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NOTE

CONCERNING THE SEMANTIC CONSEQUENCE RELATION IN FIRST-ORDER TEMPORAL LOGIC

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Abstract. In this paper we consider the first-order temporal logic with linear and discrete time. We prove that the set of tautologies of this logic is not arithmetical (i.e., it is neither Σ_n^0 nor Π_n^0 for any natural number n). Thus we show that there is no finitistic and complete axiomatization of the considered logic.

Key words. First-order temporal logic, semantic consequence, syntactic consequence, model, completeness, soundness, proof system.

1. Introduction and preliminaries

For the last few years temporal logic has been considered a powerful tool for specifying and verifying properties of programs (see, e.g., [2, 3, 4, 5]). When writing a program that computes over some data structure, one needs, at least, a first-order language to describe properties of operations and individuals. In the context of temporal logic such a language is presented, e.g., in [3, 4]. The proof system for the first-order temporal logic is studied in [2, 3]. Up to now, however, it was an open question whether the logic is complete or not (cf, e.g., [3, p. 69]). In our paper we shall show that the semantic consequence relation of the first-order temporal logic with operators $\Box$ (always), $\Diamond$ (sometimes) and $\bigcirc$ (next) has no sound axiomatization which is complete and finitistic (i.e., with decidable set of axioms) provided the language is sufficiently rich. As a corollary, we shall prove that Kröger's proof system given in [2, 3] is not complete.

Now let us briefly recall those notions of first-order temporal logic which are important for our presentation. For more precise definitions see [3, 4].

Let L be a first-order language with equality $=$, usual boolean connectives $\vee$, $\wedge$, $\neg$, $\rightarrow$, $\leftrightarrow$, universal quantifier $\forall$, and the additional syntactic rule that if A is a

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formula, then $\bigcirc A$ and $\Diamond A$ are also formulas. We define $\exists xA$ to be: $\neg\forall x\neg A$, and $\Box A$ to be $\neg\Diamond\neg A$.

A signature of the language consists of the set of predicate and functor symbols, and two sets of variable symbols, the so-called global and local variables. All predicates, functors, and global variables have a uniform interpretation which does not depend on states. The local variables may change their values from state to state. In the sequel, x and y will always denote global variables and z will denote a local variable.

Definition 1.1. A *Kripke structure* $\langle I, S\rangle$ for our language consists of

(1) a global interpretation I which specifies a domain (denoted by $\mathrm{dom}(I)$) and assigns concrete functions and relations to the functor and predicate symbols;

(2) an infinite sequence of states $S = s_0, s_1, \ldots,$ where each state s_i assigns a value $s_i(z) \in \mathrm{dom}(I)$ to each local variable z. For a sequence S and a natural number k, S/k stands for the suffix $s_k, s_{k+1}, \ldots$ of S.

Definition 1.2. Let v be a valuation of global variables. The semantic consequence operation for formulas without temporal modalities is defined exactly as in classical logic. For formulas containing temporal operators we define

(1) $\langle I, S\rangle, v \models \Diamond A$ iff there exists a natural number k such that $\langle I, S/k\rangle, v \models A$,

(2) $\langle I, S\rangle, v \models \bigcirc A$ iff $\langle I, S/1\rangle, v \models A$,

(3) $\langle I, S\rangle, v \models \forall xA$ iff for every $d \in \mathrm{dom}(I)$, $\langle I, S\rangle, v' \models A$ where v' is the valuation obtained from v by assigning d to x.

If $\langle I, S/k\rangle, v \models A$ for every global valuation v and every natural number k, we say that $\langle I, S\rangle$ is a *model* for A and denote this by $\langle I, S\rangle \models A$. We say that $\langle I, S\rangle$ is a model for a set F of formulas, and denote this by $\langle I, S\rangle \models F$, iff $\langle I, S\rangle \models A$ for each formula $A \in F$. A formula B is a *semantic consequence* of a set F of formulas ($F \models B$) iff every model $\langle I, S\rangle$ for F is also a model for B. B is a *tautology* ($\models B$) iff B is a semantic consequence of the empty set of formulas. If A is a classical first-order formula (without temporal modalities and local variables), then we sometimes write $I \models A$ instead of $\langle I, S\rangle \models A$.

As usually, $\vdash$ stands for the syntactic consequence operation given by a set of axioms and inference rules. An axiomatization is complete iff, for every formula A, $\models A$ implies $\vdash A$.

2. Main result

In the sequel, we shall assume that the first-order language contains a zero-argument functor 0, a unary functor s, and two-argument functors + and $*$. The following set of formulas will play an important role in the rest of the paper:

(N1) $\forall x \ 0 \neq \mathrm{s}(x)$,
(N2) $\forall x, y \ \mathrm{s}(x) = \mathrm{s}(y) \to x = y$,
(N3) $\forall x \ x + 0 = x$,
(N4) $\forall x, y \ x + \mathrm{s}(y) = \mathrm{s}(x + y)$,
(N5) $\forall x \ x * 0 = 0$,
(N6) $\forall x, y \ x * \mathrm{s}(y) = x * y + x$,
(N7) $\forall x \ \Box\Diamond x = z$,
(N8) $\forall x \ \Box(z = x \to \bigcirc(z = 0 \vee z = \mathrm{s}(x)))$.

We denote this set by FN, and a subset consisting of (N1) to (N6) by FC.

Lemma 2.1. *There is a model* $\langle \mathrm{Nat}, \mathrm{Seq}\rangle$ *for the set* FN.

Proof. Let $\mathrm{Nat} = \langle N, 0, \mathrm{s}, +, *\rangle$ be the standard model of natural numbers and let C be the concatenation of sequences $c_0, c_1, \ldots$ defined as follows: $c_0 = 0$ and, for $i > 0$, $c_i = c_{i-1}, \mathrm{s}^i(0)$, where $\mathrm{s}^i(0)$ is the composition $\mathrm{s}(\mathrm{s}(\ldots\mathrm{s}(0)\ldots))$ in which s is applied i times.

Let Seq be a sequence of states $s_0, s_1, \ldots$ such that $s_0(z), s_1(z), \ldots$ is the sequence C. This means that the values of z in the states $s_0, s_1, \ldots$ are

$$0, 0, \mathrm{s}(0), 0, \mathrm{s}(0), \mathrm{s}(\mathrm{s}(0)), 0, \ldots, 0, \mathrm{s}(0), \mathrm{s}(\mathrm{s}(0)), \ldots, \mathrm{s}^i(0), 0, \ldots.$$

It is obvious that $\langle \mathrm{Nat}, \mathrm{Seq}\rangle \models \mathrm{FC}$ since the formulas of FC are simply the Peano axioms for Nat.

Also, $\langle \mathrm{Nat}, \mathrm{Seq}\rangle \models \mathrm{FN}$ since

$\langle \mathrm{Nat}, \mathrm{Seq}\rangle \models \text{(N7)}$ iff $\langle \mathrm{Nat}, \mathrm{Seq}\rangle \models \forall x \ \Box\Diamond x = z$, which is true since each natural number occurs in C infinitely often;
$\langle \mathrm{Nat}, \mathrm{Seq}\rangle \models \text{(N8)}$ iff for every $d \in N$ and for every natural number k, $\langle \mathrm{Nat}, \mathrm{Seq}/k\rangle \models z = d \to \bigcirc(z = 0 \vee z = \mathrm{s}(d))$ which follows from the definition of $\bigcirc$ and Seq. □

Lemma 2.2. *If* $\langle I, S\rangle \models \mathrm{FN}$, *then, for every* $d \in \mathrm{dom}(I)$, *there exists a natural number* i *such that* $d = \mathrm{s}^i(0)$.

Proof. Assume there exists a $d \in \mathrm{dom}(I)$ such that, for every i, $d \neq \mathrm{s}^i(0)$. Note that, by (N7), $\Box\Diamond z = 0$. Let k be the smallest natural number for which $\langle I, S/k\rangle \models z = 0$. Thus, by (N8), $\langle I, S/k\rangle \models \bigcirc(z = 0 \vee z = \mathrm{s}(0))$ and, by induction, it can be proved that, for all $m \geq k$, there exists an i such that $\langle I, S/m\rangle \models z = \mathrm{s}^i(0)$, i.e., $\langle I, S/k\rangle \models \Box z \neq d$. On the other hand, by (N7), $\forall x \Box\Diamond z = x$, and so $\Box\Diamond z = d$. This implies that $\langle I, S/k\rangle \models \Diamond z = d$, which contradicts the fact that $\langle I, S/k\rangle \models \Box z \neq d$. □

Lemma 2.3. *If* $\langle I, S\rangle \models \mathrm{FN}$, *then, for every* $d \in \mathrm{dom}(I)$ *and every natural number* $i > 0$, $d \neq \mathrm{s}^i(d)$.

Proof. Let $i > 0$. Assume there exists a $d \in \mathrm{dom}(I)$ such that $d = \mathrm{s}^i(d)$. From Lemma 2.2 it follows that $d = \mathrm{s}^k(0)$ for some k. Thus, $\mathrm{s}^k(0) = \mathrm{s}^i(\mathrm{s}^k(0)) = \mathrm{s}^k(\mathrm{s}^i(0))$ and, by applying (N2) k times, we obtain $0 = \mathrm{s}^i(0)$ which contradicts (N1). □

Lemma 2.4 (fundamental lemma). If $\langle \mathrm{I}, \mathrm{S}\rangle \models \mathrm{FN}$, *then I is isomorphic to the standard model of natural numbers*, $\mathrm{Nat} = \langle N, 0, \mathrm{s}, +, *\rangle$.

Proof. We define $f: \mathrm{dom}(I) \to N$ as follows: $f(\mathrm{s}^i(0)) = \mathrm{s}^i(0)$, where $\mathrm{s}^i(0)$ on the left-hand side of the equality is a term built over I, and $\mathrm{s}^i(0)$ on the right-hand side of the equality is the ith successor of 0 in Nat.

Note that, due to Lemma 2.2, each element of $\mathrm{dom}(I)$ has the form $\mathrm{s}^i(0)$ for some i, and so f is defined for all $x \in \mathrm{dom}(I)$.

Now, let us prove that f is a function, i.e., $f(x) \neq f(y)$ implies that $x \neq y$. Let $f(x) = \mathrm{s}^i(0)$, $f(y) = \mathrm{s}^j(0)$, and $i > j$. Thus, $x = \mathrm{s}^i(0)$ and $y = \mathrm{s}^j(0)$. Assuming $x = y$ we obtain $\mathrm{s}^i(0) = \mathrm{s}^j(0)$ and, since $i > j$, there exists $k > 0$ such that $\mathrm{s}^j(0) = \mathrm{s}^i(0) = \mathrm{s}^{j+k}(0) = \mathrm{s}^k(\mathrm{s}^j(0))$. This means that there exists an $x = \mathrm{s}^j(0)$ for which $x = \mathrm{s}^k(x)$. This contradicts Lemma 2.3.

To prove that f is one-to-one, i.e., $x \neq y$ implies $f(x) \neq f(y)$, it suffices to note that if $x = \mathrm{s}^i(0)$ and $y = \mathrm{s}^j(0)$ for $i \neq j$, then $f(x) = \mathrm{s}^i(0) \neq \mathrm{s}^j(0) = f(y)$.

The fact that f is 'onto' immediately follows from the definition of f and the fact that $\mathrm{dom}(I)$ is closed with respect to s and 0.

What remains to be shown is that f preserves operations 0, s, +, and $*$. Obviously, $f(0) = f(\mathrm{s}^0(0)) = \mathrm{s}^0(0) = 0$.

Let $x = \mathrm{s}^i(0)$; then

$$f(\mathrm{s}(x)) = f(\mathrm{s}(\mathrm{s}^i(0))) = \mathrm{s}^{i+1}(0) = \mathrm{s}(\mathrm{s}^i(0)) = \mathrm{s}(f(\mathrm{s}^i(0))) = \mathrm{s}(f(x)).$$

Assuming $\langle I, S\rangle \models \mathrm{FN}$, one can easily prove that, for every $d, e \in \mathrm{dom}(I)$, if $d = \mathrm{s}^i(0)$ and $e = \mathrm{s}^j(0)$, then $d + e = \mathrm{s}^{i+j}(0)$ and $d * e = \mathrm{s}^{i*j}(0)$. Let $x = \mathrm{s}^i(0)$ and $y = \mathrm{s}^j(0)$. Thus,

$$f(x+y) = f(\mathrm{s}^i(0) + \mathrm{s}^j(0)) = f(\mathrm{s}^{i+j}(0)) = \mathrm{s}^{i+j}(0) = \mathrm{s}^i(0) + \mathrm{s}^j(0) = f(x) + f(y),$$

$$f(x*y) = f(\mathrm{s}^i(0) * \mathrm{s}^j(0)) = f(\mathrm{s}^{i*j}(0)) = \mathrm{s}^{i*j}(0) = \mathrm{s}^i(0) * \mathrm{s}^j(0) = f(x) * f(y).$$

□

Theorem 2.5. *If the language contains a zero-argument functor* 0, *a unary functor* s *and two-argument functors* + *and* $*$, *then there is no sound and complete finitistic axiomatization (with* Σ_0^0 *as set of axioms) of the first-order temporal logic.*

Proof. Let C be a classical first-order formula without free variables. Denote by B the conjunction $\mathrm{N1} \wedge \mathrm{N2} \wedge \cdots \wedge \mathrm{N8}$. We shall prove the following equivalence:

$B \to C$ is a tautology iff C is true in the standard model of natural numbers Nat (in symbols, $\models B \to C$ iff $\mathrm{Nat} \models C$).

$(\to)$: Obviously, $\mathrm{FN} \models B$. Assuming $\models B \to C$ we obtain $\mathrm{FN} \models B \to C$, and so, from the definition of semantic consequence, it follows that $\mathrm{FN} \models C$. Since C is a classical formula and $\langle \text{Nat}, \text{Seq} \rangle$, defined in the proof of Lemma 2.1, is a model for FN, C is true in Nat.

($\leftarrow$): Let $\text{Nat} \models C$. From Lemma 2.4 it follows that if $\langle I, S \rangle \models \text{FN}$, then I is isomorphic to Nat. By Lemma 2.1, there exists a Kripke structure $\langle I, S \rangle$ such that $\langle I, S \rangle \models \text{FN}$. Thus, C is true in I, and (since C is a classical formula) $\langle I, S \rangle \models C$. This means that $\text{FN} \models C$, i.e., since B is the conjunction of all formulas from FN, $\models \Box B \rightarrow C$ (cf. [3, p. 23]). On the other hand, since the formulas (N1), ..., (N6) are classical first-order formulas and (N7) and (N8) are universally closed by the modality $\Box$, we have $\langle I, S \rangle \models \Box B$ iff $\langle I, S \rangle \models B$. Thus, $\models \Box B \rightarrow C$ implies $\models B \rightarrow C$.

Now, let us prove the main result. Assume we are given a finite (with Σ_0^0 as set of axioms), sound, and complete axiomatization of first-order temporal logic. Then the set of theorems is Σ_1^0. We have just proved that $\text{Nat} \models C$ iff $\models B \rightarrow C$. From the soundness and completeness of the axiom system it follows that $\text{Nat} \models C$ iff $\vdash B \rightarrow C$. Thus, for a given C, the problem whether C is true in Nat is Σ_1^0 and a contradiction is reached. $\Box$

Since Kröger's axiom system for first-order temporal logic is finitistic, we have the following corollary.

Corollary 2.6. *If the language contains a zero-argument functor* 0, *a unary functor* s *and two-argument functors* + *and* $*$, *then Kröger's proof system for first-order temporal logic given in* [2, 3] *is not complete.*

3. Final remarks

(1) Assume we are given a natural number n. Then the set of formulas that are true in the standard model of natural numbers is neither Σ_n^0 nor Π_n^0. Thus, in fact, we have proved that if the language is sufficiently rich, then there is no complete axiomatization of first-order temporal logic with a set of axioms which is Σ_n^0 or Π_n^0 for some natural number n.

(2) A similar technique to the one used in the proof of Theorem 2.5 can be applied to show that if the language is sufficiently rich, then, for any finitistic axiomatization of the first-order temporal logic, the model existence theorem does not even hold for finite sets of formulas (i.e., there exists a finite set F of formulas such that $F \nvdash \mathbf{false}$, but $F \models \mathbf{false}$).

(3) A claim that first-order temporal logic is not complete appeared independently in [1], however, without any proof. There the authors considered the first-order temporal logic with until-operator U. The logic we have investigated here is essentially less expressive, and so we have obtained a stronger result. A difference between the satisfaction relation assumed in [1, 4] and in our paper (as well as in [2, 3]) should also be noticed. Namely, we define a formula A to be true in some Kripke structure iff A is true in all its states, while in [1, 4] it is sufficient that A be true in

the first state of the structure. Some other incompleteness results on logic with the operator U can be found in [6].

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