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Ambitable topological groups

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ABSTRACT

A topological group is said to be *ambitable* if every uniformly bounded uniformly equicontinuous set of functions on the group with its right uniformity is contained in an ambit. For $n = 0, 1, 2, \ldots$, every locally $\aleph_n$-bounded topological group is either precompact or ambitable. In the familiar semigroups constructed over ambitable groups, topological centres have an effective characterization.

1. Overview

A topological group G may be naturally embedded in larger spaces, algebraically and topologically. Two such spaces of particular interest in abstract harmonic analysis are

- the norm dual of the space of bounded right uniformly continuous functions on G, denoted here $\mathbf{U_b}(rG)^*$ (also known as $\mathrm{LUC}(G)^*$); and
- the uniform compactification of G with its right uniformity, denoted here $\overline{rG}$ (also known as the greatest ambit $\mathcal{S}(G)$, alternatively denoted G^{LUC} or $G^{\mathcal{LC}}$).

It is customary to study these "right" versions of the two spaces; the properties of the corresponding "left" versions are obtained by symmetry.

Both $\mathbf{U_b}(rG)^*$ and $\overline{rG}$ are right topological semigroups. In investigating their structure it is very helpful to have a tractable characterization of their topological centres. Feasible candidates for such characterizations are the space of uniform measures $\mathbf{M_u}(rG)$ and the completion $\widehat{rG}$ of the right uniformity on G.

When G is locally compact, $\mathbf{M_u}(rG)$ is the space of finite Radon measures on G [2,6,16], and $\widehat{rG}$ is G itself. In this case, Lau [11] and Lau and Pym [12] proved that $\mathbf{M_u}(rG)$ and $\widehat{rG} = G$ are the topological centres of $\mathbf{U_b}(rG)^*$ and $\overline{rG}$. These characterizations generalized a number of previous results for special classes of locally compact groups.

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More recently, Neufang [14] applied his factorization method to simplify the proof of Lau's result. Then Ferri and Neufang [7] used a variant of the factorization method to prove that $\mathbf{M_u}(rG)$ and $\widehat{rG}$ are the topological centres of $\mathbf{U_b}(rG)^*$ and $\overline{rG}$ for $\aleph_0$-bounded (not necessarily locally compact) topological groups.

This paper deals with another variant of the factorization method, similar to that used by Ferri and Neufang. By definition, ambitable topological groups are those in which a certain factorization theorem holds; equivalently, in the language of topological dynamics, those in which every uniformly bounded uniformly equicontinuous set of functions is contained in an ambit. In such groups $\mathbf{M_u}(rG)$ and $\widehat{rG}$ are the topological centres of $\mathbf{U_b}(rG)^*$ and $\overline{rG}$. Several classes of topological groups are shown to be ambitable. In particular, if n is a positive integer then every locally $\aleph_n$-bounded group is either precompact or ambitable, which yields a common generalization of the aforementioned results by Lau, Lau and Pym, and Ferri and Neufang.

2. Basic definitions

All topological groups considered in this paper are assumed to be Hausdorff, and all linear spaces to be over the field $\mathbb{R}$ of reals.

Let G be a group, f a real-valued function on G and $x \in G$. The *right translation of f by x*, denoted $\rho^x(f)$, is the function $z \mapsto f(zx)$. The set $\mathrm{orb}(f) = \{\rho^x(f) \mid x \in G\}$ is the (*right*) *orbit* of f. Denote by $\overline{\mathrm{orb}(f)}$ the closure of $\mathrm{orb}(f)$ in the product space $\mathbb{R}^G$ (the set of real-valued functions on G with the topology of pointwise convergence).

When Δ is a pseudometric on G, define

$$\mathrm{BLip}^+(\Delta) = \bigl\{ f : G \to \mathbb{R} \bigm| 0 \leqslant f(x) \leqslant 1 \text{ and } \bigl|f(x) - f(y)\bigr| \leqslant \Delta(x, y) \text{ for all } x, y \in G \bigr\}.$$

Then $\mathrm{BLip}^+(\Delta)$ is a compact subset of the product space $\mathbb{R}^G$; it will be always considered with this compact topology.

When G is a topological group, denote by $\mathcal{RP}(G)$ the set of all continuous right-invariant pseudometrics on G. The *right uniformity* on G is the uniform structure generated by $\mathcal{RP}(G)$. The uniform space rG is the set G with the right uniformity, and the space of all bounded uniformly continuous real-valued functions on rG is denoted $\mathbf{U_b}(rG)$. The group G is said to be *precompact* if the uniform space rG is precompact.

The following lemma summarizes several properties of $\rho^x(f)$ needed in this paper. De Vries [5, IV.5] provides a comprehensive treatment of the role of $\rho^x(f)$ in topological dynamics.

Lemma 1. *Let G be any topological group and $\Delta \in \mathcal{RP}(G)$.*

1. *If f is a real-valued function on G and $x, y \in G$ then $\rho^{xy}(f) = \rho^x(\rho^y(f))$.*
2. *If $f \in \mathrm{BLip}^+(\Delta)$ and $x \in G$ then $\rho^x(f) \in \mathrm{BLip}^+(\Delta)$.*
3. *The mapping $(x, f) \mapsto \rho^x(f)$ is continuous from $G \times \mathrm{BLip}^+(\Delta)$ to $\mathrm{BLip}^+(\Delta)$.*

Proof. 1. $\rho^{xy}(f)(z) = f(zxy) = \rho^y(f)(zx) = \rho^x(\rho^y(f))(z)$.

2. If $f \in \mathrm{BLip}^+(\Delta)$ then $|\rho^x(f)(z) - \rho^x(f)(z')| = |f(zx) - f(z'x)| \leqslant \Delta(zx, z'x) = \Delta(z, z')$.

3. To prove that the mapping $(x, f) \mapsto \rho^x(f)$ to $\mathrm{BLip}^+(\Delta)$ is continuous, it is sufficient to prove that the mapping $(x, f) \mapsto \rho^x(f)(z)$ to $\mathbb{R}$ is continuous for each $z \in G$.

Take any $z \in G$, $(x_0, f_0) \in G \times \mathrm{BLip}^+(\Delta)$ and $\varepsilon > 0$. The set

$$U = \bigl\{ (x, f) \in G \times \mathrm{BLip}^+(\Delta) \bigm| \Delta(zx, zx_0) < \varepsilon \text{ and } \bigl|f(zx_0) - f_0(zx_0)\bigr| < \varepsilon \bigr\}$$

is a neighbourhood of (x_0, f_0) in $G \times \mathrm{BLip}^+(\Delta)$. If $(x, f) \in U$ then

$$\begin{aligned} \bigl|\rho^x(f)(z) - \rho^{x_0}(f_0)(z)\bigr| &= \bigl|f(zx) - f_0(zx_0)\bigr| \\ &\leqslant \bigl|f(zx) - f(zx_0)\bigr| + \bigl|f(zx_0) - f_0(zx_0)\bigr| < 2\varepsilon. \end{aligned}$$

Thus the mapping $(x, f) \mapsto \rho^x(f)(z)$ is continuous at (x_0, f_0). □

When G and Δ are as in the lemma, $\mathrm{BLip}^+(\Delta)$ with the action $(x, f) \mapsto \rho^x(f)$ is a compact G-flow, in the terminology of topological dynamics [5]. If $f \in \mathbf{U_b}(rG)$ then there exist $\Delta \in \mathcal{RP}(G)$ and $s, t \in \mathbb{R}$ such that $f + t \in s\mathrm{BLip}^+(\Delta)$. Thus $\overline{\mathrm{orb}(f)} + t = \overline{\mathrm{orb}(f + t)} \subseteq s\mathrm{BLip}^+(\Delta)$ and therefore the set $\overline{\mathrm{orb}(f)}$ is compact in the topology of pointwise convergence, and $\overline{\mathrm{orb}(f)}$ with the action $(x, f) \mapsto \rho^x(f)$ is also a compact G-flow.

Recall that a compact G-flow is an *ambit* if it contains an element with dense orbit [5, IV.4.1]. For a fixed G, all ambits can be constructed from those of the form $\overline{\mathrm{orb}(f)}$, where $f \in \mathbf{U_b}(rG)$ [5, IV.5.8]. For example, the greatest ambit (i.e. the uniform compactification $\overline{rG}$) is the closure of the canonical image of G in the product space $\prod\{\overline{\mathrm{orb}(f)} \mid f \in \mathbf{U_b}(rG)\}$.

Say that a topological group G is *ambitable* if every $\mathrm{BLip}^+(\Delta)$, where $\Delta \in \mathcal{RP}(G)$, is contained in an ambit within $\mathbf{U_b}(rG)$. In other words, G is ambitable if for each $\Delta \in \mathcal{RP}(G)$ there exists $f \in \mathbf{U_b}(rG)$ such that $\mathrm{BLip}^+(\Delta) \subseteq \overline{\mathrm{orb}(f)}$.

Theorem 2. *No precompact topological group is ambitable.*

Proof. Let G be a precompact group, and fix any $f \in \mathbf{U_b}(rG)$. There are $\Delta \in \mathcal{RP}(G)$ and $\theta > 0$ such that if $x, x' \in G$, $\Delta(x, x') < \theta$ then $|f(x) - f(x')| < 1/3$. If $x, x', y \in G$, $\Delta(x, x') < \theta$ then $\Delta(xy, x'y) < \theta$ and therefore $|\rho^y(f)(x) - \rho^y(f)(x')| < 1/3$. Since G is precompact, there is a finite set $F \subseteq G$ such that for every $x \in G$ there is $z \in F$ with $\Delta(x, z) < \theta$, and thus $|\rho^y(f)(x) - \rho^y(f)(z)| < 1/3$ for every $y \in G$.

Consider the constant functions 0 and 1. If $0 \in \overline{\mathrm{orb}(f)}$ then there is $y \in G$ such that $\rho^y(f)(z) < 1/3$ for every $z \in F$, hence $\rho^y(f)(x) < 2/3$ for every $x \in G$. Thus $f(x) < 2/3$ for every $x \in G$, and $1 \notin \overline{\mathrm{orb}(f)}$. This proves that there is no $f \in \mathbf{U_b}(rG)$ for which $0, 1 \in \overline{\mathrm{orb}(f)}$. □

Question 1. Is every topological group either precompact or ambitable?

This question is motivated by investigations of topological centres in certain semigroups arising in functional analysis. The connection is explained in Section 5 below.

Partial answers to Question 1 are given in Section 4. They show that the topological groups that are neither precompact nor ambitable, if they exist at all, are quite rare.

3. Cardinal functions

The reader is referred to Jech [10] for definitions regarding cardinals. The cardinality of a set X is $|X|$. The cardinal successor of a cardinal κ is κ^+. The least infinite cardinal is $\aleph_0$, and $\aleph_{n+1} = \aleph_n^+$. The least cardinal larger than $\aleph_n$ for $n = 0, 1, 2, \ldots$ is $\aleph_\omega$.

Let G be a group and Δ a pseudometric on G. Sufficient conditions in the next section are expressed in terms of three cardinal functions:

- $d(\Delta)$, the Δ-*density* of G (the least cardinality of a Δ-dense subset of G);
- $\eta^\sharp(\Delta)$, the least cardinality of a set $P \subseteq G$ such that

$$G = \bigcup_{p \in P} \{x \in G \mid \Delta(p, x) \leqslant 1\};$$

- $\eta(\Delta)$, the least cardinality of a set $P \subseteq G$ for which there exists a finite set $Q \subseteq G$ such that

$$G = \bigcup_{q \in Q} \bigcup_{p \in P} \{x \in G \mid \Delta(p, qx) \leqslant 1\}.$$

The following lemma collects basic facts about these three functions. Proofs follow directly from the definition.

Lemma 3. *Let Δ be a pseudometric on a group G. Let $B = \{x \in G \mid \Delta(e, x) \leqslant 1\}$, where e is the identity element of G.*

1. $\eta(\Delta) \leqslant \eta^\sharp(\Delta) \leqslant d(\Delta)$.
2. $d(\Delta) = \lim_{k \to \infty} \eta^\sharp(k\Delta)$.
3. *If Δ' is another pseudometric on G such that $\Delta \leqslant \Delta'$ then $\eta(\Delta) \leqslant \eta(\Delta')$, $\eta^\sharp(\Delta) \leqslant \eta^\sharp(\Delta')$ and $d(\Delta) \leqslant d(\Delta')$.*
4. *If Δ is left-invariant and $\eta(\Delta) \geqslant \aleph_0$ then $\eta(\Delta) = \eta^\sharp(\Delta)$.*
5. *If Δ is right-invariant then $\eta^\sharp(\Delta)$ is the least cardinality of a set $P \subseteq G$ such that $G = BP$ and $\eta(\Delta)$ is the least cardinality of a set $P \subseteq G$ for which there exists a finite set Q such that $G = QBP$.*

Clearly a topological group G is precompact if and only if $\eta^\sharp(\Delta)$ is finite for each $\Delta \in \mathcal{RP}(G)$. Part 1 in Theorem 5 below yields a stronger statement: G is precompact if and only if $\eta(\Delta)$ is finite for each $\Delta \in \mathcal{RP}(G)$. This is equivalent to the theorem of Uspenskij [18, p. 338], [19, p. 1581], for which a simple proof was given by Bouziad and Troallic [4]. Ferri and Neufang [7] gave another proof using a result of Protasov [17, Theorem 11.5.1].

The case of finite P in the next lemma is due to Bouziad and Troallic [4, Lemma 4.1]. The proof below is a straightforward generalization of their approach, which in turn was adapted from Neumann [15].

Lemma 4. *Let G be a group, $P \subseteq G$, and $A_k \subseteq G$ for $1 \leqslant k \leqslant n$. If $G = \bigcup_{k=1}^n A_k P$ then there are a set $P' \subseteq G$ and j, $1 \leqslant j \leqslant n$, such that $G = A_j^{-1} A_j P'$ and*

(a) *if P is finite then so is P'; and*
(b) *if P is infinite then $|P'| \leqslant |P|$.*

Proof. Proceed by induction in n. When $n = 1$, the statement is true with $j = 1$ and $P' = P$.

For the induction step, let $m \geqslant 1$ and assume that the statement in the lemma is true for $n = m$. Let $P \subseteq G$ and $A_1, A_2, \ldots, A_{m+1} \subseteq G$ be such that $G = \bigcup_{k=1}^{m+1} A_k P$.

If $G = A_{m+1}^{-1} A_{m+1} P$ then set $j = m+1$ and $P' = P$.

On the other hand, if $G \neq A_{m+1}^{-1} A_{m+1} P$ then take any $x \in G \setminus A_{m+1}^{-1} A_{m+1} P$. Then $A_{m+1} x \cap A_{m+1} P = \emptyset$, and $A_{m+1} \subseteq \bigcup_{k=1}^{m} A_k P x^{-1}$. Thus

$$G = \bigcup_{k=1}^{m} A_k \big(P \cup P x^{-1} P\big).$$

By the induction hypothesis, there are $P' \subseteq G$ and j, $1 \leqslant j \leqslant m$, such that $G = A_j^{-1} A_j P'$, P' is finite if P is, and $|P'| \leqslant |P|$ if P is infinite.

Thus in either case the statement holds for $n = m+1$. □

Theorem 5. *Let G be a topological group, and $\Delta \in \mathcal{RP}(G)$.*

1. *If $\eta(\Delta)$ is finite then $\eta^{\sharp}(\frac{1}{2}\Delta)$ is finite.*
2. *If $\eta(\Delta)$ is infinite then $\eta^{\sharp}(\frac{1}{2}\Delta) \leqslant \eta(\Delta)$.*

Proof. Let $B = \{x \in G \mid \Delta(e, x) \leqslant 1\}$, where e is the identity element of G. For any $y, z \in B$ we have $y^{-1} z \in \{x \in G \mid \frac{1}{2}\Delta(e, x) \leqslant 1\}$, because

$$\Delta\big(e, y^{-1} z\big) = \Delta\big(z^{-1}, y^{-1}\big) \leqslant \Delta\big(z^{-1}, e\big) + \Delta\big(y^{-1}, e\big) = \Delta(e, z) + \Delta(e, y) \leqslant 2.$$

By part 5 of Lemma 3, there are sets $P, Q \subseteq G$ such that Q is finite, $|P| = \eta(\Delta)$ and $G = QBP$. By Lemma 4, there are $q \in Q$ and $P' \subseteq G$ such that P' is finite if P is, $|P'| \leqslant |P|$ if P is infinite, and

$$G = (qB)^{-1} qBP' = B^{-1} BP' \subseteq \left\{ x \in G \;\middle|\; \frac{1}{2}\Delta(e, x) \leqslant 1 \right\} P'$$

which shows that $\eta^{\sharp}(\frac{1}{2}\Delta) \leqslant |P'|$. If $\eta(\Delta)$ is finite then $|P|$ and $|P'|$ are finite and therefore $\eta^{\sharp}(\frac{1}{2}\Delta)$ is finite. If $\eta(\Delta)$ is infinite then $\eta^{\sharp}(\frac{1}{2}\Delta) \leqslant |P'| \leqslant |P| = \eta(\Delta)$. □

Corollary 6. *Let G be a topological group, and $\Delta \in \mathcal{RP}(G)$. If $d(\Delta) > \aleph_0$ then*

$$d(\Delta) = \lim_{k \to \infty} \eta(k\Delta).$$

Proof. Combine parts 1 and 2 of Lemma 3 with Theorem 5. □

Let κ be an infinite cardinal. Following Guran [9], say that a topological group G is *κ-bounded* if for every neighbourhood U of the identity element in G there exists a set $P \subseteq G$ such that $|P| \leqslant \kappa$ and $UP = G$. See also Section 9 in [1].

Lemma 7. *Let κ be an infinite cardinal. The following conditions for a topological group G are equivalent*:

(i) *G is κ-bounded*;
(ii) *$d(\Delta) \leqslant \kappa$ for every $\Delta \in \mathcal{RP}(G)$*;
(iii) *$\eta^{\sharp}(\Delta) \leqslant \kappa$ for every $\Delta \in \mathcal{RP}(G)$*;
(iv) *$\eta(\Delta) \leqslant \kappa$ for every $\Delta \in \mathcal{RP}(G)$.*

Proof. The family of all sets $\{x \in G \mid \Delta(e, x) \leqslant 1\}$, where $\Delta \in \mathcal{RP}(G)$, is a basis of neighbourhoods of the identity element e in G. Therefore (i) ⇔ (iii), by part 5 in Lemma 3.

(ii) ⇒ (iii) ⇒ (iv) by part 1 in Lemma 3, and (iv) ⇒ (ii) by Corollary 6. □

Let κ be an infinite cardinal. Say that a topological group G is *locally κ-bounded* if its identity element has a neighbourhood U such that for each $\Delta \in \mathcal{RP}(G)$ there is a Δ-dense subset H of U, $|H| \leqslant \kappa$. Every κ-bounded group is locally κ-bounded. Every locally compact group is locally $\aleph_0$-bounded, and therefore also locally κ-bounded for $\kappa \geqslant \aleph_0$.

4. Sufficient conditions

This section contains several sufficient conditions for a topological group to be ambitable. For each such condition a slightly stronger property than ambitability is proved; namely, that for every $\Delta \in \mathcal{RP}(G)$ there is $\Delta' \in \mathcal{RP}(G)$, $\Delta' \geqslant \Delta$ such that $\mathrm{BLip}^{+}(\Delta')$ is an ambit. The key result is Lemma 10, which is another form of the factorization theorems of Neufang [14] and Ferri and Neufang [7].

When Δ is a pseudometric on a set X and $Y, Z \subseteq X$, define $\Delta(Y, Z) = \inf\{\Delta(y, z) \mid y \in Y, z \in Z\}$.

Lemma 8. *Let Δ be a pseudometric on a group G such that $\eta(\Delta) \geqslant \aleph_0$. Let A be a set of cardinality $\eta(\Delta)$, and for each $\alpha \in A$ let F_α be a non-empty finite subset of G. Then there exist elements $x_\alpha \in G$ for $\alpha \in A$ such that $\Delta(F_\alpha x_\alpha, F_\beta x_\beta) > 1$ whenever $\alpha, \beta \in A$, $\alpha \neq \beta$.*

Proof. Without loss of generality, assume that A is the set of ordinals smaller than the first ordinal of cardinality $\eta(\Delta)$. The construction of x_α proceeds by transfinite induction. For $\gamma \in A$, let $S(\gamma)$ be the statement "there exist elements $x_\alpha \in G$ for all $\alpha \leqslant \gamma$ such that $\Delta(F_\alpha x_\alpha, F_\beta x_\beta) > 1$ whenever $\alpha < \beta \leqslant \gamma$."

Any choice of $x_0 \in G$ makes $S(0)$ true. Now assume that $\gamma \in A$, $\gamma > 0$, and $S(\gamma')$ is true for all $\gamma' < \gamma$. We want to prove $S(\gamma)$.

Since $\eta(\Delta) \geqslant \aleph_0$ and the cardinality of γ is less than $\eta(\Delta)$, from the definition of $\eta(\Delta)$ we get

$$G \neq \bigcup_{q \in F_\gamma} \bigcup_{\alpha < \gamma} \bigcup_{p \in F_\alpha x_\alpha} \{x \in G \mid \Delta(p, qx) \leqslant 1\}.$$

Thus there exists $x_\gamma \in G$ such that $\Delta(p, qx_\gamma) > 1$ for all $q \in F_\gamma$ and all $p \in F_\alpha x_\alpha$ where $\alpha < \gamma$. That means $\Delta(F_\alpha x_\alpha, F_\gamma x_\gamma) > 1$ for all $\alpha < \gamma$. □

Lemma 9. *Let G be a topological group, $\Delta \in \mathcal{RP}(G)$ and $\eta(\Delta) \geqslant \aleph_0$. If $\mathcal{O}$ is a collection of non-empty open subsets of* $\mathrm{BLip}^+(\Delta)$ *and $|\mathcal{O}| \leqslant \eta(\Delta)$, then there exists $f \in \mathrm{BLip}^+(\Delta)$ such that* $\mathrm{orb}(f)$ *intersects every set in $\mathcal{O}$.*

Proof. Without loss of generality, assume that every set in $\mathcal{O}$ is a basic neighbourhood. Thus each $U \in \mathcal{O}$ is of the form

$$U = \{f \in \mathrm{BLip}^+(\Delta) \mid |f(x) - h_U(x)| < \varepsilon_U \text{ for } x \in F_U\},$$

where $F_U \subseteq G$ is a finite set, $h_U \in \mathrm{BLip}^+(\Delta)$, and $\varepsilon_U > 0$.

By Lemma 8 with $\mathcal{O}$ in place of A, there are elements $x_U \in G$ for $U \in \mathcal{O}$ such that $\Delta(F_U x_U, F_V x_V) > 1$ whenever $U, V \in \mathcal{O}$, $U \neq V$. Define the function $f : G \to \mathbb{R}$ by

$$f(x) = \sup_{V \in \mathcal{O}} \max_{y \in F_V} \big(h_V(y) - \Delta(x, yx_V)\big)^+ \quad \text{for } x \in G.$$

Each function $x \mapsto (h_V(y) - \Delta(x, yx_V))^+$ belongs to $\mathrm{BLip}^+(\Delta)$, and thus $f \in \mathrm{BLip}^+(\Delta)$. It remains to be proved that $f(xx_U) = h_U(x)$ for every $U \in \mathcal{O}$ and $x \in F_U$. Once that is established, it will follow that $\rho^{x_U}(f) \in U$ for every $U \in \mathcal{O}$.

Take any $U \in \mathcal{O}$ and $x \in F_U$. From the definition of f we get $f(xx_U) \geqslant h_U(x)$. To prove the opposite inequality, consider any $V \in \mathcal{O}$ and any $y \in F_V$.

Case I: $V = U$. From $h_U(y) - h_U(x) \leqslant |h_U(y) - h_U(x)| \leqslant \Delta(x, y) = \Delta(xx_U, yx_U)$ and $h_U = h_V$, $x_U = x_V$, we get $(h_V(y) - \Delta(xx_U, yx_V))^+ \leqslant h_U(x)$.

Case II: $V \neq U$. From $\Delta(xx_U, yx_V) > 1$ we get $(h_V(y) - \Delta(xx_U, yx_V))^+ = 0 \leqslant h_U(x)$.

Thus $(h_V(y) - \Delta(xx_U, yx_V))^+ \leqslant h_U(x)$ in both cases, and now $f(xx_U) \leqslant h_U(x)$ follows from the definition of f. □

Lemma 10. *Let G be a topological group and $\Delta \in \mathcal{RP}(G)$. If $d(\Delta) = \eta(\Delta) \geqslant \aleph_0$ then there exists $f \in \mathrm{BLip}^+(\Delta)$ such that* $\mathrm{BLip}^+(\Delta) = \overline{\mathrm{orb}(f)}$.

Proof. Let H be a Δ-dense subset of G such that $|H| = d(\Delta) = \eta(\Delta)$. On $\mathrm{BLip}^+(\Delta)$, the topology of pointwise convergence on G coincides with the topology of pointwise convergence on H. Thus $\mathrm{BLip}^+(\Delta)$ is homeomorphic to a subset of the product space $\mathbb{R}^H$, its topology has a base of cardinality at most $\eta(\Delta)$, and by Lemma 9 there is $f \in \mathrm{BLip}^+(\Delta)$ whose orbit intersects every non-empty open set in $\mathrm{BLip}^+(\Delta)$. □

In this paper, Lemma 10 is the key for finding sufficient conditions for ambitability. The group G is ambitable whenever for every $\Delta \in \mathcal{RP}(G)$ there is $\Delta' \in \mathcal{RP}(G)$ such that $\Delta' \geqslant \Delta$ and $d(\Delta') = \eta(\Delta') \geqslant \aleph_0$.

Theorem 11. *Let κ be an infinite cardinal, and G a locally κ-bounded topological group. If there exists $\Delta_0 \in \mathcal{RP}(G)$ such that $\eta^\sharp(\Delta_0) \geqslant \kappa$ then G is ambitable.*

Proof. Take any $\Delta \in \mathcal{RP}(G)$. Let $B = \{x \in G \mid \Delta(e, x) \leqslant 1\}$, where e is the identity element of G. Without loss of generality, assume that $\eta^\sharp(\Delta) \geqslant \kappa$ and B has a Δ-dense subset H such that $|H| \leqslant \kappa$. (If Δ does not have these properties then replace Δ by a larger pseudometric in $\mathcal{RP}(G)$ that does.)

By Lemma 3, there is $P \subseteq G$ such that $|P| = \eta^\sharp(\Delta)$ and $G = BP$. The set HP is Δ-dense in G, therefore $d(2\Delta) = d(\Delta) \leqslant \kappa \cdot \eta^\sharp(\Delta) = \eta^\sharp(\Delta)$. Thus $d(2\Delta) \leqslant \eta(2\Delta)$ by Theorem 5, and by Lemma 10 there is $f \in \mathrm{BLip}^+(2\Delta)$ such that $\mathrm{BLip}^+(2\Delta) = \overline{\mathrm{orb}(f)}$. □

Corollary 12. *Let κ be an infinite cardinal. If a topological group is locally κ^+-bounded and not κ-bounded then it is ambitable.*

Proof. Let G be locally κ^+-bounded and not κ-bounded. By Lemma 7 there exists $\Delta \in \mathcal{RP}(G)$ for which $\eta^\sharp(\Delta) \geqslant \kappa^+$, and Theorem 11 applies with κ^+ in place of κ. □

Corollary 13. *Let G be a topological group, and let λ be the least infinite cardinal for which G is λ-bounded. If λ is a successor cardinal then G is ambitable.*

Proof. Let $\lambda = \kappa^+$. Then G is κ^+-bounded, therefore also locally κ^+-bounded, and it is not κ-bounded. Thus G is ambitable by Corollary 12. □

Theorem 14. *When n is a positive integer, every locally $\aleph_n$-bounded topological group is either precompact or ambitable.*

Proof. Let G be locally $\aleph_n$-bounded for some n. Let $m \geqslant 0$ be the least integer for which G is locally $\aleph_m$-bounded. If $m \geqslant 1$ then G is ambitable by Corollary 12 with $\kappa = \aleph_{m-1}$. If $m = 0$ and G is not precompact then there exists $\Delta \in \mathcal{RP}(G)$ such that $\eta^\sharp(\Delta) \geqslant \aleph_0$ and G is ambitable by Theorem 11. □

It is an open question whether every $\aleph_\omega$-bounded topological group is either precompact or ambitable.

Corollary 15. *Every locally compact topological group is either compact or ambitable.*

Corollary 16. *Every $\aleph_0$-bounded topological group is either precompact or ambitable.*

Lemma 10 yields also other classes of ambitable groups, such as those in the next two theorems. Say that a linear space is *null* if it is the one-element space $\{0\}$.

Theorem 17. *The additive group of every non-null normed linear space is ambitable.*

Proof. Let G be the additive group of a non-null normed space with the norm $\|\cdot\|$. The topology of G is defined by the metric Δ_0, where $\Delta_0(x, y) = \|x - y\|$, $x, y \in G$. If Δ is any pseudometric in $\mathcal{RP}(G)$ then $d(\Delta_0) \geqslant d(\Delta) \geqslant \eta(\Delta)$.

The metric Δ_0 is left- and right-invariant, $\eta(\Delta_0) = \eta(k\Delta_0) \geqslant \aleph_0$ for $k = 1, 2, \ldots,$ and $d(\Delta_0) = \eta(\Delta_0)$ by Lemma 3. If $\Delta \in \mathcal{RP}(G)$ then there exists $\Delta' \in \mathcal{RP}(G)$ such that $\Delta' \geqslant \Delta$ and $\eta(\Delta') \geqslant d(\Delta_0)$ (for example, $\Delta' = \Delta_0 + \Delta$). By Lemma 10 there exists $f \in \mathrm{BLip}^+(\Delta')$ such that $\mathrm{BLip}^+(\Delta') = \overline{\mathrm{orb}(f)}$. □

Let κ be an infinite cardinal. Define $\mathrm{cf}(\kappa)$, the *cofinality* of κ, to be the least cardinality of a set $\mathcal{A}$ of sets such that $|E| < \kappa$ for every $E \in \mathcal{A}$ and $|\bigcup \mathcal{A}| = \kappa$. Jech [10, 1.3] discusses cofinality in detail. Note that $\mathrm{cf}(\kappa) \leqslant \kappa$ for every κ, and $\mathrm{cf}(\aleph_\omega) = \aleph_0$.

Theorem 18. *Let G be a topological group, and assume that for every $\Delta \in \mathcal{RP}(G)$ there exists $\Delta' \in \mathcal{RP}(G)$ such that $\Delta' \geqslant \Delta$ and $\mathrm{cf}(d(\Delta')) > \aleph_0$. Then G is ambitable.*

Proof. Let $\Delta \in \mathcal{RP}(G)$ be such that $\mathrm{cf}(d(\Delta)) > \aleph_0$. Since $d(\Delta) > \aleph_0$, we have $d(\Delta) = \lim_{k\to\infty} \eta(k\Delta)$ by Corollary 6. From $\mathrm{cf}(d(\Delta)) > \aleph_0$ it follows that $d(\Delta) = \eta(k\Delta)$ for some k. Thus $d(k\Delta) = \eta(k\Delta)$ and by Lemma 10 there exists $f \in \mathrm{BLip}^+(k\Delta)$ such that $\mathrm{BLip}^+(k\Delta) = \overline{\mathrm{orb}(f)}$. □

5. Topological centres

In this section, preceding results are applied to the study of topological centres in convolution algebras. We start with a summary of necessary definitions and notation. A more detailed treatment may be found in [16].

Let G be a topological group. The Banach-space dual of $\mathbf{U_b}(rG)$ is $\mathbf{U_b}(rG)^*$, and the *weak* topology* on $\mathbf{U_b}(rG)^*$ is the weak topology of the duality $\langle \mathbf{U_b}(rG)^*, \mathbf{U_b}(rG)\rangle$.

If X, Y and Z are sets and p is a mapping from $X \times Y$ to Z then $\backslash_x p(x, y)$ is the mapping $x \mapsto p(x, y)$ from X to Z and $\backslash_y p(x, y)$ is the mapping $y \mapsto p(x, y)$ from Y to Z.

The *convolution* operation $\star$ on $\mathbf{U_b}(rG)^*$ is defined by $\mu \star \nu(f) = \mu(\backslash_x \nu(\backslash_y f(xy)))$ for $\mu, \nu \in \mathbf{U_b}(rG)^*$ and $f \in \mathbf{U_b}(rG)$.

When $x \in G$ and $f \in \mathbf{U_b}(rG)$, write $\delta_x(f) = f(x)$. The mapping $\delta : x \mapsto \delta_x$ is a topological embedding of G to $\mathbf{U_b}(rG)^*$ with the weak* topology.

The subspace $\mathbf{M_u}(rG)$ of $\mathbf{U_b}(rG)^*$ is defined as follows: $\mu \in \mathbf{M_u}(rG)$ iff μ is continuous on $\mathrm{BLip}^+(\Delta)$ for each $\Delta \in \mathcal{RP}(G)$. Here, as always, $\mathrm{BLip}^+(\Delta)$ is considered with the topology of pointwise convergence on G.

The *uniform semigroup compactification of* G, denoted $\overline{rG}$, is the weak* closure of $\delta(G)$ in $\mathbf{U_b}(rG)^*$, with the weak* topology and the convolution operation $\star$. The set $\widehat{rG} = \overline{rG} \cap \mathbf{M_u}(rG)$ is identified with the completion of rG.

The elements of the space $\mathbf{M_u}(rG)$ are called *uniform measures* in the literature, but one must be careful with the terminology: The functionals in $\mathbf{M_u}(rG)$ are represented by countably additive measures on $\overline{rG}$, but not necessarily on G.

A semigroup S with a topology on S is a *right topological semigroup* if the mapping $x \mapsto xy$ from S to S is continuous for each $y \in S$ [3, 1.3]. For any right topological semigroup S, define its *topological centre*

$$\Lambda(S) = \{x \in S \mid \text{ the mapping } y \mapsto xy \text{ is continuous on } S\}.$$

The spaces studied here are also denoted by other symbols in the literature. Some of the more common notations are:

- $\mathrm{LUC}(G)$ or $RUC^*(G)$ or $\mathcal{U}_r(G)$ or $\mathcal{LC}(G)$ instead of $\mathbf{U_b}(rG)$;
- $\mathcal{S}(G)$ or G^{LUC} or $G^{\mathcal{LC}}$ instead of $\overline{rG}$;
- $Z(G)$ or $Z_t(G)$ instead of $\Lambda(\mathbf{U_b}(rG)^*)$.

Let G be a topological group. Then $\mathbf{U_b}(rG)^*$ with the $\star$ operation and the weak* topology is a right topological semigroup. This semigroup and its subsemigroup $\overline{rG}$ have a prominent role in harmonic analysis on G. Significant research efforts have been devoted to characterizing their topological centres. In the rest of this section we will see how the known results follow from results about ambitable groups. The same approach yields a characterization of topological centres not only in $\mathbf{U_b}(rG)^*$ and $\overline{rG}$ but also in any intermediate semigroup between $\mathbf{U_b}(rG)^*$ and $\overline{rG}$.

Research in abstract harmonic analysis is often concerned with linear spaces over $\mathbb{C}$, the field of complex numbers, rather than the field $\mathbb{R}$ used here. However, it is an easy exercise to derive the $\mathbb{C}$-version of any result in this paper from its $\mathbb{R}$-version.

For every topological group G we have $\mathbf{M_u}(rG) \subseteq \Lambda(\mathbf{U_b}(rG)^*)$; see Proposition 4.2 in [7] or Section 5 in [16]. If G is precompact then $\mathbf{M_u}(rG) = \mathbf{U_b}(rG)^*$ and therefore $\mathbf{M_u}(rG) = \Lambda(\mathbf{U_b}(rG)^*)$.

Question 2. Is $\mathbf{M_u}(rG) = \Lambda(\mathbf{U_b}(rG)^*)$ for every topological group G?

As is noted in Section 1, the positive answer was proved for locally compact groups by Lau [11] and for $\aleph_0$-bounded groups by Ferri and Neufang [7]. By Corollary 22 below, the answer is positive for every ambitable group.

The situation is similar for $\Lambda(\overline{rG})$. We have $\widehat{rG} \subseteq \Lambda(\overline{rG})$ for every topological group G. If G is precompact then $\widehat{rG} = \overline{rG}$ and therefore $\widehat{rG} = \Lambda(\overline{rG})$.

Question 3. Is $\widehat{rG} = \Lambda(\overline{rG})$ for every topological group G?

The positive answer was proved for locally compact groups by Lau and Pym [12] and for $\aleph_0$-bounded groups by Ferri and Neufang [7]. Again the answer is positive for every ambitable group, by Corollary 22.

Lemma 19. *Let G be a topological group and $f \in \mathbf{U_b}(rG)$).*

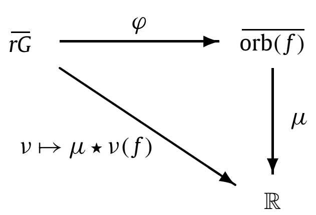
1. *The mapping $\varphi : \nu \mapsto \backslash_x \nu(\backslash_y f(xy))$ is continuous from $\overline{rG}$ to the product space $\mathbb{R}^G$.*
2. $\varphi(\overline{rG}) = \overline{\mathrm{orb}(f)}$.

Proof. 1. As noted above, $\delta_x \in \Lambda(\overline{rG})$ for each $x \in G$, and thus the mapping $\nu \mapsto \delta_x \star \nu$ is weak* continuous from $\overline{rG}$ to itself. Since $\delta_x \star \nu(f) = \nu(\backslash_y f(xy))$, this means that the mapping $\nu \mapsto \nu(\backslash_y f(xy))$ from $\overline{rG}$ to $\mathbb{R}$ is continuous for each $x \in G$, and therefore the mapping $\nu \mapsto \backslash_x \nu(\backslash_y f(xy))$ is continuous from $\overline{rG}$ to $\mathbb{R}^G$.

2. $\varphi(\delta_x) = \rho^x(f)$ for all $x \in G$, and therefore $\varphi(\delta(G)) = \mathrm{orb}(f)$. The mapping φ is continuous by part 1, $\overline{rG}$ is compact, and $\delta(G)$ is dense in $\overline{rG}$. It follows that $\varphi(\overline{rG}) = \overline{\mathrm{orb}(f)}$. □

Lemma 20. *Let G be a topological group, $\mu \in \mathbf{U_b}(rG)^*$ and $f \in \mathbf{U_b}(rG)$. If the mapping $\nu \mapsto \mu \star \nu$ from $\overline{rG}$ to $\mathbf{U_b}(rG)^*$ is weak* continuous then μ is continuous on $\overline{\mathrm{orb}(f)}$.*

Proof. As in Lemma 19, define $\varphi(\nu) = \backslash_x \nu(\backslash_y f(xy))$ for $\nu \in \overline{rG}$.



By the definition of convolution, $\mu \star \nu(f) = \mu(\backslash_x \nu(\backslash_y f(xy))) = \mu(\varphi(\nu))$. Thus $\mu \circ \varphi$ is continuous from $\overline{rG}$ to $\mathbb{R}$.

By Lemma 19, φ is continuous from $\overline{rG}$ to $\overline{\mathrm{orb}(f)}$, and $\varphi(\overline{rG}) = \overline{\mathrm{orb}(f)}$. Since $\overline{rG}$ is compact, it follows that μ is continuous on $\overline{\mathrm{orb}(f)}$. □

Theorem 21. *If G is an ambitable topological group, $\overline{rG} \subseteq S \subseteq \mathbf{U_b}(rG)^*$, and S with the $\star$ operation is a semigroup, then $\Lambda(S) = \mathbf{M_u}(rG) \cap S$.*

Proof. As was noted above, $\mathbf{M_u}(rG) \subseteq \Lambda(\mathbf{U_b}(rG)^*)$. Therefore $\mathbf{M_u}(rG) \cap S \subseteq \Lambda(S)$ for every semigroup $S \subseteq \mathbf{U_b}(rG)^*$.

To prove the opposite inclusion, take any $\mu \in \Lambda(S)$ and any $\Delta \in \mathcal{RP}(G)$. Since $\overline{rG} \subseteq S$, the mapping $\nu \mapsto \mu \star \nu$ from $\overline{rG}$ to $\mathbf{U_b}(rG)^*$ is weak* continuous by the definition of $\Lambda(S)$. Since G is ambitable, $\mathrm{BLip}^+(\Delta) \subseteq \overline{\mathrm{orb}(f)}$ for some $f \in \mathbf{U_b}(rG)$. By Lemma 20, μ is continuous on $\overline{\mathrm{orb}(f)}$ and therefore also on $\mathrm{BLip}^+(\Delta)$. Thus $\mu \in \mathbf{M_u}(rG)$. □

Corollary 22. *If G is an ambitable topological group then $\mathbf{M_u}(rG) = \Lambda(\mathbf{U_b}(rG)^*)$ and $\widehat{rG} = \Lambda(\overline{rG})$.*

Proof. Apply 21 with $S = \mathbf{U_b}(rG)^*$ and with $S = \overline{rG}$. □

Note that Theorem 21 applies not only to the semigroups $\overline{rG}$ and $\mathbf{U_b}(rG)^*$ in Corollary 22, but also to many other semigroups between $\overline{rG}$ and $\mathbf{U_b}(rG)^*$ — for example, the semigroup of all positive elements in $\mathbf{U_b}(rG)^*$, or the semigroup of all finite linear combinations of elements of $\overline{rG}$ with integral coefficients.

Corollary 23. *If G is a locally $\aleph_0$-bounded topological group then $\mathbf{M_u}(rG) = \Lambda(\mathbf{U_b}(rG)^*)$ and $\widehat{rG} = \Lambda(\overline{rG})$.*

Proof. By Theorem 14, every locally $\aleph_0$-bounded G is precompact or ambitable. As is noted above, if G is precompact then $\mathbf{M_u}(rG) = \Lambda(\mathbf{U_b}(rG)^*)$ and $\widehat{rG} = \Lambda(\overline{rG})$. Thus the statement follows from Corollary 22. □

Corollary 23 generalizes the previously published results mentioned above, for locally compact and for $\aleph_0$-bounded groups.

It is interesting to note that Questions 1 and 2 in this paper are related to a question about uniquely amenable groups asked by Megrelishvili, Pestov and Uspenskij [13]. By Theorem 5.2 in [16], every uniquely amenable topological group G such that $\mathbf{M_u}(rG) = \Lambda(\mathbf{U_b}(rG)^*)$ is precompact. Therefore, a positive answer to Question 2 would imply that every uniquely amenable topological group is precompact, thus answering Question 3.5 in [13]. The same reasoning yields the following corollary.

Corollary 24. *No uniquely amenable topological group is ambitable.*

Proof. Apply Theorem 2 and Corollary 22 in this paper, and Theorem 5.2 in [16]. □

When G is a discrete group, the uniform compactification $\overline{rG}$ is the Čech–Stone compactification βG. For any countable discrete group G, Glasner [8] strengthened the result of Lau and Pym cited in Section 1 as follows: If $\mu \in \overline{rG} = \beta G$ and the mapping $\nu \mapsto \mu \star \nu$ from βG to itself is Borel measurable then $\mu \in G$. Results of this type, where the usual continuity condition is replaced by measurability, may be also obtained from a modified version of Lemma 20 above. Details of the required modification will be described elsewhere.

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