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Note

An upper bound for the number of planar lattice triangulations

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Abstract

We prove an exponential upper bound for the number $f(m,n)$ of all maximal triangulations of the $m \times n$ grid:

$$f(m,n) < 2^{3mn}.$$

In particular, this improves a result of S.Yu. Orevkov [2].

We consider lattice polygons P (with vertices in $\mathbb{Z}^2$), for example the convex hull of the grid $P_{m,n} := \{0,1,\ldots,m\} \times \{0,1,\ldots,n\}$. We want to estimate the number of *maximal* lattice triangulations of P, i.e., triangulations using all integer points $P \cap \mathbb{Z}^2$ in P. These are exactly the *unimodular* triangulations, in which all the triangles have integer vertices and area $\frac{1}{2}$. From now on we will talk only about unimodular triangulations. Denote by $f(P)$ the number of (unimodular) triangulations of P and by $f(m,n)$ the number of triangulations of $P_{m,n}$. S.Yu. Orevkov's upper bound [2] is $f(m,n) \leqslant 4^{3mn}$.

Theorem 1. *The number $f(P)$ of maximal triangulations of a lattice polygon P is bounded by*

$$f(P) \leqslant 2^{|E'|},$$

where $|E'|$ is the cardinality of the set E' of inner (non-boundary) edges of an arbitrary *unimodular triangulation of P.*

In particular, the number of unimodular triangulations of the grid $P_{m,n}$ is bounded by

$$f(m,n) \leqslant 2^{3mn-m-n} < 2^{3mn}.$$

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The haystack approach

Let P be a closed, not necessarily convex lattice polygon and $\text{int}(P)$ its interior. Define $M := (\frac{1}{2}\mathbb{Z}^2 \backslash \mathbb{Z}^2) \cap \text{int}(P)$, the possible midpoints of the inner edges of a lattice triangulation of P.

Lemma 2. *For any unimodular triangulation T of P, there is a canonical bijection from the set E' of inner edges to the set M' of half-integral but not integral points in P, which sends each edge in E' to its midpoint.*

Proof. The injection from E' to M is clear.

On the other hand all unimodular triangles are $SL(2,\mathbb{Z})$-equivalent to $\mathbb{Z}^2$-translates of $\text{conv}\{\mathbf{0}, e_1, e_2\}$, so they do not contain interior points from M. □

Notation. For a subcomplex S of a triangulation of P and $r \in M$, if there is an edge through r in S we denote it by $e_S(r)$. We use a lexicographic order on $(\frac{1}{2}\mathbb{Z})^2$:

$$(x_1, y_1) \prec (x_2, y_2) :\Leftrightarrow [y_1 < y_2] \quad \text{or} \quad [y_1 = y_2 \text{ and } x_1 < x_2].$$

Definition 3. A *haystack* H (with respect to some $r \in M$) is a subcomplex of a triangulation of P that consists of the boundary of the polygon and of a set of interior edges whose midpoints are the points $r' \in M$ with $r' \prec r$.

Proof of Theorem 1. The idea is to run through M lexicographically, and at each step to add an edge through $r \in M$. We will see that in each step there are at most two possibilities to put the new edge through r.

We proceed by induction on the totally ordered set $(M, \prec)$, thus proving that the number of haystacks with respect to some $r \in M$ is $\leqslant 2^{e_r}$, where e_r is the number of predecessors of r in M. Thus after the final step (that is, after processing the largest r in $(M, \prec)$) we have obtained that there are at most $2^{|M|} = 2^{|E'|}$ unimodular triangulations of P.

Now for some $r \in M$ consider a haystack H with respect to r (Fig. 1). We want to add a "needle" to our haystack so that the resulting subcomplex will again be a haystack. So we consider the set A_r of possible endpoints v of edges through r, with $v \prec r$:

$$A_r := \{v \in \mathbb{Z}^2 \mid v \prec r \text{ and } H \cup \{[v, v + 2\vec{vr}]\} \text{ is a haystack}\}.$$

We want to prove that $|A_r| \leqslant 2$ for all $r \in M$.

We say that v is *visible* from r if the edge $[v, r]$ crosses no other edge or integral point. Consider

$$A := \{v \in \text{conv}(\{r\} \cup A_r) \cap \mathbb{Z}^2 \mid v \text{ is visible from } r\}.$$

As $v \in A_r$ is visible from r we have $A \supseteq A_r$. Furthermore $v \prec r$ holds for all $v \in A$.

We now order A by the angles $\alpha(v)$ of $\vec{rv}$ with the x-axis turning counter-clockwise and starting by π, so that we have $A = \{v_1, v_2, \ldots, v_k\}$, $\alpha_i = \alpha(v_i)$, $\alpha_1 < \alpha_2 < \cdots < \alpha_k$.

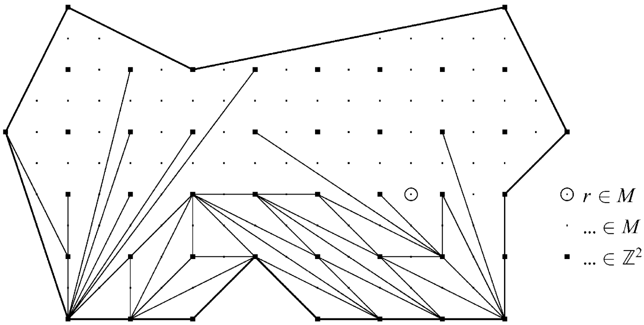


Fig. 1. A haystack with respect to r.

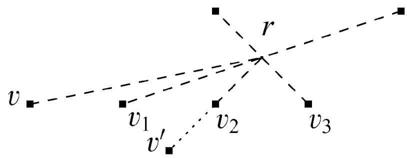


Fig. 2. Here $A_r = \{v_1, v_3\}$ and $A = \{v_1, v_2, v_3\}$, while by definition $v, v' \notin A$ and $\alpha_1 < \alpha_2 < \alpha_3$.

Indeed, we never have $\alpha_i = \alpha_j$, otherwise r, v_i, v_j would lie on a line, but then one of the two points v_i, v_j could not be visible from r, because both are $\prec r$.

Observe that $v_1 \in A_r$: We have $v \prec r$ for all $v \in A$, so a point v with a smaller angle to the x-axis than the first one in A_r cannot be in $\mathrm{conv}(A_r \cup \{r\}) \supset A$ (Fig. 2).

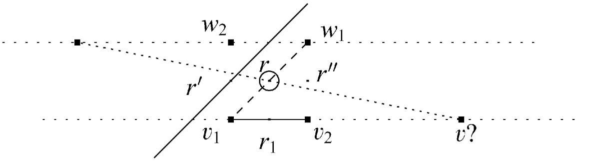
Now we consider any triangle $[v_i, v_{i+1}, r]$. Its boundary edges $[v_i, r]$ and $[v_{i+1}, r]$ do not intersect any vertices or edges of the haystack except for the endpoints v_i, v_{i+1}, since this would obstruct the visibility. Also the interior of $[v_i, v_{i+1}, r]$ does not contain any part of an edge of the haystack nor any integral or half-integral points, since this would immediately yield an integral point visible from r between v_i and v_{i+1}. (Indeed, any haystack edge meeting the interior of $[r, v_i, v_{i+1}]$ must also have an (integral) endpoint in the interior. At least one vertex of the convex hull of the integral points in the interior would be visible from r.) Thus we also get that the midpoint $s_i := \frac{1}{2}(v_i + v_{i+1})$ is visible from r, so it must be half-integer, $s_i \in M$. We also have $s_i \prec r$, and so $e_H(s_i) = [v_i, v_{i+1}]$ is an edge of the haystack, since the triangle $[v_i, v_{i+1}, r]$ does not admit any alternative integral endpoints. We also derive from this that the triangle $[v_i, v_{i+1}, r]$ has area $\frac{1}{4}$.

Define $w_i := r + \vec{v_i r}$ and $r' := \frac{1}{2}(v_1 + w_2)$, $r'' := \frac{1}{2}(v_2 + w_1)$. Then v_1, w_2, v_2, w_1 form a parallelogram with center r, and r, r', r'' are on a line (parallel to $(v_1 v_2)$). So either $r' \prec r$ or $r'' \prec r$.

Case 1: Suppose first that $r' \prec r$.

The triangle $\Delta = [v_1, v_2, w_2]$ is unimodular as $\mathrm{area}(\Delta) = 2\,\mathrm{area}[v_1, v_2, r] = \frac{1}{2}$; so there are no integer points between the line $(w_1 w_2)$ and the line $(v_1 v_2)$. The edge

$e_H(r')$ has nonempty intersection with these two lines (but does not cross $[v_1, w_1]$, since $v_1 \in A_r$).



But where could a third point $v \in A_r$ (other than v_1, v_2) be? The line $(r'r)$ is parallel to (v_1v_2), we have $\alpha(r') < \alpha_1 \leqslant \alpha_i$; and $r' \prec r$, $v \prec r$ for all $v \in A_r$. So all points of A are on the same side of $(r'r)$ as v_1 and v_2. So v is on or beyond the line (v_1v_2) and hence the edge through r starting at v would necessarily cross the edge $e_H(r')$. So there can be no other point v in A_r, that is, $|A_r| \leqslant 2$.

Case 2: The situation for $r'' \prec r$ is similar:

The edge through r'' must be $e(r'') = [v_2, w_1]$, otherwise it would cut $[v_1, w_1]$ or $[v_1, v_2]$; in the first case we would have $v_1 \notin A_r$ and in the second case v_2 would not be visible from r. And $[v_1, v_2, w_1]$ is again unimodular, so there is no possibility for a third $v \in A_r$. □

Our Theorem 1 and its proof clearly extend to a more general situation, namely the case of a not necessarily simply connected lattice polygon (which may have holes), possibly with additional, fixed inner edges.

We can define the capacities $c_{m,n} := \frac{\log_2 f(m,n)}{mn}$; see [1]. From sublinearity of $f(m,n)$ it follows by Fekete's lemma [3, p. 85] that the limit capacities

$$c_m := \lim_{n\to\infty} \frac{\log_2 f(m,n)}{mn}, \quad c_\Delta := \lim_{n\to\infty} \frac{\log_2 f(n,n)}{n^2}$$

exist. Theorem 1 yields the upper bounds

$$c_m \leqslant 3 - \frac{1}{m},$$

which includes the best known upper bounds for all c_m (compare [1]).

In generating triangulations with the "haystack approach" as in the proof of Theorem 1, one will in many situations have $|A_r| = 1$. So probably our upper bound $c_\Delta \leqslant 3$ for the limit capacity c_Δ is not sharp.

As for lower bounds, the recursion formulas for narrow strips as given in [1], together with submultiplicativity, show that $c_\Delta \geqslant c_4 > 2.055$.

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