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Ain Shams Engineering Journal (2015) 6, 613–620

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ENGINEERING PHYSICS AND MATHEMATICS

Analytical solution of nonlinear space–time fractional differential equations using the improved fractional Riccati expansion method

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Received 26 July 2014; revised 24 September 2014; accepted 19 October 2014
Available online 8 December 2014

KEYWORDS

Improved fractional Riccati expansion method;
Nonlinear fractional differential equation;
Modified Riemann–Liouville derivative;
Exact solution

Abstract In this paper, the improved fractional Riccati expansion method is proposed to solve fractional differential equations. The method is applied to solve space–time fractional modified Korteweg–de Vries equation, space–time fractional modified regularized long-wave equation, time fractional biological population model, and space–time fractional Klein–Gordon equation. The obtained solutions include generalized trigonometric and hyperbolic functions solutions. Among these solutions, some are found for the first time.
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1. Introduction

Fractional differential equations (FDEs) are generalization of differential equation of integer order. Recently, FDEs have gained much attention because it is used to describe many important phenomena in engineering, physics and chemistry. For example, fluid flows, signal processing, control theory, fiber optics, cosmology and material science [1–7]. As mentioned in [8], the most important advantage of using FDEs in many applications is their non-local property. It is known that some real physical phenomena are depend not only on its current state, but also upon its historical state (non-local property), which can be successfully modeled by using FDEs.

In the literature, many powerful and efficient methods have been proposed to obtain numerical and analytical solutions for FDEs. Examples include, Adomian decomposition method, variational iteration method, fractional difference method, differential transform method, homotopy perturbation method, the exp-function method, the (G'/G)-expansion method, the fractional sub-equation method, and generalized fractional sub-equation method [9–19]. Abdel-Salam and Yousif [20] introduced the fractional Riccati expansion method to obtain analytical solutions of FDEs to solve the space–time fractional Korteweg–de Vries (KdV) equation, regularized long-wave

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Peer review under responsibility of Ain Shams University.

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http://dx.doi.org/10.1016/j.asej.2014.10.014
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(RLW) equation, Boussinesq equation and Klein–Gordon equation. Also, Abdel-Salam et al. [21] studied the moving boundary space–time fractional Burger's equation. In this paper, we introduce the improved fractional Riccati expansion method to solve FDEs with the modified Riemann–Liouville derivative defined by Jumarie [22–24]. Recently, Abdel-Salam and Al-Muhiameed introduced the fractional mapping method by solving the fractional elliptic equation $D_x^\alpha F(x) = \sqrt{A + BF(x)^2 + CF(x)^4}$ and studied the space–time fractional combined KdV–mKdV equation.

The rest of the paper is organized as follows: some basic definitions of fractional calculus and the description of the improved fractional Riccati expansion method is presented in Section 2. The proposed method in Section 2 applied to solve the problems: space–time fractional modified KdV (mKdV) equation, space–time fractional modified RLW (mRLW) equation, time fractional biological population model and space–time fractional Klein–Gordon equation, in Section 3. The paper ends in Section 4 with conclusion.

2. Description of the improved fractional Riccati expansion method

In this section, we present the improved fractional Riccati expansion method with constant coefficients to find exact analytical solutions of nonlinear FDEs. The fractional derivatives are described in sense of the following modified Riemann–Liouville derivative defined by Jumarie [22–24] as

$$D_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{-\alpha-1}[f(\xi)-f(0)]d\xi, & \alpha<0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha}[f(\xi)-f(0)]d\xi, & 0<\alpha<1, \\ [f^{(\alpha-n)}(x)]^{(n)}, & n\leqslant\alpha<n+1, \quad n\geqslant 1, \end{cases} \tag{1}$$

which has merits over the original one, for example, the α-order derivative of a constant is zero. Some properties of the Jumarie's modified Riemann–Liouville derivative are

$$D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \quad \gamma>0, \tag{2}$$

$$D_x^\alpha[f(x)g(x)] = g(x)D_x^\alpha f(x) + f(x)D_x^\alpha g(x), \tag{3}$$

$$D_x^\alpha f[g(x)] = f_g'[g(x)]D_x^\alpha g(x) = D_g^\alpha f[g(x)](g_x')^\alpha. \tag{4}$$

The above properties play an important role in the improved fractional Riccati expansion method. The formulas (3) and (4) follow from the fractional Leibniz rule and the fractional Barrow's formula [25]. In addition, Kolwankar obtained the same formula (3) by using an approach on Cantor space [26]. Jumarie [27] gave detailed proofs of the above formulas (see Proposition 3.1 page 1746 and Section 4 (Some Basic Formulae for Fractional Derivative and Integral) page 1748). That is direct results of the equality $D_x^\alpha f(x) = \Gamma(\alpha+1)D_x f(x)$, which holds for non-differentiable functions. For a given nonlinear FDE, say, in two variables x and t

$$P(u, D_t^\alpha u, D_x^\alpha u, D_t^{2\alpha} u, D_x^{2\alpha} u, \ldots) = 0, \tag{5}$$

where $D_t^\alpha u$ and $D_x^\alpha u$ are Jumarie's modified Riemann–Liouville derivatives of u, $u = u(x,t)$, is an unknown function, P is a polynomial in u and its various partial derivatives. Using the traveling wave transformation:

$$u(x,t) = u(\xi), \quad \xi = x + \omega t, \tag{6}$$

where ω is a constant to be determined later, the nonlinear FDE (5) is reduced to the following nonlinear fractional ordinary differential equation (FODE) for $u = u(\xi)$:

$$\tilde{P}(u, \omega^\alpha D_\xi^\alpha u, D_\xi^\alpha u, \omega^{2\alpha} D_\xi^{2\alpha} u, D_\xi^{2\alpha} u, \ldots) = 0. \tag{7}$$

Suppose that $u(\xi)$ can be expressed by a finite power series of $F(\xi)$, then

$$u(\xi) = a_0 + \sum_{i=1}^{n} a_i F^i, \quad a_n \neq 0, \tag{8}$$

where a_i $(i = 0, 1, 2, \ldots, n)$ are constants to be determined later, n is a positive integer determined by balancing the linear term of the highest order with the nonlinear term in Eq. (7) and $F = F(\xi)$ satisfies the following fractional Riccati equation:

$$D_\xi^\alpha F = A + BF + CF^2, \quad 0 < \alpha \leqslant 1, \tag{9}$$

where A, B and C are constants. Using the Mittag–Leffler function in one parameter $E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+k\alpha)}$ $(\alpha > 0)$, we obtain the following solution of Eq. (9) (detailed proof of these cases is in Appendix A).

Case 1: If $B^2 - 4AC > 0$ and $BC \neq 0$, then

$$F_1 = -\frac{1}{2C}\left[B + \sqrt{B^2-4AC}\tanh\left(0.5\sqrt{B^2-4AC}\xi, \alpha\right)\right],$$
$$F_2 = -\frac{1}{2C}\left[B + \sqrt{B^2-4AC}\coth\left(0.5\sqrt{B^2-4AC}\xi, \alpha\right)\right].$$

Case 2: If $B^2 - 4AC < 0$ and $BC \neq 0$, then

$$F_3 = \frac{1}{2C}\left[-B + \sqrt{4AC-B^2}\tan\left(0.5\sqrt{4AC-B^2}\xi, \alpha\right)\right],$$
$$F_4 = -\frac{1}{2C}\left[B + \sqrt{4AC-B^2}\cot\left(0.5\sqrt{4AC-B^2}\xi, \alpha\right)\right].$$

Case 3: If $B^2 - 4AC > 0$ and $AC \neq 0$, then

$$F_5 = \frac{2A\cosh\left(0.5\sqrt{B^2-4AC}\xi,\alpha\right)}{\sqrt{B^2-4AC}\sinh\left(0.5\sqrt{B^2-4AC}\xi,\alpha\right) - B\cosh\left(0.5\sqrt{B^2-4AC}\xi,\alpha\right)},$$
$$F_6 = -\frac{2A\sinh\left(0.5\sqrt{B^2-4AC}\xi,\alpha\right)}{B\sinh\left(0.5\sqrt{B^2-4AC}\xi,\alpha\right) - \sqrt{B^2-4AC}\cosh\left(0.5\sqrt{B^2-4AC}\xi,\alpha\right)}.$$

Case 4: If $B^2 - 4AC < 0$, $AC \neq 0$, then

$$F_7 = -\frac{2A\cos\left(0.5\sqrt{4AC-B^2}\xi,\alpha\right)}{\sqrt{4AC-B^2}\sin\left(0.5\sqrt{4AC-B^2}\xi,\alpha\right) + B\cosh\left(0.5\sqrt{4AC-B^2}\xi,\alpha\right)},$$
$$F_8 = -\frac{2A\sin\left(0.5\sqrt{4AC-B^2}\xi,\alpha\right)}{B\sinh\left(0.5\sqrt{4AC-B^2}\xi,\alpha\right) - \sqrt{4AC-B^2}\cos\left(0.5\sqrt{4AC-B^2}\xi,\alpha\right)}.$$

Case 5: If $A = 0$, $BC \neq 0$, then

$$F_9 = \frac{-Br}{C[r + \cosh(B\xi,\alpha) - \sinh(B\xi,\alpha)]},$$
$$F_{10} = \frac{-B[\cosh(B\xi,\alpha) + \sinh(B\xi,\alpha)]}{C[r + \cosh(B\xi,\alpha) + \sinh(B\xi,\alpha)]},$$

where r is an arbitrary constant and the generalized hyperbolic and trigonometric functions are defined as

$$\cosh(\xi,\alpha) = \frac{E_\alpha(\xi^\alpha) + E_\alpha(-\xi^\alpha)}{2}, \quad \sinh(\xi,\alpha) = \frac{E_\alpha(\xi^\alpha) - E_\alpha(-\xi^\alpha)}{2},$$
$$\cos(\xi,\alpha) = \frac{E_\alpha(i\xi^\alpha) + E_\alpha(-i\xi^\alpha)}{2}, \quad \sin(\xi,\alpha) = \frac{E_\alpha(i\xi^\alpha) - E_\alpha(-i\xi^\alpha)}{2i},$$
$$\tanh(\xi,\alpha) = \frac{\sinh(\xi,\alpha)}{\cosh(\xi,\alpha)}, \quad \tan(\xi,\alpha) = \frac{\sin(\xi,\alpha)}{\cos(\xi,\alpha)},$$
$$\coth(\xi,\alpha) = \frac{1}{\tanh(\xi,\alpha)}, \quad \cot(\xi,\alpha) = \frac{1}{\tan(\xi,\alpha)}.$$

Determining the integer n, then substituting Eq. (8) with Eq. (9) into Eq. (7), collecting all terms with the same order of $F(\xi)$, and setting each coefficient of $F(\xi)$ to zero. This yields a system of over-determined algebraic equations for $a_0, a_1 \ldots, a_n$, and ω. Solving this system the constants $a_0, a_1, \ldots, a_n$, and ω, can be expressed in terms of constants A, B, and C. Depending on the chosen values of A, B, and C, the function $f(\xi)$ possesses the traveling wave solutions as given above; then the improved fractional Riccati expansion method (8) has the traveling wave solution of the nonlinear FDEs (5).

Remark 1. If we take $A = \sigma$, $B = 0$, and $C = 1$ in [19], then the obtained results coincide to the results in [19].

Remark 2. If we take $A = A$, $B = 0$, and $C = B$ in [20], the obtained results recover the results presented in [20].

Remark 3. When $\alpha = 1$, then the results are similar to those obtained by [28].

3. Application

3.1. Space–time fractional modified Korteweg–de Vries equation

The KdV equation is the earliest soliton equation that was firstly derived by Korteweg and de Vries to model the evolution of shallow water wave in 1895. The mKdV equation has many applications, such as quantum field theory, solid-state physics, plasma physics and fluid physics [29–33]. The space–time fractional mKdV equation is

$$D_t^\alpha u + \mu u^2 D_x^\alpha u + \tau D_x^{3\alpha} u = 0, \quad 0 < \alpha \leqslant 1, \tag{10}$$

where μ and τ are constants. In order to solve Eq. (10) by the fractional Riccati expansion method, we use the traveling wave transformation $u(x, t) = u(\xi)$, $\xi = x + \omega t$, where ω is the dimensionless velocity of the wave. Then, Eq. (10) is reduced to the following nonlinear FODE:

$$\omega^\alpha D_\xi^\alpha u + \mu u^2 D_\xi^\alpha u + \tau D_\xi^{3\alpha} u = 0. \tag{11}$$

By balancing $D_\xi^{3\alpha} u$ with $u^2\ D_\xi^\alpha u$ gives $n = 1$. Therefore, the solution of Eq. (11) can be expressed as

$$u = a_0 + a_1 F. \tag{12}$$

Substituting (12) into (11) using (9) and setting the coefficients of $F^i (i = 0, 1, 2, 3, 4)$ to zero, we get

$$a_0 = \pm B\sqrt{-\frac{3\tau}{2\mu}}, \quad a_1 = \pm C\sqrt{-\frac{6\tau}{\mu}}, \quad \omega^\alpha = \frac{\tau}{2}(B^2 - 4AC). \tag{13}$$

The general formulae for the traveling wave solution of the space–time fractional mKdV Eq. (10) is

$$u(x,t) = \pm B\sqrt{-\frac{3\tau}{2\mu}} \pm C\sqrt{-\frac{6\tau}{\mu}} F(x + \omega t), \quad \omega^\alpha = \frac{\tau}{2}(B^2 - 4AC). \tag{14}$$

By selecting the special values of the A, B, C and the corresponding function $F(\xi)$, we get the following generalized hyperbolic solutions of (10):

when $B^2 - 4AC > 0$ and $BC \neq 0$, we have

$$u_1 = \pm\sqrt{-\frac{3\tau(B^2 - 4AC)}{2\mu}} \tanh\left(0.5\sqrt{B^2 - 4AC}\xi, \alpha\right), \tag{15}$$

$$u_2 = \mp\sqrt{-\frac{3\tau(B^2 - 4AC)}{2\mu}} \coth\left(0.5\sqrt{B^2 - 4AC}\xi, \alpha\right). \tag{16}$$

If $B^2 - 4AC > 0$ and $AC \neq 0$, we have

$$u_3 = \pm\sqrt{-\frac{3\tau}{2\mu}}\left[B + \frac{4CA\cosh\left(0.5\sqrt{B^2 - 4AC}\xi, \alpha\right)}{\sqrt{B^2 - 4AC}\sinh\left(0.5\sqrt{B^2 - 4AC}\xi, \alpha\right) - B\cosh\left(0.5\sqrt{B^2 - 4AC}\xi, \alpha\right)}\right], \tag{17}$$

$$u_4 = \pm\sqrt{-\frac{3\tau}{2\mu}}\left[B - \frac{4CA\sinh\left(0.5\sqrt{B^2 - 4AC}\xi, \alpha\right)}{B\sinh\left(0.5\sqrt{B^2 - 4AC}\xi, \alpha\right) - \sqrt{B^2 - 4AC}\cosh\left(0.5\sqrt{B^2 - 4AC}\xi, \alpha\right)}\right]. \tag{18}$$

If $A = 0$ and $BC \neq 0$, we get

$$u_5 = \pm B\sqrt{-\frac{3\tau}{2\mu}}\left[1 - \frac{2r}{r + \cosh(B\xi,\alpha) - \sinh(B\xi,\alpha)}\right], \tag{20}$$

$$u_6 = \pm B\sqrt{-\frac{3\tau}{2\mu}}\left[1 - \frac{2\cosh(B\xi,\alpha) + 2\sinh(B\xi,\alpha)}{r + \cosh(B\xi,\alpha) + \sinh(B\xi,\alpha)}\right]. \quad (21)$$

Also, we can obtain the following generalized trigonometric solutions:

When $B^2 - 4AC < 0$ and $BC \neq 0$,

$$u_7 = \pm\sqrt{-\frac{3\tau(4AC - B^2)}{2\mu}}\tan\left(0.5\sqrt{4AC - B^2}\xi,\alpha\right), \quad (22)$$

$$u_8 = \pm\sqrt{-\frac{3\tau(4AC - B^2)}{2\mu}}\cot\left(0.5\sqrt{4AC - B^2}\xi,\alpha\right). \quad (23)$$

If $B^2 - 4AC < 0$ and $AC \neq 0$, we have

$$u_9 = \pm\sqrt{-\frac{3\tau}{2\mu}}\left[B - \frac{4AC\cos\left(0.5\sqrt{4AC - B^2}\xi,\alpha\right)}{\sqrt{4AC - B^2}\sin\left(0.5\sqrt{4AC - B^2}\xi,\alpha\right) + B\cos\left(0.5\sqrt{4AC - B^2}\xi,\alpha\right)}\right], \quad (24)$$

$$u_{10} = \pm\sqrt{-\frac{3\tau}{2\mu}}\left[B - \frac{4AC\sin\left(0.5\sqrt{4AC - B^2}\xi,\alpha\right)}{B\sinh\left(0.5\sqrt{4AC - B^2}\xi,\alpha\right) - \sqrt{4AC - B^2}\cos\left(0.5\sqrt{4AC - B^2}\xi,\alpha\right)}\right], \quad (25)$$

where $\omega^\alpha = \frac{\tau}{2}(B^2 - 4AC)$. When $\alpha = 1$ Eq. (10) reduced to the well known mKdV equation

$$u_t + \mu u^2 u_x + \tau u_{xxx} = 0. \quad (26)$$

The solutions (15) and (16) take the form of the following kink-shaped and singular soliton solutions of the mKdV equation

$$u_{1mKdV} = \pm\sqrt{-\frac{3\tau(B^2 - 4AC)}{2\mu}} \times \tanh\left(0.5\sqrt{B^2 - 4AC}(x + \omega t)\right), \quad (27)$$

$$u_{2mKdV} = \pm\sqrt{-\frac{3\tau(B^2 - 4AC)}{2\mu}} \times \coth\left(0.5\sqrt{B^2 - 4AC}(x + \omega t)\right), \quad (28)$$

where $\omega = \frac{\tau}{2}(B^2 - 4AC)$. The remaining solutions can be obtained in a similar manner.

3.2. *Space–time fractional modified regularized long-wave equation*

The RLW equation describes approximately the unidirectional propagation of long waves in certain nonlinear dispersive systems, has been proposed by Benjamin et al. in 1972. The RLW and mRLW equations are considered as an alternative to the KdV and mKdV equations, which are modeled to govern a large number of physical phenomena such as shallow waters and plasma waves [34–36]. The space–time fractional mRLW equation has the form

$$D_t^\alpha u + vD_x^\alpha u + \mu u^2 D_x^\alpha u - \tau D_t^\alpha D_x^{2\alpha} u = 0, \quad 0 < \alpha \leqslant 1, \quad (29)$$

where v, μ and τ are constants. By using the traveling wave transformation $u(x, t) = u(\xi)$, $\xi = x + \omega t$, Eq. (29) is reduced to the following nonlinear FODE:

$$(\omega^\alpha + v)D_\xi^\alpha u + \mu u^2 D_\xi^\alpha u - \tau\omega^\alpha D_\xi^{3\alpha} u = 0. \quad (30)$$

Thus, the solution of Eq. (29) has the form

$$u = a_0 + a_1 F. \quad (31)$$

Substituting (31) into (30) using (9) and setting the coefficients of F^i to zero, we have

$$a_0 = \pm B\sqrt{-\frac{3v\tau}{\mu(2 + B^2\tau - 4AC\tau)}}, \quad a_1 = \frac{2a_0 C}{B}, \quad \omega^\alpha = \frac{2v}{4AC\tau - 2 - B^2\tau}. \quad (32)$$

Then the general formulae of the traveling wave solution of the space–time fractional mRLW Eq. (29) is

$$u = \pm B\sqrt{-\frac{3v\tau}{\mu(2 + B^2\tau - 4AC\tau)}}\left[1 + \frac{2C}{B}F(x + \omega t)\right], \quad \omega^\alpha = \frac{2v}{4AC\tau - 2 - B^2\tau}. \quad (33)$$

By selecting the special value of the A, B, C and the corresponding function $F(\xi)$, we get the following generalized hyperbolic solutions of (29):

when $B^2 - 4AC > 0$ and $BC \neq 0$, we have

$$u_1 = \pm\sqrt{-\frac{3v\tau(B^2 - 4AC)}{\mu(2 + B^2\tau - 4AC\tau)}}\tanh\left(0.5\sqrt{B^2 - 4AC}\xi,\alpha\right), \quad (34)$$

$$u_2 = \pm\sqrt{-\frac{3v\tau(B^2 - 4AC)}{\mu(2 + B^2\tau - 4AC\tau)}}\coth\left(0.5\sqrt{B^2 - 4AC}\xi,\alpha\right), \quad (35)$$

where $\omega^\alpha = \frac{2v}{4AC\tau - 2 - B^2\tau}$. The remaining solutions can be obtained in a similar way. When $\alpha = 1$ Eq. (29) reduced to the well known mRLW equation

$$u_t + vu_x + \mu u^2 u_x - \tau u_{xxt} = 0, \quad (36)$$

The solutions (34) and (35) take the form of the following kink-shaped and singular soliton solutions of the mRLW equation

$$u_{1mRLW} = \pm\sqrt{-\frac{3v\tau(B^2 - 4AC)}{\mu(2 + B^2\tau - 4AC\tau)}}\tanh\left(0.5\sqrt{B^2 - 4AC}\xi\right), \quad (37)$$

$$u_{2mRLW} = \pm\sqrt{-\frac{3v\tau(B^2-4AC)}{\mu(2+B^2\tau-4AC\tau)}}\coth\left(0.5\sqrt{B^2-4AC}\xi\right), \quad (38)$$

where $\omega = \frac{2v}{4AC\tau-2-B^2\tau}$.

3.3. Time fractional biological population model

The problem of biological diffusion is an issue of increasing significance in contemporary ecology [37,38]. Mathematical aspects of the biological problem have been considered in many papers [39–41]. The time fractional biological population model has the form

$$D_t^\alpha u = D_x(u^2) + D_y(u^2) + h(u^2 - r), \quad 0 < \alpha \leqslant 1, \quad (39)$$

where h, r are constants, u represents the population density and $h(u^2 - r)$ represents the population supply due to births and deaths. By using the traveling wave transformation $u(x, y, t) = u(\xi)$, $\xi = kx + iky + \omega t$, $i^2 = -1$ [19], Eq. (39) is reduced to the following nonlinear FODE:

$$\omega^\alpha D_\xi^\alpha u - h(u^2 - r) = 0. \quad (40)$$

Thus, the solution of Eq. (40) has the form

$$u = a_0 + a_1 F. \quad (41)$$

Substituting (41) into (40) using (9) and setting the coefficients of F^i to zero, we have

$$a_0 = B\sqrt{\frac{r}{B^2-4AC}}, \quad a_1 = 2C\sqrt{\frac{r}{B^2-4AC}}, \quad \omega^\alpha = 2h\sqrt{\frac{r}{B^2-4AC}}. \quad (42)$$

The general formulae of the traveling wave solution of the time fractional biological population model (39) is

$$u = \sqrt{\frac{r}{B^2-4AC}}[B + 2CF(x+\omega t)], \quad \omega^\alpha = 2h\sqrt{\frac{r}{B^2-4AC}}. \quad (43)$$

By selecting the special values of the A, B, C and the corresponding function $F(\xi)$, we get the following generalized hyperbolic solutions of (39):

when $B^2 - 4AC > 0$ and $BC \neq 0$, we have

$$u_1 = -\sqrt{r}\tanh\left(0.5\sqrt{B^2-4AC}\xi, \alpha\right), \quad (44)$$

$$u_2 = -\sqrt{r}\coth\left(0.5\sqrt{B^2-4AC}\xi, \alpha\right) \quad (45)$$

where $\omega^\alpha = 2h\sqrt{\frac{r}{B^2-4AC}}$. The remaining solutions can be obtained in a similar way. When $\alpha = 1$ Eq. (39) reduced to the well known biological population model

$$u_t = (u^2)_{xx} + (u^2)_{yy} + h(u^2 - r), \quad (46)$$

And the solutions (44) and (45) take the form

$$u_{1b} = -\sqrt{r}\tanh\left(0.5\sqrt{B^2-4AC}\xi\right), \quad (47)$$

$$u_{2b} = -\sqrt{r}\coth\left(0.5\sqrt{B^2-4AC}\xi\right), \quad (48)$$

where $\omega = 2h\sqrt{\frac{r}{B^2-4AC}}$.

3.4. Space–time fractional Klein–Gordon equation

The nonlinear Klein–Gordon equation appears in relativistic quantum mechanics [20,42]. It describes the processes involving particles of spin zero. The nonlinear space–time fractional Klein–Gordon equation is

$$D_t^{2\alpha} u - vD_x^{2\alpha} u + \mu u - \tau u^3 = 0, \quad 0 < \alpha \leqslant 1, \quad (49)$$

where v, μ and τ are constants. By using the traveling wave transformation $u(x, t) = u(\xi)$, $\xi = x + \omega t$, Eq. (49) is reduced to the following nonlinear FODE:

$$(\omega^{2\alpha} - v)D_\xi^{2\alpha} u + \mu u - \tau u^3 = 0. \quad (50)$$

Thus, the solution of Eq. (49) has the form

$$u = a_0 + a_1 F. \quad (51)$$

Substituting (31) into (30) using (9) and setting the coefficients of F^i to zero, we have

$$a_0 = \pm B\sqrt{\frac{\mu}{\tau(B^2-4AC)}}, \quad a_1 = \frac{2a_0 C}{B}, \quad \omega^{2\alpha} = \frac{2\mu + v(B^2-4AC),}{B^2-4AC}. \quad (52)$$

The general formulae of the traveling wave solution of nonlinear space–time fractional Klein–Gordon Eq. (49) is

$$u = \pm B\sqrt{\frac{\mu}{\tau(B^2-4AC)}}\left[1 + \frac{2C}{B}F(x+\omega t)\right], \quad \omega^{2\alpha} = \frac{2\mu + v(B^2-4AC)}{B^2-4AC}. \quad (53)$$

By selecting the special values of the A, B, C and the corresponding function $F(\xi)$, we get the following generalized hyperbolic solutions of (49):

when $B^2 - 4AC > 0$ and $BC \neq 0$, we have

$$u_1 = \pm\sqrt{\frac{\mu}{\tau}}\tanh\left(0.5\sqrt{B^2-4AC}\xi, \alpha\right), \quad (54)$$

$$u_2 = \pm\sqrt{\frac{\mu}{\tau}}\coth\left(0.5\sqrt{B^2-4AC}\xi, \alpha\right), \quad (55)$$

where $\omega^{2\alpha} = \frac{2\mu + v(B^2-4AC)}{B^2-4AC}$. The remaining solutions can be obtained in a similar way. When $\alpha = 1$ Eq. (49) reduced to the well known nonlinear Klein–Gordon equation

$$u_t - vu_{xx} + \mu u - \tau u^3 = 0, \quad (56)$$

and the solutions (54) and (44) take the form

$$u_{1KG} = \pm\sqrt{\frac{\mu}{\tau}}\tanh\left(0.5\sqrt{B^2-4AC}\xi\right), \quad (57)$$

$$u_{2KG} = \pm\sqrt{\frac{\mu}{\tau}}\coth\left(0.5\sqrt{B^2-4AC}\xi\right), \quad (58)$$

where $\omega^2 = \frac{2\mu + v(B^2-4AC)}{B^2-4AC}$.

4. Conclusions

In this paper, the improved fractional Riccati expansion method is presented to find the analytical solutions of nonlinear space–time FDEs. The obtained solutions are expressed through Mittag–Leffler type functions. Four examples are studied to illustrate the efficiency of the method. With the best of our knowledge, some of the obtained results are appear for the first time. The improved fractional Riccati expansion method can be applied to other FDEs.

Appendix A

The product of two power series are given by

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n, \tag{A-1}$$

where $c_n = \sum_{k=0}^{\infty} a_k b_{n-k}$. If n is a natural number, then

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^n = \sum_{k=0}^{\infty} c_k x^k, \tag{A-2}$$

where $c_o = a_0^n,\ c_m = \frac{1}{m a_0}\sum_{k=1}^{\infty}(kn - m + k) a_k c_{m-k}$ [43,44]. For simplicity, we suppose that

$$E_\alpha(x^\alpha)E_\alpha(-x^\alpha) = \left(\sum_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma(1+\alpha k)}\right)\left(\sum_{k=0}^{\infty}\frac{(-x^\alpha)^k}{\Gamma(1+\alpha k)}\right) = M. \tag{A-3}$$

From the definition of $\cosh(x,\alpha)$ and $\sinh(x,\alpha)$, we can get the following inequality

$$\begin{aligned}\cosh^2(x,\alpha) - \sinh^2(x,\alpha) = \frac{1}{4}\Big[&E_\alpha(x^\alpha)^2 + 2E_\alpha(x^\alpha)E_\alpha(-x^\alpha)\\ &+ E_\alpha(-x^\alpha)^2 - E_\alpha(x^\alpha)^2 + 2E_\alpha(x^\alpha)E_\alpha(-x^\alpha)\\ &- E_\alpha(-x^\alpha)^2\Big] = E_\alpha(x^\alpha)E_\alpha(-x^\alpha) = M.\end{aligned} \tag{A-4}$$

Dividing by $\cosh^2(x,\alpha)$ and $\sinh^2(x,\alpha)$, we have

$$1 - \tanh^2(x,\alpha) = M\,\mathrm{sech}^2(x,\alpha), \tag{A-5}$$

$$\coth^2(x,\alpha) - 1 = M\,\mathrm{csch}^2(x,\alpha). \tag{A-6}$$

Similarly, we suppose that

$$E_\alpha(ix^\alpha)E_\alpha(-ix^\alpha) = \left(\sum_{k=0}^{\infty}\frac{(ix^\alpha)^k}{\Gamma(1+\alpha k)}\right)\left(\sum_{k=0}^{\infty}\frac{(-ix^\alpha)^k}{\Gamma(1+\alpha k)}\right) = \tilde{M}. \tag{A-7}$$

$$\cos^2(x,\alpha) + \sin^2(x,\alpha) = E_\alpha(ix^\alpha)E_\alpha(-ix^\alpha) = \tilde{M}. \tag{A-8}$$

$$1 + \tan^2(x,\alpha) = \tilde{M}\ \sec^2(x,\alpha), \tag{A-9}$$

$$\cot^2(x,\alpha) + 1 = \tilde{M}\csc^2(x,\alpha). \tag{A-10}$$

The fractional derivatives of the Mittag–Leffler function take the form

$$\begin{aligned}D_x^\alpha E_\alpha(x^\alpha) &= \sum_{k=0}^{\infty}\frac{D_x^\alpha x^{\alpha k}}{\Gamma(1+\alpha k)}\\ &= \sum_{k=1}^{\infty}\frac{\Gamma(1+\alpha k)x^{\alpha k-\alpha}}{\Gamma(1+\alpha k)\Gamma(\alpha k+1-\alpha)}\\ &= \sum_{k=1}^{\infty}\frac{x^{\alpha(k-1)}}{\Gamma(\alpha(k-1)+1)}\\ &= \sum_{s=0}^{\infty}\frac{x^{\alpha s}}{\Gamma(\alpha s+1)}\\ &= E_\alpha(x^\alpha),\end{aligned} \tag{A-11}$$

$$\begin{aligned}D_x^\alpha E_\alpha(-x^\alpha) &= \sum_{k=0}^{\infty}\frac{(-1)^k D_x^\alpha x^{\alpha k}}{\Gamma(1+\alpha k)}\\ &= \sum_{k=1}^{\infty}\frac{(-1)^k\Gamma(1+\alpha k)x^{\alpha k-\alpha}}{\Gamma(1+\alpha k)\Gamma(\alpha k+1-\alpha)}\\ &= \sum_{k=1}^{\infty}\frac{(-1)^k x^{\alpha(k-1)}}{\Gamma(\alpha(k-1)+1)}\\ &= \sum_{s=0}^{\infty}\frac{(-1)^{s+1}x^{\alpha s}}{\Gamma(\alpha s+1)}\\ &= -E_\alpha(-x^\alpha),\end{aligned} \tag{A-12}$$

$$D_x^\alpha E_\alpha(ix^\alpha) = iE_\alpha(ix^\alpha), \tag{A-13}$$

$$D_x^\alpha E_\alpha(-ix^\alpha) = -iE_\alpha(-ix^\alpha). \tag{A-14}$$

From Eqs. (A-11) to (A-14), we can get the derivatives of the generalized hyperbolic and trigonometric functions

$$\begin{aligned}D_x^\alpha[\sinh(x,\alpha)] &= \frac{D_x^\alpha[E_\alpha(x^\alpha)] - D_x^\alpha[E_\alpha(-x^\alpha)]}{2}\\ &= \frac{E_\alpha(x^\alpha) + E_\alpha(-x^\alpha)}{2} = \cosh(x,\alpha),\end{aligned} \tag{A-15}$$

$$\begin{aligned}D_x^\alpha[\cosh(x,\alpha)] &= \frac{D_x^\alpha[E_\alpha(x^\alpha)] + D_x^\alpha[E_\alpha(-x^\alpha)]}{2}\\ &= \frac{E_\alpha(x^\alpha) - E_\alpha(-x^\alpha)}{2} = \sinh(x,\alpha),\end{aligned} \tag{A-16}$$

$$\begin{aligned}D_x^\alpha[\sin(x,\alpha)] &= \frac{D_x^\alpha[E_\alpha(ix^\alpha)] - D_x^\alpha[E_\alpha(-ix^\alpha)]}{2i}\\ &= \frac{E_\alpha(ix^\alpha) + E_\alpha(-ix^\alpha)}{2} = \cos(x,\alpha),\end{aligned} \tag{A-17}$$

$$\begin{aligned}D_x^\alpha[\cos(x,\alpha)] &= \frac{D_x^\alpha[E_\alpha(ix^\alpha)] + D_x^\alpha[E_\alpha(-ix^\alpha)]}{2}\\ &= -\frac{E_\alpha(x^\alpha) - E_\alpha(-x^\alpha)}{2i} = -\sin(x,\alpha),\end{aligned} \tag{A-18}$$

By using these inequalities, we proof the five cases.

Case 1a: If $B^2 - 4AC > 0$ and $BC \neq 0$, $\Delta = \sqrt{B^2 - 4AC}$, then

$$F_1 = -\frac{1}{2C}[B + \Delta\tanh(0.5\Delta x,\alpha)],$$

$$\begin{aligned}\mathrm{L.H.S} &= D_x^\alpha\left[-\frac{1}{2C}[B + \Delta\tanh(0.5\Delta x,\alpha)]\right]\\ &= -\frac{\Delta}{4C}D_x^\alpha\left[\frac{\sinh(0.5\Delta x,\alpha)}{\cosh(0.5\Delta x,\alpha)}\right]\\ &= -\frac{\Delta}{4C}D_x^\alpha[\cosh(0.5\Delta x,\alpha)^{-1}\sinh(0.5\Delta x,\alpha)]\\ &= -\frac{\Delta^2}{4C}\Big[\cosh(0.5\Delta x,\alpha)^{-1}D_x^\alpha[\sinh(x,\alpha)]\\ &\quad - \sinh(x,\alpha)\cosh(0.5\Delta x,\alpha)^{-2}D_x^\alpha[\cosh(x,\alpha)]\Big]\\ &= -\frac{\Delta^2}{4C}\left[\frac{\cosh^2(0.5\Delta x,\alpha) - \sinh^2(0.5\Delta x,\alpha)}{\cosh^2(0.5\Delta x,\alpha)}\right]\\ &= -\frac{\Delta^2}{4C}\left[\frac{M}{\cosh^2(0.5\Delta x,\alpha)}\right],\\ &= -\frac{\Delta^2 M}{4C}\mathrm{sech}^2(0.5\Delta x,\alpha),\end{aligned}$$

$$\begin{aligned}
\text{R.H.S} &= A + BF + CF^2 \\
&= A - \frac{B}{2C}[B + \Delta \tanh(0.5\Delta x, \alpha)] \\
&\quad + \frac{1}{4C}[B + \Delta \tanh(0.5\Delta x, \alpha)]^2 \\
&= A - \frac{B^2}{2C} - \frac{\Delta B}{2C}\tanh(0.5\Delta x, \alpha) + \frac{B^2}{4C} \\
&\quad + \frac{\Delta B}{2C}\tanh(0.5\Delta x, \alpha) + \frac{\Delta^2}{4C}\tanh(0.5\Delta x, \alpha)^2 \\
&= A - \frac{B^2}{4C} + \frac{\Delta^2}{4C}\tanh(0.5\Delta x, \alpha)^2 \\
&= -\frac{B^2 - 4AC}{4C} + \frac{\Delta^2}{4C}\tanh(0.5\Delta x, \alpha)^2 \\
&= -\frac{\Delta^2}{4C}\left[1 - \tanh(0.5\Delta x, \alpha)^2\right] \\
&= -\frac{\Delta^2 M}{4C}\operatorname{sech}^2(0.5\Delta x, \alpha),
\end{aligned}$$

Then, the two sides are equal.

Case 2a: If $B^2 - 4AC < 0$ and $BC \neq 0, \Delta = \sqrt{4AC - B^2}$, then

$$F_3 = \frac{1}{2C}[-B + \Delta \tan(0.5\Delta\xi, \alpha)],$$

$$\begin{aligned}
\text{L.H.S} &= D_x^\alpha\left[\frac{1}{2C}[-B + \Delta \tan(0.5\Delta x, \alpha)]\right] \\
&= \frac{\Delta}{4C}D_x^\alpha\left[\frac{\sin(0.5\Delta x, \alpha)}{\cos(0.5\Delta x, \alpha)}\right] \\
&= \frac{\Delta}{4C}D_x^\alpha[\cos(0.5\Delta x, \alpha)^{-1}\sin(0.5\Delta x, \alpha)] \\
&= \frac{\Delta^2}{4C}\Big[\cos(0.5\Delta x, \alpha)^{-1}D_x^\alpha[\sin(0.5\Delta x, \alpha)] \\
&\quad - \sin(0.5\Delta x, \alpha)\cos(0.5\Delta x, \alpha)^{-2}D_x^\alpha[\cos(0.5\Delta x, \alpha)]\Big] \\
&= \frac{\Delta^2}{4C}\left[\frac{\cos^2(0.5\Delta x, \alpha) + \sin^2(0.5\Delta x, \alpha)}{\cos^2(0.5\Delta x, \alpha)}\right] \\
&= \frac{\Delta^2}{4C}\left[\frac{\tilde{M}}{\cos^2(0.5\Delta x, \alpha)}\right] = \frac{\Delta^2\tilde{M}}{4C}\sec^2(0.5\Delta x, \alpha),
\end{aligned}$$

$$\begin{aligned}
\text{R.H.S} &= A + BF + CF^2 \\
&= A + \frac{B}{2C}[-B + \Delta \tan(0.5\Delta x, \alpha)] \\
&\quad + \frac{1}{4C}[-B + \Delta \tan(0.5\Delta x, \alpha)]^2 \\
&= A - \frac{B^2}{2C} + \frac{\Delta B}{2C}\tan(0.5\Delta x, \alpha) \\
&\quad + \frac{B^2}{4C} - \frac{\Delta B}{2C}\tan(0.5\Delta x, \alpha) + \frac{\Delta^2}{4C}\tan(0.5\Delta x, \alpha)^2 \\
&= A - \frac{B^2}{4C} + \frac{\Delta^2}{4C}\tan(0.5\Delta x, \alpha)^2 \\
&= \frac{4AC - B^2}{4C} + \frac{\Delta^2}{4C}\tan(0.5\Delta x, \alpha)^2 \\
&= \frac{\Delta^2}{4C}\left[1 + \tan(0.5\Delta x, \alpha)^2\right] = \frac{\Delta^2\tilde{M}}{4C}\sec^2(0.5\Delta x, \alpha),
\end{aligned}$$

Then, the two sides are equal. By the same manner, the other formulas can be proved.

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