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Lower bounds for weak sense of direction

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Abstract

A graph with n vertices and maximum degree Δ cannot be given weak sense of direction using less than Δ colours. It is known that n colours are always sufficient, but it has been conjectured that just $\Delta + 1$ are really needed. On the contrary, we show that for sufficiently large n there are graphs requiring $\Delta + \Omega((n \log\log n)/\log n)$ colours. Moreover, we prove that, in terms of the maximum degree, $\Omega(\Delta\sqrt{\log\log \Delta})$ colours are necessary.

Keywords: Distributed computing; Sense of direction; Random graphs

1. Introduction

Sense of direction and *weak sense of direction* [5] are properties of global consistency of the colouring of a network that can be used to reduce the complexity of many distributed algorithms. Although there are polynomial algorithms for checking whether a given coloured graph has (weak) sense of direction [2], the polynomial bounds are rather high, and, moreover, there are no results (besides the obvious membership to NP) about finding a colouring that is a (weak) sense of direction using the *smallest* number of colours. Even from a theoretical point of view, there is no known way to bound from below the number of colours that are necessary to give (weak) sense of direction to the graph representing the network.

The number of vertices n in a graph is a trivial upper bound for the number of colours, and the maximum degree Δ is a trivial lower bound. If the graph is regular and it is not a Cayley graph, then $\Delta + 1$ is a lower bound [1,7], but this is in fact the only nontrivial lower bound known so far. The problem of determining more stringent lower bounds is made particularly difficult by the time needed to determine the minimum number of colours

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needed by a given graph when the number of vertices grows. It is not easy to find examples of graphs needing actually more than $\Delta + 1$ colours, and indeed the question about whether this lower bound was optimal was raised in [6].

However, extensive tests performed using `optwsod`, a tool[1] for making experimental research with weak sense of direction [3], ended up in a number of counterexamples, showing that some graphs require $\Delta + 2$ colours. For instance, in Fig. 1 we show a (regular) directed graph of degree 2 that requires 4 colours (displayed as patterns of the arcs), and in Fig. 2 we show a (regular) undirected graph of degree 3 requiring 5 colours (note that checking the previous claims by hand would be almost impossible). The colourings displayed are thus optimal. It is interesting to note that the colouring of Fig. 2 is *symmetric*, that is, colours are paired at the endpoints of an edge always in the same way; there is however no better colouring even among the nonsymmetric ones. It is now natural to wonder whether there exists a constant gap g such that every graph of maximum degree Δ can be given (weak) sense of direction using no more than $\Delta + g$ colours.

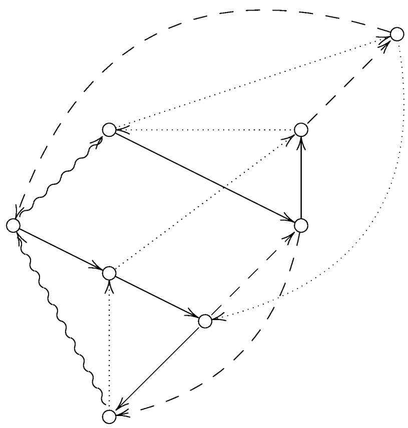

Fig. 1. A 2-regular directed graph requiring 4 colours.

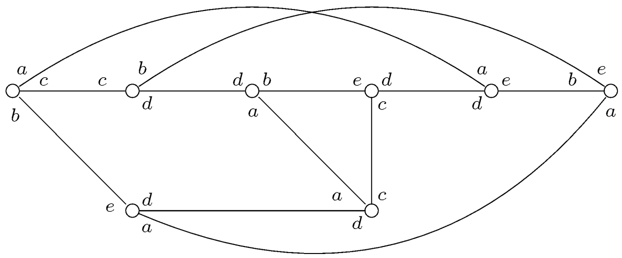

Fig. 2. A 3-regular undirected graph requiring 5 colours.

[1] `optwsod` uses implicit enumeration techniques to explore the entire search space of possible (weak) senses of direction; easy-to-compute upper and lower bounds on the chromatic number are used to restrict the search. The tool, written in C, is free and available under the Gnu Public License.

Looking at the graphs in Figs. 1 and 2, one is tempted to ask what makes them “special”. Extensive experimentation with `optwsod` showed that they are not special at all, and that instead it is rare for a graph to have a (weak) sense of direction with about Δ colours. It is indeed by analyzing properties of typical graphs (using random graph theory) that we shall prove that even if we allow g to be a function of the number of vertices (in fact, of the graph), we can always find examples of graphs needing more than $\Delta + g(n)$ colours, provided that the growth of g is mildly bounded (i.e., that $g(n) = \mathrm{o}(n \log\log n / \log n)$). This result answers in the most definitely negative way to the question above.

2. Definitions

A (*directed*) *graph* G is given by a set $V = [n] = \{0, 1, \dots, n-1\}$ of n vertices and a set $A \subseteq V \times V$ of arcs. We write $P[x, y] \subseteq A^*$ for the set of paths from vertex x to vertex y. A graph is *symmetric* if $\langle y, x\rangle$ is an arc whenever $\langle x, y\rangle$ is.

Note that in this paper we shall always manipulate symmetric loopless directed graphs, which are really nothing but undirected simple graphs (an edge is identified with a pair of opposite arcs). However, the directed symmetric representation allows us to handle more easily the notion of weak sense of direction and the related proofs. In turn, when using results from random graph theory we shall confuse a symmetric loopless directed graph with its undirected simple counterpart.

A *colouring* of a graph G is a function $\lambda : A \to \mathcal{L}$, where $\mathcal{L}$ is a finite set of colours; the map $\lambda^* : A^* \to \mathcal{L}^*$ is defined by $\lambda^*(a_1 a_2 \cdots a_p) = \lambda(a_1)\lambda(a_2)\cdots\lambda(a_p)$. We write $\mathcal{L}_x = \{\lambda(\langle x, y\rangle) \mid \langle x, y\rangle \in A\}$ for the set of colours that x assigns to its outgoing arcs.

Given a graph G coloured by λ, let

$$L = \bigcup_{\langle x,y\rangle \in V^2} \bigl\{\lambda^*(\pi) \mid \pi \in P[x, y]\bigr\};$$

be the set of all strings that colour paths of G.

A *local naming* for G is a family of injective functions $\beta = \{\beta_x : V \to \mathcal{S}\}_{x\in V}$, with $\mathcal{S}$ a finite set, called the *name space*. Intuitively, each vertex x of G gives to each other vertex y a name $\beta_x(y)$ taken from the name space.

Given a coloured graph endowed with a local naming, a function $f : L \to \mathcal{S}$ is a *coding function* iff

$$\forall x, y \in V\ \forall \pi \in P[x, y] \quad f\bigl(\lambda^*(\pi)\bigr) = \beta_x(y).$$

A coding function translates the colouring of a path along which two vertices x, y are connected into the name that x gives to y. A colouring λ is a *weak sense of direction* for a graph G iff for some local naming there is a coding function.[2] We shall also say that a coloured graph *has* weak sense of direction, or that λ *gives* weak sense of direction to G.

[2] In [5] a slightly different definition is given, in which the empty string is not part of L. The examples and the results of this paper are not affected by this difference.

3. Representing graphs using weak sense of direction

The main idea of this paper is that a coding function f *represents compactly a great deal of information about a graph*, and that the values of f on a small set of strings, together with some additional information, can be used to reconstruct the graph. This happens because f tells whether two paths outgoing from the same vertex have the same ending vertex.

Suppose now that we want to exploit this feature to code compactly a (strongly) connected graph G with weak of sense of direction, and to this purpose assume without loss of generality that $\beta_0(x) = x$ for all vertices x, that is, vertex 0 locally gives to all other vertices their real names. To code G, first specify for each vertex the set of colours of outgoing arcs. Then, give the values of f on every string of colours having length at most $D + 1$, where D is the diameter of G.

To rebuild G from the above data, we proceed as follows: first of all we compute the targets of the arcs out of 0 using f on strings of length one, thus obtaining the set of coloured paths of length one going out of 0. Then, since we know the colours of the arcs going out of the targets of such paths, we can build the set of coloured paths of length two out of 0, and compute their targets using f on strings of length two. When, continuing in this way, we compute all coloured paths of length $D + 1$ out of 0 we are done, since every arc of G must appear in some of these paths. An example of this process for a very simple graph is given in Fig. 3.

In the process above, all the information used was the value of f on paths of length $D + 1$ at most. If we concentrate on graphs of diameter 2, then we need approximately $(|\mathcal{L}| + |\mathcal{L}|^2 + |\mathcal{L}|^3)\lceil 2\log n\rceil$ bits to code f (the name space has cardinality at most n^2, so $\lceil 2\log n\rceil$ bits are sufficient to specify a name), and $n|\mathcal{L}|$ bits to code the colours of the outgoing arcs. However, this is still too much for our purposes, so we are going to use a

Label Sets

x	$\mathscr{L}_x$
0	$\{a, b\}$
1	$\{a\}$
2	$\{a, b\}$
3	$\{a\}$
4	$\{a\}$

Coding Function

w	$f(w)$	w	$f(w)$
ϵ	0	aab	4
a	1	aba	4
b	2	abb	0
aa	2	baa	4
ab	3	bab	0
ba	3	bba	0
bb	4	bbb	1
aaa	3		

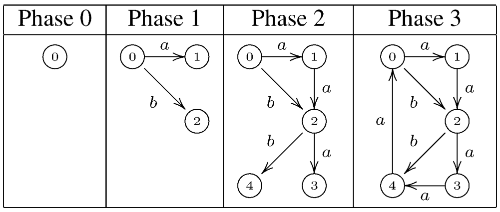

Fig. 3. An example of graph reconstruction.

more sophisticated approach; nonetheless, the basic idea is more easily grasped using the above description.

Let $\mathcal{C}(n,k)$ be the class of all symmetric graphs with n vertices, maximum degree at most k and enjoying the following property, which we shall call *property* A_3:

> For every set X of at most three vertices, there is a vertex $x \notin X$ that is adjacent to all vertices in X.

Intuitively property A_3 is "a bit more than having diameter two" (which would be implied by the existence of a vertex adjacent to every *pair* of vertices). Somehow this is the key point: we want to get down to $|\mathcal{L}|^2\lceil \log n\rceil$ bits to code f, but this would mean to restrict to graphs of diameter one—that is, just complete graphs. On the contrary, we shall see that almost all graphs enjoy property A_3, and that nonetheless graphs satisfying A_3 can be coded compactly.

Lemma 3.1. *Let G be a graph satisfying A_3. Let λ be a sense of direction for G with name space $\mathcal{S}$, coding function f and local naming β. Assume without loss of generality that $\mathcal{S} \supseteq [n]$ and $\beta_0(x) = x$ for all vertices x. Then $\langle x, y\rangle$ is an arc iff there exist a vertex z and colours $a \in \mathcal{L}_0$, $b, c \in \mathcal{L}_z$ and $d \in \mathcal{L}_x$ such that $f(a) = z$, $f(ab) = x$, $f(ac) = y$ and $f(bd) = f(c)$.*

Proof. For the left-to-right implication, suppose that $\langle x, y\rangle$ is an arc, and use property A_3 to find a vertex z adjacent to x, y and 0. Let $a = \lambda(\langle 0, z\rangle)$, $b = \lambda(\langle z, x\rangle)$, $c = \lambda(\langle z, y\rangle)$ and $d = \lambda(\langle x, y\rangle)$ (see Fig. 4). By weak sense of direction, we have the following identities:

$$
\begin{aligned}
f(a) &= \beta_0(z) = z,\\
f(ab) &= \beta_0(x) = x,\\
f(ac) &= \beta_0(y) = y,\\
f(bd) &= \beta_z(y) = f(c),
\end{aligned}
$$

as required.

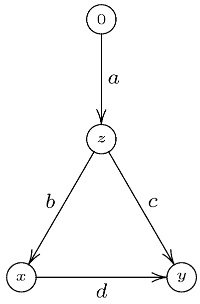

Fig. 4. Property A_3 in action.

For the other implication, let 0_a, z_b, z_c and x_d be vertices of G such that $\lambda(\langle 0, 0_a\rangle) = a$, $\lambda(\langle z, z_b\rangle) = b$, $\lambda(\langle z, z_c\rangle) = c$ and $\lambda(\langle x, x_d\rangle) = d$. Then we have:

$$0_a = \beta_0(0_a) = f(a) = z,$$
$$z_b = \beta_0(z_b) = f\bigl(\lambda\bigl(\langle 0, z\rangle\bigr)\lambda\bigl(\langle z, z_b\rangle\bigr)\bigr) = f(ab) = x,$$
$$z_c = \beta_0(z_c) = f\bigl(\lambda\bigl(\langle 0, z\rangle\bigr)\lambda\bigl(\langle z, z_c\rangle\bigr)\bigr) = f(ac) = y,$$
$$\beta_z(y) = f\bigl(\lambda\bigl(\langle z, y\rangle\bigr)\bigr) = f(c) = f(bd) = f\bigl(\lambda\bigl(\langle z, x\rangle\bigr)\lambda\bigl(\langle x, x_d\rangle\bigr)\bigr) = \beta_z(x_d).$$

Hence, $y = x_d$ and $\langle x, y\rangle$ is an arc. □

Using the above lemma, we can finally prove the promised result about compact coding:

Theorem 3.2. *Let $c = c(G) \in \mathbf{N}$ be such that every graph G can be given weak sense direction using no more than $c(G)$ colours. Then every graph with n vertices satisfying A_3 can be described*[3] *using $\alpha(cn + c^2 \log n)$ bits, for some constant α. Hence, in particular, if c is a function of n only $\mathrm{O}(cn + c^2 \log n)$ bits are sufficient.*

Proof. Let G be a graph with n vertices, satisfying A_3, and having weak sense of direction with colouring λ, name space $\mathcal{S}$, local naming β and coding function f. Assume without loss of generality that $\mathcal{S} \supseteq [n]$ and $\beta_0(x) = x$ for every vertex x. Describe G as follows:

(1) give the number of colours c;
(2) for every vertex x, use c bits to describe the set $\mathcal{L}_x$;
(3) give the values of f on every string of length one or two.

The first data require $\lceil \log c\rceil$ bits, the second one cn and the third one $(c + c^2)\lceil 2 \log n\rceil$ (as we mentioned, $\lceil 2 \log n\rceil$ bits are sufficient to specify a name). From the above description, G can be recovered using Lemma 3.1. □

4. Some random graph theory

We briefly describe two standards models of random graphs, referring the reader to [4] for further information. For the rest of the paper let $N = \binom{n}{2}$.

- For $p \in (0, 1) \subseteq \mathbf{R}$ and $n \in \mathbf{N}$, the model $\mathcal{G}(n, p)$ consists of all the labelled[4] undirected simple graphs with vertex set $[n]$ in which the edges are chosen independently and with probability p.
- For $0 \leqslant M \leqslant N$, the model $\mathcal{G}(n, M)$ consists of all the labelled undirected simple graphs with vertex set $[n]$ and M edges in which all graphs have the same probability.

[3] From now on, we shall sometimes omit the explicit dependence of functions from their argument, when the latter is clear from the context, thus writing c instead of $c(G)$, d instead of $d(n)$ and so on.

[4] The term “labelled” underlines the fact that we do not consider graphs up to isomorphisms.

One says that *almost every graph has property* Q (in a certain model) if the probability that a graph in the model satisfies Q goes to 1 when $n \to \infty$ (of course, in this case p and M might depend on n).

The following two results are trivial consequences of Theorem II.5 and Corollary III.14 of [4]:

Theorem 4.1. *Suppose* $p = p(n) : \mathbf{N} \to [0,1] \subseteq \mathbf{R}$ *is such that for all* $\varepsilon > 0$ *we have* $pn^{\varepsilon} \to \infty$ *and* $(1-p)n^{\varepsilon} \to \infty$ *for* $n \to \infty$. *Then for every fixed* $k \in \mathcal{N}$ *almost every graph in* $\mathcal{G}(n,p)$ *has the following property: for every pair of disjoint sets of at most* k *vertices* X *and* Y *there is a vertex* $z \notin X \cup Y$ *that is adjacent to all vertices in* X *but to no vertex in* Y.

Theorem 4.2. *Suppose* $pn/\log n \to \infty$. *Then almost every graph in* $\mathcal{G}(n,p)$ *has maximum degree* $\mathrm{O}(pn)$.

A property Q of graphs of order n is *convex* if H satisfies Q whenever $F \subseteq H \subseteq G$ and F, G satisfy Q, where we write $\subseteq$ to denote (partial) subgraph inclusion. A powerful meta-theorem of random graph theory (Theorem II.2, ibid.) has the following consequence:

Theorem 4.3. *Let* Q *be a convex property and* $p(1-p)N \to \infty$ *when* $n \to \infty$. *If almost every graph in* $\mathcal{G}(n,p)$ *satisfies* Q *then almost every graph in* $\mathcal{G}(n, \lfloor pN \rfloor)$ *satisfies* Q.

In other words, we can first prove convex properties of almost all graphs in the probabilistic model $\mathcal{G}(n,p)$ and then translate them into properties of the model $\mathcal{G}(n,M)$. Stating that "almost every graph satisfies property Q in $\mathcal{G}(n,M)$" is equivalent to saying that the ratio between the number of graphs having n vertices and M edges satisfying the property and the overall number of graphs with n vertices and M edges goes to 1 as n goes to ∞; hence, we will be able to use Theorem 4.3 to turn probabilistic statements into deterministic ones.

For the rest of the paper, we shall concentrate on the case in which $p \to 0$ as $n \to \infty$, and moreover $p = \Omega(1/\log n)$. It is just a matter of elementary calculus to prove that in this case p satisfies the hypotheses of the previous theorems. Thus, noting that A_3 is implied by the property of Theorem 4.1 for $k = 3$, we have

Proposition 4.4. *With* p *as above, almost every graph in* $\mathcal{G}(n,p)$ *has property* A_3 *and has maximum degree* $\mathrm{O}(pn)$.

Henceforth, by Theorem 4.3 we have that almost all graphs with $\lfloor pN \rfloor$ edges have property A_3 and have maximum degree $\mathrm{O}(pn)$ (these properties are easily shown to be convex).

Corollary 4.5. *With* p *as above, there exist a function* $M = \mathrm{O}(pn)$ *such that*

$$\left|\mathcal{G}\big(n, \lfloor pN \rfloor\big)\right| = \mathrm{O}\big(\left|\mathcal{C}(n,M)\right|\big).$$

5. Counting the number of bits required to describe a graph

Theorem 3.2 tells us that a certain number of bits are sufficient to describe all graphs out of a certain class, and thus implicitly bounds the overall number of graphs in that class. Nevertheless, we shall prove that the cardinality of the class grows too fast, hence obtaining the main theorem by contradiction.

We first prove the following asymptotic identity:

Lemma 5.1. *Let $A = A(n) : \mathbf{N} \to \mathbf{N}$ and $\alpha = \alpha(n) : \mathbf{N} \to [0, 1] \subseteq \mathbf{R}$ be such that $\alpha \to 0$ and $A\alpha \to \infty$ when $n \to \infty$. Then, we have*

$$\log \binom{A}{\lfloor \alpha A \rfloor} \sim A\alpha \log \frac{1}{\alpha}.$$

Proof. We use the factorial symbol (and consequently binomials) in its generalized meaning, that is, $x! = \Gamma(x + 1)$ for all nonnegative real x. By Stirling's approximation (i.e., $\ln n! \sim n \ln n$), and since $\alpha A < A/2$ ultimately, we have

$$\begin{aligned}
\ln \binom{A}{\lfloor \alpha A \rfloor} \leqslant \ln \binom{A}{\alpha A} &= \ln \frac{A!}{\alpha A!(A - \alpha A)!} \\
&\sim A \ln A - \alpha A \ln(\alpha A) - A(1 - \alpha) \ln(A - A\alpha) \\
&= (A - \alpha A - A + \alpha A) \ln A - \alpha A \ln \alpha - A(1 - \alpha) \ln(1 - \alpha) \\
&\sim -A\alpha \ln \alpha = A\alpha \ln \frac{1}{\alpha}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\ln \binom{A}{\lfloor \alpha A \rfloor} &\geqslant \ln \binom{A}{\alpha A - 1} \\
&= \ln \frac{A!}{(\alpha A - 1)!(A - \alpha A + 1)!} \\
&\sim A \ln A - (\alpha A - 1) \ln\left(A\left(\alpha - \frac{1}{A}\right)\right) \\
&\quad - (A - A\alpha + 1) \ln\left(A\left(1 - \alpha + \frac{1}{A}\right)\right) \\
&= -(\alpha A - 1) \ln\left(\alpha - \frac{1}{A}\right) - (A - A\alpha + 1) \ln\left(1 - \alpha + \frac{1}{A}\right),
\end{aligned}$$

so

$$\begin{aligned}
&\ln \binom{A}{\alpha A - 1} \Big/ A\alpha \ln \frac{1}{\alpha} \\
&\sim \left(1 - \frac{1}{\alpha A}\right)\left(1 + \frac{\ln(1 - \frac{1}{\alpha A})}{\ln \alpha}\right) + \left(\frac{1}{\alpha} - 1 + \frac{1}{\alpha A}\right) \frac{\ln(1 - \alpha + \frac{1}{A})}{\ln \alpha} \to 1
\end{aligned}$$

as $n \to \infty$, hence the result follows immediately (after a change of base). $\square$

Since the number of graphs with n vertices and $\lfloor pN \rfloor$ edges is $\binom{N}{\lfloor pN \rfloor}$, we have the following

Corollary 5.2. *Let $p \to 0$ and $pN \to \infty$ as $n \to \infty$. Then, the number of bits required to describe a graph with n vertices and $\lfloor pN \rfloor$ edges is asymptotic to $Np \log \frac{1}{p}$.*

6. Lower bounds on the number of colours

We are finally able to state our main results:

Theorem 6.1. *If $g(n) = \mathrm{o}(n \log\log n / \log n)$, it is impossible to give (weak) sense of direction to all regular graphs using $\Delta_G + g(n)$ colours.*

Proof. We work by contradiction, and consider two cases. If $g = \mathrm{O}(n/\log n)$, let $p = 1/\log n$. Let M be as in Corollary 4.5. Then, by Theorem 3.2 we can describe the graphs in $\mathcal{C}(n, M)$ using

$$\mathrm{O}\big(cn + c^2 \log n\big) = \mathrm{O}\left(\frac{n^2}{\log n}\right)$$

bits, since $c = \mathrm{O}(M + g) = \mathrm{O}(n/\log n)$. On the other hand, by Corollary 5.2,

$$\Theta\left(\frac{n^2 \log\log n}{\log n}\right)$$

bits are necessary, contradicting Corollary 4.5.

Otherwise, let $g(n) = nf(n)/\log n$, with $f(n) = \mathrm{o}(\log\log n)$, $p = g/n$ and again $M = \mathrm{O}(pn) = \mathrm{O}(g)$ as in Corollary 4.5. Then, by Theorem 3.2 we can describe the graphs in $\mathcal{C}(n, M)$ using

$$\mathrm{O}\big(cn + c^2 \log n\big) = \mathrm{O}\left(\frac{n^2}{\log n} f(n)^2\right) = \mathrm{o}\left(\frac{n^2 \log\log n}{\log n} f(n)\right)$$

bits, since $c = \mathrm{O}(M + g) = \mathrm{O}(g)$. On the other hand, by Corollary 5.2,

$$\Theta\left(\frac{n^2}{\log n} f(n)(\log\log n - \log f(n))\right) = \Theta\left(\frac{n^2 \log\log n}{\log n} f(n)\right)$$

bits are necessary, contradicting again Corollary 4.5. □

Theorem 6.2. *It is impossible to give (weak) sense of direction to all regular graphs using*

$$\mathrm{o}\big(\Delta_G \sqrt{\log\log \Delta_G}\,\big)$$

colours. That is, for every function $h(m) = \mathrm{o}(m\sqrt{\log\log m}\,)$ there is a graph G such that $h(\Delta_G)$ colours are not sufficient.

Proof. By contradiction, let $h(m) = \mathrm{o}(m\sqrt{\log\log m})$, $p = 1/\log n$ and M be as in Corollary 4.5. Then, by Theorem 3.2, we can describe the graphs in $\mathcal{C}(n, M)$ using

$$\mathrm{O}\bigl(cn + c^2 \log n\bigr) = \mathrm{o}\left(\frac{n^2 \log\log n}{\log n}\right)$$

bits, since $c = h(M) = \mathrm{o}(M\sqrt{\log\log M})$ and $M = \mathrm{O}(pn) = \mathrm{O}(n/\log n)$. On the other hand, by Corollary 5.2,

$$\Theta\left(\frac{n^2 \log\log n}{\log n}\right)$$

bits are necessary, contradicting Corollary 4.5. □

Of course, the bounds we obtain are *a fortiori* true for nonsymmetric graphs. However, it should be noted that the contradiction is obtained using graphs of very high degree; an extension of these techniques to obtain results for graphs of low degree seems difficult.

Anyway, the previous theorems show that in the worst case the local colouring around a processor requires $\Omega(\log n)$ bits per edge, that is, the same amount of information required to specify the identifier of the adjacent neighbour. Said otherwise, from the point of view of the amount of information used the trivial sense of direction that is obtained by assigning to the arc $\langle x, y\rangle$ the label y is asymptotically *optimal*.

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