Abstract

A remarkable result by Nordgren, Rosenthal and Wintrobe states that a positive answer to the Invariant Subspace Problem is equivalent to the statement that any minimal invariant subspace for a composition operator $C_\varphi$ induced by a hyperbolic automorphism $\varphi$ of the unit disc $\mathbb{D}$ in the Hardy space $H^2$ is one dimensional. Motivated by this result, for $f \in H^2$ we consider the space $K_f$, which is the closed subspace generated by the orbit of $f$. We obtain two results, one for functions with radial limit zero, and one for functions without radial limit zero, but tending to zero on a sequence of iterates. More precisely, for those functions $f \in H^2$ with radial limit zero and continuous at the fixed points of $\varphi$, we provide a construction of a function $g \in K_f$ such that $f$ is a cluster point of the sequence of iterates $\{g \circ \varphi^{-n}\}$. In case $f$ is in the disc algebra, we have $K_g \subseteq K_f \subseteq \text{span}\{g \circ \varphi^n : n \in \mathbb{Z}\}$.

For a function $f \in H^2$ tending to zero on a sequence of iterates $\{\varphi^n(z_0)\}$ at a point $z_0$ with $|z_0| < 1$, but having no radial limit at the attractive fixed point, we establish the existence of certain functions in the space and show that unless $f$ is constant on the sequence of iterates $\{\varphi^n(z_0)\}$, the space $K_f$ is not minimal.

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Résumé

D'après un résultat remarquable de Nordgren, Rosenthal et Wintrobe, une réponse positive au Problème du Sous-Espace Invariant est équivalent à l'énoncé que, pour tout opérateur de composition $C_\varphi$ induit par un automorphisme hyperbolique du disque unité $\mathbb{D}$ dans l'espace de Hardy $H^2$, tout sous-espace invariant minimal est de dimension un. Motivés par ce résultat, on considère pour une fonction $f \in H^2$ le sous-espace fermé $K_f$ engendré par l'orbite de $f$. On obtient deux résultats : un pour les fonctions de limite radiale nulle et un pour les fonctions convergeant vers zéro par une suite d'itérées de $\varphi$ en un point du disque. Plus précisément, pour les fonctions dont la limite radiale est nulle et qui sont continues aux points fixes de $\varphi$, nous construisons une fonction $g \in K_f$ dont $f$ est un point d'accumulation de la suite des itérées $\{g \circ \varphi^{-n}\}$. Lorsque $f$ appartient à l’algèbre du disque, nous obtenons $K_g \subseteq K_f \subseteq \text{span}\{g \circ \varphi^n : n \in \mathbb{Z}\}$. Pour une fonction $f \in H^2$ tendant vers zéro pour une suite $\{\varphi^n(z_0)\}$ et un certain $z_0 \in \mathbb{D}$, mais sans limite radiale au point fixe attractif, on établit l’existence de certaines fonctions dans l’espace et on montre que $K_f$ n’est pas minimal, à moins que $f$ soit constante sur la suite $\{\varphi^n(z_0)\}$.

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1. Introduction and preliminaries

One of the most prominent open problems in the study of linear bounded operators on separable Hilbert spaces is the Invariant Subspace Problem, which asks the following: given a complex separable Hilbert space $\mathcal{H}$ of dimension greater than one and a bounded linear operator $T$ on $\mathcal{H}$, do there exist nontrivial closed invariant subspaces for $T$? In this paper, we will study subspaces of the Hardy space $\mathcal{H}^2$ that were shown to be closely connected to this famous problem, [13,14].

Recall that $\mathcal{H}^2$ is the Hilbert space consisting of holomorphic functions $f$ on the unit disc $D$ for which the norm, 
\[
\|f\|_2 = \left( \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} \right)^{1/2},
\]
is finite. A classical result due to Fatou states that every Hardy function $f$ has radial limit at $e^{i\theta} \in \partial D$, except possibly on a set Lebesgue measure zero (see [5], for instance). Throughout this work, $f(e^{i\theta})$ will denote the radial limit of $f$ at $e^{i\theta}$.

In the eighties, Nordgren, Rosenthal and Wintrobe [13] (see also [14]) gave an equivalent formulation of the Invariant Subspace Problem, which is stated in terms of composition operators acting on the classical Hardy space $\mathcal{H}^2$. In [14, Corollary 6.3] they showed that if $\varphi$ is a hyperbolic automorphism of the unit disc $D$, then every bounded operator on a complex separable Hilbert space of dimension greater than one has a nontrivial invariant subspace if and only if the minimal nontrivial invariant subspaces for the composition operator induced by $\varphi$ in $\mathcal{H}^2$, 
\[
C_{\varphi}f = f \circ \varphi \quad (f \in \mathcal{H}^2),
\]
are one dimensional. Recall that a hyperbolic automorphism $\varphi$ of $\mathbb{D}$ can be expressed by:
\[
\varphi(z) = e^{i\theta} \frac{p - z}{1 - \overline{p}z} \quad (z \in \mathbb{D}),
\]
where $p \in \mathbb{D}$, $-\pi < \theta \leq \pi$ and $|p| > \cos(\theta/2)$. In this case, $\varphi$ fixes two points on the boundary of $\mathbb{D}$ and the sequence $\{\varphi_n\}$ of the iterates of $\varphi$, that is,
\[
\varphi_n = \varphi \circ \cdots \circ \varphi \quad (n \text{ times}),
\]
converges uniformly on compact subsets of $\mathbb{D}$ to one of the fixed points, which is called the attractive fixed point. Furthermore, the derivative of $\varphi$ at the fixed point $\alpha$ satisfies $0 < \varphi'(\alpha) < 1$. The other fixed point $\beta$ turns out to be attractive for the sequence of the iterates of $\varphi^{-1}$, and it is called the repulsive fixed point. It is easy to deduce that, in such a case, $\varphi'(\beta) > 1$. We refer to Ahlfors’ book [1] for more details.

Note that if $\varphi$ is a hyperbolic automorphism of $\mathbb{D}$ and $f \in \mathcal{H}^2$, the cyclic subspace generated by $f$,
\[
K_f = \overline{\text{span}\{C_{\varphi}^n f: n \geq 0\}}^{\mathcal{H}^2},
\]
is the minimal closed $C_{\varphi}$-invariant subspace that contains $f$. It is clear that every minimal invariant subspace of $C_{\varphi}$ has the form $K_f$ for some $f \in \mathcal{H}^2$. Thus, it is possible to restate Nordgren, Rosenthal and Wintrobe's result as follows:

The Invariant Subspace Problem has a positive solution if and only if for every $f \in \mathcal{H}^2$ it is the case that
\[
K_f \text{ is minimal for } C_{\varphi} \iff f \text{ is an eigenfunction of } C_{\varphi}.
\]

Note that for every eigenfunction $f \in \mathcal{H}^2$ of $C_{\varphi}$, one has $K_f = \{\lambda f: \lambda \in \mathbb{C}\}$.

The results above suggest that a detailed study of spaces of the form $K_f$ is of significant interest and some authors have studied conditions on $f$ under which the cyclic subspace $K_f$ is not minimal for $C_{\varphi}$. Obviously, if $f \in \mathcal{H}^2$ is a nonconstant function and $K_f$ contains a nonzero constant function, then $K_f$ is not minimal (since $\mathbb{C}$ is invariant under any composition operator).
In this direction, Matache (see [9]) showed that if \( f \in H^2 \) is a nonconstant function that extends continuously and is nonzero in a neighborhood of one of the fixed points of \( \varphi \), then \( K_f \) is not minimal. From [11, Proposition 1.2], it is possible to show that if \( f \) is a nonconstant bounded analytic function; that is, \( f \in H^\infty \), to prove that \( K_f \) is not minimal it is enough to ensure that the radial limit, \( f(e^{i\theta}) \), exists and is different from zero at one of the fixed points \( e^{i\theta} \) of \( \varphi \). Finally, in [10, Theorem 2.2] a refinement is presented: if \( f \in H^2 \) and the radial limit \( f(e^{i\theta}) \) exists and is different from zero at one of the fixed points \( e^{i\theta} \) of \( \varphi \), and \( f \) is essentially bounded on an open arc containing \( e^{i\theta} \), then \( K_f \) is minimally invariant if and only if \( f \) is constant.

For \( f \) bounded in a neighborhood of the attractive fixed point, these results show that \( K_f \) is nonminimal by producing a nonzero constant function as a cluster point of the set \( \{ f \circ \varphi_n \} \). Thus, one might think of these as minimality results, but it is also useful to view them as the first step in a description of the spaces \( K_f \). It is this viewpoint that we adopt in this paper; that is, we focus on trying to understand the behavior of functions in the space \( K_f \) in what appear to be the most extreme cases: functions tending to 0 near the attractive fixed point. In some sense, we might expect these functions to generate very small invariant subspaces, but our work will show that this is not the case, in general.

One very simple setting is the following: If \( f \) is an eigenvector of \( C_\varphi \), then \( K_f \) is one dimensional. Thus, in looking for small \( K_f \) one might be tempted to change an eigenvector only very slightly. Consider, for example, the hyperbolic automorphism \( \varphi(z) = (z + r)/(1 + rz) \), where \( r \) is real, \( 0 < r < 1 \). This fixes the points +1 and −1. In this setting, consider the Blaschke product \( B \) with zeroes invariant under \( \varphi \); that is, the zero set of \( B \) is the set \( \{ \varphi_n(z_0): n \in \mathbb{Z} \} \), for some fixed element \( z_0 \in \mathbb{D} \). Then \( C_\varphi(B) = B \circ \varphi \) is a Blaschke product with the same zeroes as \( B \) and we see that \( B \) is an eigenvector. We will assume now that \( z_0 = 0 \). (Our comments below hold for general hyperbolic automorphisms, but the proofs are not as transparent.) This particular \( B \) is also an interpolating Blaschke product: the positive zeroes of \( B \), denoted \( \{ z_n \} \), are increasing, and (as a computation shows) satisfy,

\[
\frac{1 - z_n}{1 - z_{n-1}} < c < 1, 
\]

for some constant \( c \) independent of \( n \), [8, p. 104]. Similarly, the negative zeroes form an interpolating sequence and these sequences have disjoint closures, so \( B \) is interpolating, [8, p. 208]. As a consequence, \( B \) cannot have radial limit zero, [7, Lemma 1.4, p. 404].

We take this as a starting point and consider functions \( f \) for which \( f(\varphi_n(z_0)) \to 0 \) for a fixed point \( z_0 \in \mathbb{D} \), but the radial limit of \( f \) at the attractive fixed point of \( \varphi \) is not zero. Assuming that \( f \in H^2 \) is bounded in a neighborhood of the attractive fixed point, we show (Theorem 2.4) that \( K_f \) contains a function of the form \( Bh \), where \( h \) is nonzero, and \( B \) vanishes on the set \( \{ \varphi_n(z_0) \}_{n \geq 1} \). As a consequence, if \( f(\varphi_n(z_0)) \neq 0 \) for some \( n \geq 1 \), then \( K_f \) is not minimal. In Theorem 2.5, we show that if \( f \) has nonzero limit on the sequence \( \{ \varphi_n(z_0) \}_{n \geq 1} \) and \( K_f \) is minimal, then \( f \) is constant on the sequence \( \{ \varphi_n(z_0) \} \) for \( n \geq 1 \). Thus, if \( f(\varphi_n(z_0)) \to L \neq 0 \) and \( f(\varphi_m(z_0)) \neq L \) for some \( m \geq 1 \), then \( K_f \) is not minimal, extending the results of Mortini [11, Proposition 1.2] and Matache [10, Theorem 2.2].

Turning to functions with radial limit zero, the most interesting and powerful result is due to Chkliar [2], who relates conditions on \( f \) to the point spectrum of \( C_\varphi \) acting on the doubly invariant closed subspace generated by \( f \); that is, \( \text{span}\{C_\varphi^n f: n \in \mathbb{Z}\}^{H^2} \). Chkliar’s result reads as follows:

**Chkliar’s Theorem.** Let \( \varphi \) be a hyperbolic automorphism of \( \mathbb{D} \) with fixed points \( \alpha, \beta \in \mathbb{D} \). Assume that \( \alpha \) is the attractive fixed point. Let \( f \in H^2 \) satisfying:

1. \( \lim_{z \to \beta} |f(z)| < \infty \),
2. \( |f(z)| \leq C |z - \alpha|^s \) for some constant \( C \), and some \( s > 0 \) in a neighborhood of \( \alpha \).

Then, the point spectrum of \( C_\varphi \) acting on \( \text{span}\{C_\varphi^n f: n \in \mathbb{Z}\}^{H^2} \) contains the annulus

\[
\left\{ z \in \mathbb{C}: |\varphi'(\alpha)|^{\min\{r, \frac{1}{2}\}} < |z| < 1 \right\},
\]

except possibly for some discrete subset.

Since every minimal invariant subspace is doubly invariant (see [9, Theorem 1]), Chkliar’s Theorem yields that under conditions (1) and (2), if \( K_f \) is minimal, then \( K_f = \text{span}\{C_\varphi^n f: n \in \mathbb{Z}\}^{H^2} \), and the theorem above implies that \( K_f \) contains eigenfunctions.
In view of the results above, it is clear that the most interesting case in which \( f \) has a radial limit is when that radial limit is zero. Thus, it is natural to try to understand the behavior of functions in \( K_f \) and to see how different the behavior of an arbitrary function in \( K_f \) can be from the function with which we began. In Section 3 we show that if we begin with a reasonably well-behaved function, \( f \in \mathcal{H}^2 \) continuous at both fixed points of \( \phi \) and with radial limits zero at the fixed points, then we can construct a relatively “badly-behaved” function \( g \in K_f \) in the sense that the radial limit at one of the fixed points of \( \phi \) does not exist. This is an explicit construction, allowing a more in-depth study of the relation between the function \( g \) and \( f \) than is possible with a non-constructive proof; for example, \( f \) will be a cluster point of \( \{ g \circ \varphi_n \}_{n \geq 1} \). Thus, \( K_g \subseteq K_f \) and we’ve moved the discussion of minimality from a space with no interesting cluster points to one with interesting cluster points. When \( f \) is in the disc algebra, \( g \) will be bounded and therefore \( f \in K_g \), as well. So, when \( f \) is in the disc algebra and \( K_f \) is doubly invariant, we conclude that \( K_f = K_g \). Furthermore, when \( K_f \) is doubly invariant, we can modify the proof of our main theorem (see Remark 3.2) to show that

\[
K_g \subseteq K_f \subseteq \text{span} \{ g \circ \varphi_n : n \in \mathbb{Z} \}.
\]

Finally, we show that no such function will be an eigenvector for \( C_\psi \).

### 1.1. Preliminaries

In this preliminary subsection, we collect some basic facts that will be useful throughout this work. We begin by showing that it suffices to study minimal invariant subspaces for a composition operator induced by a concrete hyperbolic disc automorphism: one that fixes 1 and \(-1\). In fact, if \( \psi \) is any hyperbolic automorphism of \( \mathbb{D} \), it is not hard to see that \( \psi \) can be conjugated under a disc automorphism \( T \) to a hyperbolic automorphism \( \varphi \) that fixes 1 and \(-1\); that is,

\[
\psi = T^{-1} \circ \varphi \circ T,
\]

where

\[
\varphi(z) = \frac{z + r}{rz + 1}, \quad (2)
\]

with \( 0 < r < 1 \). In this case, the operators \( C_\varphi \) and \( C_\psi \) are similar operators and therefore, the lattice of invariant subspaces of \( C_\varphi \) and \( C_\psi \) are in one-to-one correspondence. Hence, it is enough to study the minimal invariant subspaces for \( C_\varphi \).

Note that if \( \varphi(z) = \frac{z + r}{rz + 1} \) with \( 0 < r < 1 \) is a hyperbolic automorphism and we let \( r = (1 - \mu)/(1 + \mu) \) with \( 0 < \mu < 1 \), we deduce the following formula for the iterates of \( \varphi \),

\[
\varphi_n(z) = \frac{(1 + \mu^n)z + (1 - \mu^n)}{(1 - \mu^n)z + (1 + \mu^n)} \quad (z \in \mathbb{D}), \quad (3)
\]

which holds for \( n \in \mathbb{Z} \). Note that \( \varphi_0 \) is the identity function. Using (3), it is easy to check that \( 1 \) is the attractive fixed point of \( \varphi \) and \(-1\) is the repulsive one.

In what follows, we will assume that \( \varphi \) is a hyperbolic automorphism fixing 1 and \(-1\), and 1 is the attractive fixed point. Before proceeding, we remark that if \( f \in \mathcal{H}^2 \) is continuous at 1 and \(-1\), and either \( f(1) \neq 0 \) or \( f(-1) \neq 0 \), then by Matache’s result [9, Theorem 2] the subspace \( K_f \) is minimal if and only if \( f \) is constant, and therefore \( f \) is an eigenvector. Thus the interesting case for an \( \mathcal{H}^2 \)-function \( f \) continuous at 1, is the one in which \( f(1) = f(-1) = 0 \).

The following elementary lemma establishes a bound for the norm of \( C_{\varphi_n} f \) whenever \( f \) is an \( \mathcal{H}^2 \)-function bounded in a neighborhood of 1.

**Lemma 1.1.** Let \( f \in \mathcal{H}^2 \) be bounded in a neighborhood of 1. Then \( \sup_n \| C_{\varphi_n} f \|_2 < \infty \). Moreover, if \( f \) is continuous in a neighborhood of 1 and satisfies \( f(1) = 0 \), then \( \lim_n \| C_{\varphi_n} f \|_2 = 0 \).

**Proof.** We use the following well-known identity (see Nordgren [12]):

\[
\| C_{\varphi_n} f \|_2^2 = \left( \int_0^{2\pi} |f(e^{i\theta})|^2 P(\varphi_n(0), \theta) \, d\theta \right)^{1/2},
\]
where $P(\varphi_n(0), \theta)$ is the Poisson kernel at $\varphi_n(0)$,

$$P(\varphi_n(0), \theta) = \Re \left( \frac{1 + \varphi_n(0)e^{i\theta}}{1 - \varphi_n(0)e^{i\theta}} \right).$$

Suppose $f$ is bounded in a neighborhood of 1. There exists $\delta > 0$ such that for $0 < |\theta| < \delta$ there exists a constant $M_1$ such that $|f(e^{i\theta})| \leq M_1$ a.e.

Let $\varepsilon > 0$ and $n_0$ be such that for $n \geq n_0$ and $|\theta| > \delta$ we have $|P(\varphi_n(0), \theta)| < \varepsilon$. Then

$$\int_0^{2\pi} |f(e^{i\theta})|^2 P(\varphi_n(0), \theta) \, d\theta = \int_{-\delta}^\delta |f(e^{i\theta})|^2 P(\varphi_n(0), \theta) \, d\theta + \int_{|\theta| > \delta} |f(e^{i\theta})|^2 P(\varphi_n(0), \theta) \, d\theta \leq M_1 + \varepsilon \|f\|_2^2.$$

From here the first half of the lemma follows. The second half of the lemma is now straightforward. □

2. Functions without radial limits

Given a function $f \in \mathcal{H}^2$, one possible way to prove that $K_f$ is not minimal is to look for eigenvectors of $C\varphi$ in $K_f$. While the point spectrum of $C\varphi$ is well known (see [4, Chapter 7]), a constructive way to describe the eigenvectors has only been provided recently (see [6]). Prior information about eigenvectors can be found in other sources (see [3, p. 89, Proposition 4.4] and [10] for more about this). A theorem of Matache [10] can be modified to obtain the following useful bit of information about eigenvectors. In what follows, we assume that $\varphi$ is a hyperbolic automorphism of $\mathbb{D}$ with attractive fixed point at 1.

**Lemma 2.1.** Let $f \in \mathcal{H}^2$ be an eigenvector of $C\varphi$ and suppose that there exists a sequence of positive integers $\{n_k\}$ such that

$$\sup \left\{ |f(\varphi_{n_k}(z_0))| + |f(\varphi_{-n_k}(z_0))| \right\} < \infty,$$

for some $z_0 \in \mathbb{D}$. Then $|f(\varphi_m(z_0))| = |f(\varphi_{m}(z_0))|$ for all integers $m$ and $n$.

**Proof.** Suppose that $f$ is an eigenvector with corresponding eigenvalue $\lambda$. Note that since $C\varphi$ is invertible, $\lambda \neq 0$. Without loss of generality, we may assume that $z_0 = 0$.

If $f(\varphi_n(0)) = 0$ for all $n$, we are done. Thus, there is some $n$ for which $f(\varphi_n(0)) \neq 0$. Without loss of generality we may suppose that $f(0) \neq 0$. Then

$$(f \circ \varphi_n)(0) = \lambda^n f(0).$$

By assumption, there is a subsequence $\{n_l\}$ of $\{n_k\}$ for which $((f \circ \varphi_{n_l})(0))$ converges. Therefore $|\lambda| \leq 1$.

On the other hand, $C\varphi(f) = \lambda f$ and therefore $C\varphi^{-1}(f) = \lambda^{-1} f$. Arguing as above, we conclude that $|\lambda|^{-1} \leq 1$. Thus, $|\lambda| = 1$. □

As a consequence, we obtain the following result (see also [3] and [10]).

**Proposition 2.1.** A nonzero function $f$ in the disc algebra $A(\mathbb{D})$ is an eigenfunction for $C\varphi$ if and only if $f$ is a nonzero constant.

**Proof.** If $f$ is a nonzero constant, it is obviously an eigenvector. So assume that $f$ is an eigenvector of $C\varphi$. Note that $C\varphi f \neq 0$, since $C\varphi$ is invertible. Thus, there exists $\lambda \neq 0$ such that for each $z \in \mathbb{D}$ and integer $n \geq 0$ we have:

$$f \circ \varphi_n(z) = \lambda^n f(z) \quad \text{and} \quad f \circ \varphi_{-n}(z) = \lambda^{-n} f(z).$$

Since 1 is the attractive fixed point of $\varphi$ and $f$ is continuous at the point 1, we know that $\lim(f \circ \varphi_n)(z)$ exists. Therefore, the corresponding eigenvalue must be $\lambda = 1$ and for each $z \in \mathbb{D}$ we have $f(\varphi_n(z)) = f(\varphi_m(z))$ for all $n$ and $m$. 
Choose \(z, w \in \mathbb{D}\). Then \(\varphi_n(z) \to 1\) and \(\varphi_n(w) \to 1\) as \(n \to \infty\). Thus
\[
0 = \lim_{n \to \infty} \left| f(\varphi_n(z)) - f(\varphi_n(w)) \right| = \left| f(z) - f(w) \right|.
\]
From here, the statement of the proposition follows. \(\square\)

**Remark 2.2.** Note that the same argument in the proof of Proposition 2.1 applies to a function \(f \in \mathcal{H}^2\) with (finite) radial limit at the fixed points of \(\varphi\).

**Remark 2.3.** Note that if \(F \in \mathcal{H}^2\) is an eigenvector with \(F(\varphi_n(z_0)) \to 0\) for some \(z_0 \in \mathbb{D}\), the fact that \(|F(\varphi_n(z_0))| = |F(\varphi_m(z_0))|\) for all \(m, n\) would imply that \(F(\varphi_n(z_0)) = 0\) for all \(n\). Therefore, \(F = Bh\) where \(B\) is the Blaschke product having zero set \(\{\varphi_n(z_0)\}\), and where each zero has a zero of order 1 and \(h \in \mathcal{H}^2\).

Before stating the main result of this section, we introduce some notation. Recall that for two points \(z, w \in \mathbb{D}\), the pseudohyperbolic distance between \(z\) and \(w\) is defined as
\[
\rho(z, w) = \frac{|z - w|}{|1 - \overline{w}z|}.
\]
For the pseudohyperbolic disc of center \(a\) and radius \(r\) we write \(D_\rho(a, r)\) and we denote the Euclidean disc of center \(a\) and radius \(r\) by \(D(a, r)\).

**Theorem 2.4.** Let \(\varphi\) be a hyperbolic disc automorphism and \(f \in \mathcal{H}^2\). Suppose that there is a function \(F \in K_f = \text{span}\{C_\varphi^n f : n \geq 0\}\mathcal{H}^2\) such that for some \(z_0 \in \mathbb{D}\),
\[
(1) \text{ as } n \to \infty \text{ we have } F(\varphi_n(z_0)) \to 0 \text{ for the sequence of iterates } \{\varphi_n(z_0)\}_{n \geq 1}. \text{ Then } \|F \circ \varphi_k + B\mathcal{H}^2\|_2 \to 0,
\]
where \(B\) is the Blaschke product with zeroes \(\{\varphi_n(z_0)\}_{n \geq 1}\).

If, in addition,
\[
(2) \text{ the radial limit of } F \text{ does not exist at the attractive fixed point of } \varphi, \text{ and}
\]
\[
(3) \text{ there exists a constant } M \text{ such that } \sup \|F \circ \varphi_n\|_2 < M,
\]
then there exists a function \(g \in K_f\) that is not (identically) zero, such that \(g = Bh\). In particular, if there exists \(n \geq 1\) such that \(f(\varphi_n(z_0)) \neq 0\), then \(K_f\) is not minimal.

**Proof.** Without loss of generality, we may assume that \(z_0 = 0\) and that 1 is the attractive fixed point of \(\varphi\).

Since \(\{\varphi_n(0)\}_{n \geq 1}\) is an interpolating sequence \([4, p. 80]\), we have (see \([8, p. 199]\)),
\[
\sup \left\{ \sum_{n=1}^{\infty} |g(\varphi_n(0))|^2 \left(1 - |\varphi_n(0)|^2\right) : g \in \mathcal{H}^2, \|g\|_2 \leq 1 \right\} < \infty.
\]
Thus the map \(T : \mathcal{H}^2 \to \ell^2\) defined by,
\[
T(g) = \left\{ g(\varphi_n(0))(1 - |\varphi_n(0)|^2)^{1/2} \right\}_{n \geq 1},
\]
is a well-defined linear map. By \([8, \text{Lemma 4, p. 202}]\), there is a constant \(K\) such that for every square-summable sequence \(\{\lambda_n\}_{n \geq 1}\) there exists a function \(g \in \mathcal{H}^2\) with
\[
(1) \|g\|_2 \leq K \|\lambda_n\|_2; \quad (2) \ g(\varphi_n(0))(1 - |\varphi_n(0)|^2)^{1/2} = \lambda_n \text{ for } n = 1, 2, 3, \ldots,
\]
Moreover (see \([8, \text{p. 200}]\)),
\[
\sum_{n=1}^{\infty} |g(\varphi_n(0))|^2 \left(1 - |\varphi_n(0)|^2\right) \leq C \|g\|_2^2,
\]
for some universal constant \(C\). So \(T\) is a bounded linear map from \(\mathcal{H}^2\) onto \(\ell^2\).
If $B$ denotes the Blaschke product with zero sequence $\{\varphi_n(0)\}_{n \geq 1}$, we see that the map,

$$T_1 : \mathcal{H}^2 / B\mathcal{H}^2 \to \ell^2$$

defined by:

$$T_1 (g + B\mathcal{H}^2) = T(g)$$

is an isomorphism and therefore bounded below. Consequently, there exists $\delta > 0$ such that

$$\| T_1 (g + B\mathcal{H}^2) \|_2 \geq \delta^{1/2} \| g + B\mathcal{H}^2 \|_2.$$ 

In other words,

$$\sum_{n=1}^{\infty} |(g(\varphi_n(0)))^2 (1 - |\varphi_n(0)|^2)) \geq \delta \| g + B\mathcal{H}^2 \|_2^2,$$

for all $g \in \mathcal{H}^2$.

Therefore, for each $k \geq 0$, we deduce:

$$\sum_{n=1}^{\infty} (1 - |\varphi_n(0)|^2) < M_1 < \infty.$$

So, given $\varepsilon > 0$ we may choose a positive integer $n_1$ so that

$$\sum_{n \geq n_1} (1 - |\varphi_n(0)|^2) < (\delta \varepsilon) / M_0^2,$$

where $M_0$ is the positive constant given above.

Further, as $n \to \infty$, $F(\varphi_n(0)) \to 0$ by assumption. Consequently, we may choose $n_2$ so that $|F \circ \varphi_{n+k}(0)| < \delta / M_1$ for $n \geq n_2$ and for any $k \geq 0$.

Bearing (4) in mind, we deduce that for any $k \geq N_0 = \max\{n_1, n_2\},$

$$\| F \circ \varphi_k + B\mathcal{H}^2 \|_2^2 \leq \frac{1}{\delta} \sum_{n=1}^{N_0} |F \circ \varphi_{k+n}(0)|^2 (1 - |\varphi_n(0)|^2)$$

$$\leq \frac{1}{\delta} \left( \sum_{n=1}^{N_0_0} |F(\varphi_{k+n}(0))|^2 (1 - |\varphi_n(0)|^2) \right)$$

$$+ \sum_{n=N_0+1}^{\infty} |F(\varphi_{k+n}(0))|^2 (1 - |\varphi_n(0)|^2)$$

$$\leq 2 \varepsilon.$$

So

$$\| F \circ \varphi_k + B\mathcal{H}^2 \|_2 \to 0,$$

as $k \to \infty$, completing the proof of the first claim of Theorem 2.4.

Now suppose that, in addition to (1), we also know that the radial limit of $F$ does not exist at the point 1, and that the norms $\| F \circ \varphi_n \|_2$ are uniformly bounded.

Since the radial limit of $F$ does not exist at the point 1, we may choose a sequence of positive real numbers $\{r_k\}$ tending to 1 and a constant $\gamma$, such that $|F(r_k)| \geq \gamma > 0$ for all $k$.

By [4, Lemma 2.66, p. 82], $\{\varphi_n(0)\}$ approach the attractive fixed point nontangentially. We claim that the following is true:
Claim 1. There exists a sequence \( \{n_k\} \) and a constant \( \beta < 1 \), independent of \( k \), such that \( \rho(\varphi_{n_k}(0), r_k) < \beta < 1 \).

**Proof.** Since \( \{\varphi_n(0)\} \) approaches the fixed point +1 non-tangentially, there exists a constant \( \alpha > 1 \) so that

\[
|\varphi_n(0) - 1| < \alpha (1 - |\varphi_n(0)|).
\]

For notational convenience, we let \( y_k = \varphi_k(0) \) and \( s_k = |y_k| \).

Then

\[
1 - \rho^2(y_k, s_k) = 1 - \rho^2(y_k, s_k) = (1 - |y_k|^2)(1 - s_k^2)/|1 - s_k y_k|^2
\]

\[
= (1 - s_k^2)^2/|1 - y_k s_k|^2.
\]

But

\[
|1 - y_k s_k| \leq |1 - y_k| + |s_k| |y_k - 1| \leq |1 - y_k| + |1 - s_k|.
\]

Going back to (6) we have \( |1 - y_k s_k| \leq (1 + \alpha)|1 - s_k| \). Returning to (5) we have:

\[
1 - \rho^2(y_k, s_k) \geq (1 - s_k^2)^2/((1 + \alpha)^2(1 - s_k^2)^2) = (1 + s_k)^2/(1 + \alpha^2).
\]

In particular, it’s bounded away from zero, so

\[
\rho(y_k, |y_k|) = \rho(y_k, s_k) < \beta_1 < 1,
\]

for all \( k \). But \( t_k := \rho(|y_k|, |y_{k+1}|) \leq \rho(y_k, y_{k+1}) = \rho(\varphi_k(0), \varphi_{k+1}(0)) = |\varphi(0)| \) for all \( k \), since the pseudohyperbolic distance is invariant under \( \varphi \).

Now the discs \( D_p(|y_k|, t_k) \) are Euclidean discs and their closures contain the points \( |y_k| \) and \( |y_{k+1}| \), so the line segments between these points are contained in the closures of these discs as well. Since \( |y_k| \nrightarrow 1 \), the (closed) pseudohyperbolic discs, \( D_p(|y_k|, t_k) \) where \( t_k \leq |\varphi(0)| \), cover the set \( \{x \in \mathbb{R}: 0 < x < 1\} \). Consequently for each \( k \) we can choose \( y_{n_k} \) so that

\[
\rho(r_k, y_{n_k}) \leq |\varphi(0)| \quad \text{and from our work above,} \quad \rho(y_{n_k}, y_{n_k}) < \beta_1.
\]

The result now follows from the upper-triangle inequality for pseudohyperbolic distances, [7, Lemma 1.4, p. 4]:

\[
\rho(r_k, \varphi_{n_k}(0)) = \rho(r_k, y_{n_k}) \leq \rho(r_k, |y_{n_k}|) + \rho(|y_{n_k}|, |y_{n_k}|) = \frac{|\varphi(0)| + \beta_1}{1 + |\varphi(0)|} = \beta,
\]

using the fact that \( (x + y)/(1 + xy) \) is an increasing function of \( x \) or \( y \) when the other is held fixed. This establishes Claim 1.

Now \( \varphi_{n_k} \) maps the Euclidean disc \( D(0, \beta) \) onto the pseudohyperbolic disc \( D_p(\varphi_{n_k}(0), \beta) \). Since \( r_k \in D_p(\varphi_{n_k}(0), \beta) \) there exists \( w_k \) with \( |w_k| < \beta \) such that \( \varphi_{n_k}(w_k) = r_k \).

Since \( \{w_k\} \) is bounded, there exists \( w_0 \) with \( |w_0| \leq \beta \) such that (for some subsequence) \( w_k \to w_0 \). Thus

\[
\rho(w_k, w_0) = \frac{|w_k - w_0|}{1 - |w_0|} \leq \frac{1}{1 - \beta^2} |w_k - w_0| \to 0.
\]

By assumption sup \( \|F \circ \varphi_n\|_2 < \infty \), so there exists a subsequence of \( \{F \circ \varphi_{n_k}\} \) converging to a function \( F_1 \) weakly. We claim that \( F_1 \neq 0 \). Now (some subsequence of) \( \{F \circ \varphi_{n_k}\} \) converges uniformly on compacta. Since \( \gamma/2 > 0 \), there exists \( N \) such that for \( |w| \leq \beta \) we have:

\[
|(F \circ \varphi_{n_k})(w) - F_1(w)| < \gamma/2
\]

for all \( k \geq N \). Since \( |w_k| \leq \beta \) for all \( k = 0, 1, 2, \ldots \), we have:

\[
|F \circ \varphi_{n_k}(w_k) - F_1(w_k)| < \gamma/2
\]
for \( k \geq N \). Thus \(|F_1(w_k)| \geq \gamma / 2\) for \( k \geq N \), and therefore \(|F_1(w_0)| \geq \gamma / 2\). In particular, \( F_1 \neq 0\).

Using condition (1) of Theorem 2.4, choose \( h_k \in \mathcal{H}^2 \) such that
\[
\| F \circ \varphi_k + Bh_k \|_2 \to 0,
\]
as \( k \to \infty \). Since \( \sup_n \| F \circ \varphi_n \|_2 < \infty \) by hypothesis, \( \{Bh_n\} \) is also bounded and (the appropriate subsequence) converges to \( F_1 \) weakly as well. Since \( \|h_n\|_2 \) is bounded, there exists \( h \in \mathcal{H}^2 \) such that a subsequence of \( \{h_n\} \) converges to \( h \) weakly. Consequently, \( F_1 = Bh \).

Finally, since \( \sup_n \| F \circ \varphi_n \|_2 < \infty \), convex combinations of \( \{ F \circ \varphi_n \} \) converge to \( F_1 \) in \( \mathcal{H}^2 \) and therefore \( F_1 \in K_f \).

Suppose that, in addition to the other assumptions on \( f \), we have \( f(\varphi_{h_k}(0)) \neq 0 \) for some \( k_0 \) and \( K_f \) minimal. Then \( K_f = K_{F_1} = K_{Bh} \). But \( B \circ \varphi_n \) vanishes on \( \{\varphi_k(0)\} \) for all \( k \geq 1 \) and \( n \geq 0 \), so \( f \) would have to vanish as well, a contradiction. \( \square \)

We note that the proof above shows that if \( F \in \mathcal{H}^2 \) satisfies \( F(\varphi_n(0)) \to 0 \) as \( n \to \infty \) and \( \sup_n \| F \circ \varphi_n \|_2 \) is finite, then there exists \( h_k \in \mathcal{H}^2 \) such that \( \| F \circ \varphi_k + Bh_k \|_2 \to 0 \), where \( B \) is the Blaschke product with zeros \( \{\varphi_n(0)\}_{n \geq 1} \).

Thus, we obtain the following theorem:

**Theorem 2.5.** Let \( \varphi \) be a hyperbolic automorphism of \( \mathbb{D} \). Let \( F \in \mathcal{H}^2 \) satisfy \( \lim_{n \to \infty} F(\varphi_n(0)) = L \neq 0 \) and \( \sup_n \| F \circ \varphi_n \|_2 \leq M \) for some constant \( M \). If \( K_F \) is minimal, then \( F(\varphi_n(0)) = L \) for all \( n \geq 1 \). In particular, if \( F(\varphi_n(0)) \neq L \) for some \( n \geq 1 \), then \( K_F \) cannot be minimal.

**Proof.** Let \( z_n = \varphi_n(0) \), where \( n \geq 0 \). Note that \( L - F \in \mathcal{H}^2 \), \( (L - F)(\varphi_n(0)) \to 0 \), and
\[
\sup_n \| (L - F) \circ \varphi_n \|_2 \leq M + L.
\]
Therefore, by the comments preceding the proof of this theorem, there exists \( h_k \in \mathcal{H}^2 \) such that \( \|(L - F) \circ \varphi_k - Bh_k \|_2 \to 0 \).

Now, because \( \| F \circ \varphi_k \|_2 \) is bounded, \( \|h_k\|_2 \) is bounded as well, so (some subsequence) converges weakly to some \( h \in \mathcal{H}^2 \). Therefore, the corresponding subsequence (denoted again by \( F \circ \varphi_k \)) of \( F \circ \varphi_k \) satisfies \( F \circ \varphi_k \) converges weakly to some \( F_1 \).

Thus we have (for both subsequences)
\[
F \circ \varphi_k \text{ converges weakly to } F_1 \text{ and } L - Bh_k \text{ converges weakly to } L - Bh.
\]

But \( \| F \circ \varphi_k \circ (L - Bh_k) \|_2 \to 0 \), so \( F_1 = L - Bh \). But \( \{ F \circ \varphi_k \} \) is bounded and converges weakly to \( L - Bh \), so there exist convex combinations of \( F \circ \varphi_k \) that converge in norm to \( L - Bh \). Therefore \( L - Bh \in K_f \).

In particular, if \( K_F \) is minimal, then \( K_F = K_{L - Bh} \). So if \( g \in K_{F} \), then \( g \in \mathcal{H}^2 \) and \( g \) is a limit of sums of the form:
\[
f_n := \sum_{k=0}^{M_n} \alpha_{k,n}(L - (B \circ \varphi_k)(h \circ \varphi_k)).
\]
Evaluating at \( \varphi_l(0) \), where \( l \geq 0 \) we get:
\[
f_n(\varphi_l(0)) \to g(\varphi_l(0)).
\]
But \( f_n(\varphi_l(0)) = (\sum_{k=1}^{M_n} \alpha_{k,n})L \) and so \( f_n(\varphi_l(0)) = f_n(\varphi_m(0)) \) for all \( l, m \geq 1 \). Therefore,
\[
g(\varphi_l(0)) = \lim_n f_n(\varphi_l(0)) = \lim_n f_n(\varphi_m(0)) = g(\varphi_m(0)),
\]
for all \( l, m \geq 1 \). In particular, for \( F \) we have that \( F(\varphi_n(0)) = F(\varphi_m(0)) \) for \( n, m \geq 1 \), and the conclusion of the theorem follows. \( \square \)

### 3. Functions with radial limits

In this section, we will show that if \( f \) is not an eigenvector of \( C_\varphi \), the subspace \( K_f = \operatorname{span}(C_\varphi f; n \geq 0) \mathcal{H}^2 \)
generated by \( f \) can contain functions with very different behavior than the original function; that is, we will show
that if we begin with a disc algebra function $f$ with radial limit 0 at $+1$ and $-1$, there exists a function $g \in \mathcal{H}^\infty \cap K_f$ such that $g$ does not have radial limit 0 at $-1$. In this case, then, if we assume that $K_f$ is minimal, we can replace $f$ by $g$ and assume that our function does not have radial limit zero at $-1$ without losing the fact that our generator is bounded.

**Theorem 3.1.** Let $\varphi$ be a hyperbolic automorphism of $\mathbb{D}$ fixing $1$ and $-1$. Assume that $1$ is the attractive fixed point. Let $f$ be a nonzero function in $\mathcal{H}^2$ that is continuous at $1$ and $-1$ and such that $f(1) = f(-1) = 0$. Then there exists $g \in \mathcal{H}^2$ satisfying the following conditions:

(a) $g \in K_f := \text{span}\{C^n f : n \geq 0\}^{H^2}$.
(b) There exists a subsequence $\{\varphi_{n_k}\}$ such that $g \circ \varphi_{-n_k}$ converges to $f$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$.
(c) $g$ has no radial limit at $-1$.

Furthermore, if $f$ belongs to the disc algebra $A(\mathbb{D})$, then $g \in \mathcal{H}^\infty$ and, consequently, if $K_f$ is minimal, then $K_g = K_f$.

In what follows, given a set $M$ contained in the unit circle, $\partial\mathbb{D}$, the Lebesgue measure of $M$ will be denoted by $|M|$.

**Proof.** Since $f$ is a nonconstant function, there exists $z_0 \in \mathbb{D}$ such that $f(z_0) \neq 0$. Without loss of generality, we may assume that $0 \neq f(0) = 1$. Let $\{r_n\}$ be an increasing sequence of points with $0 < r_n < 1$, $r_n \to 1$, and denote the closed disc centered at $0$ of radius $r_n$ by $D_{r_n} = \{z \in \mathbb{C} : |z| \leq r_n\}$.

For each $n \geq 1$, let $a_n = \varphi_{-n}(0)$. Using Lemma 1.1, we may choose a subsequence $\{m_k\}$ from the positive integers for which $\|f \circ \varphi_{m_k}\|_2 \leq 1/2^k$. In what follows we will choose a sequence $\{n_k\}$ from the subsequence $\{m_k\}$ and define a sequence of analytic functions $\{g_j\}_{j \geq 0}$ on $\mathbb{D}$, recursively, as follows:

- (Step 0.) For $j = 0$, let $g_0 = f$.
- (Step 1.) For $j = 1$, define
  
  $g_1 = f + f \circ \varphi_{n_1}$,

where $n_1$ is a positive integer chosen satisfying the following conditions:

(i) For every integer $k$ with $|k| \geq n_1$, we have $\sup_{z \in D_{r_1}} |f(\varphi_k(z))| \leq \frac{1}{2}$. Since $f$ is continuous at $-1$ and $f(-1) = 0$, we may choose $B_1 \subseteq \partial \mathbb{D}$, a neighborhood of $-1$, such that $|B_1| < \frac{1}{2}$ and $\text{ess sup}_{e^{i\theta} \in B_1} |f(e^{i\theta})| < \frac{1}{2}$.

Let $A_1 = \partial \mathbb{D} \setminus B_1$. Since $f$ is continuous at $1$ and $-1$, $f(1) = f(-1) = 0$, and both sequences $\{\varphi_n\}$ and $\{\varphi_{-n}\}$ converge uniformly on compact subsets of $\mathbb{D} \setminus \{-1\}$ to $1$ and on compact subsets of $\mathbb{D} \setminus \{1\}$ to $-1$, respectively, as $n \to \infty$, we may also require that

(ii) for every $k \geq n_1$, we have $\text{ess sup}_{e^{i\theta} \in A_1} |f(\varphi_k(e^{i\theta}))| \leq \frac{1}{2}$. (iii) $|f(a_{n_1})| = |f(\varphi_{-n_1}(0))| < \frac{1}{2\pi}$.

Observe that $g_1$ is also continuous at $1$ and $-1$, and satisfies the following conditions:

1. $|g_1(0)| \geq 1 - \frac{1}{2\pi}$ since $f(0) = 1$ and (i) holds.
2. $|g_1(a_{n_1})| = |f(a_{n_1}) + f(0)| \geq 1 - \frac{1}{2\pi}$, since (iii) holds.

For the sake of clarity, we include the case $j = 2$.

- (Step 2.) For $j = 2$, we define
  
  $g_2 = f + f \circ \varphi_{n_1} + f \circ \varphi_{n_2}$,

where $n_2 > n_1$ is a positive integer chosen to satisfy conditions (i)–(v) described below:
(i) For every $k$ with $|k| \geq n_2$, we have $\sup_{z \in D_{n_2}} |f(\phi_k(z))| \leq \frac{1}{2^n}$.

We choose $B_2 \subset B_1 \subset \partial D$ a neighborhood of $-1$ such that

$$|B_2| < \frac{1}{2^n},$$

$$\text{ess sup}_{e^{i\theta} \in B_2} |f(e^{i\theta})| < \frac{1}{2^n},$$

and

$$\text{ess sup}_{e^{i\theta} \in B_2} |f(\varphi_n(\phi_k(z)))| < \frac{1}{2^n}.$$

Let $A_2 = \partial D \setminus B_2$. Further, we require that:

(ii) For every $k \geq n_2$, we have $\text{ess sup}_{e^{i\theta} \in A_2} |f(\phi_k(e^{i\theta}))| < \frac{1}{2^n}$.

(iii) $|f(a_{n_2})| = |f(\varphi_{-n_2}(0))| < \frac{1}{2^n}$.

(iv) $\sup_{z \in D_{n_2}} |f(\varphi_{n_2-n_1}(z))| \leq \frac{1}{2^n}$.

(v) $\sup_{z \in D_{n_2}} |f(\varphi_{n_1-n_2}(z))| \leq \frac{1}{2^n}$.

Note that conditions (i)–(v) are fulfilled because of the hypotheses on $f$. In addition, observe that $g_2$ is in $\mathcal{H}^2$, continuous at 1 and $-1$, $g_2(1) = g_2(-1) = 0$, and $g_2$ satisfies the following conditions:

1. $|g_2(0)| \geq 1 - \frac{1}{2^n} - \frac{1}{2^n}$ since $f(0) = 1$ and (i) holds in step 1 and step 2.
2. $|g_2(a_{n_2})| \geq 1 - \frac{1}{2^n} - \frac{1}{2^n}$ since (iii) holds.

Assume that $g_1, g_2, \ldots, g_{N-1}$ have been defined so that $g_j \in \mathcal{H}^2$, $g_j$ is continuous at 1 and $-1$, and $g_j(1) = g_j(-1) = 0$. We proceed to define $g_N$.

• (Step N.) For $j = N$, we define

$$g_N = f + f \circ \varphi_{n_1} + f \circ \varphi_{n_2} + \cdots + f \circ \varphi_{n_N},$$

where $n_N > n_{N-1}$ is a positive integer satisfying conditions (i)–(v) described below:

(i) For every integer $k$ with $|k| \geq n_N$, we have $\sup_{z \in D_{n_N}} |f(\phi_k(z))| \leq \frac{1}{2^n}$.

From the construction, we obtain neighborhoods of $B_1$, $B_1, \ldots, B_{N-1}$, with $B_{N-1} \subset B_{N-2} \subset \cdots \subset B_2 \subset B_1 \subset \partial D$ such that for each $k = 1, \ldots, N - 1$,
\[ |B_k| < \frac{1}{2^k}, \]
\[ \text{ess sup}_{\epsilon^\theta \in B_k} |f(e^{i\theta})| < \frac{1}{2^k}, \]
\[ \text{and ess sup}_{\epsilon^\theta \in B_k} |f(\varphi_n(e^{i\theta}))| < \frac{1}{2^{k+1}}, \]
for \( j = 1, \ldots, k - 1. \)

Proceeding in the same fashion we obtain a neighborhood of \(-1\), denoted \( B_N \), with \( B_N \subset B_{N-1} \) such that
\[ |B_N| < \frac{1}{2^N}, \]
\[ \text{ess sup}_{\epsilon^\theta \in B_N} |f(e^{i\theta})| < \frac{1}{2^N}, \]
\[ \text{and ess sup}_{\epsilon^\theta \in B_N} |f(\varphi_n(e^{i\theta}))| < \frac{1}{2^{N+1}}, \]
for \( j = 1, \ldots, N - 1. \)

Let \( A_N = \partial \mathbb{D} \setminus B_N \). We require the following:

(ii) For every \( k \geq nN \) we have \( \text{ess sup}_{\epsilon^\theta \in A_N} |f(\varphi_k(e^{i\theta}))| < \frac{1}{2^{k+1}}. \)

(iii) \( |f(a_{nN})| = |f(\varphi_{-nN}(0))| < \frac{1}{2^{N+1}}. \)

(iv) \( \sup_{z \in D_{Nj}} |f(\varphi_{nN-n_j}(z))| \leq \frac{1}{2^{N+1}} \) for \( j = 1, \ldots, N - 1. \)

(v) \( \sup_{z \in D_{Nj}} |f(\varphi_{n_j-n_N}(z))| \leq \frac{1}{2^{N+1}} \) for \( j = 1, \ldots, N - 1. \)

Note that conditions (i)–(v) are fulfilled because of hypotheses on \( f \). In addition, observe that \( g_N \in \mathcal{H}^2 \), \( g_N \) is continuous at \( 1 \) and \( -1 \), \( g(1) = g(-1) = 0 \) and \( g_N \) satisfies the following conditions:

(1) \( |g_N(0)| \geq 1 - \sum_{k=1}^{N} \frac{1}{2^{k+1}} \) since \( f(0) = 1 \) and (i) holds.

(2) \( |g_N(a_{nN})| \geq 1 - \sum_{k=1}^{N} \frac{1}{2^{k+1}}. \) Moreover, for \( j = 1, \ldots, N, \)
\[ |g_N(a_{n_j})| \geq |f(\varphi_{n_j}(a_{n_j}))| - \sum_{k=1}^{N} |f(\varphi_{n_k}(a_{n_j}))| = 1 - \sum_{k=1}^{N} |f(\varphi_{n_k-n_j}(0))| \geq 1 - \frac{1}{2^j}. \]

Since we chose \( \varphi_{n_j} \) so that \( \|f \circ \varphi_{n_j}\|_2 < 1/2^j \), the sequence \( \{g_l\}_{l \geq 0} \) converges in \( \mathcal{H}^2 \) to a function \( g \in \mathcal{H}^2 \) as \( l \to \infty \). Next, we show that \( g \) satisfies the conditions stated in Theorem 3.1.

By construction, \( g \in K_f \) and therefore condition (a) of Theorem 3.1 holds trivially. Indeed, \( g \) is expressed in \( \mathcal{H}^2 \) by means of the sum,
\[ g = f + \sum_{k=1}^{\infty} f \circ \varphi_{n_k}. \] (7)

Since \( \mathcal{H}^2 \)-convergence implies uniform convergence on compacta, the series above converges to \( g \) uniformly on compact subsets of \( \mathbb{D}. \)

In order to prove condition (b) of Theorem 3.1, for any \( l \geq 1 \) let \( n_j \) denote the positive integer chosen to define the corresponding function \( g_l \). Let \( K \) be a compact subset of \( \mathbb{D} \) and let \( r_j \) be chosen so that \( K \subseteq D_{r_j}. \) We will show that
\[ \sup_{w \in K} |g \circ \varphi_{-n_j}(w) - f(w)| \to 0, \quad \text{as } l \to \infty. \] (8)

Taking into account expression (7) and conditions (iv) and (v), we deduce that for any \( z \in D_{r_1} = \{ z \in \mathbb{C}: |z| \leq r_1 \}, \)
which proves condition (c) of Theorem 3.1, and therefore the proof of (a), (b), and (c) is complete.

Directly as follows, keeping in mind the conditions (iv) and (v) of the construction of the sequence such that
\[ e^{i\theta} \]
gverges to a bounded analytic function \( g \), for every \( \{ an_l \} \) that converges to 0 as \( l \to \infty \), uniformly on compacta. This proves (8), and therefore, condition (b) of Theorem 3.1.

Now we show that \( g \) is not constant, and \( \varphi_{-n_l} \to -1 \) uniformly on compacta as \( l \to \infty \), it follows that \( g \) has no radial limit at \(-1\), which proves condition (c) of Theorem 3.1, and therefore the proof of (a), (b), and (c) is complete.

In order to check condition (c) of Theorem 3.1, recall that \( a_{n_l} = \varphi_{-n_l}(0) \) for every \( l \geq 1 \). Therefore, since we know that \( g \circ \varphi_{-n_l} \to f \) uniformly on compacta, we see that \( g(a_{n_l}) = g(\varphi_{-n_l}(0)) \to f(0) = 1 \). One can also compute this directly as follows, keeping in mind the conditions (iv) and (v) of the construction of the sequence \( \{ g_l \} \):

\[
|g(a_{n_l}) - 1| = \left| f(a_{n_l}) + \sum_{k=1}^{\infty} f(\varphi_{n_k}(a_{n_l})) - 1 \right|
\leq |f(a_{n_l})| + \sum_{k=1 \atop k \neq l}^{\infty} |f(\varphi_{n_k}(0))|
\leq |f(a_{n_l})| + \sum_{k=1 \atop k \neq l}^{\infty} \frac{1}{2^j + k},
\]

for every \( l \geq 1 \). Since \( f(a_{n_l}) \to f(-1) = 0 \) as \( l \to \infty \), it follows that
\[
\lim_{l \to \infty} g(a_{n_l}) = 1.
\]

On the other hand, from (8) we deduce that for any \( z \in \mathbb{D} \),
\[
g \circ \varphi_{-n_l}(z) \to f(z) \quad \text{as} \quad l \to \infty.
\]

Since \( f \) is not constant, and \( \varphi_{-n_l} \to -1 \) uniformly on compacta as \( l \to \infty \), it follows that \( g \) has no radial limit at \(-1\), which proves condition (c) of Theorem 3.1, and therefore the proof of (a), (b), and (c) is complete.

Now we show that \( g \in H^\infty \) if \( f \in A(\mathbb{D}) \). Fix \( e^{i\theta} \in \partial \mathbb{D} \setminus \{-1\} \). By construction, there exists a positive integer \( N \) such that \( e^{i\theta} \in A_{N+1} \setminus A_N \).

Then, taking into account expression (7), condition (ii) and the fact that \( A_{N+1} \subseteq A_k \) for all \( k \geq N + 1 \), we deduce that
\[ |g(e^{i\theta})| \leq |f(e^{i\theta})| + \sum_{k=1}^{N-1} |f \circ \varphi_{n_k}(e^{i\theta})| + |f \circ \varphi_N(e^{i\theta})| + \sum_{k=N+1}^{\infty} |f \circ \varphi_k(e^{i\theta})| \]
\[ \leq 2\|f\|_\infty + \sum_{k=1}^{N-1} |f \circ \varphi_{n_k}(e^{i\theta})| + \sum_{k=N+1}^{\infty} \frac{1}{2^k+1}. \tag{9} \]

Note that since \( e^{i\theta} \notin A_N = \partial \mathbb{D} \setminus B_N \), it follows that \( e^{i\theta} \in B_N \subset B_{N-1} \subset \cdots \subset B_1 \) by construction. Thus, using the properties of \( B_j \), we have that
\[ |f(\varphi_{n_k}(e^{i\theta}))| < \frac{1}{2^k} \]
for \( 1 \leq k \leq N - 1 \). Thus, the sum in (9) is bounded by \( \sum_{k=1}^{N-1} \frac{1}{2^k} \). This implies \( g \in H^\infty \).

**Remark 3.2.** Suppose that, in addition to the hypotheses of the theorem above, we know that \( K_f \) is doubly invariant. Then we may construct \( g \) using \( \varphi_{-n} \); in other words, we may take \( g \) to be an \( H^2 \) limit of functions of the form:
\[ f + f \circ \varphi_{-n_1} + \cdots + f \circ \varphi_{-n_k}. \]
Recall that we know, from Lemma 1.1, that \( \|f \circ \varphi_{n_j}\|_2 \to 0 \). Choosing our sequences as above, we may further assume that \( \|f \circ \varphi_{n_j-n_k}\|_2 < 1/2^{j+k} \) for \( j \neq k \). As above, we will have \( g \circ \varphi_{n_j} \to f \) and \( \sup_n \|g \circ \varphi_{n_j}\|_2 \leq M \), for some constant \( M \). Thus, if \( K_f \) is doubly invariant, we will have \( K_g = K_f \).

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After this paper was completed we learned that Joel H. Shapiro has strengthened the work of Chkliar (see [15]).

**References**