1} \). We denote it by \( v_iw \). Then \( w \) is not in \( V_1 \). The subgraph induced by the edges \( \{wv_i, v_iv_j, v_iv_{i+1}, v_{i+1}v_i, v_{i+1}v_{i+2}\} \) is forbidden.

**Subcase 2:** The graph \( G - G'_0 \) contains two isolated edges \( e_1, e_2 \). By Lemma 2.1, \( G \) contains no 3-cycles. There are three possibilities.

(i) One of the isolated edges is incident with \( v_i \), the other is incident with \( v_{i+1} \). We denote them by \( v_iw_1 \) and \( v_{i+1}w_2 \). Then the subgraph induced by the edges \( \{w_1v_i, v_iw_j, v_iv_{i+1}, v_{i+1}v_i\} \) is forbidden.

(ii) One of the isolated edges is incident with \( v_i \), the other is incident with \( v_{i} \). We denote them by \( v_iw_1 \) and \( v_iw_2 \). Then the subgraph induced by the edges \( \{w_1v_i, v_iw_j, v_iw_{i+1}, v_{i+1}v_i\} \) is forbidden.
(iii) One of the isolated edges is incident with \( v_i \), the other is incident with \( v_{i+1} \). We denote them by \( v_i w_1 \) and \( v_{i+1} w_2 \). Then the subgraph induced by the edges \( \{ w_1 v_i, v_i v_{i+1}, v_i v_{i+1} w_2 \} \) is forbidden.

Subcase 3: The graph \( G - G'_0 \) contains three isolated edges. There is only one possibility, that is, each vertex of \( \{ v_i, v_{i+1}, v_l \} \) has an incident isolated edge. We denote them by \( v_i w_1, v_i w_2, v_{i+1} w_3 \). Then the subgraph induced by the edges \( \{ w_2 v_i, v_i v_{i-1}, v_i v_{i+1}, v_i w_1 \} \) is forbidden.

Subcase 4: The graph \( G - G'_0 \) contains no isolated edge. So \( G'_0 \) is forbidden.

So \( G \) is not \( P_3 \)-equicoverable. □

Finally, we consider trees.

Except for \( P_4 \), the tree with size 3 is \( K_{1,3} \) which is clearly \( P_3 \)-equicoverable. So we consider trees with size larger than 3 in the following.

Lemma 2.9. Let \( T \) be a tree of size \( m > 3 \) that is not a path. If \( \text{diam}(T) \leq 3 \), then \( T \) is not \( P_3 \)-equicoverable.

Proof. When \( \text{diam} (T) = 2 \), \( T \) is a star \( K_{1,m} (m > 3) \) which is clearly not \( P_3 \)-equicoverable by Lemma 1.6.

When \( \text{diam} (T) = 3 \), \( T \) is a double-star which also satisfies Lemma 1.6, so \( T \) is not \( P_3 \)-equicoverable. □

We denote by \( P_2 \cdot S^*_k (k \geq 2) \) a graph obtained from a path \( P_2 \) and a \( k \)-extendedstar \( S^*_k \) by identifying an endpoint of the path \( P_2 \) with the center of the \( k \)-extendedstar \( S^*_k \). See the first graph of Fig. 5 for \( k = 5 \). We denote by \( K_{1,3} \cdot S^*_k (k \geq 1) \) a tree obtained from a star \( K_{1,3} \) and a \( k \)-extendedstar \( S^*_k \) by identifying a leaf of the star \( K_{1,3} \) with the center of the \( k \)-extendedstar \( S^*_k \). See the second graph of Fig. 5 for \( k = 3 \).

Remark 2.10. Clearly, \( P_4 \) can be denoted by \( P_2 \cdot S^*_1 \) and \( K_{1,3} \) can be denoted by \( K_{1,3} \cdot S^*_0 \).
Lemma 2.11. Let $T$ be a tree that is not a path. If $\text{diam}(T) = 4$, then $T$ is $P_3$-equicoverable if and only if $T$ belongs to one of the three families below:

1. $T$ is a tree $K_{1, 3} \cdot S_k^+(k \geq 1)$.
2. $T$ is a tree $P_2 \cdot S_k^+(k \geq 2)$.
3. $T$ is a $k$-extended star $S_k^+(k \geq 2)$.

Proof. Clearly, the trees described in the statement of the Lemma are all $P_3$-equicoverable.

Assume that $T$ is a $P_3$-equicoverable tree with $\text{diam}(T) = 4$. Let $V_1 = \{v_1, v_2, v_3, v_4, v_5\}$ be the vertices of a 4-path, then $v_1$ and $v_5$ have no neighbors outside $V_1$.

Since $\text{diam}(T) = 4$, we have two possibilities.

1. The vertex $v_2$ has at least one neighbor $u$ with degree 1 outside $V_1$. Then we can get the following:

   (a) The vertex $u$ is the only neighbor of $v_2$ outside $V_1$.
   Otherwise, $T$ contains a star $K_{1, k}(k \geq 4)$ which is forbidden.
   (b) For any adjacent vertex $u_1$ of $v_3$, $d(u_1) = 2$.
   Otherwise, if $d(u_1) = k > 2$ or $d(u_1) = 1$, we can find a forbidden subgraph induced by the edges $\{uv_2, u_1v_2, v_2v_3, v_3v_4\}$.

   (c) All the paths beginning with $v_3$ outside the 4-path $v_1v_2v_3v_4v_5$ have length no more than 2 since $\text{diam}(T) = 4$.

   (d) The vertex $v_4$ has no neighbor outside $V_1$.
   Otherwise, the subgraph induced by the edges $\{uv_2, u_1v_2, v_2v_3, v_3v_4\}$ is forbidden.

From above, $T$ is a tree $K_{1, 3} \cdot S_k^+$.

2. The vertex $v_2$ has no neighbor outside $V_1$. Similar to the former proofs, we can get the following statements:

   (i) If the vertex $v_3$ has one neighbor $u$ with degree 1 outside $V_1$, then we can get: $u$ is the only neighbor of $v_3$ with degree 1; all the paths beginning with $v_3$ outside the 4-path $v_1v_2v_3v_4v_5$ are of length no more than 2; for any other adjacent vertex $u_1$ of $v_3$, $d(u_1) = 2$; $v_4$ has no neighbor outside $V_1$.

   So $T$ is a tree $P_2 \cdot S_k^+.$

   (ii) If the vertex $v_3$ has no neighbor $u$ with degree 1 outside $V_1$, then we can get: all the paths beginning with $v_3$ outside the 4-path $v_1v_2v_3v_4v_5$ are of length no more than 2; for any adjacent vertex $u_1$ of $v_3$, $d(u_1) = 2$; $v_4$ has at most one neighbor $u$ outside $V_1$ and $d(u) = 1$.

   So $T$ is a $k$-extended star $S_k^+(k \geq 2)$ or a tree $K_{1, 3} \cdot S_k^+$.

Proposition 2.12. Let $P = v_1v_2 \ldots v_n \ (n \geq 6)$ be a longest path of a tree $T$. If $T$ is $P_3$-equicoverable, then $v_2$ and $v_{n-1}$ have no neighbors outside $P$. 


Proof. Suppose that \( v_2 \) has neighbors outside \( P = v_1v_2 \ldots v_n \). Since \( P \) is the longest path of a tree \( T \), then each neighbor \( u \) of \( v_2 \) outside \( P \) is of degree 1. Denote by \( T_0 \) the subgraph induced by the edges \( \{uv_2, v_1v_2, v_2v_3, v_3v_4\} \). Since \( n \geq 6 \), there are three cases for \( T - T_0 \).

Case 1: The graph \( T - T_0 \) has no isolated edges. Then \( T_0 \) is forbidden.

Case 2: The graph \( T - T_0 \) has only one isolated edge \( e \) which must be incident to \( v_2 \) or \( v_3 \). Then the subgraph induced by the edges \( \{e, v_1v_2, v_2v_3, u_1v_2\} \) is forbidden.

Case 3: The graph \( T - T_0 \) has exactly two isolated edges \( e_1, e_2 \). Then the subgraph induced by the edges \( \{e_1, u_2v_2, v_1v_2, v_2v_3\} \) is forbidden.

In all cases, \( T \) is not \( P_3 \)-equicoverable. This is a contradiction. So \( v_2 \) has no neighbors outside \( P \). By symmetry, \( v_{n-1} \) has no neighbors outside \( P \). □

A tree is called a double-extendedstar if it is obtained from two extendedstars \( S_{k_1}^* \) and \( S_{k_2}^* \) \((k_1 \geq 1, k_2 \geq 1)\) by adding an edge between their centers, which is denoted by \( S_{k_1}^* \cdot P_2 \cdot S_{k_2}^* \). See Fig. 6 for the case \( k_1 = 3, k_2 = 3 \).

Remark 2.13. We see that \( P_6 \) can be denoted by \( S_{1}^* \cdot P_2 \cdot S_{1}^* \).

Lemma 2.14. Let \( T \) be a tree that is not a path. If \( \text{diam}(T) = 5 \), then \( T \) is \( P_3 \)-equicoverable if and only if \( T \) is a double-extendedstar \( S_{k_1}^* \cdot P_2 \cdot S_{k_2}^* \) \((k_1 \geq 1, k_2 \geq 1)\).

Proof. Clearly, each double-extendedstar \( S_{k_1}^* \cdot P_2 \cdot S_{k_2}^* \) \((k_1 \geq 1, k_2 \geq 1)\) is \( P_3 \)-equicoverable.

Assume that \( T \) is a \( P_3 \)-equicoverable tree with \( \text{diam}(T) = 5 \). Let \( P = v_1v_2v_3v_4v_5v_6 \) be a longest 5-path, then \( v_1 \) and \( v_6 \) have no neighbors outside \( P \). By Proposition 2.12, \( v_2 \) and \( v_5 \) have no other neighbors, either. The following statements are true:

1. The vertex \( v_3 \) has no neighbor with degree 1.
Fig. 7. The tree $S^*_1 \cdot K_{1,3} \cdot S^*_1$.

Fig. 8. The tree $S^*_2 \cdot P_4 \cdot S^*_3$.

Otherwise, suppose that $v_3$ has a neighbor $u$ with degree 1 and let $T_0$ be the subgraph induced by $\{v_1 v_2, v_2 v_3, u v_3, v_3 v_4\}$. For $T - T_0$, there are two possibilities:

(i) There exists no isolated edge. Then $T_0$ is a forbidden subgraph of $T$.

(ii) There exists one isolated edge $e$ which must be incident to $v_3$. Then the subgraph induced by $\{v_1 v_2, v_2 v_3, u v_3, e\}$ is forbidden.

So $v_3$ has no neighbor with degree 1.

(2) In $T - P$, all the 2-paths beginning with $v_3$ are edge-disjoint. Otherwise, $T$ contains a forbidden graph again.

(3) By symmetry, $v_4$ has no neighbor with degree 1 and all the 2-paths beginning with $v_4$ in $T - P$ are edge-disjoint. So $T$ must be a double-extendedstar $S^*_k \cdot P_2 \cdot S^*_k (k_1 \geq 1, k_2 \geq 1)$.

We denote a tree by $S^*_k \cdot K_{1,3} \cdot S^*_k$ which is the union of two extendedstars $S^*_k \cdot S^*_k (k_1 \geq 1, k_2 \geq 1)$ and a star $K_{1,3}$ satisfying two leaves of $K_{1,3}$ are, respectively, the centers of the two extendedstars. See Fig. 7 for the case $k_1 = 1, k_2 = 1$.

**Lemma 2.15.** Let $T$ be a tree that is not a path. If $\text{diam}(T) = 6$, then $T$ is $P_3$-equicoverable if and only if $T$ is a tree $S^*_k \cdot K_{1,3} \cdot S^*_k$.

**Proof.** Each tree $S^*_k \cdot K_{1,3} \cdot S^*_k (k_1 \geq 1, k_2 \geq 1)$ is clearly $P_3$-equicoverable.

Assume that $T$ is a $P_3$-equicoverable tree with $\text{diam}(T) = 6$.

Let $P = v_1 v_2 v_3 v_4 v_5 v_6 v_7$ be a longest 6-path. In the same way, $v_1$, $v_2$, $v_6$, $v_7$ has no neighbor with degree 1. Similar to the proof of Lemma 2.14, neither $v_2$ nor $v_5$ has neighbor with degree 1; all the paths beginning with $v_2$ and $v_5$ are of length 2 and edge-disjoint; $v_4$ has exactly one neighbor with degree 1 outside $P$ (Otherwise, $P$ is a forbidden subgraph of $T$). So $T$ is a tree $S^*_k \cdot K_{1,3} \cdot S^*_k (k_1 \geq 1, k_2 \geq 1)$.

We denote a tree by $S^*_k \cdot P_4 \cdot S^*_k$ which is the union of two extendedstars $S^*_k \cdot S^*_k (k_1 \geq 1, k_2 \geq 1)$ and a path $P_4$ satisfying two endpoints of $P_4$ are, respectively, the centers of the two extendedstars. See Fig. 8 for the case $k_1 = 2, k_2 = 3$. 
Remark 2.16. Clearly, \( P_8 \) can be denoted by \( S_1^* \cdot P_4 \cdot S_1^* \).

Lemma 2.17. Let \( T \) be a tree that is not a path. If \( \text{diam}(T) = 7 \), then \( T \) is \( P_3 \)-equicoverable if and only if \( T \) is a tree \( S_{k_1}^* \cdot P_4 \cdot S_{k_2}^* \) \( (k_1 \geq 1, k_2 \geq 1) \).

Proof. Each tree \( S_{k_1}^* \cdot P_4 \cdot S_{k_2}^* \) \( (k_1 \geq 1, k_2 \geq 1) \) is clearly \( P_3 \)-equicoverable.

Assume that \( T \) is a \( P_3 \)-equicoverable tree with \( \text{diam}(T) = 7 \).

Let \( P = v_1v_2v_3v_4v_5v_6v_7v_8 \) be a longest \( T \)-path in \( T \). As the proofs of former lemmas, \( v_1, v_2, v_7, v_8 \) has no neighbors with degree 1, \( v_3 \) and \( v_6 \) have no neighbor with degree 1 and the 2-paths beginning with \( v_3 \) and \( v_6 \) are edge-disjoint. In the following, we only need to prove that \( v_4 \) and \( v_5 \) have no neighbor outside \( P \).

If \( v_4 \) has a neighbor \( u \) outside \( P \), we denote by \( T_0 \) the subgraph induced by the edges \( \{uv_4, v_3v_4, v_4v_5, v_5v_6\} \). For \( T - T_0 \), we consider four cases.

Case 1: There exists no isolated edge in \( T - T_0 \). So \( T_0 \) is forbidden.

Case 2: There exists only one isolated edge \( e \) in \( T - T_0 \). No matter which vertex of \( \{u, v_4, v_5\} \) is incident with \( e \), the subgraph induced by the edges \( \{uv_4, v_3v_4, v_4v_5, e\} \) is forbidden.

Case 3: There exist two isolated edges \( e_1, e_2 \) in \( T - T_0 \). If they are, respectively, incident with \( u, v_4 \) or \( v_4, v_5 \), then the subgraph induced by \( \{uv_4, v_3v_4, e_1, e_2\} \) is forbidden. If \( e_1 \) is incident with \( u \) and \( e_2 \) is incident with \( v_5 \), then the subgraph induced by \( \{uv_4, v_3v_4, v_4v_5, e_1\} \) is forbidden.

Case 4: There exist three isolated edges \( e_1, e_2, e_3 \) in \( T - T_0 \). In the same way, \( T \) also contains a forbidden subgraph induced by \( \{e_1, e_2, uv_4, v_4v_5\} \).

In all cases, \( T \) is not \( P_3 \)-equicoverable. So \( v_4 \) has no neighbor outside \( P \).

By symmetry, \( v_5 \) has no neighbor outside \( P \).

From above, we know that \( T \) is a tree \( S_{k_1}^* \cdot P_4 \cdot S_{k_2}^* \) \( (k_1 \geq 1, k_2 \geq 1) \). \( \Box \)

Lemma 2.18. Let \( T \) be a tree that is not a path. If \( \text{diam}(T) = k \geq 8 \), then \( T \) is not \( P_3 \)-equicoverable.

Proof. Let \( P = v_1v_2v_3v_4v_5v_6v_7 \ldots v_kv_{k+1} \) be a longest \( k \)-path \( (k \geq 8) \) in \( T \). As the proofs of former lemmas, \( v_1, v_2, v_3 \) have no neighbors with degree 1. Let \( T_0 \) denote the path \( P_1 = v_1v_2v_3v_4v_5v_6v_7 \). For \( T - T_0 \), there are four cases.

Case 1: There exists no isolated edge in \( T - T_0 \). So \( T_0 \) is forbidden. \( T \) is not \( P_3 \)-equicoverable.
Case 2: There exists only one isolated edge $e$ in $T - T_0$. When $e$ is incident with $v_4$ or $v_5$, the subgraph induced by the edges $\{v_3v_4, v_4v_5, v_5v_6, e\}$ is forbidden. When $e$ is incident with $v_6$, the subgraph induced by the edges $\{v_4v_5, v_5v_6, v_6v_7, e\}$ is forbidden.

\[ \begin{array}{c}
\text{Case 3: There exist two isolated edges } e_1, e_2 \text{ in } T - T_0. \text{ In the same way, we can always find a forbidden subgraph induced by } \{e_1, e_2, v_4v_5, v_5v_6\} \text{ or } \{v_3v_4, v_4v_5, v_5v_6, e_1\}. \\
\end{array} \]

\[ \begin{array}{c}
\text{Case 4: There exist three isolated edges } e_1, e_2, e_3 \text{ in } T - T_0. \text{ Then the subgraph induced by } \{v_3v_4, v_4v_5, e_1 = u_1v_4, e_2 = u_2v_5\} \text{ is forbidden.} \\
\end{array} \]

In all cases, $T$ is not $P_3$-equicoverable. $\square$

We summarize the characterization in the following theorem:

**Theorem 2.19.** Let $G$ be a connected graph, then $G$ is $P_3$-equicoverable if and only if $G$ satisfies one of the following:

1. $G$ is a cycle $C_n$ ($n = 3, 4, 5, 7$);
2. $G$ is a $k$-extendedstar $S^*_k(k \geq 1)$;
3. $G$ is a graph $C_3 \cdot S^*_k(k \geq 1)$ obtained from a cycle $C_3$ and a $k$-extendedstar $S^*_k$ by identifying one vertex of $C_n$ with the center of the $k$-extendedstar $S^*_k$;
4. $G$ is a graph $C_4 \cdot P_2 \cdot S^*_k(k \geq 0)$ obtained from a cycle $C_4$ and a $k$-extendedstar $S^*_k$ by adding an edge between a vertex of the cycle $C_4$ and the center of the $k$-extendedstar;
5. $G$ is a tree $K_{1,3} \cdot S^*_k(k \geq 0)$ obtained from a star $K_{1,3}$ and a $k$-extendedstar $S^*_k$ by identifying a leaf of the star $K_{1,3}$ with the center of the $k$-extendedstar $S^*_k$;
6. $G$ is a tree $P_2 \cdot S^*_k(k \geq 1)$ obtained from a path $P_2$ and a $k$-extendedstar $S^*_k$ by identifying an endpoint of the path $P_2$ with the center of the $k$-extendedstar;
7. $G$ is a double-extendedstar $S^*_{k_1} \cdot P_2 \cdot S^*_k(k_1 \geq 1, k_2 \geq 1)$;
8. $G$ is a tree $S^*_{k_1} \cdot K_{1,3} \cdot S^*_k$, which is the union of two extendedstars $S^*_{k_1}(k_1 \geq 1), S^*_k(k_2 \geq 1)$ and a star $K_{1,3}$ satisfying two leaves of $K_{1,3}$ are, respectively, the centers of the two extendedstars;
9. $G$ is a tree $S^*_{k_1} \cdot P_4 \cdot S^*_k$, which is the union of two extendedstars $S^*_{k_1}(k_1 \geq 1), S^*_k(k_2 \geq 1)$ and a path $P_4$ satisfying two endpoints of $P_4$ are, respectively, the centers of the two extendedstars.

For disconnected graphs, we can easily get:

**Theorem 2.20.** A graph $G$ is $P_3$-equicoverable if and only if each component of $G$ is $P_3$-equicoverable.

This problem of characterizing $H$-equicoverable graphs stems from the studies of $H$-decomposable graphs [4], randomly $H$-packable graphs [1] and $H$-equipackable graphs. There have been many results for randomly $H$-packable graphs and $H$-equipackable graphs, see [5]. So there are much related work that can be done for $H$-equicoverable graphs.
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References