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Some families of generating functions for the multiple orthogonal polynomials associated with modified Bessel K-functions

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Abstract

The main object of this paper is to derive several substantially more general families of bilinear, bilateral, and mixed multilateral finite-series relationships and generating functions for the multiple orthogonal polynomials associated with the modified Bessel K-functions also known as Macdonald functions. Some special cases of the above statements are also given.

Keywords: Macdonald function; Multiple orthogonal polynomial; Rodrigues formula; Generating function; Finite-series relationship

1. Introduction

We consider multiple orthogonal polynomials associated with modified Bessel function K_ν, which were first introduced by Van Assche and Yakubovich [9] and which were recently also studied by Ben Cheikh and Douak [1]. The modified Bessel function of the second kind $K_\nu(x)$ $(\nu \geqslant 0)$ is sometimes known as the Macdonald function, especially in the Russian literature. The scaled modified Bessel K-function ρ_ν is then defined as follows:

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$$\rho_\nu(x) = 2x^{\nu/2} K_\nu(2\sqrt{x}), \quad x > 0.$$

For the multiple orthogonal polynomials, we use the weight functions

$$d\mu_1(x) = x^\alpha \rho_\nu(x)\,dx, \quad d\mu_2(x) = x^\alpha \rho_{\nu+1}(x)\,dx, \quad x \in [0, \infty),\ \alpha > -1,\ \nu \geqslant 0.$$

With these weight functions, we can define multiple orthogonal polynomials of types I and II. Let $n, m \in \mathbb{N}$; then the vector $(A^\alpha_{n,m}(x), B^\alpha_{n,m}(x))$ of multiple orthogonal polynomials of type I is such that $A^\alpha_{n,m}(x)$ is a polynomial of degree at most n, $B^\alpha_{n,m}(x)$ is a polynomial of degree at most m and $(A^\alpha_{n,m}(x), B^\alpha_{n,m}(x))$ satisfies the orthogonality conditions

$$\int_0^\infty \left[A^\alpha_{n,m}(x)\rho_\nu(x) + B^\alpha_{n,m}(x)\rho_{\nu+1}(x)\right] x^{k+\alpha}\,dx = 0, \quad k = 0, 1, 2, \ldots, n+m.$$

We use the notation

$$q^\alpha_{n,m}(x) = A^\alpha_{n,m}(x)\rho_\nu(x) + B^\alpha_{n,m}(x)\rho_{\nu+1}(x),$$
$$Q^\alpha_{2n}(x) = q^\alpha_{n,n}(x), \qquad Q^\alpha_{2n+1}(x) = q^\alpha_{n+1,n}(x).$$

The multiple orthogonal polynomials $p^\alpha_{n,m}(x)$ of type II are such that $p^\alpha_{n,m}(x)$ is a polynomial of degree at most $n+m$ that satisfies the multiple orthogonality condition

$$\int_0^\infty p^\alpha_{n,m}(x)\rho_\nu(x) x^{k+\alpha}\,dx = 0, \quad k = 0, 1, 2, \ldots, n-1,$$
$$\int_0^\infty p^\alpha_{n,m}(x)\rho_{\nu+1}(x) x^{k+\alpha}\,dx = 0, \quad k = 0, 1, 2, \ldots, m-1.$$

In [9] it was shown that the weights $(\rho_\nu, \rho_{\nu+1})$ form an AT-system on $[0, \infty)$ [6, p. 140] (in fact, they are very close to a Nikishin system or MT-system [6, p. 142]), so that the polynomials $A^\alpha_{n,m}(x)$, $B^\alpha_{n,m}(x)$, and $p^\alpha_{n,m}(x)$ have degrees exactly n, m and $n+m$, respectively. Usually, the polynomial $p^\alpha_{n,m}(x)$ is chosen to be monic. We define

$$P^\alpha_{2n}(x) = p^\alpha_{n,n}(x), \qquad P^\alpha_{2n+1}(x) = p^\alpha_{n+1,n}(x).$$

These multiple orthogonal polynomials were introduced in [9] and solve an open problem posed by Prudnikov [7]. The various differential properties of the modified Bessel K-function imply useful differential properties of the multiple orthogonal polynomials, a Rodrigues formula for type I polynomials, and explicit formulas for the recurrence coefficients in the four-term recurrence relation of type II polynomials [9]. Recently, type II polynomials were also studied in [1], who started with an explicit expression of type II multiple orthogonal polynomials and found a generating function and a third-order differential equation for these polynomials. In [3] an explicit expression for both types I and II multiple orthogonal polynomials were obtained.

2. Generating functions for the multiple orthogonal polynomials of type *I* and *II*

In this section, let us remember some theorems in order to prove our main theorems in the following section.

In [3] one can find the following theorem.

Theorem A. *We have the following generating function for* $x \in \mathbb{C} \setminus (-\infty, 0]$:

$$\sum_{n=0}^{\infty} Q_{n-1}^{\alpha}(x) \frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}} \rho_{\upsilon}\left(\frac{x}{1-t}\right), \quad \operatorname{Re} x \geqslant 0 \ (x \neq 0), \ |t| < \frac{1}{2}. \tag{1}$$

If $\operatorname{Re} x < 0$ *and* $\operatorname{Im} x \neq 0$, *then the series converges for*

$$|t| < \frac{|\operatorname{Im} x|}{|x| + |\operatorname{Im} x|}.$$

The following theorem is a very specialized case of [8, Eq. 2.6(19), p. 141] for $r = 0$, $s = 2$ and t replaced by $-t$.

Theorem B. *The generating function*

$$H(x,t) = \sum_{n=0}^{\infty} \frac{P_n^{\alpha}(x)}{(\alpha+1)_n (\alpha+\nu+1)_n} \frac{t^n}{n!}$$

is given by

$$H(x,t) = e^{-t} \, {}_0F_2(-; \alpha+1, \alpha+\nu+1; xt), \quad x, t \in \mathbb{C}. \tag{2}$$

3. Main theorems

In recent years by making use of the familiar group-theoretic (Lie algebraic) method a certain mixed trilateral finite-series relationships have been proved for orthogonal polynomials (see, for instance, [8]).

The main object of this section is to derive several substantially more general families of bilinear, bilateral, mixed multilateral finite-series relationship and generating functions for the multiple orthogonal polynomials of type I and II associated with the modified Bessel K-functions as Theorems 3.1 and 3.2, respectively.

By applying the formula (1) we can prove the following theorem for multiple orthogonal polynomials of type I associated with the modified Bessel K-function of the first kind, instead of using group theoretic method, with the help of the similar method as considered in [2,5,10].

Theorem 3.1. *Corresponding to an identically nonvanishing function* $\Omega_{\mu}(\xi_1, \xi_2, \ldots, \xi_s)$ *of* s (*real or complex*) *variables* $\xi_1, \xi_2, \ldots, \xi_s$ ($s \in \mathbb{N} := \mathbb{N}_0 \setminus \{0\}$) *and of* (*complex*) *order* μ, *let*

$$\Lambda_{\psi,\mu}^{(1)}(\xi_1, \ldots, \xi_s; \tau) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(\xi_1, \ldots, \xi_s) \tau^k \quad (a_k \neq 0, \ \psi \in \mathbb{C}). \tag{3}$$

Suppose also that

$$\Theta_{n,q}^{\mu,\lambda,\psi}(x;\xi_1,\dots,\xi_s;\zeta) := \sum_{k=0}^{[n/q]} \frac{n!}{(n-qk)!} a_k Q_{n-qk-1}^{\alpha+\lambda k}(x)\Omega_{\mu+\psi k}(\xi_1,\dots,\xi_s)\zeta^k$$
$$(n,q\in\mathbb{N};\ \alpha+\lambda k>-1). \tag{4}$$

Then

$$\sum_{n=0}^{\infty}\Theta_{n,q}^{\mu,\lambda,\psi}\left(x;\xi_1,\dots,\xi_s;\frac{\eta}{t^q}\right)\frac{t^n}{n!}$$
$$= \frac{1}{(1-t)^{\alpha+1}}\rho_\nu\left(\frac{x}{1-t}\right)\Lambda_{\psi,\mu}^{(1)}\left(\xi_1,\dots,\xi_s;\frac{\eta}{(1-t)^\lambda}\right), \tag{5}$$

provided that each member of (5) *exists and that* $\operatorname{Re}x \geqslant 0$ $(x\neq 0)$, $|t|<1/2$. *If* $\operatorname{Re}x<0$ *and* $\operatorname{Im}x\neq 0$, *then the series converges for*

$$|t| < \frac{|\operatorname{Im}x|}{|x|+|\operatorname{Im}x|}.$$

The notation $[n/q]$ means the greatest integer less than or equal to n/q.

Proof. For convenience, let $\Delta(x,t)$ denote the first member of the assertion (5). Then, upon substituting for the polynomials

$$\Theta_{n,q}^{\mu,\lambda,\psi}(x;\xi_1,\dots,\xi_s;\zeta)$$

from the definition (4) into the left-hand side of (5), we obtain

$$\Delta(x,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/q]}\frac{1}{(n-qk)!}a_k Q_{n-qk-1}^{\alpha+\lambda k}(x)\Omega_{\mu+\psi k}(\xi_1,\dots,\xi_s)\eta^k t^{n-qk}$$
$$= \sum_{k=0}^{\infty} a_k\Omega_{\mu+\psi k}(\xi_1,\dots,\xi_s)\eta^k\sum_{n=0}^{\infty}Q_{n-1}^{\alpha+\lambda k}(x)\frac{t^n}{n!},$$

which in view of (1) with

$$\alpha\to\alpha+\lambda k\quad(k\in\mathbb{N}_0)$$

yields

$$\Delta(x,t) = \sum_{k=0}^{\infty} a_k\Omega_{\mu+\psi k}(\xi_1,\dots,\xi_s)\eta^k\frac{1}{(1-t)^{\lambda k+\alpha+1}}\rho_\nu\left(\frac{x}{1-t}\right)$$
$$= \sum_{k=0}^{\infty} a_k\Omega_{\mu+\psi k}(\xi_1,\dots,\xi_s)\left\{\frac{\eta}{(1-t)^\lambda}\right\}^k\frac{1}{(1-t)^{\alpha+1}}\rho_\nu\left(\frac{x}{1-t}\right)$$

and the assertion (5) follows immediately by means of the definition (3). □

In a similar manner, by appealing to the formula (2), we are led fairly easily to

Theorem 3.2. *Corresponding to an identically nonvanishing function* $\Omega_\mu(\xi_1, \xi_2, \dots, \xi_s)$ *of* s (*real or complex*) *variables* $\xi_1, \xi_2, \dots, \xi_s$ ($s \in \mathbb{N} := \mathbb{N}_0 \setminus \{0\}$) *and of* (*complex*) *order* μ, *let*

$$\Lambda^{(2)}_{\psi,\mu}(\xi_1, \dots, \xi_s; \tau) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(\xi_1, \dots, \xi_s)\tau^k \quad (a_k \neq 0,\ \psi \in \mathbb{C}). \tag{6}$$

Suppose also that

$$\Phi^{\mu,\psi}_{n,q}(x; \xi_1, \dots, \xi_s; \zeta) := \sum_{k=0}^{[n/q]} \frac{n!}{(n-qk)!} a_k P^{\alpha}_{n-qk}(x) \frac{\Omega_{\mu+\psi k}(\xi_1, \dots, \xi_s)\zeta^k}{(\alpha+1)_{n-qk}(\alpha+\nu+1)_{n-qk}}$$
$$(n, q \in \mathbb{N},\ x \in \mathbb{C}). \tag{7}$$

Then, for $t \in \mathbb{C}$ *we have*

$$\sum_{n=0}^{\infty} \Phi^{\mu,\psi}_{n,q}\left(x; \xi_1, \dots, \xi_s; \frac{\eta}{t^q}\right)\frac{t^n}{n!}$$
$$= e^{-t}\, {}_0F_2(-; \alpha+1, \alpha+\nu+1; xt)\Lambda^{(2)}_{\psi,\mu}(\xi_1, \dots, \xi_s; \eta), \tag{8}$$

provided that each member of (8) *exists.*

4. Some special cases of Theorems 3.1 and 3.2

When the multivariable function $\Omega_{\mu+\psi k}(\xi_1, \dots, \xi_s)$ ($k \in \mathbb{N}_0,\ s \in \mathbb{N}$) is expressed in terms of simpler function of one and more variables then we can give further applications of Theorems 3.1 and 3.2. For example, if we set $s = 1$, $\xi_1 = z$ and $\Omega_{\mu+\psi k}(z) = y_m(z; \mu + \psi k, \beta)$ ($k, m \in \mathbb{N}_0$, $\mu, \psi \in \mathbb{C}$) in Theorem 3.1, where $y_m(z; \alpha, \beta)$ denotes the generalized Bessel polynomials defined by (see [4])

$$y_m(z; \alpha, \beta) = {}_2F_0\left(-m, \alpha+m-1; -; \frac{-x}{\beta}\right),$$

we shall readily obtain a class of bilateral generating functions for the generalized Bessel polynomials or the multiple orthogonal polynomials of type I associated with modified Bessel K-functions, given by

Corollary 4.1. *If*

$$\Pi^{(1)}_{\mu,\psi,m}(z; \tau) := \sum_{k=0}^{\infty} a_k y_m(z; \mu+\psi k, \beta)\tau^k \quad (a_k \neq 0,\ \psi, \mu, \beta \in \mathbb{C})$$

and

$$\Psi^{(1)}_{\mu,\psi,\lambda,n,q}(x, z, \zeta) := \sum_{k=0}^{[n/q]} \frac{n!}{(n-qk)!} a_k Q^{\alpha+\lambda k}_{n-qk-1}(x) y_m(z; \mu+\psi k, \beta)\zeta^k$$
$$(n, q \in \mathbb{N},\ \lambda \in \mathbb{C}),$$

then

$$\sum_{n=0}^{\infty} \Psi^{(1)}_{\mu,\psi,\lambda,n,q}\left(x, z, \frac{\eta}{t^q}\right)\frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}}\rho_\nu\left(\frac{x}{1-t}\right)\Pi^{(1)}_{\mu,\psi,m}\left(z; \frac{\eta}{(1-t)^\lambda}\right), \tag{9}$$

provided that each member of (9) *exists, where* $\operatorname{Re} x \geqslant 0$ $(x \neq 0)$, $|t| < 1/2$. *If* $\operatorname{Re} x < 0$ *and* $\operatorname{Im} x \neq 0$, *then the series converges for*

$$|t| < \frac{|\operatorname{Im} x|}{|x| + |\operatorname{Im} x|}.$$

Example 4.1. By using the generating relation

$$\sum_{k=0}^{\infty} \binom{\mu+m+k-2}{k} y_m(z; \mu+k, \beta)\tau^k$$
$$= (1-\tau)^{1-\mu-m} y_m\left(\frac{z}{1-\tau}; \mu, \beta\right), \quad |\tau| < 1,$$

for generalized Bessel polynomials (see [5, Eq. (4.15), p. 270]) and taking

$$\psi = 1, \qquad a_k = \binom{\mu+m+k-2}{k},$$

we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/q]} \frac{1}{(n-qk)!} \binom{\mu+m+k-2}{k} Q^{\alpha+\lambda k}_{n-qk-1}(x) y_m(z; \mu+k, \beta)\eta^k t^{n-qk}$$
$$= \frac{1}{(1-t)^{\alpha+1}}\rho_\nu\left(\frac{x}{1-t}\right)\left(1 - \frac{\eta}{(1-t)^\lambda}\right)^{1-\mu-m} y_m\left(\frac{z(1-t)^\lambda}{(1-t)^\lambda - \eta}; \mu, \beta\right),$$
$$\left|\frac{\eta}{(1-t)^\lambda}\right| < 1.$$

We can give another special case of Theorem 3.1 when we set $s = 1$, $\xi_1 = z$ and $\Omega_{\mu+\psi k}(z) = Q^{\mu+\psi k}_N(z)$ $(k, N \in \mathbb{N}_0, \mu + \psi k > -1)$ we shall readily obtain a class of bilinear generating function for the multiple orthogonal polynomials of type I associated with modified Bessel K-functions, given by

Corollary 4.2. *If*

$$\Pi^{(2)}_{\mu,\psi,N}(z; \tau) := \sum_{k=0}^{\infty} a_k Q^{\mu+\psi k}_N(z)\tau^k \quad (a_k \neq 0)$$

and

$$\Psi^{(2)}_{\mu,\psi,\lambda,n,q}(x, z, \zeta) := \sum_{k=0}^{[n/q]} \frac{n!}{(n-qk)!} a_k Q^{\alpha+\lambda k}_{n-qk-1}(x) Q^{\mu+\psi k}_N(z)\,\zeta^k$$
$$(n, q \in \mathbb{N},\ \lambda \in \mathbb{R}),$$

then

$$\sum_{n=0}^{\infty} \Psi^{(2)}_{\mu,\psi,\lambda,n,q}\left(x, z, \frac{\eta}{t^q}\right)\frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}}\rho_\nu\left(\frac{x}{1-t}\right)\Pi^{(2)}_{\mu,\psi,N}\left(z; \frac{\eta}{(1-t)^\lambda}\right), \tag{10}$$

provided that each member of (10) *exists, where* $\operatorname{Re} x \geqslant 0$ $(x \neq 0)$, $|t| < 1/2$. *If* $\operatorname{Re} x < 0$ *and* $\operatorname{Im} x \neq 0$, *then the series converges for*

$$|t| < \frac{|\operatorname{Im} x|}{|x| + |\operatorname{Im} x|}.$$

By choosing $s = 1$, $\xi_1 = z$ and $\Omega_{\mu+\psi k}(z) = Q_N^{\mu+\psi k}(z)$ (where $\mu + \psi k > -1$) in Theorem 3.2 we have the following result immediately.

Corollary 4.3. *If*

$$\Pi^{(3)}_{\mu,\psi,N}(z;\tau) := \sum_{k=0}^{\infty} a_k Q_N^{\mu+\psi k}(z)\tau^k \quad (a_k \neq 0)$$

and

$$\Psi^{(3)}_{\mu,\psi,\lambda,n,q}\left(x, z, \frac{\eta}{t^q}\right) = \sum_{k=0}^{[n/q]} \frac{n!}{(n-qk)!} a_k P^{\alpha}_{n-qk}(x) \frac{Q_N^{\mu+\psi k}(z)\,\zeta^k}{(\alpha+1)_{n-qk}(\alpha+\nu+1)_{n-qk}}$$
$$(n, q \in \mathbb{N}),$$

then

$$\sum_{n=0}^{\infty} \Psi^{(3)}_{\mu,\psi,\lambda,n,q}\left(x, z, \frac{\eta}{t^q}\right)\frac{t^n}{n!} = e^{-t}\, {}_0F_2(-; \alpha+1, \alpha+\nu+1; xt)\Pi^{(3)}_{\mu,\psi,N}(z;\tau), \tag{11}$$

provided that each member of (11) *exists.*

Moreover, for each suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable function

$$\Omega_{\mu+\psi k}(\xi_1, \ldots, \xi_s) \quad (s \in \mathbb{N})$$

is expressed as an appropriate product of several simpler functions (as it seen in the example), Theorems 3.1 and 3.2 can be shown to yield various classes of mixed multilateral generating functions for the multiple orthogonal polynomials of type I and II associated with modified Bessel K-functions.

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