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On lower semi-continuous metric projections onto finite dimensional subspaces of spaces of continuous functions

Aldric L. Brown

Department of Mathematics, University College London, UK

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Abstract

The context of the paper is: a locally compact Hausdorff space \( T \); the space \( C_0(T) \), equipped with the uniform norm, of real continuous functions on \( T \) which vanish at infinity; a linear subspace \( G \), of finite dimension \( n \), of the space \( C_0(T) \); and the set-valued metric projection \( P_G \) of \( C_0(T) \) into the family of non-empty compact convex subsets of \( G \), defined by \( P_G(f) = \) the set of best uniform approximations to \( f \) from \( G \). Those \( G \) which are Chebyshev (that is the metric projection is single point valued) were characterised by Haar (1918). Those \( G \) for which the metric projection \( P_G \) is lower semi-continuous have been characterised by Wu Li (1989) and A.L. Brown (2005); the paper interprets and exploits the characterisation. A ‘Generalised Haar Condition’ which is necessary for \( P_G \) to be lower semi-continuous is identified; in some circumstances it is also sufficient. The condition has a calculable determinantal form. The results of the paper include an essentially complete determination, in case \( T \) is a compact space in which no net of components is convergent to a single point set, of those subspaces \( G \) of \( C(T) \) for which \( P_G \) is lower semi-continuous. Simple examples show that if the space \( T \) is compact and totally disconnected, but not finite, the situation is essentially different.

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1. Introduction

Let \( T \) be a locally compact Hausdorff space; \( C_0(T) \) denotes the space of real continuous functions on \( T \) which vanish at infinity, and \( C_0(T) \) is equipped with the uniform norm.
Let $G$ denote a linear subspace of $C_0(T)$ of finite dimension $n$, and let

$$P_G : C_0(T) \to \mathcal{P}(G)$$

be the set-valued metric projection of $C_0(T)$ onto $G$ defined by

$$P_G(f) = \{ g \in G : \| f - g \| = d(f, G) \}$$

for all $f \in C_0(T)$, where $d(f, G)$ is the distance of $f$ from $G$. Our concern is with those $G$ for which the metric projection is lower semi-continuous.

Associated with $G \subseteq C_0(T)$ there is a mapping $e : T \to G^* = F$ defined by

$$e(t)(g) = g(t) \quad \text{for all } g \in G \text{ and each } t \in T.$$

If $A$ is a subset of $T$ let

$$E(A) = \text{span } e(A).$$

The notations card $A$, int $A$, bdy $A$ will denote the cardinality of $A$, the interior of $A$ and the boundary (frontier) of $A$, respectively.

The mapping $e : T \to F$ has the three properties that it is continuous, $e(t) \to 0$ as $t \to \infty$, and $E(T) = F$. Throughout the paper, $e : T \to F$ with $\dim F = n \in \mathbb{N}$, will denote a mapping with these three properties. If $e : T \to F$ is such a mapping then there exists a subspace $G$ of $C_0(T)$ such that $e : T \to F$ is (a copy of) the mapping associated with $G$.

In this paper properties of a subspace $G$ of $C_0(T)$ will be expressed almost exclusively as properties of $e : T \to F$.

Best uniform approximation in spaces of continuous functions were first considered by P.L. Chebyshev in the 1850s and he discovered the phenomenon of uniqueness of best approximations. The subspace $G$ of $C_0(T)$ is now said to be a Chebyshev subspace of $C_0(T)$ if $\text{card } P_G(f) = 1$ for all $f \in C_0(T)$, that is if there is a unique best approximation from $G$ to each $f \in C_0(T)$. Chebyshev famously considered approximation to functions on an interval by polynomials, and approximation to periodic functions on $\mathbb{R}$ by trigonometric polynomials. Early in the 20th century more general situations were considered, including spaces $T$ which were subspaces of Euclidean spaces. A. Haar [4] characterised those subspaces $G$ which are Chebyshev subspaces. The following conditions are equivalent.

(1) $G$ is a Chebyshev subspace of $C_0(T)$.
(2) $\text{card } g^{-1}(0) \leq \dim G - 1$ for each $g \in G \setminus \{0\}$.
(3) If $M$ is a hyperplane of $F = G^*$ then $\text{card } e^{-1}(M) \leq \dim F - 1$.
(4) If $N$ is a proper subspace of $F = G^*$ then $\text{card } e^{-1}(N) \leq \dim N$.
(5) If $t_1, \ldots, t_n$ are distinct points of $T$ then the points $e(t_1), \ldots, e(t_n)$ of $F$ are linearly independent.
(6) If $N$ is a proper subspace of $F = G^*$ and $N = E(e^{-1}(N))$ then

$$N = \bigoplus_{t \in e^{-1}(N)} E(\{t\}).$$
The equivalence of (1) and (2) is Haar’s Theorem. (3) is a dual translation of (2). If (5) does not hold, \( t_1, \ldots, t_n \) are distinct points of \( T \) and \( e(t_1), \ldots, e(t_n) \) are linearly dependent, then \( N = E([t_1, \ldots, t_n]) \) is of dimension less than \( n \), but \( \text{card } e(N) \geq n \), so that (4) does not hold. If (4) does not hold, \( N \) is a proper subspace of \( F \) and \( \text{card } e^{-1}(N) > \dim N \), then, using the fact that \( E(T) = F \), it follows that there is a hyperplane \( M \) of \( F \), containing \( N \), which does not satisfy (3). Thus (3) implies (4), which implies (5). If (5) holds, \( M \) is a subspace of \( F \) and \( \text{card } e^{-1}(M) \geq n \), then \( M \supseteq E(e^{-1}(M)) = F \); thus (5) implies (4). If the condition \( N = E(e^{-1}(N)) \) of (6), which will recur throughout the paper, is satisfied by a subspace \( N \) of \( F \), then \( \text{card } e^{-1}(N) \geq \dim N \), and if (4) is satisfied then \( (e(t) : t \in e^{-1}(N)) \) is a basis of \( N \) and \( N \) has the direct sum decomposition of (6). So (4) implies (6). If \( N \) is a proper subspace of \( F \) let \( N' = E(e^{-1}(N)) \). Then \( N' \subseteq N \), \( e^{-1}(N') = e^{-1}(N) \) and \( N' \) satisfies the condition \( N' = E(e^{-1}(N')) \) of (6). It now follows from (6), applied to \( N' \) that \( N \) satisfies the condition of (4); thus (6) implies (4).

The condition \( N = E(e^{-1}(N)) \) is clearly necessary for the conclusion of (6). Note, for example, that if \( T \) is finite (\( \text{card } T \geq \dim F \)) then the condition \( N = E(e^{-1}(N)) \) is satisfied by only a finite number of subspaces \( N \).

Any and all of (2)–(6) will be referred to as the Haar Condition (HC). Appeal will be made to condition (6) in the final section. Note that if (HC) is satisfied and \( e^{-1}(0) \neq \emptyset \) then \( F = \{0\} \).

If a real continuous function \( g \) on the square \([0, 1] \times [0, 1]\) changes sign then the zero set of \( g \) disconnects the plane and must be infinite, so the condition (HC) is not satisfied. The existence of non-trivial Chebyshev subspaces \( G \) of \( C_0(T) \) is a restriction on the space \( T \).

**Mairhuber’s Theorem (10).** If there exists a Chebyshev subspace \( G \) of \( C_0(T) \), of dimension at least two, then \( T \) is homeomorphic to a subspace of a circle.

Actually, Mairhuber, in 1956, considered spaces \( T \) which were compact subspaces of Euclidean spaces; the general result was obtained by Lutts [9]. It may be noted that, if \( G \) is Chebyshev and \( \dim G = 2 \), then the conclusion of the theorem is obtained almost immediately by considering the mapping \( e : T \rightarrow G^* \).

The rest of this introduction will give a streamlined account of the results of the paper, which can be regarded as a continuation of the early results which have been described.

We begin with a mapping \( e : T \rightarrow F \) as described above.

**Definition 1.1.** If \( N \) is a linear subspace of \( F \) let

\[
\text{s}(N) = \text{card bdy } e^{-1}(N) + \dim E(\text{int } e^{-1}(N)) - \dim N.
\]

Usually subspaces \( N \) such that \( N = E(e^{-1}(N)) \) are considered, and then, as \( e^{-1}(N) \) is a closed set,

\[
N = E(\text{bdy } e^{-1}(N) \cup \text{int } e^{-1}(N)) = E(\text{bdy } e^{-1}(N)) + E(\text{int } e^{-1}(N)).
\]

Our starting point is the following characterisation of those \( G \) for which the metric projection is lower semi-continuous; it was obtained in [1] and is equivalent to the characterisation obtained by Wu Li [7].

**Theorem 1.2.** The metric projection \( P_G : C_0(T) \rightarrow \mathcal{P}(G) \) is lower semi-continuous if and only if \( G \) satisfies the condition:

\((s_0)\) if \( N \) is a subspace of \( F \) and \( N = E(e^{-1}(N)) \) then \( \text{s}(N) = 0 \).
We are concerned with the consequences of the condition \((s_0)\), and with the question for which spaces \(T\) do there exist subspaces \(G\) of \(C_0(T)\) which are not Chebyshev but do have a lower semi-continuous metric projection? It should be noted that if \(G\) is Chebyshev then the single valued metric projection is continuous, and the Haar Condition implies \((s_0)\), and that if the space \(T\) is discrete (so boundaries are empty and \(e^{-1}(N)\) is always open) then \((s_0)\) is always satisfied. The first consequences are relatively simple results.

**Proposition 1.3.** (1) The condition \((s_0)\) is satisfied if and only if \(G\) satisfies the condition:

\[
\text{(DSD) } N = \left( \bigoplus_{t \in \text{bdy } e^{-1}(N)} E(\{t\}) \right) \oplus E(\text{int } e^{-1}(N))
\]

whenever \(N\) is a subspace of \(F\) such that \(N = E(e^{-1}(N))\).

(2) If condition \((s_0)\) is satisfied then the set \(\text{bdy } e^{-1}(N)\) is finite for every linear subspace \(N\) of \(F\).

(3) If \((s_0)\) is satisfied and \(N\) is a linear subspace of \(F\) then \(\text{int } e^{-1}(N)\) is an open and closed subset of \(T\).

The next consequence is pivotal.

The family of components of \(T\) is a partition (pairwise disjoint cover) of \(T\). We will work with indexed partitions of \(T\). In particular, let \(\Gamma\) be an index set for the family of components of \(T\), that is \((T_\gamma : \gamma \in \Gamma)\) is the family of components of \(T\), such that \(T_\gamma \neq T_{\gamma'}\) if \(\gamma \neq \gamma'\). For each \(\gamma \in \Gamma\) let \(S_\gamma = E(T_\gamma)\). This notation is used throughout the paper. There are situations in which other partitions of \(T\) will be considered.

**Theorem 1.4.** If \(e : T \rightarrow F\) has the property \((s_0)\) then for each component \(T_\gamma\) of \(T\) the restriction

\[e|_{T_\gamma} : T_\gamma \rightarrow E(T_\gamma) = S_\gamma\]

satisfies the Haar Condition \((HC)\).

The case of this theorem in which the space \(T\) is itself connected is a theorem of Wu Li [7].

It is convenient here to assume that \(e^{-1}(0) = \emptyset\). It will be seen in Section 2 that the assumption does not involve a restriction.

It is a consequence of the direct sum decomposition (DSD) of Proposition 1.3 that if \(N\) is a subspace of \(F\) and \(N = E(e^{-1}(N))\) then

\[
\text{(wDSD) } N = \left( \bigoplus_{\{\gamma \in \Gamma : S_\gamma \subseteq N\}} E(T_\gamma \cap e^{-1}(N)) \right) \oplus \left( \sum_{\{\gamma \in \Gamma : S_\gamma \subseteq N\}} S_\gamma \right).
\]

If \(T\) has the property that each one point component of \(T\) is an isolated point of \(T\) then the two direct sum decompositions are equivalent; otherwise they may not be.

Consider a partition \((V_b : b \in \mathcal{V})\) of \(T\) by non-empty closed sets, let \(U_b = E(V_b)\) for each \(b \in \mathcal{V}\), and consider the two conditions.
For each $b \in \mathcal{V}$ the restriction $e|_{V_b} : V_b \to U_b$ satisfies the Haar Condition, and 

$$(1_{\text{HC}})(\mathcal{V})$$

for each linear subspace $N$ of $F$ such that $N = E(e^{-1}(N))$ there is a direct sum decomposition

$$N = \bigoplus_{\{b \in \mathcal{V} : U_b \subseteq N\}} E(e^{-1}(N) \cap V_b) \bigoplus_{\{b \in \mathcal{V} : U_b \subseteq N\}} U_b.$$ 

Note that the direct sum notation allows trivial summands, as $E(\emptyset) = \{0\}$ and if $V_b \subseteq e^{-1}(0)$ then $U_b = \{0\}.$

If $(V_b : b \in \mathcal{V}) = (T_{\gamma} : \gamma \in \Gamma)$ and the condition $(s_0)$ is satisfied then the two conditions $(1_{\text{HC}})(\mathcal{V})$ and $(2_{\text{wDSD}})(\mathcal{V})$ are satisfied.

The two conditions, together, will be recast in a calculable form (Theorem 1.7). Consider a finite dimensional real linear space $F$ and a family $(U_b : b \in \mathcal{V})$ of linear subspaces of $F$ such that $\sum_{b \in \mathcal{V}} U_b = F.$

**Definition 1.5.** If $A \subseteq \mathcal{V}$ let $U_A = \sum_{b \in A} U_b.$ Thus $U_\mathcal{V} = F.$ A function $l : \mathcal{V} \to \{0\} \cup \mathbb{N}$ will be called an *admissible function* for the family $(U_b : b \in \mathcal{V})$ if, for every subset $A$ of $\mathcal{V},$

$$l(A) := \sum_{b \in A} l(b) \leq \dim U_A.$$ 

If $l : \mathcal{V} \to \{0\} \cup \mathbb{N}$ is a function and, for each $b \in \mathcal{V}, l(b)$ distinct points $x_{b1}, \ldots, x_{bl(b)}$ are chosen from $U_b,$ then for the chosen points

$$(x_{b1} : t = 1, \ldots, l(b), b \in \mathcal{V})$$

to be linearly independent it is *necessary* that $l$ be an admissible function for the family $(V_b : b \in \mathcal{V}).$

**Definition 1.6.** A mapping $e : T \to F$ and partition $(V_b : b \in \mathcal{V}),$ both as above, are said to satisfy the *Generalised Haar Condition* (GHC) if for any function $l : \mathcal{V} \to \{0\} \cup \mathbb{N}$ which is admissible for the family $(U_b = E(V_b) : b \in \mathcal{V})$ and any choice, for each $b \in \mathcal{V},$ of $l(b)$ distinct points $t_{b1}, \ldots, t_{bl(b)}$ in $V_b,$ the family $(e(t_{bt}) : t = 1, \ldots, l(b), b \in \mathcal{V})$ is linearly independent.

The central result of the paper is the following theorem.

**Theorem 1.7.** A pair $(e : T \to F, (V_b : b \in \mathcal{V}))$ satisfies the conditions $(1_{\text{HC}})(\mathcal{V})$ and $(2_{\text{wDSD}})(\mathcal{V})$ if and only if it satisfies the Generalised Haar Condition.

If $g_1, \ldots, g_n$ is a basis of the subspace $G$ of $C_0(T)$ then the mapping

$$T \xrightarrow{\text{\scriptsize{\begin{array}{l} t \mapsto (g_1(t), \ldots, g_n(t)) \end{array}}}} \mathbb{R}^n$$

is a copy of the mapping $e : T \to F.$ Using the former mapping one obtains a determinantal form of (GHC), which is calculable.

The theorem, applied to $(e : T \to F, (T_\gamma : \gamma \in \Gamma))$ has two straightforward consequences.

**Theorem 1.8.** If $e : T \to F$ satisfies $(s_0)$ and $S_\gamma = E(T_\gamma) = F$ for each $\gamma \in \Gamma$ then $e : T \to F$ satisfies the Haar Condition (HC).
Proof. By (s0) the conditions (1HC)(Γ) and (2nDSD)(Γ) are satisfied, and so, by the theorem (GHC) is also. Given that all \( S_\gamma = F, I \) is admissible if and only if \( l(\Gamma) \leq \dim F \) so that (GHC) reduces to the Haar Condition (HC).

Determinantal arguments are now available. An argument by which one proves that a Chebyshev subspace \( G \) of \( C(S^1) \) must be of odd dimension yields the following structure theorem. The details of the proof will be given in Section 4.

Theorem 1.9. Suppose that \( (e : T \to F, (S_\gamma : \gamma \in \Gamma)) \) satisfies (GHC), and that \( \alpha \in \Gamma, T_\alpha \) is a circle and that \( \dim S_\alpha \geq 2 \). Then

\[
F = S_\alpha \oplus E(T \setminus T_\alpha),
\]

and \( T_\alpha \) is an open and closed subset of \( T \). (It follows that the corresponding subspace \( G \) of \( C_0(T) \) is also a direct sum.)

Among the spaces that we consider are the subspace

\[
T_0 = \{0\} \cup \left\{ \frac{1}{m} : m \in \mathbb{N} \right\}
\]

of \( \mathbb{R} \) and the subspace \( T_0 \times [0, 1] \) of \( \mathbb{R}^2 \). If \( e : T_0 \times [0, 1] \to F \) has the property (s0) it is relatively simple to prove that, for some \( M \in \mathbb{N}, E(\{\frac{1}{m} : m \geq M\}) \) is of dimension one. One must work harder for the following general result. It will be proved in Section 5.

Theorem 1.10. Let \( T \) be a compact Hausdorff space with the property that no component of \( T \) is a circle or a single point, and that no net of components of \( T \) is convergent to a single point. If \( e : T \to F \) has the property (s0) and \( e^{-1}(0) = \emptyset \), then there exists a finite partition \( (V_b : b \in \mathcal{V} = \{1, \ldots, K\}) \) by closed subsets of \( T \) such that for each \( b \in \mathcal{V} \) either \( V_b \) is an interval component of \( T \) or \( V_b \) is not embeddable in a circle and the subspace \( U_b \) of \( F \) is of dimension one.

Note that if \( T \) is a compact metric space then the essential condition on \( T \) is equivalent to the existence of \( \delta > 0 \) such that each component of \( T \) is of diameter \( \geq \delta \).

If the space \( T \), the mapping \( e : T \to F \) and the families \( (V_b : b \in \mathcal{V}) \) and \( (U_b : b \in \mathcal{V}) \) are as described in the conclusion of Theorem 1.10, then (GHC)(\( \mathcal{V} \)) is a necessary and sufficient condition for (s0) to be satisfied. Given such \( T \) and \( (V_b : b \in \mathcal{V}) \), for any family \( (U_b : b \in \mathcal{V}) \) such that \( \sum_{b \in \mathcal{V}} U_b = F \), it is possible to construct a mapping \( e : T \to F \) such that \( U_b = E(V_b) \) for each \( b \in \mathcal{V} \) and the condition (s0) is satisfied. The construction and verification in Section 6 depend on ideas in [5].

Non-isolated one-point components of \( T \) are a significant complication. If \( T \) is totally disconnected (every component is a single point) then the results which have been described add no information to the condition (s0) (the condition (GHC) is in this case a triviality). A simple example is given of a mapping \( e : T_0 \to F \), which is not Chebyshev, has the property (s0) but for which no finite partition \( (V_b : b \in \mathcal{V}) \) by closed sets has the property that \( e|_{V_b} : V_b \to U_b \) satisfies (HC) for every \( b \in \mathcal{V} \). The case of spaces which have non-isolated single point components remains open.

There is however a larger related open problem. The question whether a metric projection \( P_G : C_0(T) \to \mathcal{P} \) of \( C_0(T) \) onto a finite dimensional subspace \( G \) of \( C_0(T) \) admits a continuous selection has been studied for more than 50 years. Wu Li [8] has characterised those \( G \) for which the answer is ‘Yes’. The present author [2] has obtained an alternative, equivalent,
characterisation, using work of Thomas Fischer [3]. For the background the references of these papers should be consulted. It follows from the characterisations of those $G$ for which $P_G$ admits a continuous selection that the mapping $e : T \to F$ then satisfies the condition:

(s$_1$) if $M$ is a hyperplane of $F$ then $s(M) \leq 1$.

It remains to be see whether the programme of the present paper can be extended to the continuous selection problem and the characterisation conditions which are involved. However, interesting results concerning the condition (s$_1$) can be found in [8].

2. The condition (s$_0$)

In this section $F$ is a finite dimensional real linear space of dimension $n$, and $e : T \to F$ is a mapping with the three properties we always assume. For a linear subspace $N$ of $F$ the cardinal number $s(N)$ is as in Definition 1.1. Here we begin the investigation of the consequences of the condition (s$_0$) of Theorem 1.2.

The first theorem describes facts about restrictions of the mapping $e : T \to F$. The proofs are straightforward.

**Theorem 2.1.** (1) If $e : T \to F$ satisfies the Haar Condition (HC), $T'$ is a closed subset of $T$ and $F' = E(T')$ then $e|_{T'} : T' \to F'$ also satisfies the Haar Condition.

(2) If $e : T \to F$ satisfies the condition (s$_0$), $T'$ is an open and closed subset of $T$ and $F' = E(T')$ then $e|_{T'} : T' \to F'$ satisfies the condition (s$_0$).

(3) If $e_1 : T_1 \to F_1$ and $e_2 : T_2 \to F_2$ have the property (s$_0$) then

$$e_1 \oplus e_2 : T_1 \sqcup T_2 \to F_1 \oplus F_2,$$

defined by $(e_1 \oplus e_2)|_{T_j} = e_j$ for $j = 1, 2$, also has the property (s$_0$).

Suppose that $N = E(e^{-1}(N))$. Then

$$N = E(\text{bdy } e^{-1}(N)) + E(\text{int } e^{-1}(N))$$

and

$$\dim N \leq \dim E(\text{bdy } e^{-1}(N)) + \dim E(\text{int } e^{-1}(N)) \leq \text{card bdy } e^{-1}(N) + \dim E(\text{int } e^{-1}(N)) = s(N) + \dim N.$$ 

Therefore $s(N) = 0$ if and only if the two weak inequalities are actually equalities, that is if and only if

$$N = E(\text{bdy } e^{-1}(N)) \oplus E(\text{int } e^{-1}(N)), \quad (2.1)$$

and

$$\dim E(\text{bdy } e^{-1}(N)) = \text{card bdy } e^{-1}(N). \quad (2.2)$$

These two equations are together equivalent to (DSD) of Proposition 1.3. Thus part (1) of Proposition 1.3 is proved. Part (2) of Proposition 1.3 is immediate from the definition of $s(N)$.

**Proposition 2.2.** If (s$_0$) is satisfied then $e^{-1}(0)$ is an open and closed subset of $T$. 

**Proof.** By (s0), applied to $N = \{0\}$, card bdy $e^{-1}(0) = 0$, bdy $e^{-1}(0) = \emptyset$ and so $e^{-1}(0)$ is open.

**Proof of Part (3) of Proposition 1.3.** Suppose that (s0) is satisfied and that $N = E(e^{-1}(N))$. If $t \in \text{bdy } e^{-1}(N)$ then, by Proposition 2.2, $e(t) \neq 0$ and so by (2.1) $e(t) \notin E(\text{int } e^{-1}(N))$ so that $t \not\in (\text{int } e^{-1}(N))^{-}$ and $t \notin \text{bdy } e^{-1}(N)$. \hfill $\Box$

The next theorem follows from Theorem 2.1 and Proposition 2.2. It shows that we may restrict attention to mappings $e : T \to F$ for which $e^{-1}(0) = \emptyset$.

**Theorem 2.3.** The mapping $e : T \to F$ has the property (s0) if and only if

1. $e^{-1}(0)$ is an open and closed subset of $T$, and
2. the restriction $e|_{T \setminus e^{-1}(0)} : T \setminus e^{-1}(0) \to F$ has the property (s0).

**Theorem 2.4.** Suppose that the condition (s0) is satisfied.

1. If $N = E(e^{-1}(N))$, $\tau \in e^{-1}(N)$ and $N = E(e^{-1}(N) \setminus \{\tau\})$ then $\tau \in \text{int } e^{-1}(N)$.
2. If $T_\gamma$ is a component of $T$ and $\text{card } T_\gamma > 1$ then $T_\gamma \subseteq e^{-1}(S_\gamma)$.
3. If $T_\gamma$ is a one-point component of $T$ and $S_\gamma \subseteq N = E(e^{-1}(N))$ then either $T_\gamma \subseteq e^{-1}(N)$ or $T_\gamma$ is an isolated point of $e^{-1}(N)$.

**Proof.** (1) follows immediately from (DSD).

2. The component $T_\gamma$ is an infinite subset of $e^{-1}(S_\gamma)$ so, by Proposition 1.3(2), $T_\gamma \not\subseteq \text{bdy } e^{-1}(S_\gamma)$ and $T_\gamma \cap e^{-1}(S_\gamma) \neq \emptyset$. But $T_\gamma$ is connected, and $\text{int } e^{-1}(S_\gamma)$ is open and closed, so that $T_\gamma \subseteq e^{-1}(S_\gamma)$.

3. Suppose $T_\gamma \not\subseteq \text{int } e^{-1}(N)$ so that, being a single point, $T_\gamma \subseteq \text{bdy } e^{-1}(N)$, $T_\gamma$ is not a surplus point for $N$, and so $T_\gamma$ is an isolated point of $e^{-1}(N)$. \hfill $\Box$

The letter ‘s’ is for ‘surplus’. Part (1) of the theorem can be read as saying that no boundary point of $e^{-1}(N)$ is surplus for $N$.

The following five conditions are equivalent. They relate this paper to the papers [7,1].

1. $P_G : C_0(T) \to \mathcal{P}(G)$ is lower semi-continuous.
2. For each $g \in G \setminus \{0\}$
   \[ \text{card bdy } g^{-1}(0) \leq \dim \{p \in G : \text{int } g^{-1}(0) \subseteq p^{-1}(0)\} - 1. \]
3. For each hyperplane $M$ of $G^*$, $s(M) \leq 0$.
4. For each subspace $N$ of $G^*$, if $N = E(e^{-1}(N))$, then $s(N) = 0$.
5. For each linear subspace $N$ of $G^*$, $s(N) \leq 0$.

In (4) and (5), $s(N)$ is as in Definition 1.1.

The equivalence of (1) and (2) is a theorem of Wu Li [7]. The equivalence of (2) and (3) is a matter of translation, taking $M = \{g\}^\perp$. The equivalence of (1) and (4) is Theorem 1.2. (4) is the condition (s0).

(4) and (5) are equivalent because, if $N$ is a linear subspace of $G^*$ and we let $N' = E(e^{-1}(N))$ then $N' \subseteq N$, and $e^{-1}(N') = e^{-1}(N)$ so that $N' = E(e^{-1}(N'))$ and
\[ s(N) = s(N') + \dim N' - \dim N \leq s(N'). \]

A direct proof will be given that (3) implies (5). Suppose that (5) does not hold, that there exists an $N = E(e^{-1}(N))$ such that $s(N) > 0$. Then there exists $\tau \in \text{bdy } e^{-1}(N)$ which is
suppose that condition (3) does not hold. Then $e^{-1}(N) = \cap_{j=1}^{r} e^{-1}(M_j)$ and so $\tau \in \cup_{j=1}^{r} \text{bdy } e^{-1}(M_j)$. If $\tau \in \text{bdy } e^{-1}(M_k)$ then $\tau$ is surplus for the hyperplane $M_k$ and $s(M_k) \geq 1$, so that (3) does not hold. \hfill \Box

**Proof of Theorem 1.4.** Suppose that condition $(s_0)$ is satisfied. If $T_\gamma \subseteq e^{-1}(0)$, then $E(T_\gamma) = \{0\}$ and the condition (HC) is satisfied. The set $e^{-1}(0)$ is open and closed in $T$ so if a component $T_\gamma \not\subseteq e^{-1}(0)$ then $T_\gamma \cap e^{-1}(0) = \emptyset$.

Suppose $\gamma \in \Gamma$ and $T_\gamma \cap e^{-1}(0) = \emptyset$. Then dim $S_\gamma \geq 1$. Let $r = \text{dim } S_\gamma$ and suppose that $t_1, \ldots, t_r$ are distinct points of $T_\gamma$ such that $e(t_1), \ldots, e(t_r)$ are not linearly independent. All of $e(t_1), \ldots, e(t_r)$ are non-zero so there exist real $\alpha_1, \ldots, \alpha_r$, not all zero, such that $\sum \alpha_j e(t_j) = 0$. Let

$$N = \text{span}\{e(t_j) : \alpha_j \neq 0\} \subseteq S_\gamma.$$

Then $N = E(e^{-1}(N))$, dim $N < r$ and $N \neq S_\gamma$. If $\alpha_j \neq 0$ then $t_j$ is surplus for $N$, so that $t_j \in \text{int } e^{-1}(N)$. Thus $T_\gamma \cap \text{int } e^{-1}(N) \neq \emptyset$ and therefore $T_\gamma \subseteq \text{int } e^{-1}(N)$. It follows that $S_\gamma \subseteq N$, which is a contradiction. This proves that $e(t_1), \ldots, e(t_r)$ are linearly independent. \hfill \Box

The final theorem in this section is a refinement of the direct sum decomposition (DSD) of Proposition 1.3.

**Theorem 2.5.** Suppose that $e : T \to F$ is such that $e^{-1}(0) = \emptyset$. For a subspace $N = E(e^{-1}(N))$ of $F$ let

$$P_N^1 = \{\gamma \in \Gamma : T_\gamma \cap e^{-1}(N) \neq \emptyset, S_\gamma \not\subseteq N\},$$

$$P_N^2 = \{\gamma \in \Gamma : T_\gamma \cap \text{bdy } e^{-1}(N) \neq \emptyset, S_\gamma \subseteq N\},$$

$$P_N^3 = \{\gamma \in \Gamma : T_\gamma \cap \text{int } e^{-1}(N) \neq \emptyset, S_\gamma \subseteq N\}.$$

Then $e : T \to F$ satisfies the condition $(s_0)$ if and only if it satisfies the following four conditions.

1. $(\text{HC})$ (I) For each $\gamma \in \Gamma$ the restriction $e|_{T_\gamma} : T_\gamma \to S_\gamma$ satisfies the Haar Condition.

2. $(\text{DSD})$ (I) If $N = E(e^{-1}(N))$ then

$$N = \left( \bigoplus_{\gamma \in \Gamma ; S_\gamma \not\subseteq N} E(T_\gamma \cap e^{-1}(N)) \right) \oplus \left( \sum_{\gamma \in \Gamma ; S_\gamma \subseteq N} S_\gamma \right).$$

$(3_{s_0})$ (I)

$$\sum_{\gamma \in \Gamma ; S_\gamma \subseteq N} S_\gamma = \left( \bigoplus_{\gamma \in P_N^2} S_\gamma \right) \oplus \left( \sum_{\gamma \in P_N^3} S_\gamma \right).$$

4. If $\gamma \in \Gamma$ and card $T_\gamma > 1$ then $T_\gamma \subseteq \text{int } e^{-1}(S_\gamma)$.

If the condition $(s_0)$ is satisfied, $N = E(e^{-1}(N))$ and $\gamma \in P_N^2$ then $T_\gamma$ is a single point which is a boundary point of $e^{-1}(N)$.
Proof. The assumption that \( e^{-1}(0) = \emptyset \) will be used without comment.

Note that, by their definitions, the sets \( P_1^N, P_2^N, P_3^N \) are such that

\[
\begin{align*}
P_1^N & \cup P_2^N \cup P_3^N = \{ \gamma \in \Gamma : T_\gamma \cap e^{-1}(N) \neq \emptyset \}, \\
P_1^N \cap (P_2^N \cup P_3^N) & = \emptyset, \\
P_2^N \cup P_3^N & = \{ \gamma \in \Gamma : S_\gamma \subseteq N \}.
\end{align*}
\]

Suppose only that conditions \((1_{HC})(\Gamma)\) and \((4)\) are satisfied. Suppose that \( N = E(e^{-1}(N)) \).

If \( \gamma \in P_1^N \) then \( N \cap S_\gamma \neq \emptyset \) and \( S_\gamma \subseteq N \) so that \( \dim S_\gamma \geq 2 \) and \( \card T_\gamma > 1 \). By \((1_{HC})(\Gamma)\) and Mairhuber’s Theorem the component \( T_\gamma \) is either an interval or a circle. Also, \( T_\gamma \cap e^{-1}(N) \) is finite, so that \( T_\gamma \cap e^{-1}(N) \subseteq \bdy e^{-1}(N) \), and \( T_\gamma \cap \inter INT \gamma^{-1}(N) = \emptyset \).

If card \( T_\gamma \geq 1 \) and \( S_\gamma \subseteq N \) then, by \((4)\), \( T_\gamma \subseteq \inter INT \gamma^{-1}(N) \) so that \( \gamma \in P_1^N \setminus P_2^N \).

It follows that, if \( \gamma \in P_2^N \) then card \( T_\gamma = 1 \) so that \( T_\gamma \subseteq \bdy e^{-1}(N) \), \( T_\gamma \cap \inter INT \gamma^{-1}(N) = \emptyset \) and \( P_2^N \cap P_3^N = \emptyset \). So

\[
P_1^N, P_2^N \text{ and } P_3^N \text{ are pairwise disjoint.}
\]

If \((\text{s}_0)\) is satisfied then by Theorems 1.4, 2.4(2), the conditions \((1_{HC})(\Gamma)\) and \((4)\) are satisfied, so the final statement of the theorem is proved.

If \( \gamma \in P_3^N \) and card \( T_\gamma = 1 \) then \( T_\gamma \subseteq \inter INT \gamma^{-1}(N) \). So, by the previous paragraph, if \( \gamma \in P_3^N \) then \( T_\gamma \subseteq \inter INT \gamma^{-1}(N) \).

It follows that

\[
\begin{align*}
\bigcup \{ T_\gamma \cap e^{-1}(N) : \gamma \in P_3^N \} & = \inter INT \gamma^{-1}(N), \\
\bigcup \{ T_\gamma \cap e^{-1}(N) : \gamma \in P_1^N \cup P_2^N \} & = \bdy e^{-1}(N).
\end{align*}
\]

Consider the following sequence of statements.

\[
N = \left( \bigoplus_{t \in \bdy e^{-1}(N)} E(\{t\}) \right) \oplus E(\inter INT \gamma^{-1}(N)), \tag{2.5}
\]

\[
N = \left( \bigoplus_{\gamma \in P_1^N \cup P_2^N} E(T_\gamma \cap e^{-1}(N)) \right) \oplus \left( \sum_{\gamma \in P_3^N} S_\gamma \right), \tag{2.6}
\]

\[
N = \left( \bigoplus_{\gamma \in P_1^N} E(T_\gamma \cap e^{-1}(N)) \right) \oplus \left( \left( \bigoplus_{\gamma \in P_2^N} S_\gamma \right) \oplus \left( \sum_{\gamma \in P_3^N} S_\gamma \right) \right), \tag{2.7}
\]

\[
N = \left( \bigoplus_{\gamma : S_\gamma \subseteq N} E(T_\gamma \cap e^{-1}(N)) \right) \oplus \left( \left( \bigoplus_{\gamma \in P_2^N} S_\gamma \right) \oplus \left( \sum_{\gamma \in P_3^N} S_\gamma \right) \right). \tag{2.8}
\]

Statement \((2.5)\) is the direct sum decomposition (DSD) of Proposition 1.3; the condition \((\text{s}_0)\) is satisfied if and only if \((2.5)\) holds for all \( N = E(e^{-1}(N)) \subseteq F \). The statement \((2.8)\), for all \( N = E(e^{-1}(N)) \subseteq F \), is an amalgamation of, and so equivalent to, the two conditions \((2_{\text{wDSD}})(\Gamma)\) and \((3_{\text{wDSD}})(\Gamma)\).

Given \((1_{HC})(\Gamma)\) and \((4)\), \((2.5) \iff (2.6)\) by \((2.3)\) and \((2.4)\), \((2.6) \iff (2.5)\) by the Haar Condition \((1_{HC}), (2.6) \iff (2.7) \iff (2.8)\). Thus the four statements are equivalent.

The proof of the theorem is now complete. \(\Box\)
Proof. It is easily seen that, if \( \bigcup A \) for \( l \) is an admissible function for \( \text{Lemma 3.1} \). If \( T \) is totally disconnected then \( P_N^1 = \emptyset \) for all \( N = E(e^{-1}(N)) \), and the condition \( (2_{\text{wDSD}}) \) is vacuous. The space \( T_0 \) is the simplest infinite totally disconnected compact Hausdorff space. Section 6, and the paper, conclude with some simple examples of functions \( e : T_0 \to F = \mathbb{R}^n \), \( n \geq 2 \), which satisfy \( (s_0) \), but not, in a non-trivial way, the Haar Condition.

3. Families of linear subspaces and admissible functions

In this section \( F \) is a real linear space of finite dimension \( n \). \( \mathcal{V} \) is an index set and \((U_b : b \in \mathcal{V})\) is a family of linear subspaces of \( F \) such that \( \sum_{b \in \mathcal{V}} U_b = F \). An admissible function \( l : \mathcal{V} \to \{0\} \cup \mathbb{N} \) has been defined in the Introduction (Definition 1.5) but here we extend the definition a little. The topological spaces \( T \) are absent from this section.

Given a function \( l : \mathcal{V} \to \{0\} \cup \mathbb{N} \) and a subset \( A \) of \( \mathcal{V} \) we write \( l_A = \sum_{b \in A} l(b) \) and \( U_A = \sum_{b \in A} U_b \). The function is admissible for the family \((U_b : b \in \mathcal{V})\) if \( l_A \leq \dim U_A \) for every subset \( A \) of \( \mathcal{V} \). Such a function \( l \) is then said to be admissible for a subset \( B \) of \( \mathcal{V} \) if \( l(b) = 0 \) for \( b \notin B \). An admissible function \( l \) is maximal if \( l_A = n \). A function \( l \) which is admissible for \( B \subseteq \mathcal{V} \) is maximal for \( B \) if \( l_B = \dim U_B \). If \( l \) is an admissible function, \( A \subseteq \mathcal{V} \) and \( l_A = \dim U_A \) then \( A \) will be called an equality set for \( l \). Thus an admissible function \( l \) is maximal if \( \mathcal{V} \) is an equality set for \( l \).

In this section two results, concerning maximal admissible functions and the existence of ‘coherent systems’ of bases of the subspaces \( U_b \), \( b \in \mathcal{V} \), are obtained.

**Lemma 3.1.** If \( l \) is an admissible function for \((U_b : b \in \mathcal{V})\) and subsets \( A \) and \( B \) are equality sets for \( l \) then \( A \cup B \) is an equality set for \( l \).

**Proof.** It is easily seen that, if \( l \) is an admissible function,

\[
 l_{A \cup B} = \sum_{b \in A \cup B} l(b) \\
= \sum_{b \in A} l(b) + \sum_{b \in B} l(b) - \sum_{b \in A \cap B} l(b) \\
\geq \dim U_A + \dim U_B - \dim U_{A \cap B} \\
= \dim(U_A + U_B) + \dim U_A \cap U_B - \dim U_{A \cap B} \\
\geq \dim(U_A + U_B) = \dim U_{A \cup B},
\]

from which the conclusion follows. \( \square \)

It follows from the lemma that if \( l \) is admissible for \( A \subseteq \mathcal{V} \) and each member of \( A \) belongs to some equality subset of \( A \) then \( l \) is maximal for \( A \).

**Theorem 3.2.** If \( A \subseteq \mathcal{V} \), \( l \) is a function which is admissible for \( A \) then there exists a function \( l' \geq l \) which is maximal for \( A \).

**Proof.** If \( l \) is maximal for \( A \) then (taking \( l' = l \)) there is nothing to prove. If \( l \) is not maximal for \( A \) then, by the lemma, there exists \( b \in \mathcal{V} \) which belong to no equality set for \( l \); that is, if \( b \in B \subseteq \mathcal{V} \) then \( l_B < \dim U_B \). Define \( l' : \mathcal{V} \to \{0\} \cup \mathbb{N} \) by

\[
 l''(\beta) = \begin{cases} 
 l(\beta), & \text{if } \beta \neq b, \\
 l(b) + 1, & \text{if } \beta = b.
\end{cases}
\]
Clearly \( l'' \geq l, l'' \) is an admissible function for \( A \) and \( l_A \neq l'' \). Repetition of this step a finite number of times yields an admissible function \( l' \) with the required properties.

In the remainder of this section attention is restricted to a finite index set \( \mathcal{V} \) and a family \((U_b : b \in \mathcal{V})\) of linear subspaces of \( F \) such that \( F = \sum_{b \in \mathcal{V}} U_b \). Let \( d(b) = \dim U_b \), and \( d(\mathcal{V}) = \sum_{b \in \mathcal{V}} d(b) \). □

**Definition 3.3.** A family \(((x_{b_t} : t = 1, \ldots, d(b)), b \in \mathcal{V})\) \( \in \prod_{b \in \mathcal{V}} U_{b}^{d(b)} \) will be called a coherent system of bases for the family of subspaces \((U_b : b \in \mathcal{V})\) if, for any function \( l : \mathcal{V} \to \{0\} \cup \mathbb{N} \) which is admissible for \((U_b : b \in \mathcal{V})\), the family \((x_{b_t} : t = 1, \ldots, l(b), b \in \mathcal{V})\) of elements of \( F \) is linearly independent (and so, if \( l \) is maximal for \( \mathcal{V} \), it is a basis of \( F \)).

Note that if \( b \in \mathcal{V} \) and \( l \) is the function with \( l(b) = d(b) \) and \( l(\beta) = 0 \) for \( \beta \neq b \), then the defining condition says that \( x_{b_1}, \ldots, x_{b_{d(b)}} \) is a basis of \( U_b \). It must be shown that coherent systems of bases exist. There are straightforward proofs of the following preliminary lemma.

**Lemma 3.4.** Let \( X \) be a real linear space of finite dimension \( d \) and let \( 1 \leq k \leq d \). Then the set of those \( k \)-tuples \((x_1, \ldots, x_k)\) in \( X^k \) that are linearly independent is a dense open subset of \( X^k \).

**Theorem 3.5.** If \((U_b : b \in \mathcal{V})\) is a finite family of subspaces of \( F \) then the set of those \((x_{b_t} : t = 1, \ldots, d(b)), b \in \mathcal{V}\) in \( \prod_{b \in \mathcal{V}} U_{b}^{d(b)} \) which are coherent systems of bases for \((U_b : b \in \mathcal{V})\) is a dense open subset of \( \prod_{b \in \mathcal{V}} U_{b}^{d(b)} \).

**Proof.** The set of all admissible functions \( l \) for \((U_b : b \in \mathcal{V})\), where \( \mathcal{V} \) is finite, is finite. For each admissible function \( l \) let \( \pi_l : \prod_{b \in \mathcal{V}} U_{b}^{d(b)} \to F_{l(\mathcal{V})} = \prod_{b \in \mathcal{V}} F_{l(b)} \) be the projection defined by

\[ \pi_l((x_{b_t} : t = 1, \ldots, d(b)), b \in \mathcal{V})) = (x_{b_t} : t = 1, \ldots, l(b), b \in \mathcal{V}). \]

Let \( \mathcal{B}_l \) be the dense open subset of \( F_{l(\mathcal{V})} = \prod_{b \in \mathcal{V}} F_{l(b)} \) which consists of all linearly independent \( l(\mathcal{V}) \)-tuples \((y_{b_t} : t = 1, \ldots, l(b)), b \in \mathcal{V}\) of elements of \( F \). Then \( \pi_l^{-1}(\mathcal{B}_l) \) is a dense open subset of \( \prod_{b \in \mathcal{V}} U_{b}^{d(b)} \). The finite intersection

\[ \cap \{\pi_l^{-1}(\mathcal{B}_l) : l \text{ is admissible for } (U_b : b \in \mathcal{V})\} \]

is a dense open subset of \( \prod_{b \in \mathcal{V}} U_{b}^{d(b)} \) and it is the set of coherent systems of bases for the family \((U_b : b \in \mathcal{V})\). □

### 4. The Generalised Haar Condition

In this section Theorems 1.7 and 1.9 will be proved.

Note first that the condition \((1_{\mathcal{V}H})(\mathcal{V})\) coincides with the restriction of the Generalised Haar Condition (GHC) to those admissible functions for the family \((U_b : b \in \mathcal{V})\) whose supports are single members of \( \mathcal{V} \).

Now suppose that conditions \((1_{\mathcal{V}H})(\mathcal{V})\) and \((2_{\text{wDSD}})(\mathcal{V})\) are satisfied.

Let \( l : \mathcal{V} \to \{0\} \cup \mathbb{N} \) be admissible for \((U_b : b \in \mathcal{V})\) and let the family \((t_{b_t} : t = 1, \ldots, l(b)), b \in \mathcal{V}\) be as in the statement of (GHC), that is, for each \( b \in \mathcal{V} \), \( t_{b_1}, \ldots, t_{b_l(b)} \) are distinct.
points of $V_b$. It must be shown that the family of image points $(e(t_{b_i}) : t = 1, \ldots, l(b), b \in \mathcal{V})$ is linearly independent. Let $N = \sum_{b \in \mathcal{V}} E([t_{b1}, \ldots, t_{bl(b)}])$. Then, by $(2_{\text{wDSD}})(\mathcal{V})$,

$$
N = \left( \bigoplus_{\{b \in \mathcal{V}: U_b \subseteq N\}} E(V_b \cap e^{-1}(N)) \right) \oplus \left( \sum_{\{b \in \mathcal{V}: U_b \subseteq N\}} U_b \right)
$$

$$
\supseteq \left( \bigoplus_{\{b \in \mathcal{V}: U_b \subseteq N\}} E([t_{b1}, \ldots, t_{bl(b)}]) \right) \oplus \left( \sum_{\{b \in \mathcal{V}: U_b \subseteq N\}} E([t_{b1}, \ldots, t_{bl(b)}]) \right)
$$

$$
= N. \quad (4.1)
$$

Therefore, if $b \in \mathcal{V}$ and $U_b \not\subseteq N$ then

$$
E(V_b \cap e^{-1}(N)) = E([t_{b1}, \ldots, t_{bl(b)}]) \quad (4.2)
$$

and

$$
\sum_{\{b \in \mathcal{V}: U_b \subseteq N\}} U_b = \sum_{\{b \in \mathcal{V}: U_b \subseteq N\}} E([t_{b1}, \ldots, t_{bl(b)}]). \quad (4.3)
$$

By (4.3) and the admissibility of $l$, the family $(e(t_{b_i}) : t = 1, \ldots, l(b), U_b \subseteq N)$ spans $\sum_{\{b \in \mathcal{V}: U_b \subseteq N\}} U_b$ and

$$
\dim \sum_{\{b \in \mathcal{V}: U_b \subseteq N\}} U_b \leq \sum_{\{b \in \mathcal{V}: U_b \subseteq N\}} l(b) \leq \dim \sum_{\{b \in \mathcal{V}: U_b \subseteq N\}} U_b,
$$

so there is equality, and it follows that $(e(t_{b_i}) : t = 1, \ldots, l(b), U_b \subseteq N)$ is a basis of $\sum_{\{b \in \mathcal{V}: U_b \subseteq N\}} U_b$.

If $b \in \mathcal{V}$ and $U_b \not\subseteq N$ then $U_b \cap N \neq U_b$ and, by (1HC) for $e|_{V_b} : V_b \rightarrow U_b$ and the admissibility of $l, l(b) \leq \dim U_b$ and $e(t_{b1}), \ldots, e(t_{bl(b)})$ are linearly independent. This proves that (GHC) is satisfied.

It remains to prove that if (GHC) is satisfied then so is $(2_{\text{wDSD}})$. Suppose that (GHC) is satisfied, that $N$ is a subspace of $F$ and $N = E(e^{-1}(N))$. Then

$$
N = E \left( \bigcup_{b \in \mathcal{V}} V_b \cap e^{-1}(N) \right)
$$

$$
= \sum_{b \in \mathcal{V}} E(V_b \cap e^{-1}(N))
$$

$$
= \left( \sum_{\{b \in \mathcal{V}: U_b \subseteq N\}} E(V_b \cap e^{-1}(N)) \right) + \sum_{\{b \in \mathcal{V}: U_b \subseteq N\}} U_b.
$$

A set of points in $e^{-1}(N)$ will be chosen so that their images in $F$ span $N$; then $l(b)$ will denote the number of chosen points which lie in $V_b$, so that a function $l : \mathcal{V} \rightarrow [0] \cup \mathbb{N}$ is obtained. The chosen set will be indexed as $(t_{b_i} : t = 1, \ldots, l(b), b \in \mathcal{V})$. It will then be shown that as a consequence of the choice the function $l$ is admissible for the family $(U_b : b \in \mathcal{V})$. It will then follow, by (GHC) that $(e(t_{b_i}) : t = 1, \ldots, l(b), b \in \mathcal{V})$ are linearly independent, and so is a basis of $N$. It then follows immediately that $N$ has the direct sum decomposition of $(2_{\text{wDSD}})$, and the proof will be complete.
If \( U_b = \{0\} \) or \( E(V_b \cap e^{-1}(N)) = \{0\} \) let \( l(b) = 0 \). Choose points of \( \bigcup_{b \in V : U_b \subseteq N} V_b \) such that their images are a basis of \( \sum_{b \in V : U_b \subseteq N} U_b \); for each \( b \) such that \( U_b \subseteq N \) let \( l(b) \) be the number of chosen points which lie in \( V_b \). Then for each subset \( A' \) of \( A = \{ b \in V : U_b \subseteq N \} \)

\[
\sum_{b \in A'} l(b) \leq \dim \left( \sum_{b \in A'} U_b \right). \tag{4.4}
\]

For each \( b \in V \) such that \( U_b \not\subseteq N \) and \( E(V_b \cap e^{-1}(N)) \neq \{0\} \) let \( l(b) = \dim E(V_b \cap e^{-1}(N)) \) and choose points \( t_{b1}, \ldots, t_{bl(b)} \) in \( V_b \cap e^{-1}(N) \) such that \( e(t_{b1}), \ldots, e(t_{bl(b)}) \) are a basis of \( E(V_b \cap e^{-1}(N)) \). By the condition (1GHC), \( t_{b1}, \ldots, t_{bl(b)} \) are an enumeration of \( V_b \cap e^{-1}(N) \).

It must be shown that the function \( l \) is admissible for \( (U_b : b \in V) \). Suppose that \( l \) is not admissible. Then there exists a minimal subset \( B \) of \( V \) such that

\[
\sum_{b \in B} l(b) > \dim \sum_{b \in B} U_b.
\]

Then, by the minimality of \( B \), \( l(b) > 0 \) for each \( b \in B \) and \( B \) is finite. Also, by (4.4), \( B \not\subseteq A \), so there exists \( b_0 \in B \) such that \( U_{b_0} \not\subseteq N \) and \( l(b_0) \neq 0 \).

Now define \( l' : V \to \{0\} \cup \mathbb{N} \) so that, first, \( l'(b) = 0 \) for \( b \not\in B \), second, \( 0 \leq l'(b) \leq l(b) \) for \( b \in B \), and, third,

\[
\sum_{b \in B} l'(b) = \dim \sum_{b \in B} U_b. \tag{4.5}
\]

(by choosing \( 0 \leq l'(b) < l(b) \) for one or more \( b \in B \)). Now for every proper subset \( B' \) of \( B \), by the minimality of \( B \),

\[
\sum_{b \in B'} l'(b) \leq \sum_{b \in B} l(b) \leq \dim \sum_{b \in B} U_b.
\]

Therefore \( l' \) is admissible for \( (U_b : b \in B) \) and by (GHC) the family \( (e(t_{b_i}) : t = 1, \ldots, l'(b), b \in B) \) is linearly independent. By (4.5) the span of this set must be \( \sum_{b \in B} U_b \) so that \( U_{b_0} \subseteq \sum_{b \in B} U_b \subseteq N \), which is a contradiction. Therefore \( l \) is admissible for \( (U_b : b \in V) \). \( \square \)

A determinantal form of the Generalised Haar Condition

Let \( e : T \to F = G^* \), \( (V_b : b \in V) \) and \( U_b = E(V_b) \) for \( b \in V \), be as before.

When considering (GHC) it is sufficient, by Theorem 3.2, to consider maximal admissible functions for the family \( (U_b : b \in V) \).

Let \( g_1, \ldots, g_n \) be a basis of \( G \). Then \( \phi : F \to \mathbb{R}^n \), defined by \( \phi(x) = (x(g_1), \ldots, x(g_n)) \) for \( x \in F \), is a linear isomorphism. Then \( (\phi \circ e)(t) = (g_1(t), \ldots, g_n(t)) \). If \( t_1, \ldots, t_n \) are points of \( T \) then the images \( e(t_1), \ldots, e(t_n) \) are linearly independent if and only if the vectors \( (\phi \circ e)(t_1), \ldots, (\phi \circ e)(t_n) \) are linearly independent. The latter condition is equivalent to

\[ \det(g_j(t_i)) \neq 0. \]

Let \( l \) be a maximal admissible function for \( (U_b : b \in V) \). Let \( k = k(l) = \text{card}\{b \in V : l(b) > 0\} \) and let the set \( \{b \in V : l(b) > 0\} \) be given an order, let it be \( \{b_1, \ldots, b_k\} \). Suppose that, for each \( b \in V \) with \( l(b) > 0 \), \( t_{b1}, \ldots, t_{bl(b)} \) are distinct points of \( V_b \). Let \( M_l(g_j(t_{b_i})) \) denote the \( n \times n \) matrix obtained from the \( n \) vectors \( ((\phi \circ e)(t_{b_i}) : i = 1, \ldots, l(b), b \in V, l(b) > 0) \). Thus the matrix has \( k \) blocks (indexed by \( b_1, \ldots, b_k \)). If \( b \in V \) and \( l(b) > 0 \) then the \( b \)th block is an
$l(b) \times n$ matrix. Now (GHC) can be expressed as
\[
\det M_j(g_j(t_{b_1})) \neq 0.
\]

The proof of Theorem 1.9. Suppose that $(e : t \to F, (S_\gamma : \gamma \in \Gamma))$ satisfies (GHC) and that
$
\alpha \in \Gamma$, $T_\alpha$ is a circle, and dim $S_\alpha \geq 2$. We suppose that $F = G^*$ and that a basis $g_1, \ldots, g_n$ has
been chosen.

By Theorem 1.7 the condition $(1_{HC})$ is satisfied; in particular the restriction $e|_{T_\alpha} : T_\alpha \to S_\alpha$
satisfies the Haar Condition and, as $T_\alpha$ is a circle, dim $S_\alpha$ is an odd integer and is \(\geq 3\).

Suppose that, contrary to the statement of the theorem,
\[
S_\alpha \cap \left( \sum_{\gamma \neq \alpha} S_\gamma \right) \neq \{0\}
\]
(and so $\Gamma \neq \{\alpha\}$). Let dim $S_\alpha = 2p + 1$, so $p \in \mathbb{N}$. It is possible to choose $2p$
distinct points of $T_\alpha$ and $n - 2p$ distinct points of $T \setminus T_\alpha = \cup_{\gamma \neq \alpha} T_\gamma$ such that their images in $F$ are a basis of $F$;
let $l$ be the maximal admissible function for $(S_\gamma : \gamma \in \Gamma)$ determined by the choice.

The component $T_\alpha$ will be identified with the circle
\[
S^1 = \{\exp(2\pi i \theta) : 0 \leq \theta \leq 1\}.
\]
Let $\gamma_1 = \alpha, \gamma_2, \ldots, \gamma_r$ be an enumeration of $\{\gamma \in \Gamma : l(\gamma) > 0\}$. So $r \geq 2$ and $l(1) = l(\gamma_1) = 2p$.

For all $\theta \in [0, 1]$ choose
\[
t_{\alpha t}(\theta) = \exp(\pi i (t - 1 + \theta) / p) \quad \text{for } t = 1, \ldots, 2p.
\]
Choose distinct points $t_{\gamma_1}, \ldots, t_{\gamma_{l(\rho)}}$ of $T_{\gamma_{\rho}}$, for each $\rho = 2, \ldots, r$, and let
\[
t_{\gamma_{\rho t}}(\theta) = t_{\gamma_{\rho t}} \quad \text{for all } \theta \in [0, 1], \text{ all } t = 1, \ldots, l(\rho) \text{ and all } \rho = 2, \ldots, r.
\]

Then, by (GHC), for each $\theta \in [0, 1]$,
\[
f(\theta) = \det M_j(g_j(t_{\gamma_{\rho t}}(\theta))) \neq 0.
\]

The first block of the matrix of (4.6) is $(g_j(t_{\alpha t}(\theta)))$. Now
\[
(t_{\alpha t}(1), \ldots, t_{\alpha t}(2p)) = (t_{\alpha t}(0), \ldots, t_{\alpha t}(2p - 1), t_{\alpha t}(1)).
\]
The right hand side of this equation is a cyclic odd permutation of
\[
(t_{\alpha t}(1), \ldots, t_{\alpha t}(2p))
\]
So the matrix of $f(1)$ is obtained from the matrix of $f(0)$ by an odd permutation of the $2p$ rows of
the first block. Therefore $f(1) = -f(0)$. The function $f$ is continuous and so $f(\theta) = 0$ for
some $\theta \in (0, 1)$ which contradicts (4.6).

It remains to show that the open set $T \setminus T_\alpha$ is also a closed subset of $T$. Suppose that $T \setminus T_\alpha$
is not closed, so that $T_\alpha \cap (T \setminus T_\alpha)^- \neq \emptyset$. If $\tau \in T_\alpha \cap (T \setminus T_\alpha)^-$ then $e(\tau) \neq 0$ (by $(1_{HC})$ for $T_\alpha$
and dim $S_\alpha \geq 2$) and
\[
eq 0 \quad \text{by $(1_{HC})$ for $T_\alpha$ and dim $S_\alpha \geq 2$) and}
\]
which contradicts (1.1). \(\Box\)

Corollary 4.1. If $(e : T \to F, (S_\gamma : \gamma \in \Gamma))$ satisfies (GHC) then $\{\gamma \in \Gamma : T_\gamma \cong S^1, \text{dim } S_\gamma \geq 2\}$ is finite.
5. Compact spaces in which no net of components is convergent to a single point

In this section spaces $T$ which are compact Hausdorff spaces are considered. If $T_\alpha$ is a component of $T$ let $\mathcal{W} = \mathcal{W}_\alpha$ be the directed set of open and closed neighbourhoods of $T_\alpha$. The result of this section depends upon the fact that $T_\alpha = \cap \mathcal{W}$ [6, Section 47, Vol. 2, Theorem 2]; if $T$ is not compact the result may not hold. Suppose that $T_\alpha$ is not an open subset of $T$. For each $W \in \mathcal{W}$ choose a component $C_W \subseteq W \setminus T_\alpha$. The net $(C_W : W \in \mathcal{W})$ will be considered. The aim now is the proof of the following theorem from which Theorem 1.10 will follow.

**Theorem 5.1.** Suppose that the following two conditions are satisfied.

1. $T$ is a compact Hausdorff space, and $e : T \to F$, where $\dim F \geq 2$, is a mapping satisfying the standard assumptions and the condition $(s_0)$.
2. Each component of $T$ is a non-degenerate interval and no net of components is convergent to a single point.

If $T_\alpha$ is a component of $T$ which is not an open subset of $T$ then there exists an open and closed neighbourhood $W$ of $T_\alpha$ such that $e|_W : W \to E(W)$ satisfies the Haar Condition (HC) and $\dim E(W) = 1$.

The first step of the proof is a topological lemma in which the condition $(s_0)$ is not involved.

**Lemma 5.2.** Suppose that the conditions of the theorem are satisfied and that $T_\alpha$ is a non-open component of $T$; let $(C_W : W \in \mathcal{W})$ be a net of components as described above.

If $v \in \mathbb{N}$ then there exist distinct points $t_1, \ldots, t_v$ in $T_\alpha$, a subnet $(C_W : W \in \mathcal{W}')$ of $(C_W : W \in \mathcal{W})$, and points $t_{W_1}, \ldots, t_{W_v}$ in $C_W$ for each $W \in \mathcal{W}'$, such that

$$t_j = \lim_{W \in \mathcal{W}'} t_{W_j}, \quad \text{for } j = 1, \ldots, v.$$  

**Proof.** By the compactness of $T$, if $t_W \in C_W$ for each $W \in \mathcal{W}$ there is a subnet of $(t_W : W \in \mathcal{W})$ which is convergent to a point $t_1 \in \cap \mathcal{W} = T_\alpha$. Suppose that $1 \leq \mu < v$ and that there exist distinct points $t_1, \ldots, t_\mu$ of $T_\alpha$, and points $t_{W_1}, \ldots, t_{W_\mu}$ in $C_W$ for each $W \in \mathcal{W}'$ (a cofinal subset of $\mathcal{W}$) such that

$$(t_1, \ldots, t_\mu) = \lim_{W \in \mathcal{W}'} (t_{W_1}, \ldots, t_{W_\mu}).$$

Choose pairwise disjoint closed neighbourhoods $V_1, \ldots, V_\mu$ of $t_1, \ldots, t_\mu$, respectively. For each $W \in \mathcal{W}'$ such that $C_W \cap V_j \neq \emptyset$ for $j = 1, \ldots, \mu$, the component $C_W$ must contain a point $t_{W(\mu+1)} \in W \setminus \bigcup_{j=1}^\mu V_j$. Then the net $((t_{W_1}, \ldots, t_{W_\mu}, t_{W(\mu+1)})) : W \in \mathcal{W}')$ has a subnet convergent to some point $(t_1, \ldots, t_\mu, t_{\mu+1})$ where $t_{\mu+1} \in T_\alpha \setminus \bigcup_{j=1}^\mu \text{int } V_j$. Repeating this step $v - 1$ times we obtain points $(t_1, \ldots, t_v)$ with the required properties. \hfill \Box

**Corollary 5.3.** If $W \in \mathcal{W}$ then $W$ cannot be embedded in a circle.

**Proof.** The case $v = 3$ of the lemma yields three distinct points $t_1, t_2, t_3$ in $T_\alpha$ at least one of which is not an end point of the interval $T_\alpha$, but is in the closure of $W \setminus T_\alpha$, and so the corollary is proved. \hfill \Box

**Lemma 5.4.** If the hypotheses of the theorem are satisfied then there exists $W \in \mathcal{W}_\alpha$ such that if $b \in \Gamma$ and $T_b \subseteq W$ then $S_b = S_\alpha$. 

Proof. By Theorem 2.5(4) there exists $W \in \mathcal{W}_a$ such that $S_b \subseteq S_a$ if $T_b \subseteq W$. The conclusion of the lemma will follow if it is shown that, for some $W \in \mathcal{W}_a$, $\dim S_b = \dim S_a$ whenever $T_b \subseteq W$. Suppose that this is false. Then for each $W \in \mathcal{W}$ there exists a component $C_W \in W$ such that $\dim (C_W) < \dim S_a =: v$ (and so $C_W \subseteq W \setminus T_a$). Consider the net $(C_W : W \in \mathcal{W})$. Let $t_1, \ldots, t_r$ in $T_a$ and $t_{W_1}, \ldots, t_{W_r}$ in $C_W$, for each $W \in \mathcal{W}'$ be as in Lemma 5.2. Then, by (1HC)($F$), $e(t_1), \ldots, e(t_r)$ are linearly independent, and, by the continuity of $e : T \to F$

$$e(t_1), \ldots, e(t_r) = \lim_{W \in \mathcal{W}'} (e(t_{W_1}), \ldots, e(t_{W_r})).$$

It follows that, for all $W$ contained in some $W_0$, $e(t_{W_1}), \ldots, e(t_{W_r})$ are linearly independent; and thus $\dim (C_W) \geq v$, which is a contradiction. \hfill \Box

Completion of the proof of Theorem 1.10. If $W \in \mathcal{W}_a$ is as in Lemma 5.4 then $e|_W : W \to E(W)$ satisfies condition ($s_0$) and $S_V = S_a$ for each component $T_V$ of $W$. Then by Theorem 1.8 $e|_W : W \to E(W)$ satisfies (HC). By Corollary 5.3 $W$ is not embeddable in a circle and so by Mairhuber’s Theorem, $\dim E(W) = 1$. \hfill \Box

The following theorem follows simply from Theorem 1.10.

Theorem 5.5. If the hypotheses of Theorem 1.10 are satisfied and $e^{-1}(0) = \emptyset$ then there exists a finite pairwise disjoint cover of $T$ by closed sets $V_1, \ldots, V_k$ such that for each $b = 1, \ldots, k$ either $V_b$ is a component of $T$ or $\dim E(V_b) = 1$.

Proof. By Theorem 1.10, the family of open and closed subsets $V$ of $T$ such that either $V$ is a component of $T$ or $\dim E(V) = 1$, is a cover of $T$, which has a finite subcover. If $V_1, V_2$ are two members of such a subcover and $V_1 \cap V_2$ is non-empty then the pair $V_1, V_2$ can be replaced by $V_1 \cup V_2$. \hfill \Box

The proof of Theorem 1.10 is now complete.

6. Construction of examples with property ($s_0$)

In this section we assume the conclusion of Theorem 1.10. Let $T = V_1 \cup \cdots V_k$ be a compact Hausdorff space in which $V_1, \ldots, V_k$ are pairwise disjoint non-empty closed (and so also open) subsets of $T$ such that, for each $b = 1, \ldots, k$, either $V_b$ is an interval or $V_b$ is not embeddable in a circle. For each $b = 1, \ldots, k$ let $U_b$ be a subspace of $\mathbb{R}^n$ ($n \geq 2$) with $d(b) = \dim U_b \geq 1$, such that $\dim U_b = 1$ if $V_b$ is not an interval, and $\sum_{b=1}^k U_b = \mathbb{R}^n$. Let $V = \{1, \ldots, k\}, V_1 = \{b \in V : V_b$ is an interval$\}$ and $V_2 = V \setminus V_1$. The principal aim of this section is the following theorem.

Theorem 6.1. If the space $T$ and subspaces $(U_b : b \in V)$ are as above then there exists a subspace $G = \text{span}\{g_1, \ldots, g_n\} \subseteq C(T)$, of dimension $n$, such that the metric projection

$$P_G : C(T) \to \mathcal{P}(G)$$

is lower semi-continuous, the associated mapping

$$e : T \to \mathbb{R}^n,$$

with $e(t) = (g_1(t), \ldots, g_n(t))$ for all $t \in T$, satisfies the condition ($s_0$) and

$$E(V_b) = U_b \quad \text{for } b = 1, \ldots, k.$$  \hfill (6.1)
The first step in the proof of this theorem is to show that in this context the (GHC) is a sufficient condition for (s0).

Suppose that the (GHC) is satisfied by \((e : T \to \mathbb{R}^n, (U_b : b \in \mathcal{V}))\). Then, by Theorem 1.7, the conditions \((1_{HC})(\mathcal{V})\) and \((2_{wDSD})(\mathcal{V})\) are satisfied. It must be shown that conditions (1)–(4) of Theorem 2.5 are satisfied.

If \(b \in \mathcal{V}\) then, by the (HC) and the condition \(\dim U_b \geq 1, e^{-1}(0) \cap V_b = \emptyset\). Thus our assumptions imply that \(e^{-1}(0) = \emptyset\).

By \((1_{HC})(\mathcal{V})\), \(e|_{V_b} : V_b \to U_b\) satisfies the (HC), and so, by Theorem 2.1(1), \(e|_{T_y} : T_y \to S_y\) satisfies the (HC) if \(b \in \mathcal{V}\) and \(T_y \subseteq V_b\). So \((1_{HC})(\mathcal{I})\) of Theorem 2.5 is satisfied.

If \(b \in \mathcal{V}, \gamma \in \mathcal{I}\) and \(T_y \subseteq V_b\) then \(S_y = U_b\) (either \(T_y = V_b\), or \(T_y \subseteq V_b, e^{-1}(0) = \emptyset\), so \(\dim S_y \geq 1\), and \(\dim U_b = 1\)). Then

\[ T_y \subseteq V_b \subseteq e^{-1}(U_b) = e^{-1}(S_y) \]

and the set \(V_b\) is open so that \(T_y \subseteq \text{int } e^{-1}(S_y)\). Thus condition (4) of Theorem 2.5 is satisfied.

Now suppose that \(N \subseteq \mathbb{R}^n\) and that \(N = E(e^{-1}(N))\). It must be shown that in the present context the direct sum decompositions \((2_{wDSD})(\mathcal{V})\) and \((2_{wDSD})(\mathcal{I})\) are equivalent.

If \(T_y \subseteq V_b\) then \(S_y = U_b\), and \(S_y \not\subseteq N\) if and only if \(U_b \not\subseteq N\). If \(U_b \not\subseteq N\) and \(E(V_b \cap e^{-1}(N)) \neq \emptyset\) then \(U_b \cap N \neq \emptyset\), and it follows that \(\dim U_b \geq 2\) and \(b \in \mathcal{V}_1\).

Therefore the two expressions

\[ \bigoplus_{\gamma \in \mathcal{I}: S_y \not\subseteq N} E(T_y \cap e^{-1}(N)) \quad \text{and} \quad \bigoplus_{b \in \mathcal{V}: U_b \not\subseteq N} E(V_b \cap e^{-1}(N)) \]

coincide apart from possible zero summands, while

\[ \sum_{\gamma \in \mathcal{I}: S_y \subseteq N} S_y = \sum_{b \in \mathcal{V}: U_b \subseteq N} U_b, \]

although the expressions may differ—the left hand sum may have repetitions which do not correspond to repetitions on the right. It follows that \((2_{wDSD})(\mathcal{I})\) of Theorem 2.5 is satisfied.

Finally it will be shown that the set \(P^2_N\) of Theorem 2.5 is empty. Suppose \(\gamma \in P^2_N\) that is, \(\gamma \in \mathcal{I}, T_y \not\subseteq \text{bdy } e^{-1}(N) = \emptyset\) and \(S_y \not\subseteq N\) (so that \(T_y \subseteq e^{-1}(N)\)). Then \(T_y\) is not open and \(T_y \subseteq V_b\) for some \(b \in \mathcal{V}_2\). Then \(S_y = U_b\) (both of dimension 1) and

\[ T_y \subseteq V_b \subseteq e^{-1}(U_b) \subseteq e^{-1}(N); \]

the set \(V_b\) is open and so \(T_y \cap \text{bdy } e^{-1}(N) = \emptyset\), which is a contradiction. Thus \(P^2_N\) is empty and \((3_{rsq})\) is vacuously satisfied.

To prove the theorem, functions \(g_1, \ldots, g_n\) must be constructed so that the (GHC) and the conditions (6.1) are satisfied. The arguments which follow are mainly straightforward extensions, to the block matrices which concern us, of the arguments of [5, pp. 6,7].

The functions \(g_1, \ldots, g_n\) will be required to be indefinitely differentiable on each of the closed intervals \(V_b, b \in \mathcal{V}_1\); in the examples which are constructed the restrictions of \(g_1, \ldots, g_n\) to the intervals will be polynomial functions. The proof depends upon the following theorem.

**Theorem 6.2.** If, for every maximal admissible function \(l\) for the family \((U_b : b = 1, \ldots, k)\), the functions \(g_1, \ldots, g_n\) in \(C(T)\) have the property that

\[ \det M_l(g_i^{(r)}(\tau_{b_i})) \neq 0, \]

(6.2)
for every choice of \(0 \leq \tau_{b_1} < \cdots < \tau_{b_l(b)} \leq 1\) for \(b \in \mathcal{V}_1\), and every choice of \(\tau_b \in V_b\) for \(b \in \mathcal{V}_2\), then the pair
\[
(e : T \to \mathbb{R}^n, (U_1, \ldots, U_k))
\]
satisfies \((\text{GHC})(\mathcal{V})\).

**Proof.** Suppose that the condition (6.2) is satisfied. The proof is elementary and consists in showing that for each maximal admissible function \(l\) for the family \((U_b : b = 1, \ldots, k)\) and for each choice of
\[
0 \leq t_{b_1} < \cdots < t_{b_l(b)} \leq 1 \quad \text{for } b \in \mathcal{V}_1, \quad \text{and } t_{b_1} \in V_b \quad \text{for } b \in \mathcal{V}_2,
\]
there exist points
\[
0 \leq \tau_{b_1} < \cdots < \tau_{b_l(b)} \leq 1 \quad \text{for } b \in \mathcal{V}_1, \quad \text{and } \tau_{b_1} = t_{b_1} \in V_b \quad \text{for } b \in \mathcal{V}_2,
\]
such that \(\det M_l(g_{j(b)}^{(l-1)}(\tau_{b_1}))\) is a positive multiple of \(\det M_l(g_{j(b)}(t_{b_1}))\) in the two preceding matrices, if \(b \in \mathcal{V}_2\) then \(l(b) = 1\), and if \(b \in \mathcal{V}\) and \(l(b) = 1\) then the two \(b\)th blocks are the same single row.

It will be shown that there is a sequence of matrices of which the first is \(M_l(g_{j(b)}(t_{b_1}))\), the last is of the form in (6.2), and each of the quotients of the determinants of successive matrices is positive.

If \(l(b) \leq 1\) for all \(b \in \mathcal{V}\) there is nothing to be done. Consider \(\beta \in \mathcal{V}_1\) such that \(l(\beta) > 1\). Let \(f(t)\) be the function obtained by replacing \(t_{\beta_{l(\beta)}}\) in \(\det M_l(g_{j(b)}(t_{b_1}))\) by the variable \(t\). Thus \(f(t_{\beta_{l(\beta)}}) = \det M_l(g_{j(b)}(t_{b_1}))\) and \(f(t_{\beta_{l(\beta)}} - 1) = 0\). Apply the Mean Value Theorem to \(f\) and the interval \([t_{\beta_{l(\beta)}}, t_{\beta_{l(\beta)-1}}]\). For some \(\xi_{\beta_{l(\beta)}}\), in the open interval,
\[
\det M_l(g_{j(b)}(t_{b_1})) = (t_{\beta_{l(\beta)}} - t_{\beta_{l(\beta)-1}}) f'(\xi_{\beta_{l(\beta)}})
\]
where \(f'(\xi_{\beta_{l(\beta)}})\) is obtained by replacing the last row of the \(\beta\)th block of the matrix \(M_l(g_{j(b)}(t_{b_1}))\), by the row \((g^{(1)}_{\beta_1}(\xi_{\beta_{l(\beta)}}), \ldots, g^{(1)}_{\beta_1}(\xi_{\beta_{l(\beta)}}))\). This step (for the last row of the block) is repeated for rows \(l(\beta) - 1, \ldots, 2\) in turn, to reach a matrix in which the first row of block \(\beta\) is unchanged, but subsequent rows all involve \(g^{(1)}_{\beta_1}, \ldots, g^{(1)}_{\beta_l}\) and, for \(t = 2, \ldots, l(\beta), t_{\beta_{t}}\) is replaced by \(\xi_{\beta_{t}} \in (t_{\beta_{t-1}}, t_{\beta_{t}})\). The process so far is now repeated for the last \(l(\beta) - 1\) rows of the block, then for the last \(l(\beta) - 2\) rows, and so on, until a matrix is reached the \(\beta\)th block of which is of the form required in (6.2). The process is then repeated for each \(b \in \mathcal{V}\) such that \(l(b) > 1\). The proof of Theorem 6.2 is complete. \(\square\)

**Proof of Theorem 6.1.** By Theorem 3.5 there exists a coherent system of bases \((x_{b_1} : t = 1, \ldots, d(b), b \in \mathcal{V})\) for the family of subspaces \((U_b : b \in \mathcal{V})\). Each \(x_{b_t}\) is a row vector in \(U_b \subseteq \mathbb{R}^n\) and we may write
\[
x_{b_t} = (x_{b_t}(1), \ldots, x_{b_t}(n)).
\]
Define functions \(g_1, \ldots, g_n\) in \(C(T)\) by
\[
g_j(b, t) = \left(\sum_{i=1}^{d(b)} \frac{t^{i-1}}{(t-1)!} x_{b_t}\right) (j) = \sum_{i=1}^{d(b)} \frac{t^{i-1}}{(t-1)!} x_{b_t}(j),
\]
for \(b \in \mathcal{V}_1\) and \((t, b) \in [0, 1] \times \{b\} = V_b\), and
\[
g_j(\tau) = x_{b_1}(j)
\]
for \( b \in \mathcal{V}_2 \) and \( t \in \mathcal{V}_b \). Thus, for \( b \in \mathcal{V}_1 \), the function \( g_{jb} : [0, 1] \to \mathbb{R} \) (where \( g_{jb}(t) = g_j(t, b) \)) is a polynomial function, and for \( b \in \mathcal{V}_2, g_{jb} = g_j \bigr|_{\mathcal{V}_b} \) is a constant function.

Let \( l \) be a maximal admissible function for \( (U_b : b \in \mathcal{V}) \). The matrix \( M_l(g^{l(-1)}_{ji}(\tau_{bi})) \) is now defined for \( 0 \leq \tau_{b1} \leq \cdots \leq \tau_{bl(b)} \leq 1 \) for \( b \in \mathcal{V}_1 \) and for \( \tau_{bi} \in \mathcal{V}_b \) for \( b \in \mathcal{V}_2 \). If \( \tau_{bi} = 0 \) for \( t = 1, \ldots, l(b) \), for \( b \in \mathcal{V}_1 \), and for \( \tau_{b1} \in \mathcal{V}_b \) for \( b \in \mathcal{V}_2 \), then

\[
\det M_l(g^{l(-1)}_{ji}(\tau_{bi})) = \det M_l(x_{bi}(j)) \neq 0. \tag{6.5}
\]

The matrices involved are all continuous functions of their \( n \) variables and the set of maximal admissible functions for the finite family \( (U_b : b \in \mathcal{V}) \) is finite, so there exists \( \delta > 0 \) such that

\[
\det M_l(g^{l(-1)}_{ji}(\tau_{bi})) \neq 0 \tag{6.6}
\]

for all \( \tau_{bi}, t = 1, \ldots, l(b) \), \( b \in \mathcal{V}_1 \) with \( 0 \leq \tau_{b1} \leq \cdots \leq \tau_{bl(b)} \leq \delta \), and all \( \tau_{bi} \in \mathcal{V}_b \) for \( b \in \mathcal{V}_2 \), and for each function \( l \) which is maximal admissible for \( (U_b : b \in \mathcal{V}) \).

Now consider the restrictions of the functions \( g_1, \ldots, g_n \) to the subspace

\[
T(\delta) = \left( \bigcup_{b \in \mathcal{V}_1} [0, \delta] \times \{ b \} \right) \cup \left( \bigcup_{b \in \mathcal{V}_2} \mathcal{V}_b \right) = \bigcup_{b \in \mathcal{V}} \mathcal{V}_b' \quad \text{(say)};
\]

the symbols \( g_1, \ldots, g_n \) will be used for the restrictions also. For \( b \in \mathcal{V}_1 \) let

\[
\mathcal{U}_b' = \text{span}\{(g_{1b}(\tau), \ldots, g_{nb}(\tau)) : 0 \leq \tau \leq \delta\}
\]

and for \( b \in \mathcal{V}_2 \) let \( \mathcal{U}_b = U_b \).

In order to apply Theorem 6.2 to the space \( T(\delta) \) it must be shown that \( \mathcal{U}_b' = U_b \) for all \( b \in \mathcal{V}_1 \), also, so that a maximal admissible function \( l \) for \( (U_b : b \in \mathcal{V}) \) is also maximal admissible for \( (\mathcal{U}_b' : b \in \mathcal{V}) \).

Consider \( \beta \in \mathcal{V}_1 \). For each \( t \in [0, 1] \) the vector \( (g_1(t, \beta), \ldots, g_n(t, \beta)) \) is a linear combination of the vectors \( x_{\beta1}, \ldots, x_{\beta d(\beta)} \) which are a basis of \( U_\beta \). Thus \( (g_1(t, \beta), \ldots, g_n(t, \beta)) \in U_\beta \) for all \( t \in [0, 1] \), and \( U_\beta' \subseteq U_\beta \).

If functions \( h_1, \ldots, h_n \) are differentiable and \( (h_1(t), \ldots, h_n(t)) \in U_\beta' \) for all \( t \in [0, \delta] \) then \( (h_1'(t), \ldots, h_n'(t)) \in U_\beta' \) for all \( t \in [0, \delta] \). It follows that \( (g_1^{(l-1)}(t, b), \ldots, g_n^{(l-1)}(t, b)) \in U_b' \) for all \( t \in [0, \delta] \), for all \( l = 1, \ldots, d(b) \), for all \( b \in \mathcal{V}_1 \).

Let \( l \) be a maximal admissible function for \( (U_b : b \in \mathcal{V}) \) such that \( l(\beta) = d(\beta) \) (such an \( l \) exists by Theorem 3.2). Then, by (6.5), the rows of the matrix \( M_l(g^{(l-1)}_{ji}(\tau_{bi})) \) are linearly independent. Then the span of the rows of the \( \beta \) th block is a subspace of \( U_\beta' \) and is of dimension \( d(\beta) = \dim U_\beta \). Therefore \( U_\beta' = U_\beta \).

It now follows from Theorem 6.2 that the pair \( (e : T(\delta) \to \mathbb{R}^n, (U_b : b \in \mathcal{V})) \) where

\[
e(t, b) = (g_1(b, t), \ldots, g_n(b, t)) \quad \text{for} \quad 0 \leq t \leq \delta, \quad b \in \mathcal{V}_1,
\]

\[
e(\tau) = (g_1(\tau), \ldots, g_n(\tau)) \quad \text{for} \quad \tau \in \mathcal{V}_b, \quad b \in \mathcal{V}_2,
\]

satisfies both (GHC) and the condition that \( \text{span} e(V_b') = U_b \) for \( b \in \mathcal{V} \). The spaces \( T(\delta) \) and \( T \) are homeomorphic and so Theorem 6.1 is proved. \(\square\)

Finally, simple examples are constructed, for the space

\[
T_0 = \{ 0 \} \cup \left\{ \frac{1}{m} : m \in \mathbb{N} \right\} \subseteq \mathbb{R},
\]

to show that the conclusion of Theorem 1.10 may not be satisfied.
Theorem 6.3. For each $F$ of dimension $>1$, there exists a continuous mapping $e : T_0 \rightarrow F$, with $E(T_0) = F$ such that the condition $(s_0)$ is satisfied, but there exists no (open and closed) neighbourhood $W$ of 0 in $T_0$ such that the restriction $e|_W : W \rightarrow F$ satisfies (HC).

Proof. Let

$$T'_0 = \{0\} \cup \left\{ \frac{1}{m} : m \in 2\mathbb{N} - 1 \right\} \subseteq T_0.$$ 

Let $e : T'_0 \rightarrow F$ be a continuous mapping such that $E(T'_0) = F$ and the condition (HC) is satisfied (it follows that $e^{-1}(0) = \emptyset$). Let $e : T_0 \rightarrow F$ denote the extension of $e : T'_0 \rightarrow F$ defined by

$$e(2k) = e(2k - 1) \quad \text{for } k \in \mathbb{N}.$$ 

It is obvious that there is no neighbourhood $W$ of 0 in $T_0$ on which the restriction $e|_W$ satisfies (HC).

It remains to verify that $(s_0)$ is satisfied. Suppose that $N = E(e^{-1}(N))$ and $N \neq F$. By the Haar Condition for $e|_{T'_0}$, $0 \notin \text{int} e^{-1}(N)$. If $0 \notin \text{bdy} e^{-1}(N)$ then $e^{-1}(N)$ is open and $s(N) = 0$. Suppose that $0 \in \text{bdy} e^{-1}(N)$. Then $\text{bdy} e^{-1}(N) = \{0\}$, and $e^{-1}(N) \setminus \{0\} = \text{int} e^{-1}(N)$. Now, by the definition of $e : T_0 \rightarrow F$, and by the (HC) satisfied by $e|_{T'_0}$, $E(e^{-1}(N) \cap T'_0) = N$ and

$$N = \mathbb{R}e(0) \oplus E((e^{-1}(N) \cap T'_0) \setminus \{0\}) = \mathbb{R}e(0) \oplus E(e^{-1}(N) \setminus \{0\}).$$

The proof is complete. $\square$

References


