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# Bubbles in anti-de Sitter space

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## Abstract

We explore the bubble spacetimes which can be obtained from double analytic continuations of static and rotating black holes in anti-de Sitter space. In particular, we find that rotating black holes with elliptic horizon lead to bubble spacetimes only in dimension greater than five. For dimension greater than seven, the topology of the bubble can be non-spherical. However, a bubble spacetime is shown to arise from a rotating de Sitter black hole in four dimensions. In all cases, the evolution of the bubble is of de Sitter type. Double analytic continuations of hyperbolic black holes and branes are also discussed.

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## 1. Introduction

The formulation of string theory in time-dependent backgrounds presents a particularly challenging problem, although progress can be achieved by considering some simple time-dependent solutions. As a step in this direction, a class of time-dependent backgrounds has been investigated recently in [1]. The spacetimes considered are obtained from a double analytic continuation of asymptotically flat black holes, and describe the Lorentzian evolution of a bubble. The technique of double analytic continuation was originally developed for the study of the stability of the Kaluza–Klein vacuum [2], see also [3,4]. This technique has also been used in the formulation of a positive energy theorem for anti-de Sitter space [5], and discussed within the context of brane world scenarios [6,7], and M-theory [8,9].

In general, the evolution pattern of the bubble is determined by the form of the original black hole metric. The analysis of [1] focussed on a class of bubbles which arise from asymptotically flat Kerr black holes. Indeed, the presence of the rotation parameters is crucial in order to obtain interesting time dependent behavior. It was found that at early times the bubble undergoes a de Sitter phase of exponential expansion, while at late times the evolution is governed by a milder Milne phase.

It is of interest to explore the possible time-dependent spacetimes which can arise from double analytic continuation of more general black holes. Our aim here is to investigate the bubble spacetimes which can be obtained from the analytic continuation of black holes in de Sitter and anti-de Sitter space. In the latter case, there is wide class of topological black hole spacetimes available, due to the possibility of non-trivial horizon topology [10–17]. In particular, the topology of the horizon can be elliptic (which includes the standard spherical case), toroidal, or hyperbolic.

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The rotating counterparts of these solutions have also been constructed [18–21]. We find that the static and rotating black holes with elliptic horizon lead to bubbles which have only a de Sitter phase of evolution. Furthermore, the rotating black holes lead to acceptable bubble spacetimes only in dimension  $d \geq 6$ . Moreover, it is possible to have a bubble with non-spherical (elliptic) topology in dimension  $d \geq 8$ . For the case of hyperbolic horizon, one can indeed perform the double analytic continuation. However, in all cases, one finds that the evolution of the bubble is determined by the embedding of an anti-de Sitter space, and thus does not lead to an evolving bubble situation. Finally, we show that a bubble spacetime does arise from the rotating de Sitter black hole in four dimensions.

### 2. Static anti-de Sitter black holes

It is useful to begin with the static black hole in  $d$  dimensions, with line element given by

$$ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 h_{ij}(x) dx^i dx^j, \tag{1}$$

with

$$f = k + \frac{r^2}{l^2} - \frac{2M}{r^{d-3}}. \tag{2}$$

The topology of the black hole horizon is labelled by the parameter  $k$ , which can be normalized to take the values  $\pm 1, 0$ . The above metric is a solution of Einstein’s equations with negative cosmological constant,  $R_{\mu\nu} = -(d - 1)/l^2 g_{\mu\nu}$ . We consider the class of locally asymptotically anti-de Sitter black holes, for which the topology of the horizon is either elliptic ( $k = 1$ ), toroidal ( $k = 0$ ), or hyperbolic ( $k = -1$ ). The possibility of anti-de Sitter black hole solutions with non-trivial topology  $k = 0, -1$  was first discussed in four dimensions [10–16], and generalized to arbitrary dimensions in [17].

For the case of  $k = 1$ , the line element of the horizon space is given by  $h_{ij}(x) dx^i dx^j = d\Omega_{d-2}^2$ . For definiteness, we consider the spherical case where  $d\Omega_{d-2}^2 = d\theta^2 + \sin^2 \theta d\Omega_{d-3}^2$ . The associated bubble spacetime is obtained in the standard manner [2] by performing the double analytic continuation  $t = i\chi$ ,

$\theta = \frac{\pi}{2} + i\tau$ . This leads to the bubble spacetime

$$ds^2 = f d\chi^2 + f^{-1} dr^2 + r^2(-d\tau^2 + \cosh^2 \tau d\Omega_{d-3}^2). \tag{3}$$

The radial variable is now restricted to the range  $r \geq r_+$ , where  $r_+$  is the largest real zero of  $f$ . Regularity of the metric at  $r_+$  then requires that  $\chi$  be identified as a periodic variable with period given by [17]

$$\beta = \frac{4\pi l^2 r_+}{(d - 1)r_+^2 + (d - 3)kl^2}. \tag{4}$$

In the usual way [2], this spacetime now describes a bubble at  $r = r_+$  evolving in an asymptotically anti-de Sitter spacetime. We see that the bubble grows exponentially in time and the geometry traced out by the  $r = r_+$  surface is a  $(d - 2)$ -dimensional de Sitter spacetime.

The horizon metric for the  $k = -1$  case can be written in the form  $h_{ij}(x) dx^i dx^j = d\Sigma_{d-2}^2 = d\theta^2 + \sinh^2 \theta d\Omega_{d-3}^2$ . If we continue  $\theta = i\tau$ , we obtain a metric that clearly has non-Lorentzian signature. However, we can obtain a Lorentzian bubble spacetime if we continue an angular variable in  $d\Omega_{d-3}^2 = (d\psi^2 + \sin^2 \psi d\Omega_{d-4}^2)$ . Upon the substitution  $\psi = i\tau + \pi/2$ , the metric becomes

$$ds^2 = f d\chi^2 + f^{-1} dr^2 + r^2[d\theta^2 + \sinh^2 \theta(-d\tau^2 + \cosh^2 \tau d\Omega_{d-4}^2)]. \tag{5}$$

In this case, one notes that the term in square brackets describes a  $(d - 2)$ -dimensional anti-de Sitter space, which can be written in globally static coordinates. Thus, the bubble spacetime in this case is static. This case has also been considered in the context of a positive energy theorem in [5].

### 3. Rotating anti-de Sitter black holes

In [1], a more interesting class of time-dependent backgrounds was obtained through the analytic continuation of rotating black holes in four and higher dimensions. In particular, the presence of the rotating parameters allowed an easing of the exponential expansion of the bubble into a milder Milne phase. In

order to check whether such behavior is also present in the anti-de Sitter case, we begin with the line element of the rotating black hole with elliptic horizon. This takes the form [18–21]

$$ds^2 = -\frac{\Delta_r}{\rho^2} \left[ dt - \frac{a}{\mathcal{E}} \sin^2 \theta d\phi \right]^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta}{\rho^2} \sin^2 \theta \left[ a dt - \frac{r^2 + a^2}{\mathcal{E}} d\phi \right]^2 + r^2 \cos^2 \theta d\Omega_{d-4}^2, \tag{6}$$

where  $a$  is the angular momentum. Again, for definiteness, we shall consider the case of spherical horizon, for which  $d\Omega_{d-4}^2$  describes the metric of a  $(d - 4)$ -sphere. Of course, this  $(d - 4)$ -dimensional part of the spacetime can be replaced by a general elliptic space, for example, a lens space. In the above, we have

$$\begin{aligned} \Delta_r &= (r^2 + a^2) \left( 1 + \frac{r^2}{l^2} \right) - \frac{2M}{r^{d-5}}, \\ \Delta_\theta &= 1 - \frac{a^2}{l^2} \cos^2 \theta, \\ \mathcal{E} &= 1 - \frac{a^2}{l^2}, \\ \rho^2 &= r^2 + a^2 \cos^2 \theta. \end{aligned} \tag{7}$$

The standard Euclidean section of the black hole is defined by the analytic continuation  $t = i\chi$  and  $a = -i\alpha$ . The resulting spacetime is defined for  $r \geq r_+$ , where  $r_+$  is now the largest real zero of  $\Delta_r$ . In the usual way, regularity of the metric at  $r_+$  then requires the coordinate identifications

$$(\chi, \phi) \equiv (\chi + 2\pi R n_1, \phi + 2\pi R \Omega n_1 + 2\pi n_2), \tag{8}$$

where  $n_1, n_2 \in \mathbf{Z}$ . We have

$$R = \frac{2(r_+^2 - \alpha^2)}{\Delta_r'(r_+)}, \quad \Omega = \frac{\alpha \mathcal{E}}{(r_+^2 - \alpha^2)}, \tag{9}$$

where  $R$  is the inverse of the surface gravity [18] after the continuation. Let us now apply the analytic continuation  $\theta = i\tau + \pi/2$  in four dimensions. The main observation here is that the metric has a time-dependent signature. Indeed, the induced metric on the bubble, at  $r = r_+$ , is given by

$$ds_{\text{bubble}}^2 = -\left[ \frac{r_+^2 + \alpha^2 \sinh^2 \tau}{1 - \frac{\alpha^2}{l^2} \sinh^2 \tau} \right] d\tau^2$$

$$+ \left[ \frac{1 - \frac{\alpha^2}{l^2} \sinh^2 \tau}{r_+^2 + \alpha^2 \sinh^2 \tau} \right] \left( \frac{\alpha}{\Omega} \right)^2 \cosh^2 \tau d\tilde{\phi}^2, \tag{10}$$

where  $\tilde{\phi} = \phi - \Omega\chi$ . One then sees that there exists a value  $\tau_{\text{crit}}$  such that the induced metric has signature  $(-, +)$  for  $|\tau| < \tau_{\text{crit}}$ , and  $(+, -)$  otherwise. Thus, the four-dimensional case does not yield a suitable bubble spacetime. Clearly, the influence of the anti-de Sitter radius  $l^2$  is crucial to this behavior, and the metric reduces to the one found in [1] in the limit  $l^2 \rightarrow \infty$ .

Proceeding to higher dimensions, one notes that an acceptable double analytic continuation is possible by choosing to continue one of the coordinates in the  $d\Omega_{d-4}^2$  part of the metric. This leads to the bubble spacetime

$$ds^2 = \frac{\Delta_r}{\rho^2} \left[ d\chi + \frac{\alpha}{\mathcal{E}} \sin^2 \theta d\phi \right]^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta}{\rho^2} \sin^2 \theta \left[ \alpha d\chi - \frac{(r^2 - \alpha^2)}{\mathcal{E}} d\phi \right]^2 + r^2 \cos^2 \theta [-d\tau^2 + \cosh^2 \tau d\Omega_{d-5}^2], \tag{11}$$

where

$$\begin{aligned} \Delta_r &= (r^2 - \alpha^2) \left( 1 + \frac{r^2}{l^2} \right) - \frac{2M}{r^{d-5}}, \\ \Delta_\theta &= 1 + \frac{\alpha^2}{l^2} \cos^2 \theta, \quad \mathcal{E} = 1 + \frac{\alpha^2}{l^2}, \\ \rho^2 &= r^2 - \alpha^2 \cos^2 \theta. \end{aligned} \tag{12}$$

The time-dependent part of the bubble geometry is now described by the embedding of a  $(d - 4)$ -dimensional de Sitter space. Therefore, the bubble spacetime is time-dependent only if  $d \geq 6$ . Clearly, the  $(\theta, \phi)$  part of the spacetime does not take part in this evolution. Thus, in contrast to the models obtained in [1], the anti-de Sitter case does not yield phases of milder evolution. Furthermore, one sees that for  $d \geq 8$ , the  $d\Omega_{d-5}^2$  part of the bubble metric can be a general elliptic space.

The analogous rotating solution with hyperbolic topology is given by [20]

$$ds^2 = -\frac{\Delta_r}{\rho^2} \left[ dt + \frac{a}{\mathcal{E}} \sinh^2 \theta d\phi \right]^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta}{\rho^2} \sinh^2 \theta \left[ a dt - \frac{(r^2 + a^2)}{\mathcal{E}} d\phi \right]^2 + r^2 \cosh^2 \theta d\Sigma_{d-4}^2, \tag{13}$$

where

$$\begin{aligned} \Delta_r &= (r^2 + a^2) \left( -1 + \frac{r^2}{l^2} \right) - \frac{2M}{r^{d-5}}, \\ \Delta_\theta &= 1 + \frac{a^2}{l^2} \cosh^2 \theta, \quad \mathcal{E} = 1 + \frac{a^2}{l^2}, \\ \rho^2 &= r^2 + a^2 \cosh^2 \theta, \end{aligned} \tag{14}$$

and  $d\Sigma_{d-4}^2 = d\psi^2 + \sinh^2 \psi d\Omega_{d-5}^2$  is the metric on a  $(d - 4)$ -dimensional hyperbolic space. In this case, the  $(\theta, \phi)$  sector of the spacetime is necessarily non-compact [20]. Replacing  $t = i\chi$  and  $a = -i\alpha$ , and imposing the appropriate identifications on  $\chi$  and  $\phi$ , a further continuation of  $\theta$  again leads to a metric with time-dependent signature for any  $d \geq 4$ . However, we can perform a continuation of an angular variable in the spherical section embedded in  $d\Sigma_{d-4}^2$ , and we find

$$\begin{aligned} ds^2 &= \frac{\Delta_r}{\rho^2} \left[ d\chi - \frac{\alpha}{\mathcal{E}} \sinh^2 \theta d\phi \right]^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 \\ &+ \frac{\Delta_\theta}{\rho^2} \sinh^2 \theta \left[ \alpha d\chi - \frac{(r^2 - \alpha^2)}{\mathcal{E}} d\phi \right]^2 \\ &+ r^2 \cosh^2 \theta \\ &\times [d\psi^2 + \sinh^2 \psi (-d\tau^2 + \cosh^2 \tau d\Omega_{d-6}^2)] \end{aligned} \tag{15}$$

with

$$\begin{aligned} \Delta_r &= (r^2 - \alpha^2) \left( -1 + \frac{r^2}{l^2} \right) - \frac{2M}{r^{d-5}}, \\ \Delta_\theta &= 1 - \frac{\alpha^2}{l^2} \cosh^2 \theta, \quad \mathcal{E} = 1 - \frac{\alpha^2}{l^2}, \\ \rho^2 &= r^2 - \alpha^2 \cosh^2 \theta. \end{aligned} \tag{16}$$

As in the static case with hyperbolic horizon, the bubble spacetime is again described simply by the embedding of a  $(d - 4)$ -dimensional anti-de Sitter space.

One can also consider the 5-dimensional anti-de Sitter black hole with two rotational parameters [18]. In this case, one finds that the only consistent continuation involves the angular variable not related to rotations, which again leads to a time-dependent signature. Finally, the analysis of the double analytical continuation of a cylindrical rotating black hole [20] is very similar to the  $k = 1$  case. In four dimensions, the only possible continuation leads to a time-dependent

signature, while in higher dimensions one can continue an angular variable of the spherical section.

#### 4. Rotating de Sitter black holes

A rotating black hole in de Sitter space with one angular momentum can be found by replacing  $l^2 \rightarrow -l^2$  in the metric (6), see, for example, [22,23]. In four dimensions, we find behavior which contrasts the anti-de Sitter case. Indeed, we find that the metric obtained after the double analytic continuation has a constant signature. To see this, we consider the metric (6) for  $d = 4$ , and make the continuations  $l \rightarrow il$ ,  $t = i\chi$ ,  $a = -i\alpha$  and  $\theta = i\tau + \pi/2$  (the choice  $\theta = i\tau$  leads to a time-dependent signature). The resulting metric takes the form

$$\begin{aligned} ds^2 &= \frac{\Delta_r}{\rho^2} \left[ d\chi + \frac{\alpha}{\mathcal{E}} \cosh^2 \tau d\phi \right]^2 + \frac{\rho^2}{\Delta_r} dr^2 - \frac{\rho^2}{\Delta_\tau} d\tau^2 \\ &+ \frac{\Delta_\tau}{\rho^2} \cosh^2 \tau \left[ \alpha d\chi - \frac{(r^2 - \alpha^2)}{\mathcal{E}} d\phi \right]^2, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \Delta_r &= (r^2 - \alpha^2) \left( 1 - \frac{r^2}{l^2} \right) - 2Mr, \\ \Delta_\tau &= 1 + \frac{\alpha^2}{l^2} \sinh^2 \tau, \quad \mathcal{E} = 1 - \frac{\alpha^2}{l^2}, \\ \rho^2 &= r^2 + \alpha^2 \sinh^2 \tau. \end{aligned} \tag{18}$$

The induced metric on the bubble for small  $\tau$  is given by

$$ds^2 \simeq -r_+^2 d\tau^2 + \frac{\alpha^2}{\Omega^2 r_+^2} \cosh^2 \tau d\tilde{\phi}^2, \tag{19}$$

where  $\tilde{\phi} = \phi - \Omega\chi$  with  $\Omega = \alpha\mathcal{E}/(r_+^2 - \alpha^2)$ . We see that the bubble metric behaves like a 2-dimensional de Sitter space. For large  $\tau$ , we have

$$ds^2 \simeq -l^2 d\tau^2 + \frac{\alpha^2}{\Omega^2 l^2} e^{2\tau} d\tilde{\phi}^2, \tag{20}$$

which again looks like a de Sitter space at late times. Thus, the evolution of the bubble is described by a de Sitter phase only.

## 5. Conclusion

We have examined the double analytic continuations of static and rotating black holes in anti-de Sitter and de Sitter spacetime. The set of coordinates which can be continued is constrained by the requirement that the resulting bubble spacetime be real Lorentzian and time-dependent. The examples studied only led to a de Sitter phase of time dependence, and do not appear to support the milder phase of Milne evolution found in the asymptotically flat case [1]. It would be worthwhile to consider the classical and quantum stability of these spacetimes along the lines of discussed in [1].

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