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The semiclassical resolvent and the propagator for non-trapping scattering metrics

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Abstract

Consider a compact manifold with boundary M with a scattering metric g or, equivalently, an asymptotically conic manifold (M°, g). (Euclidean $\mathbb{R}^n$, with a compactly supported metric perturbation, is an example of such a space.) Let Δ be the positive Laplacian on (M, g), and V a smooth potential on M which decays to second order at infinity. In this paper we construct the kernel of the operator $(h^2\Delta + V - (\lambda_0 \pm i0)^2)^{-1}$, at a non-trapping energy $\lambda_0 > 0$, uniformly for $h \in (0, h_0)$, $h_0 > 0$ small, within a class of Legendre distributions on manifolds with codimension three corners. Using this we construct the kernel of the propagator, $e^{-it(\Delta/2+V)}$, $t \in (0, t_0)$ as a quadratic Legendre distribution. We also determine the global semiclassical structure of the spectral projector, Poisson operator and scattering matrix.

Keywords: Resolvent; Semiclassical; Scattering matrix; Propagator; Legendrian; Scattering manifold

Contents

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Part 1. Introduction

1. Overview

In this paper we analyze the structure of the semiclassical resolvent on a class of non-compact manifolds with asymptotically conic ends. The class of asymptotically conic, or 'scattering,' manifolds, introduced by Melrose [24], consists of those Riemannian manifolds that can be described as the interior of a manifold M with boundary, such that in terms of some boundary defining function x, we can write the metric g near ∂M as

$$g = \frac{dx^2}{x^4} + \frac{k}{x^2}$$

where k is a smooth 2-cotensor on M with $k\restriction_{\partial M}$ a non-degenerate metric on ∂M; there is no loss of generality in assuming that k has no dx component, so that $k = k(x, y, dy)$ [16]. In terms of $r = 1/x$ this reads

$$g = dr^2 + r^2 k\left(\frac{1}{r}, y, dy\right)$$

and is thus asymptotic to the exact conic metric $dr^2 + r^2 k(0, y, dy)$ as $r \to \infty$. The interior M° of M is thus metrically complete, with the boundary of M 'at infinity'. An important class of examples is that of asymptotically Euclidean spaces, pictured in a radial compactification: here M is the unit ball, and $k\restriction_{S^{n-1}}$ is the standard metric on the sphere. More generally, collar neighborhoods of boundary components are large conic ends of the scattering manifold.

We are concerned here with the operator $H = h^2\Delta + V$, where $\Delta = \Delta_g$ is the Laplacian on M with respect to the metric g, $h \in (0, h_0)$ is a small parameter ('Planck's constant') and $V \in x^2\mathcal{C}^\infty(M)$ is a real potential function, smooth on M and vanishing to second order at the boundary (hence, V is $O(r^{-2})$ and thus short-range). The bulk of this paper is concerned with the analysis of the semiclassical resolvent, i.e. the operator $(h^2\Delta + V - \lambda_0^2)^{-1}$, for λ_0 real, or more precisely the limit of this as λ_0 approaches the real axis from above or below, denoted $(h^2\Delta + V - (\lambda_0 \pm i0)^2)^{-1}$.

For λ_0 non-real, the resolvent $(H - \lambda_0^2)^{-1}$ is a relatively simple object, as $H - \lambda_0^2$ is then an elliptic operator in the 'semiclassical scattering calculus' of pseudodifferential operators, hence a parametrix, and indeed the inverse itself, lies in this calculus [24,37]; also see Section 10. In the limit as $\operatorname{Im}\lambda_0 \to 0$, ellipticity, in the strengthened sense required by the scattering calculus, fails, and the resolvent becomes more complicated. Hassell and Vasy [10,11] analyzed the resolvent in this regime for a fixed $h > 0$. In this paper, we analyze the resolvent $(H - (\lambda_0 \pm i0)^2)^{-1}$ uniformly as $h \to 0$. We assume throughout that the energy level λ_0^2 is *non-trapping*. That is, we assume that every null bicharacteristic of the operator $H - \lambda_0^2$ reaches the boundary ∂M in both directions, or equivalently, every null bicharacteristic eventually leaves each compact set $K \subset M^\circ$. In the case $V \equiv 0$, bicharacteristics are simply geodesics and the condition is that there are no trapped geodesics: every maximally extended geodesic reaches infinity in both directions.

Our main result is the identification of the Schwartz kernel $R_\pm$ of $(H - (\lambda_0 \pm i0)^2)^{-1}$ as a *Legendrian distribution*. We now informally describe Legendrian distributions, and how those arising in the Schwartz kernel of $R_\pm$ are associated to the underlying geometry of the problem. First, a Legendrian distribution on a manifold N with boundary is a smooth function on the interior of N with singular, oscillatory behavior at ∂N. It can locally be written as a sum of oscillatory integrals of the form $\int a(x, y, v)e^{i\phi(y,v)/x}\,dv$ where x is a boundary defining function, y are variables in ∂N, ϕ satisfies a non-degeneracy condition, and the variable v ranges over a compact set in some Euclidean space. Associated to such a distribution (indeed, parametrized by ϕ much as in Hörmander's theory of Lagrangian distributions) is a Legendrian manifold in the *scattering cotangent bundle*, a rescaled version of the cotangent bundle of N, restricted to ∂N; this bundle has a natural contact structure (see Definition 3.2). Legendre distributions, introduced in [23], were generalized to the setting of manifolds with codimension-two corners, with fibered boundaries, by Hassell–Vasy [10]. Here we further generalize to codimension-three boundaries with fibrations; these arise naturally as the Schwartz kernel of $R_\pm$ lies in the manifold with codimension-three corners $M \times M \times [0, h_0)_h$, while the fibrations on the various faces arise from projection operators on this product. Indeed, we need a further refinement: a class of 'Legendrian conic pairs' generalizing that constructed by Melrose–Zworski [23] and Hassell–Vasy [10]; these distributions are associated to pairs of Legendrian manifolds, one of which is allowed to have a conic singularity at its intersection with the other.

The manifold $M \times M \times [0, h_0)$ is too crude a space on which to describe the structure of the resolvent kernel. In particular, the asymptotic behavior of the kernel at the corner $\partial M \times \partial M \times [0, h_0)$ will depend in a complicated way on the angle of approach. We work on the space X which is obtained from $M \times M \times [0, h_0)$ by blowing up[1] $\partial M \times \partial M \times [0, h_0)$. That is, $X = M_b^2 \times [0, h_0)$ where M_b^2, the b-double space, introduced in [22], is the blowup of M^2 at $(\partial M)^2$. The space X has four boundary hypersurfaces: the 'main face' $M_b^2 \times \{0\}$, denoted mf; the left and right boundaries lb and rb, which are $\partial M \times M \times [0, h_0)$ and $M \times \partial M \times [0, h_0)$, respectively, lifted to X; and the boundary hypersurface created by the blowup, which we denote bf, which is a quarter-circle bundle over $(\partial M)^2 \times [0, h_0)$. (See Fig. 1 in Section 2.)

[1] This operation amounts analytically to the introduction of polar coordinates in the transverse coordinates, or geometrically to the introduction of a new boundary hypersurface replacing the corner; see [19] or Section 6 for details.

The resolvent kernel is most naturally described in two pieces, $R_\pm = K_\psi + K'$. The first of these, K_ψ, is a semiclassical scattering pseudodifferential operator, and thus has the same microlocal structure as the (true) resolvent kernel when $\operatorname{Im}\lambda_0 \neq 0$. It captures the diagonal singularity of $R_\pm$ in the interior of X, but not uniformly as the boundary of X is approached.

The other piece, K', is Legendrian in nature; in particular, it is smooth in the interior of X, but is oscillatory as the boundary is approached. There are in fact *three* Legendrian submanifolds associated to K'. The first is the conormal bundle at the diagonal, denoted $N^*\Delta_b$, reflecting the fact that the pseudodifferential part K_ψ cannot capture the singularities at the diagonal near its intersection with the characteristic variety $\Sigma(H - \lambda_0^2)$; this intersection is non-trivial at the boundary of the diagonal, both at mf and at bf. The second is what we call the 'propagating Legendrian' L, which is obtained by flowout from the intersection of the characteristic variety and $N^*\Delta_b$ under the Hamilton vector field associated to $H - \lambda_0^2$. In fact, L is divided into two halves, $L = L_+ \cup L_-$ by $N^*\Delta_b$, and the incoming $(-)$/outgoing $(+)$ resolvent $R_\pm$ is singular only at $L_\pm$. The geometry of $(N^*\Delta_b, L_\pm)$ is that of a pair of cleanly intersecting Legendre submanifolds, and $K'_\pm$ microlocally lies in a calculus of 'intersecting Legendre distribution' associated to this pair, analogous to the class of intersecting Lagrangian distributions of Melrose and Uhlmann [20]. The propagating Legendrian L turns out to have conic singularities at bf, and another Legendrian, $L_2^\sharp$, appears, to 'carry off' the singularities at the conic intersection. This latter Legendrian consists of those points in phase space over bf which point in pure outgoing/incoming directions in both variables. We thus state our first main theorem as follows (relevant classes of Legendrian distributions are defined below in Sections 4–8, while the particular Legendrian manifolds referred to are discussed in Section 11; also here and below h lies in some interval $(0, h_0]$ with h_0 sufficiently small).

Theorem 1.1. *The semiclassical outgoing resolvent kernel R_+, multiplied by the density factor $|dh|^{1/2}$, is the sum of a semiclassical pseudodifferential operator of order $(-2, 0, 0)$, an intersecting Legendre distribution associated to the diagonal Legendrian $N^*\Delta_b$ and the propagating Legendrian L_+, and a conic Legendrian pair associated to L_+ and the outgoing Legendrian $L_2^\sharp$. The orders of the Legendrians are $3/4$ at $N^*\Delta_b$, $1/4$ at L_+, $(2n-3)/4$ at $L_2^\sharp$, $-1/4$ at* bf, *and $(2n-1)/4$ at* lb *and* rb.

This is rather similar in nature to the main result of [10]. The main difference is that in [10] only propagation inside bf needed to be considered; this is closely related to geodesic flow 'at infinity,' and only involves exact conic geometry. Here, by contrast (and in the case $V \equiv 0$) the geodesic flow over the entire manifold M is relevant.

From Theorem 1.1 we can obtain analogous results for other fundamental operators in scattering theory, including the spectral projections, Poisson operator and scattering matrix, since their kernels can be obtained from the resolvent in a straightforward way. The simplest one is the spectral measure $dE(h^{-2})$ which is $1/2\pi i$ times the difference between the incoming and outgoing resolvents. In taking this difference the diagonal singularity disappears and we obtain, in the notation of Section 6.5,

Corollary 1.2. *The spectral measure $dE(h^{-2})$ times $|dh/h^2|^{-1/2}$ is for $h \in [0, h_0)$ an intersecting Legendre distribution associated to the conic pair $(L, L_2^\sharp)$:*

$$dE\big(h^{-2}\big) \otimes \big|dh/h^2\big|^{-1/2} \in I^{1/4, n/2-3/4; n/2-1/4, -1/4}\big(\big(L, L_2^\sharp\big), X; {}^{s\Phi}\Omega^{\frac{1}{2}}\big).$$

This generalizes results (in the globally non-trapping setting) of Vainberg [34] and Alexandrova [3].

The Poisson operator $P(h^{-1})$, as defined in [23], takes a function f on the boundary ∂M and maps it to that generalized eigenfunction with eigenvalue h^{-2} with outgoing data f. It can be obtained from the resolvent kernel by restriction to rb after the removal of an oscillatory factor. The Legendrians L and $L_2^\sharp$ themselves have 'boundary values' at rb, which are denoted SR and $G^\sharp$ respectively. Here SR stands for 'sojourn relation'; it is the twisted graph of a contact transformation identified in [13], and is related to the sojourn time of Guillemin [8] (see Section 2.3 for further discussion).

Corollary 1.3. *The Poisson kernel $P(h^{-1})$, times $|dh/h^2|^{1/2}$, is a Legendre distribution associated to the conic pair* $(\mathrm{SR}, G^\sharp)$ *of order* $(0, (n-1)/2; 0)$.

The scattering matrix $S(h^{-1})$ is, in turn, obtained by restricting the kernel of $P(h^{-1})$ to the boundary, now at $\mathrm{rb} \cap \mathrm{bf}$, although the limit here is more subtle, compared to that for the Poisson operator, as it only makes sense distributionally—this was explained in [23]. The sojourn relation SR has a 'boundary value' at bf which we denote T and call the 'total sojourn relation.' We obtain a global characterization of the S-matrix as an oscillatory function. It has two kinds of behavior: for fixed $h > 0$ it was shown by Melrose–Zworski to be a Fourier integral operator on ∂M, i.e. a Lagrangian distribution on $\partial M \times \partial M$. On the other hand, *away from these singularities,* it has been shown by Alexandrova to be a semiclassical FIO [1,2]. Our structure theorem is that the semiclassical scattering matrix globally lies in a calculus of 'Legendrian–Lagrangian' distributions (defined in Section 8) that combine these two different behaviors.

Theorem 1.4. *The scattering matrix $S(h^{-1})$, times $|dh/h^2|^{1/2}$, is a Legendrian–Lagrangian distribution of order* $(-1/4, -1/4)$ *associated to the total sojourn relation T.*

In a prior paper, the authors constructed a partial parametrix for the Schrödinger propagator on non-trapping scattering manifolds; this parametrix was valid in regions where one variable may range out to ∂M (i.e. out to 'infinity') but the other is restricted to lie in a compact set in M°. Here, by integrating over the spectrum and using Corollary 1.2, we are able to extend our description of the Schrödinger propagator to a global one. To state the theorem, we note that, based e.g. on the form of the free propagator $(2\pi i t)^{-n/2} e^{i|z-z'|^2/2t}$ on $\mathbb{R}^n$, we expect the propagator to be Legendrian, with semiclassical parameter t, but with *quadratic* oscillations at spatial infinity. We define such a class of quadratic scattering-fibered Legendre distributions. Corresponding to the Legendre submanifolds L, $L_2^\sharp$ introduced earlier are quadratic Legendre submanifolds $Q(L)$, $Q(L_2^\sharp)$ (see Section 8). Instead of the semiclassical non-trapping hypothesis at a fixed energy level, we now need to assume that *geodesics* (that is to say, bicharacteristics with $V = 0$) are non-trapped: V is no longer relevant. Our result is

Theorem 1.5. *The Schrödinger propagator $e^{-it(\Delta/2+V)}$ is for $0 < t < t_0 < \infty$ a quadratic Legendre distribution associated to the conic pair $(\tilde{L}, \tilde{G}_2^\sharp)$:*

$$e^{-it((1/2)\Delta+V)} \in I^{3/4, n/2+1/4; 1/4, -n/2+1/4}\big(M_{\mathrm{b}}^2 \times [0, t_0), \big(Q(L), Q\big(L_2^\sharp\big)\big); {}^{\mathrm{qs}\Phi}\Omega^{\frac{1}{2}}\big). \quad (1.1)$$

The resolvent construction described here is a direct generalization of work of Hassell–Vasy [10,11] on the fixed-energy resolvent. This work was in turn motivated by the paper [23] of Melrose–Zworski on the Poisson operator and scattering matrix for scattering metrics. All these works are ultimately based on the original paper of Melrose [24]. The construction is also related to the parametrix construction of Isozaki–Kitada [15], which is valid in the outgoing region.

Our results on the scattering matrix have many antecedents. The description of the behavior of the scattering matrix in the semiclassical regime, away from singularities of the kernel (which occur at the diagonal in $\mathbb{R}^n$ with the usual normalizations) originates with Majda [18] for the case of obstacles and for compactly-supported metric perturbations of $\mathbb{R}^n$ by Guillemin [8]. The semiclassical limit on $\mathbb{R}^n$ with potential has been studied by Protas [26], Vainberg [33], Yajima [38], Robert–Tamura [27], and Alexandrova [1], in varying degrees of generality. (See [1] for a clear summary of this literature.)

Numerous authors have studied the structure of the Schrödinger kernel on flat space (with a potential). In this setting parametrices have been constructed by Fujiwara [7], Zelditch [40], Trèves [32] and Yajima [39]. For a compactly-supported non-trapping perturbation, Kapitanski–Safarov [17] have constructed a parametrix modulo $\mathcal{C}^\infty(\mathbb{R}^n)$, but without control over asymptotics at infinity. More recently, Tataru [31] has completed a construction of a frequency-localized outgoing parametrix, valid for C^2 time-dependent coefficients that are only rather weakly asymptotically flat; this construction, while not giving a global description of the Schwartz kernel, suffices for obtaining global-in-time Strichartz estimates.

The paper is divided into four parts. In the following section we give some heuristic motivation for our geometric approach, particularly for the choice of the space X and the 'scattering fibered structure' on it. The fundamental mathematical objects involved in this structure, namely the Lie algebra of vector fields, scattering-fibered tangent and cotangent bundles, and contact structures at the boundary, are introduced more formally in Section 3.

In Part 2, we give the definitions of Legendre distributions of various sorts. Unfortunately, although this follows a well-worn path (via [10,11,14,23,24]), there is little we can use directly from previous literature, since we need to generalize to manifolds with codimension three corners, so this part is rather long and technical. Each section follows a similar template: we define the relevant Legendre submanifolds, explain how to parametrize them, show that parametrizations always exist and the equivalence of parametrizations, define Legendre distributions and give a symbol calculus. The reader should perhaps skip this part on a first reading and return to it as needed.

In Part 3, we construct the semiclassical resolvent, thereby proving Theorem 1.1, using the machinery from part 2, following [11] rather closely.

In Part 4, we prove Corollary 1.2, Theorem 1.5, Corollary 1.3 and Theorem 1.4.

2. Geometric motivation

Before getting into details we make some additional motivational remarks about the geometric ingredients of this paper.

2.1. The space X

In the Overview we introduced the space X, which is the blowup of $M \times M \times [0, h_0)$ at the corner $\partial M \times \partial M \times [0, h_0)$, or in other words, $X = M_b^2 \times [0, h_0)$. The space M_b^2 has boundary hypersurfaces $\mathrm{lb} = \partial M \times M$ (the left boundary), $\mathrm{rb} = M \times \partial M$ (the right boundary) and the

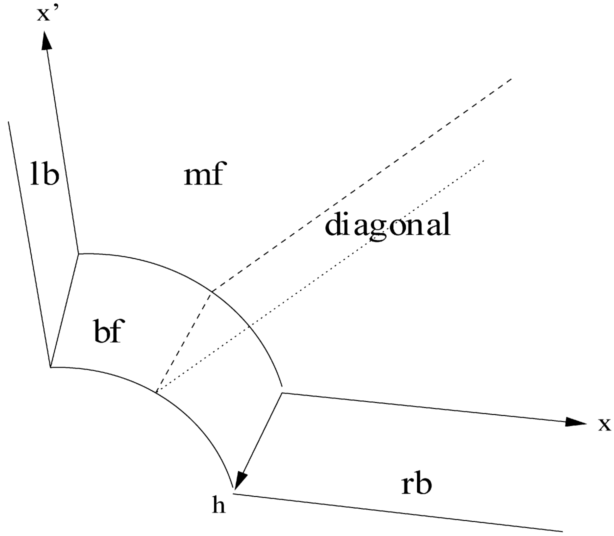

Fig. 1. The space X; in this figure dimensions in the direction of ∂M, in either factor, are not shown.

blowup face bf (the 'b-face'), which is a quarter-circle bundle over $(\partial M)^2$. The boundary hypersurfaces of X are then $\mathrm{bf} \times [0, h_0)$, $\mathrm{lb} \times [0, h_0)$, $\mathrm{rb} \times [0, h_0)$ and $M_{\mathrm{b}}^2 \times \{0\}$. We shall denote these hypersurfaces (by abuse of notation) bf, lb, rb and mf. The diagonal in X is the submanifold $\Delta_b \times [0, h_0)$, where $\Delta_b \subset M_{\mathrm{b}}^2$ is the lift of the diagonal in M^2 to M_{b}^2. In a further abuse of notation we shall denote $\Delta_b \times \{0\} \subset X$ simply by Δ_b. See Fig. 1.

A total boundary defining function for a manifold with corners is, by definition, a product of defining functions for each boundary hypersurface. The total boundary defining function for X can be taken to be $\mathbf{x} = h\rho$, where ρ, a total boundary defining function for M_{b}^2, is given by $\rho^{-2} = x^{-2} + (x')^{-2}$. Here x is a boundary defining function for M, lifted to M^2 by the left projection and then to M_{b}^2 by the blowdown map, while x' is the same boundary defining function on M lifted via the right projection.

In this subsection we give some motivation for the choice of X as the space on which to analyze the kernel of the resolvent $(H - (\lambda_0^2 \pm i0))^{-1}$. We first point out that it allows us to decouple the diagonal singularities from the long-range behavior far from the diagonal, i.e. the lack of decay in the kernel at spatial infinity (at bf, lb and rb) and as $h \to 0$. Indeed, on X the diagonal is separated from lb and rb, while it meets bf transversally. This allows us to solve for the resolvent kernel by first determining the conormal singularity at the diagonal using standard pseudodifferential techniques, and then solving away the remaining error as a separate step.

Consider the free semiclassical resolvent kernel on $\mathbb{R}^n$, say for $n = 3$,

$$\frac{1}{4\pi h^2} \frac{e^{i\lambda_0|z-z'|/h}}{|z-z'|}.$$

Let us ignore the diagonal singularity in the remainder of this section, in view of the remarks above, for example by multiplying by a function on X that vanishes in a neighborhood of Δ_b. Considered as a function on X, the resulting kernel is the product of a function *conormal* at the boundary of X, times an explicit oscillatory factor $e^{i\lambda_0|z-z'|/h}$. Here, 'conormal at the boundary' means that the function has stable regularity, i.e. remains in a some fixed weighted Sobolev space, under repeated application of vector fields tangent to each boundary face. This would not be true if the corner were not blown up, i.e. $|z - z'|^{-1}$ (or even a smooth non-vanishing function of $z - z'$) is not conormal at the boundary on the space $M^2 \times [0, h_0)$. In this sense the singularities of the resolvent kernel at the boundary (and away from the diagonal) are 'resolved' when lifted to the blowup space X.

More crucially, the blowup is needed so that we can analyze the resolvent kernel as a Legendre distribution at spatial infinity. A Legendre distribution of the simplest sort is given by an oscillatory function

$$e^{i\Phi/\mathbf{x}}a$$

where $\mathbf{x}$ is the total boundary defining function for X as above and a is conormal on X. The phase function Φ should be smooth (and have certain properties with respect to fibrations at the boundary—see the following subsection, and Section 4.2). The function $\Phi = \Phi(z, z')$ is given, loosely speaking (and for $V \equiv 0$), by the geodesic distance between z and z', at least in the region where this is smooth; thus we want a compactification of $(M^\circ)^2$ where $\mathbf{x}d(z, z')$ is a smooth function up to the boundary (at least in this region, and away from the diagonal). The b-double space M_b^2 has this property [9], and the blowup is essential here.

2.2. *Scattering-fibered structure*

The space X comes equipped with fibrations on its boundary hypersurfaces, and a corresponding Lie algebra of vector fields, which dictate the type of Legendre distributions we expect to find comprising the semiclassical resolvent. This is dealt with in detail in Section 3, but we give an informal motivation here. We begin by noting the vector fields out of which our operator is built. Near the boundary of M, the vector fields of unit length with respect to our metric g are $\mathcal{C}^\infty(M)$-linear combinations of the vector fields

$$x^2\partial_x \quad \text{and} \quad x\partial_{y_i}.$$

(Note that in polar coordinates on Euclidean space, $\partial_r = -x^2\partial_x$ and $\partial_\omega/r = x\partial_\omega$ are of approximately unit length as $r \to \infty$.) These vector fields generate the scattering Lie algebra of vector fields introduced by Melrose [24]. In the semiclassical setting, we multiply each derivative by h, so we can think of our operator $H = h^2\Delta$ (acting in either the left or the right set of variables) as being 'built' out of the vector fields

$$hx^2\partial_x, \quad hx\partial_{y_i}, \quad h(x')^2\partial_{x'}, \quad hx'\partial_{y_i'}$$

where the left set of variables is indicated without, and the right set with, a prime.

Motivated by the program proposed by Melrose [21], we should add one more vector field to this set in order to obtain $N = \dim X$ vector fields, so that it can generate a vector bundle which can be taken to replace the tangent bundle of X. It is not obvious what this extra vector field should be, but in hindsight we can observe that the vector field

$$h(x\partial_x + x'\partial_{x'} - h\partial_h)$$

fits the bill. In fact, on Euclidean space, the self-adjoint operator corresponding to this is

$$-ih(r\partial_r + r'\partial_{r'} + h\partial_h + n)$$

and this *annihilates* the semiclassical resolvent kernel on $\mathbb{R}^n$ (this follows immediately from the fact that it is h^{-n} times a function of $(z - z')/h$).

We now have a set of vector fields generating a Lie algebra, and *we can expect that the semiclassical resolvent has fixed regularity under the repeated application of these vector fields* (away from the diagonal); this means that it remains in some *fixed* weighted Sobolev space throughout. At $\mathrm{mf} \subset X$, i.e. at the interior of the $h = 0$ face, we obtain from these vector fields all the scattering vector fields, i.e. those of the form

$$h^2\partial_h, \quad h\partial_{z_i}, \quad h\partial_{z'_j}$$

where we use $z = (z_1, \dots, z_n)$ as a local coordinate on M°, here and throughout this paper. Thus the resolvent can be expected to be Legendre at the interior of mf (equivalently, a semiclassical Lagrangian distribution). At the other boundary hypersurfaces, the situation is a little different. Our vector fields do not vanish at bf, lb or rb; rather they are tangent to the leaves of a fibration on each of these boundaries. At bf, all the vector fields vanish except the last one introduced above, which restricts to $h^2\partial_h$ at bf. At rb, the vector fields $h(x')^2\partial_{x'}$ and $hx'\partial_{y'_j}$ vanish, but the others restrict to $h\partial_{z_i}$ and $h(x\partial_x - h\partial_h)$, which do not. These statements can be rephrased by saying that on bf, the vector fields are constrained to be tangent to the leaves of the fibration that projects off the h factor, while at rb the vector fields are constrained to be tangent to the leaves of the fibration $\mathrm{rb} = M \times \partial M \times [0, h_0) \to \partial M$ which projects to the second factor. We finally end up with a characterization of our vector fields in terms of these boundary fibrations and the total boundary defining function $\mathbf{x}$ (see Definition 3.3, and also Example 3.8). *Our ansatz in this paper—justified by Theorem* 1.1*—is that the semiclassical resolvent is Legendrian with respect to this Lie algebra structure*, which we call the scattering-fibered structure, on X.

2.3. Sojourn relations

The sojourn time was introduced by Guillemin [8], motivated by a result of Majda [18], in connection with metric or obstacle scattering on $\mathbb{R}^n$. Let γ be a geodesic with asymptotic incoming direction y and asymptotic outgoing direction y' ($y, y' \in S^{n-1}$), and suppose that it is non-degenerate, meaning that locally it is the only such geodesic (in a quantitative sense, so that the corresponding Jacobian is non-zero). Guillemin defined the sojourn time $T(y, y')$ to be the limit $l(R) - 2R$ where $l(R)$ is the length of the part of the geodesic lying inside $B(R, 0)$. He then showed that the scattering matrix locally took the form

$$S(\lambda, y, y') = \sigma(y, y')^{-1/2}\lambda^{(n-1)/2}e^{i\lambda T(y,y')} + O\bigl(\lambda^{(n-3)/2}\bigr) \tag{2.1}$$

(or a sum of such terms if there are finitely many such geodesics) where σ is a Jacobian factor. This has been generalized by Alexandrova, who removed the non-degeneracy assumption and proved that the scattering matrix is a semiclassical Fourier integral operator away from the diagonal. The Lagrangian to which the scattering matrix is associated, which Alexandrova calls the scattering relation, is parametrized by Guillemin's sojourn time whenever it is projectable, i.e.

whenever (y, y') locally form coordinates on it. Thus, the sojourn time is better thought of as a *Lagrangian submanifold* rather than as a function.

In our Corollary 1.3 and Theorem 1.4 we see the sojourn time show up naturally, in two different guises. For simplicity we explain this in the case of zero potential. First we consider a geodesic emanating from a point $(z, \hat{\zeta})$ in the cosphere bundle of M°. By the non-trapping assumption, this geodesic $\gamma(s)$ tends to infinity as $s \to \infty$, and it does so in such away that the limit

$$\nu = \lim_{s\to\infty} s - r\big(\gamma(s)\big)$$

exists, where s is arc-length along the geodesic and $r = 1/x$ is the radial coordinate. In [13] we showed that there is a contact transformation, which we called the sojourn relation, taking the point $(z, \hat{\zeta}) \in S^*M^\circ$ to (y_0', ν, μ), where y_0' is the asymptotic direction of γ as $s \to \infty$ and μ is the limiting value of $s^{-2}\,dy'/ds$ as $s \to \infty$. The image point (y_0', ν, μ) can be taken to lie in the boundary of the scattering cotangent bundle[2] (see Definition 3.2). We show in Corollary 1.3 that the Poisson operator is a Legendre distribution associated to a Legendre submanifold SR which is the twisted graph of the sojourn relation. Just as the Poisson operator is a boundary value of the resolvent kernel (divided by $e^{ir'/h}$), so the sojourn relation appears as the ‘boundary value’ of the Legendrian L associated to the resolvent (see Section 15). The function ν appears as the boundary value of $\psi - r'$ where ψ is the function parametrizing L, with the renormalizing term r' coming directly from the removal of the oscillatory factor $e^{ir'/h}$. Moreover, whenever (z, y') locally form coordinates on SR, the function $\nu(z, y')$ locally parametrizes SR.

When the point z itself tends to infinity, say $z = \gamma(s')$ along a fixed geodesic γ, with $s' \to -\infty$, the coordinate ν itself diverges as $1/r$ and we can take a limit

$$\tau = \lim_{s'\to\infty} \nu - s' = \lim_{s,s'\to\infty} s - s' - r\big(\gamma(s)\big) - r\big(\gamma(s')\big)$$

which is precisely Guillemin’s sojourn time. We obtain the kernel of the scattering matrix as a boundary value of the Poisson operator, divided by $e^{ir/h}$, and in doing so, we find the total sojourn relation T appearing as the ‘boundary value’ of the sojourn relation, with the sojourn time τ as the (renormalized) limit of ν. Whenever (y, y') locally form coordinates on T (the non-degeneracy condition of Guillemin) then $\tau(y, y')$ locally parametrizes T, and we recover the description (2.1) of the scattering matrix.

Our Theorem 1.4 improves upon results already in the literature in two ways. First, we treat (non-trapping) asymptotically *conic*, rather than flat, metrics, and second it is completely global. In particular, we do not need to localize away from the geodesics which are uniformly close to infinity (corresponding to the localization away from the diagonal in Alexandrova’s result). Indeed it is this limiting regime which provides the transition between the Legendre (or semiclassical Lagrangian) behavior of the scattering matrix in the limit $h \to 0$ and the Lagrangian behavior of the scattering matrix for fixed h as proved by Melrose–Zworski [23], since the latter is related to the geodesics ‘at infinity.’ Our class of Legendrian–Lagrangian distributions unifies these two regimes into a single microlocal object.

[2] To be completely invariant it should be thought of as lying in an affine bundle identified in [13].

3. Scattering-fibered structure

In this section we shall define the scattering-fibered structure on manifolds with corners. Although we only need the case of manifolds with corners of codimension at most three, this structure can be defined on manifold with corners of arbitrary codimension, and there is some conceptual gain in considering the general case. So we shall give the basic definitions for corners of arbitrary codimension, but rapidly specialize to the case of codimension three corners for most of the exposition. The basic definitions are based on unpublished work [12] by the first-named author and András Vasy, and we thank him for permission to use this material. Note that the case of corners of codimension two has been explicitly worked out in [10]. To begin, we review the scattering structure on a manifold with boundary.

3.1. Scattering structures

Let X be an n-dimensional manifold with boundary, and let x denote a boundary defining function on X. Denote by $\mathcal{V}_{\mathrm{b}}(X)$ the Lie algebra of vector fields on X tangent to ∂X.

Definition 3.1. The Lie algebra of scattering vector fields $\mathcal{V}_{\mathrm{sc}}$ is defined by

$$\mathcal{V}_{\mathrm{sc}}(X) = x\mathcal{V}_{\mathrm{b}}(X), \quad \text{i.e. } V \in \mathcal{V}_{\mathrm{sc}}(X) \quad \text{iff } V = xW \quad \text{for some } W \in \mathcal{V}_{\mathrm{b}}(X). \tag{3.1}$$

It is easy to verify that if y are coordinates in ∂X, extended to a collar neighborhood of the boundary, we may write a scattering vector field locally near the boundary as a $\mathcal{C}^\infty(X)$-linear combination of $x^2\partial_x$ and $x\partial_{y_i}$, whilst away from the boundary a scattering vector field is simply a smooth vector field. It follows that $\mathcal{V}_{\mathrm{sc}}(X)$ is the space of sections of a vector bundle over X.

Definition 3.2. We define ${}^{\mathrm{sc}}T(X)$, the *scattering tangent bundle over* X, to be the vector bundle of which $\mathcal{V}_{\mathrm{sc}}(X)$ is the space of sections; explicitly, the fiber ${}^{\mathrm{sc}}T_p(X)$ at $p \in X$ is given by $\mathcal{V}_{\mathrm{sc}}(X)/I_p \cdot \mathcal{V}_{\mathrm{sc}}(X)$, where $I_p(\mathcal{V}_{\mathrm{sc}}(X))$ is the set of vector fields of the form fV, where $f \in \mathcal{C}^\infty(X)$ vanishes at p and $V \in \mathcal{V}_{\mathrm{sc}}(X)$. We define ${}^{\mathrm{sc}}T^*(X)$, the *scattering cotangent bundle over* X, to be the dual vector bundle to ${}^{\mathrm{sc}}T(X)$.

Locally near the boundary, the scattering cotangent space is spanned by the sections $d(1/x) = -dx/x^2$ and dy_i/x. Thus any point in ${}^{\mathrm{sc}}T^*X$ can be written

$$\nu\, d\left(\frac{1}{x}\right) + \sum_i \mu_i \frac{dy_i}{x}$$

and this defines linear coordinates (ν, μ_i) on each fiber of ${}^{\mathrm{sc}}T^*X$. In these coordinates, the natural symplectic form on ${}^{\mathrm{sc}}T^*X$ takes the form

$$\omega = d\nu\, d\left(\frac{1}{x}\right) + \sum_i d\left(\frac{\mu_i}{x}\right) dy_i.$$

There is a natural structure on ${}^{\mathrm{sc}}T^*_{\partial X}X$ defined by contracting the symplectic form with $x^2\partial x$ and restricting to the boundary, taking the form

$$d\nu - \sum_i \mu_i \, dy_i$$

in these coordinates.

Further details about scattering structures, and in particular of the "scattering algebra" of pseudodifferential operators that microlocalize the scattering vector fields, can be found in [24].

3.2. Scattering-fibered structures on manifolds with corners

Let X be a compact manifold with corners of codimension d.

Definition 3.3. A scattering-fibered structure on X consists of

(a) an ordering of the boundary hypersurfaces $\{H_1, H_2, \ldots, H_d\}$ of M, where we allow H_i to be disconnected, i.e. to be a union of a disjoint collection of connected boundary hypersurfaces,
(b) fibrations $\phi_{H_i} : H_i \to Z_i$, $1 \leqslant i \leqslant d$, to a compact manifold Z_i with corners of codimension $i-1$, and
(c) a total boundary defining function $\mathbf{x}$ (that is, a product of boundary defining functions $\prod_i \rho_i$ where ρ_i is a boundary defining function for H_i) which is distinguished up to multiplication by positive $\mathcal{C}^\infty$ functions which are constant on the fibers of ∂X.

The fibrations ϕ_i are assumed to satisfy the following conditions:

(i) if $i < j$, then $H_i \cap H_j$ is transverse to the fibers of ϕ_i, and thus ϕ_i is a fibration from $H_i \cap H_j$ to Z_i, and
(ii) $H_i \cap H_j$ is a union of fibers of ϕ_j and thus ϕ_j is a fibration from $H_i \cap H_j$ to $\partial_i Z_j \equiv \phi_j(H_i \cap H_j)$, where $\partial_i Z_j$ is a boundary hypersurface of Z_j. In addition,
(iii) there is a fibration $\phi_{ij} : \partial_i Z_j \to Z_i$ such that when restricted to $H_i \cap H_j$, $\phi_i = \phi_{ij} \circ \phi_j$; in other words, there is a commutative diagram

$$\begin{array}{ccccc} H_i & \xleftarrow{\ inc\ } & H_i \cap H_j & \xrightarrow{\ inc\ } & H_j \\ \downarrow{\scriptstyle \phi_i} & \swarrow{\scriptstyle \phi_i} & \downarrow{\scriptstyle \phi_j} & & \downarrow{\scriptstyle \phi_j} \\ Z_i & \xleftarrow[\ \phi_{ij}\ ]{} & \partial_i Z_j & \xrightarrow[\ inc\ ]{} & Z_j. \end{array} \tag{3.2}$$

In this paper, we shall always assume the following additional condition:

(iv) The manifold Z_d coincides with H_d and the fibration ϕ_d is the identity map.

The hypersurface H_d will often be denoted mf (for 'main face').

We have a local model for this structure. Let $p \in M$ be a point on the codimension d corner of M.

Proposition 3.4. *Near p there are local coordinates $x_1, \dots, x_d, y_1, \dots, y_d$ where $x_i \geqslant 0$ is a boundary defining function for H_i and y_i lies in a neighborhood of zero in $\mathbb{R}^{d_i}$, such that $p = (0, \dots, 0)$, and there are coordinates $(x_1, \dots, x_{i-1}, y_1, \dots, y_i)$ on Z_i near $\phi_i(p)$ such that, locally, each ϕ_i takes the form*

$$(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d, y_1, \dots, y_d) \mapsto (x_1, \dots, x_{i-1}, y_1, \dots, y_i). \tag{3.3}$$

Moreover, the coordinates can be chosen so that x_i is constant on the fibers of H_j for $j > i$, and $\prod_i x_i = \mathbf{x}$.

Proof. We begin by choosing coordinates on the Z_i, in a neighborhood of $\phi_i(p)$. We start with coordinates y_1 for Z_1, where y_1 lies in a neighborhood of 0 in $\mathbb{R}^{k_1}$ and $y_1(\phi_i(p)) = 0$. Using the implicit function theorem, we may choose coordinates (y_1, y_2) on $\partial_1 Z_2$ so that the projection from $\partial_1 Z_2$ to Z_1 takes the form $(y_1, y_2) \to y_1$. We choose an arbitrary boundary defining function x_1 for Z_2 and extend the coordinates (y_1, y_2) to a neighborhood of $\partial_1 Z_2$, and in this way have coordinates x_1, y_1, y_2 on a neighborhood of $\phi_2(p)$ in Z_2. Inductively, given coordinates $(x_1, \dots, x_{j-1}, y_1, \dots, y_j)$ near $\phi_j(p)$ in Z_j, we choose coordinates on $\partial_j Z_{j+1}$ of the form $(x_1, \dots, x_{j-1}, y_1, \dots, y_j, y_{j+1})$ so that the projection from $\partial_j Z_{j+1}$ to Z_j is the coordinate projection off y_{j+1}. We then choose an arbitrary boundary defining function for $\partial_j Z_{j+1}$ and extend the coordinates from the boundary into the interior, and in this way have coordinates on a neighborhood of $\phi_{j+1}(p)$ in Z_{j+1}.

We can lift the coordinates from Z_i to H_i by the fibration ϕ_i in a neighborhood of p. Thus x_j and y_j are defined on the union of $H_j, \dots, H_d$. These functions agree on intersections $H_j \cap H_k$ due to the way they are defined on Z_i and due to the commutativity of the diagram (3.2). Hence they extend to smooth functions on a neighborhood of p. Finally we define $x_d = \mathbf{x}/(x_1 \dots x_{d-1})$ and all conditions are satisfied. □

Thus, in the codimension three case, there are local coordinates near the corner in which the fibrations take the form

$$\begin{aligned} \phi_1 &: (x_2, x_3, y_1, y_2, y_3) \mapsto y_1, \\ \phi_2 &: (x_1, x_3, y_1, y_2, y_3) \mapsto (x_1, y_1, y_2), \\ \mathrm{id} = \phi_3 &: (x_1, x_2, y_1, y_2, y_3) \mapsto (x_1, x_2, y_1, y_2, y_3). \end{aligned} \tag{3.4}$$

We proceed to give the main example of the scattering-fibered structure for the purposes of this paper.

Example 3.5. Let Y be a scattering-fibered manifold with codimension 2 corners. Thus Y has two boundary hypersurfaces K_1 and K_2 with boundary defining functions x_1, x_2 together with fibrations $\psi_i : K_i \to Z_i$; moreover, $Z_2 = K_2$ and ψ_2 is the identity, while Z_1 is a manifold without boundary and the fibers of ψ_1 are transverse to the boundary.

Then the space $X = Y \times [0, \epsilon)_{x_3}$ is, in a natural way, a scattering-fibered manifold with codimension 3 corners. The boundary hypersurfaces are now $H_1 = K_1 \times [0, \epsilon)$, $H_2 = K_2 \times [0, \epsilon)$ and $H_3 = \mathrm{mf} = Y \times \{0\}$. The structure is specified as follows: a distinguished boundary defining function is given by ρx_3 where ρ is a distinguished boundary defining function for Y; the bases of the fibrations are given by Z_1 and Z_2 and $Z_3 = Y$; and the fibrations are given by

$$\phi_1 : K_1 \times [0, \epsilon) \to Z_1 = \psi_1 \circ \Pi_1,$$
$$\phi_2 : K_2 \times [0, \epsilon) \to Z_2 = \Pi_2,$$
$$\phi_3 = \mathrm{id} \tag{3.5}$$

where $\Pi_i : K_i \times [0, \epsilon) \to K_i$ is projection onto the first factor. It is easily checked that this satisfies all conditions of a scattering-fibered structure on X.

(We remark that we are ignoring the fact that X is non-compact, contrary to the above definition; this is harmless since we will in practice only be concerned with compactly supported distributions on X, supported in say $x_3 \leqslant \epsilon/2$.)

A special case of this is of course $Y = M_{\mathrm{b}}^2$, the b-double space of a manifold with boundary M; we have discussed this space already in Section 2. In this case, $H_1 = \mathrm{lb} \cup \mathrm{rb}$, $H_2 = \mathrm{bf}$ and $H_3 = \mathrm{mf}$. Let us further consider the boundary fibration structure in this case. The fibrations are given by the identity on mf, by the projection off the h factor on bf, and by the projection to ∂M on lb and rb.

Consider a point on the codimension three face $\mathrm{bf} \cap \mathrm{rb} \cap \mathrm{mf}$, which is naturally diffeomorphic to $(\partial M)^2$. Recall that the total boundary defining function ρ for M_{b}^2 is given by $\rho = (x^{-2} + (x')^{-2})^{-1/2}$ where x is a boundary defining function for M lifted via the left projection, and x' is the lift of the same boundary defining function via the right projection. Local coordinates near this point are $x_1 = \rho/x$, $x_2 = x$, $x_3 = h$, $y_1 = y'$, $y_2 = y$. (Notice that $x_1 = x'/x(1 + (x'/x)^2)^{-1/2}$ is equivalent to x'/x for x'/x small.) Then the fibrations take the form

$$\phi_1 : (x, h, y', y) \mapsto y' \quad \text{on rb},$$
$$\phi_2 : (x_1, h, y', y) \mapsto (x_1, y', y) \quad \text{on bf},$$
$$\phi_3 : (x_1, x, y', y) \mapsto (x_1, x, y', y) \quad \text{on mf}.$$

Moreover, the product of the three boundary defining functions satisfies

$$x_1 \cdot x \cdot h = \mathbf{x},$$

so these coordinates satisfy the conditions of Proposition 3.4.

3.3. *Scattering-fibered tangent and cotangent bundles*

We return briefly to the case of corners of arbitrary codimension.

Definition 3.6. The space $C^\infty_\Phi(X)$ is the space of $\mathcal{C}^\infty$ functions f on X which are constant on the fibers of Φ.

It is not hard to check that changing the total boundary defining function $\mathbf{x}$ to $f\mathbf{x}$, where $f \in C^\infty_\Phi(X) > 0$, leads to the same scattering-fibered structure. Hence the total boundary defining function is distinguished up to multiplication by elements of $C^\infty_\Phi(X)$.

Definition 3.7. The Lie algebra of scattering-fibered vector fields $\mathcal{V}_{\mathrm{s}\Phi}$ is defined by

$$V \in \mathcal{V}_{\mathrm{s}\Phi}(X) \quad \text{iff } V \in \mathcal{V}_{\mathrm{b}}(X), \qquad V(\mathbf{x}) = O\big(\mathbf{x}^2\big) \quad \text{and} \quad V(f) = O(\mathbf{x}) \quad \text{for all } f \in C^\infty_\Phi(X). \tag{3.6}$$

Here we recall that $\mathcal{V}_b(X)$ is the Lie algebra of smooth vector fields on X which are tangent to each boundary hypersurface. We remark that the condition $V(f) = O(\mathbf{x})$ is equivalent to $V|H_i$ being tangent to the fibers of ϕ_i.

It is easy to check that this *is* a Lie algebra. For if $V, W \in \mathcal{V}_{s\Phi}(X)$ then $V(W\mathbf{x}) = V(\mathbf{x}^2 g)$ for some smooth g, and this is $O(\mathbf{x}^2)$ since V is a b-vector field. Thus $[V, W]\mathbf{x} = VW\mathbf{x} - WV\mathbf{x} = O(\mathbf{x}^2)$. Similarly, if $V(f) = O(\mathbf{x})$ and $W(f) = O(\mathbf{x})$ then $[V, W]f = O(\mathbf{x})$. It is equally clear that $\mathcal{V}_{s\Phi}(X)$ is invariant under multiplication by smooth functions on X, and thus can be localized in any open set.

Using coordinates as in Proposition 3.4, it may be checked that the Lie algebra $\mathcal{V}_{s\Phi}(X)$ is the $\mathcal{C}^\infty(X)$-span of the vector fields

$$\begin{array}{ll} (x_1x_2x_3\dots x_d)x_1\partial_{x_1}, & (x_1x_2x_3\dots x_d)\partial_{y_1}, \\ (x_2x_3\dots x_d)(x_1\partial_{x_1} - x_2\partial_{x_2}), & (x_2x_3\dots x_d)\partial_{y_2}, \\ (x_3\dots x_d)(x_2\partial_{x_2} - x_3\partial_{x_3}), & (x_3\dots x_d)\partial_{y_3}, \\ \vdots & \vdots \\ x_d(x_{d-1}\partial_{x_{d-1}} - x_d\partial_{x_d}), & x_d\partial_{y_d} \end{array} \tag{3.7}$$

(where we write ∂_{y_i} for the k_i-tuple of vector fields $\partial_{y_i^j}$, $1 \leqslant j \leqslant k_i$, if $\dim y_i = k_i$). Thus, in the codimension three case, any vector field in $\mathcal{V}_{s\Phi}(X)$ is a linear combination of

$$\begin{array}{ll} (x_1x_2x_3)x_1\partial_{x_1}, & (x_1x_2x_3)\partial_{y_1}, \\ x_2x_3(x_1\partial_{x_1} - x_2\partial_{x_2}), & (x_2x_3)\partial_{y_2}, \\ x_3(x_2\partial_{x_2} - x_3\partial_{x_3}), & x_3\partial_{y_3}. \end{array} \tag{3.8}$$

Therefore, locally near any point in X, the vector fields in $\mathcal{V}_{s\Phi}(X)$ are arbitrary linear combinations (over $\mathcal{C}^\infty(X)$) of $N = \dim X$ vector fields. It follows that $\mathcal{V}_{s\Phi}(X)$ is the space of sections of a vector bundle over X.

Example 3.8. At the corner $\mathrm{bf} \cap \mathrm{rb} \cap \mathrm{mf}$ of the space X from Section 2, we have $x_1 = x'/x$, $x_2 = x$, $x_3 = h$, $y' = y_1$, $y = y_2$; in these coordinates, we have

$$\begin{aligned} h(x')^2\partial_{x'} &= (x_1x_2x_3)x_1\partial_{x_1}, & hx'\partial_{y'} &= (x_1x_2x_3)\partial_{y_1}, \\ hx^2\partial_x &= x_2x_3(x_1\partial_{x_1} - x_2\partial_{x_2}), & hx\partial_y &= x_2x_3\partial_{y_2}, \\ h(x\partial_x + x'\partial_{x'} - h\partial_h) &= x_3(x_2\partial_{x_2} - x_3\partial_{x_3}), \end{aligned} \tag{3.9}$$

so the vector fields arising in the discussion of Section 2.2 generate the scattering-fibered Lie algebra.

Definition 3.9. We define ${}^{s\Phi}T(X)$, the *scattering-fibered tangent bundle over* X, to be the vector bundle of which $\mathcal{V}_{s\Phi}(X)$ is the space of sections; explicitly, the fiber ${}^{s\Phi}T_p(X)$ at $p \in X$ is given by $\mathcal{V}_{s\Phi}(X)/I_p \cdot \mathcal{V}_{s\Phi}(X)$, where $I_p(\mathcal{V}_{s\Phi}(X))$ is the set of vector fields of the form fV, where $f \in \mathcal{C}^\infty(X)$ vanishes at p and $V \in \mathcal{V}_{s\Phi}(X)$. We define ${}^{s\Phi}T^*(X)$, the *scattering-fibered cotangent bundle over* X, to be the dual vector bundle to ${}^{s\Phi}T(X)$.

We define ${}^{s\Phi}\mathrm{Diff}(X)$ to be the ring of differential operators generated by $\mathcal{V}_{s\Phi}(X)$ over $\mathcal{C}^\infty(X)$.

The vector bundle ${}^{s\Phi}T^*X$ is spanned by one-forms of the form $d(f/\mathbf{x})$ where $f \in C^\infty_\Phi(M)$. To see the duality between scattering-fibered vector fields and differentials $d(f/\mathbf{x})$ for $f \in C^\infty_\Phi(X)$, first observe that there is a pairing between scattering-fibered vector fields and such differentials for each $p \in X$ given by

$$\left\langle d\left(\frac{f}{\mathbf{x}}\right), V\right\rangle_p = V\left(\frac{f}{\mathbf{x}}\right)(p). \tag{3.10}$$

This is finite for every $p \in X$ since $V(f) = O(\mathbf{x})$ and $V(\mathbf{x}) = O(\mathbf{x}^2)$. In the codimension three case, choosing f equal to

$$\begin{aligned} y_1^j, \quad x_1 y_2^j, \quad x_1 x_2 y_3^j, \\ 1, \quad x_1, \quad x_1 x_2, \end{aligned} \tag{3.11}$$

in turn, and pairing with the vector fields in (3.8) gives a non-degenerate matrix. Thus, we can identify the dual space of $\mathcal{V}_{s\Phi\, p}(X)$, the scattering-fibered cotangent bundle at p, ${}^{s\Phi}T^*_p(X)$, as

$${}^{s\Phi}T^*_p(X) = \left\{ d\left(\frac{f}{\mathbf{x}}\right) \;\middle|\; f \in C^\infty_\Phi(X) \right\} \Big/ \sim_p \tag{3.12}$$

where $\sim_p$ is the equivalence relation of yielding the same pairing (3.10) at the point p.

The dual basis to the vector fields (3.8) is

$$d\left(\frac{1}{x_1x_2x_3}\right), \quad d\left(\frac{1}{x_2x_3}\right), \quad d\left(\frac{1}{x_3}\right), \quad \frac{dy_1}{x_1x_2x_3}, \quad \frac{dy_2}{x_2x_3}, \quad \frac{dy_3}{x_3}. \tag{3.13}$$

Here dy_i is shorthand for a k_i-vector of 1-forms, if $y_i \in \mathbb{R}^{k_1}$. Any element of ${}^{s\Phi}T^*X$ may therefore be written uniquely as

$$\nu_1 d\left(\frac{1}{x_1x_2x_3}\right) + \nu_2 d\left(\frac{1}{x_2x_3}\right) + \nu_3 d\left(\frac{1}{x_3}\right) + \mu_1 \cdot \frac{dy_1}{x_1x_2x_3} + \mu_2 \cdot \frac{dy_2}{x_2x_3} + \mu_3 \cdot \frac{dy_3}{x_3}. \tag{3.14}$$

The function ν_1, regarded as a linear form on the fibers of ${}^{s\Phi}T^*X$, can be identified with the vector field $(x_1x_2x_3)x_1\partial_{x_1}$, and similarly for the other fiber coordinates. The same expression can be viewed as the canonical one-form on ${}^{s\Phi}T^*X$. Taking d of (3.14) therefore gives the symplectic form on ${}^{s\Phi}T^*X$.

There is an alternative basis which is sometimes more convenient; instead of (3.13) we use the basis

$$d\left(\frac{1}{x_1x_2x_3}\right), \quad \frac{dx_1}{x_1x_2x_3}, \quad \frac{dx_2}{x_2x_3}, \quad \frac{dy_1}{x_1x_2x_3}, \quad \frac{dy_2}{x_2x_3}, \quad \frac{dy_3}{x_3}. \tag{3.15}$$

Using this basis, we can write any $q \in {}^{s\Phi}T^*X$ locally in the form

$$q = \bar{\nu}_1 d\left(\frac{1}{x_1x_2x_3}\right) + \bar{\nu}_2 \frac{dx_1}{x_1x_2x_3} + \bar{\nu}_3 \frac{dx_2}{x_2x_3} + \mu_1 \frac{dy_1}{x_1x_2x_3} + \mu_2 \frac{dy_2}{x_2x_3} + \mu_3 \frac{dy_3}{x_3}. \tag{3.16}$$

These are related to the ν_i by

$$\begin{aligned}\bar\nu_1 &= \nu_1 + x_1\nu_2 + x_1x_2\nu_3,\\ \bar\nu_2 &= \nu_2 + x_2\nu_3,\\ \bar\nu_3 &= \nu_3.\end{aligned} \tag{3.17}$$

In particular, $\bar\nu_1 = \nu_1$ at $x_1 = 0$ and $\bar\nu_2 = \nu_2$ at $x_2 = 0$.

3.4. Induced bundles and fibrations

There is a natural subbundle of ${}^{s\Phi}T^*_{H_i}(X)$, namely[3] equivalence classes of differentials $d(f/\mathbf{x})$ where $f \in C^\infty_\Phi(X)$ *vanishes* at H_i. Let us denote this subbundle ${}^{s\Phi}T^*(F_i, H_i)$; the reason for this notation will become evident below. Notice that any $f \in C^\infty_\Phi(X)$ has a representation

$$f = f_1(y_1) + x_1 f_2(x_1, y_1, y_2) + x_1x_2 f_3(x_1, x_2, y_1, y_2, y_3) + \cdots + x_1x_2x_3\ldots x_d\tilde f \tag{3.18}$$

where f_i and $\tilde f$ are smooth. Thus the ith subbundle corresponds to f with $f_1 = \cdots = f_i = 0$, while the f_j, $j > i$, are arbitrary. A point in the quotient bundle ${}^{s\Phi}T^*_{H_i}(X)/{}^{s\Phi}T^*(F_i, H_i)$ is therefore given by a differential $d(f/\mathbf{x})$ where only $f_1, \ldots, f_i$ are relevant. Since these functions are constant on the fibers of H, they may be regarded as functions on Z_i. Hence this is the lift to H_i of a bundle over Z_i, which we shall denote ${}^{s\Phi}N^*Z_i$. Therefore there is an induced fibration given by the composition

$$\tilde\phi_i : {}^{s\Phi}T^*_{H_i}X \to {}^{s\Phi}T^*_{H_i}X/{}^{s\Phi}T^*(F_i, H_i) \to {}^{s\Phi}N^*Z_i.$$

In the coordinates above, the subbundle ${}^{s\Phi}T^*(F_i, H_i)$ is given by $x_i = 0, \nu_1 = \cdots = \nu_i = 0$, $\mu_1 = \cdots = \mu_i = 0$, while $(x_1, \ldots, x_{i-1}, y_1, \ldots, y_i, \nu_1, \ldots, \nu_i, \mu_1, \ldots, \mu_i)$ furnish coordinates on ${}^{s\Phi}N^*Z_i$ in a natural way. The subbundle ${}^{s\Phi}T^*(F_i, H_i)$ can be interpreted as follows. We observe that each fixed fiber F_i of H_i has an induced scattering-fibered structure, since F_i meets $H_{i+1}\ldots H_d$ and the fibrations ϕ_j for $j > i$ restrict to fibrations from $F \cap H_j$ to a face of Z_j. Moreover, a total boundary defining function for F is given by $\mathbf{x}/(x_1\ldots x_i)$, where x_k for $k \leqslant i$ is chosen to be constant on the fibers of H_j for $j > k$. Then the bundle ${}^{s\Phi}T^*(F_i, H_i)$ restricted to a single fiber F of H_i is naturally isomorphic to the scattering-fibered cotangent bundle of F, ${}^{s\Phi}T^*F$.

For concreteness consider the codimension three case. Recall that Z_1 is a manifold without boundary, while Z_2 has a boundary which we denote $\partial_1 Z_2$, and $Z_3 = \mathrm{mf}$ has two boundary hypersurfaces which we denote $\partial_1 Z_3$ (the intersection with H_1) and $\partial_2 Z_3$ (the intersection with H_2). Moreover, there is an induced fibration $\phi_{12} : \partial_1 Z_2 \to Z_1$, as in (3.2). We claim that the commu-

[3] The restriction of ${}^{s\Phi}T^*X$ to a subset $S \subset X$ will be denoted ${}^{s\Phi}T^*_S X$.

tative diagram (3.2) with $i = 1$, $j = 2$ 'lifts' to a commutative diagram at the level of cotangent spaces

$$\begin{array}{ccccc}
{}^{s\Phi}T^*_{H_1}X & \xleftarrow{\ inc\ } & {}^{s\Phi}T^*_{H_1\cap H_2}X & \xrightarrow{\ inc\ } & {}^{s\Phi}T^*_{H_2}X \\
\tilde{\phi}_1\big\downarrow & \swarrow{\scriptstyle \tilde{\phi}_1} & \big\downarrow\tilde{\phi}_2 & & \big\downarrow\tilde{\phi}_2 \\
{}^{s\Phi}N^*Z_1 & \xleftarrow[\tilde{\phi}_{12}]{} & \partial_1{}^{s\Phi}N^*Z_2 & \xrightarrow[inc]{} & {}^{s\Phi}N^*Z_2.
\end{array} \tag{3.19}$$

In this diagram everything has been explained except the existence and properties of the map $\tilde{\phi}_{12}$. To define it, note that the subbundle ${}^{s\Phi}T^*_{H_1\cap H_2}(F_2, H_2)$ is a subbundle of ${}^{s\Phi}T^*_{H_1\cap H_2}(F_1, H_1)$. Therefore there is an induced map on the quotient bundles, which we denote

$$\tilde{\phi}_{12} : \partial_1{}^{s\Phi}N^*Z_2 \to {}^{s\Phi}N^*Z_1,$$

making the diagram (3.19) commute.

(We remark that there is also a diagram analogous to (3.19) for $(i, j) = (1, 3)$ or $(2, 3)$ as well. In these cases, the maps $\tilde{\phi}_3$ is the identity, but the map $\tilde{\phi}_{i3}$, $i = 1, 2$ is still of interest, mapping from ${}^{s\Phi}T^*_{H_i\cap H_3}X$ to ${}^{s\Phi}N^*Z_i$.)

We shall often be interested in the restriction of the fibrations $\tilde{\phi}_i$ to ${}^{s\Phi}T^*_{H_i\cap \mathrm{mf}}X \to {}^{s\Phi}N^*Z_i$; notice that this is still onto since the fibers of H_i are transverse to mf, $i < d$. We shall abuse notation slightly and call the restriction $\tilde{\phi}_i$ also. Thus, restriction to mf gives the following variant of (3.19):

$$\begin{array}{ccccc}
{}^{s\Phi}T^*_{H_1\cap H_3}X & \xleftarrow{\ inc\ } & {}^{s\Phi}T^*_{H_1\cap H_2\cap H_3}X & \xrightarrow{\ inc\ } & {}^{s\Phi}T^*_{H_2\cap H_3}X \\
\tilde{\phi}_1\big\downarrow & \swarrow{\scriptstyle \tilde{\phi}_1} & \big\downarrow\tilde{\phi}_2 & & \big\downarrow\tilde{\phi}_2 \\
{}^{s\Phi}N^*Z_1 & \xleftarrow[\tilde{\phi}_{12}]{} & \partial_1{}^{s\Phi}N^*Z_2 & \xrightarrow[inc]{} & {}^{s\Phi}N^*Z_2.
\end{array} \tag{3.20}$$

Remark. Each space in the diagram above has a simple form in terms of the coordinates x_i, y_i, ν_i, μ_i. For example, the top left space is $\{x_1 = x_3 = 0\}$, the top middle space is $\{x_1 = x_2 = x_3 = 0\}$, the top right space is $\{x_2 = x_3 = 0\}$, while on the bottom row the left space is $\{x_1 = x_2 = x_3 = 0, y_2 = y_3 = 0, \mu_2 = \mu_3 = 0, \nu_2 = \nu_3 = 0\}$, the middle space is $\{x_1 = x_2 = x_3 = 0, y_3 = 0, \mu_3 = 0, \nu_3 = 0\}$ and the right space is $\{x_2 = x_3 = 0, y_3 = 0, \mu_3 = 0, \nu_3 = 0\}$. Moreover, all the maps are the obvious coordinate projections or inclusions.

3.5. *Contact structures*

In the remainder of this paper we restrict attention to the codimension three case. We define a 1-form χ on ${}^{s\Phi}T^*_{\mathrm{mf}}X$ by contracting the symplectic form ω with $\mathbf{x}x_3\partial_{x_3}$ (where x_3 is a boundary defining function for mf) and restricting to mf. This yields a contact structure (i.e. the form χ is non-degenerate in the sense that $\chi \wedge (d\chi)^{N-1} \neq 0$, $N = \dim X$) in the interior of mf. However, this contact structure degenerates at the boundary of mf. In local coordinates (3.14), the contact structure takes the form

$$\chi = d\nu_1 + x_1\, d\nu_2 + x_1x_2\, d\nu_3 - \mu_1 \cdot dy_1 - x_1\mu_2 \cdot dy_2 - x_1x_2\mu_3\, dy_3 \tag{3.21}$$

and this degeneration is evident. Indeed, at ${}^{s\Phi}T^*_{H_i\cap \mathrm{mf}}X$, χ vanishes on the subbundle ${}^{s\Phi}T^*_{H_i\cap \mathrm{mf}}(F_i, H_i)$. However, it is not difficult to see that $\chi|{}^{s\Phi}T^*_{H_i\cap \mathrm{mf}}X$ is the lift of a one-form from ${}^{s\Phi}N^*Z_i$. This is most easily seen in local coordinates; at $x_1 = 0$, $\chi = d\nu_1 - \mu_1 \cdot dy_1$ is the lift of a one-form χ_1 from ${}^{s\Phi}N^*Z_1$ since it is expressible in terms of the coordinates y_1, ν_1, μ_1 which are the lifts of functions on ${}^{s\Phi}N^*Z_1$. Similarly, at $x_2 = 0$, $\chi = d\nu_1 + x_1\, d\nu_2 - \mu_1 \cdot dy_1 - x_1\mu_2 \cdot dy_2$ is the lift of a one-form χ_2 from ${}^{s\Phi}N^*Z_2$. Moreover, χ_1 is non-degenerate, i.e. is a contact form, on ${}^{s\Phi}N^*Z_1$, while χ_2 is non-degenerate except at $\partial_1{}^{s\Phi}N^*Z_2$.

In the coordinates (3.16) the contact form takes the form

$$\chi = d\bar\nu_1 - \bar\nu_2\, dx_1 - x_1\bar\nu_3\, dx_2 - \mu_1 \cdot dy_1 - x_1\mu_2 \cdot dy_2 - x_1x_2\mu_3\, dy_3. \tag{3.22}$$

These coordinates are more convenient when analyzing Legendre distributions (see Section 4).

The degeneration of χ at ${}^{s\Phi}T^*_{\mathrm{mf}\cap H_i}X$ and of χ_2 on ${}^{s\Phi}N^*_{\partial_1 Z_2}Z_2$ is captured by contact structures on the fibers of the maps $\tilde\phi_i$ and $\tilde\phi_{12}$. To define these we make the following definition.

Definition 3.10. Suppose that M is a manifold, $S \subset M$ a hypersurface with boundary defining function s, and α a one-form on M that vanishes at S. Thus $\alpha = s\beta$ for some one-form[4] β. We call β the *leading part* of α at S. It is well defined up to multiplication by a non-zero function. This remains true even if α itself is only well defined up to multiplication by a non-zero function.

Notice that $\chi = \tilde\phi_1^*\chi_1$ at $\partial_1{}^{s\Phi}N^*Z_3 \equiv {}^{s\Phi}T^*_{H_1\cap \mathrm{mf}}X$, that $\chi = \tilde\phi_2^*\chi_2$ at $\partial_2{}^{s\Phi}N^*Z_3 \equiv {}^{s\Phi}T^*_{H_2\cap \mathrm{mf}}X$, and that $\chi_2 = \tilde\phi_{12}^*\chi_1$ at $\partial_1{}^{s\Phi}N^*Z_2$. Using the definition we can define χ_{13} to be the leading part of $\chi - \tilde\phi_1^*\chi_1$ at ${}^{s\Phi}T^*_{H_1\cap \mathrm{mf}}X \subset {}^{s\Phi}T^*_{\mathrm{mf}}X$, χ_{23} to be the leading part of $(\chi - \tilde\phi_2^*\chi_2)/x_1$ at ${}^{s\Phi}T^*_{H_2\cap \mathrm{mf}}X \subset {}^{s\Phi}T^*_{\mathrm{mf}}X$ and χ_{12} to be the leading part of $\chi_2 - \tilde\phi_{12}^*\chi_1$ at $\partial_1{}^{s\Phi}N^*Z_2 \subset {}^{s\Phi}N^*Z_2$. Using the invariance property in the last part of the definition, we see that these one-forms are well defined up to multiplication by non-zero functions. In local coordinates, we have

$$\begin{aligned}\chi_{12} &= d\nu_2 - \mu_2 \cdot dy_2,\\ \chi_{23} &= d\nu_3 - \mu_3 \cdot dy_3,\\ \chi_{13} &= d\nu_2 - \mu_2 \cdot dy_2 + x_2(d\nu_3 - \mu_3 \cdot dy_3).\end{aligned}$$

Hence we have well-defined contact structures (i.e. χ_{12} and χ_{23} are non-degenerate) on the fibers of $\tilde\phi_{12}$ and $\tilde\phi_2$ in (3.20), while χ_{13} is non-degenerate on the fibers of $\tilde\phi_1$ for $x_2 > 0$.

[4] Note that vanishing at S is a strictly stronger condition than vanishing when restricted to S; e.g. ds does not vanish at S although it vanishes when restricted to S.

Part 2. Machinery

4. Legendrian submanifolds and distributions

In this section we define Legendre distributions on a scattering-fibered manifold X with corners of codimension 3. These will be smooth functions in the interior of X which oscillatory behavior at the boundary.

4.1. Legendre submanifolds

Definition 4.1. A Legendre submanifold is a submanifold G of dimension N of ${}^{s\Phi}T^*_{\mathrm{mf}}X$ on which the contact form χ vanishes, and such that G is transverse to each boundary ${}^{s\Phi}T^*_{\mathrm{mf}\cap H_i}X$ of ${}^{s\Phi}T^*_{\mathrm{mf}}X$.

Example 4.2. Let $f \in C^\infty_\Phi(X)$. Then the graph of $d(f/\mathbf{x})$, restricted to ${}^{s\Phi}T^*_{\mathrm{mf}}X$, is a Legendre submanifold. The condition that $f \in C^\infty_\Phi(X)$, as opposed to $\mathcal{C}^\infty(X)$, is essential; see Section 4.2.

As a consequence of this definition, G is well behaved with respect to the fibrations $\tilde{\phi}_i : {}^{s\Phi}T^*_{\mathrm{mf}\cap H_i}X \to {}^{s\Phi}N^*Z_i$. To ease notation, we write $\partial_i G$ for the boundary hypersurface of G lying over H_i, and $\partial_{12}G$ for the corner lying over $H_1 \cap H_2$.

Proposition 4.3.

(i) *The restriction of $\tilde{\phi}_i$ to $\partial_i G$ is locally a fibration*

$$\phi_i^G : \partial_i G \to G_i$$

to an immersed Legendre submanifold $G_i \subset {}^{s\Phi}N^*Z_i$, and the fibers of ϕ_i^G are Legendre submanifolds for the contact structure for the fibers of $\tilde{\phi}_i$, i.e. for the contact form χ_{i3}.

(ii) *The manifold G_2 is a manifold with boundary $\partial_1 G_2$. The restriction of $\tilde{\phi}_{12}$ to $\partial_1 G_2$ is locally a fibration*

$$\phi_{12}^G : \partial_1 G_2 \to G_1$$

and the fibers of ϕ_{12}^G are Legendre submanifolds for the contact structure for the fibers of $\tilde{\phi}_{12}$, i.e. for the contact form χ_{12}. The maps form a commutative diagram

$$\begin{array}{ccccc} \partial_1 G & \xleftarrow{inc} & \partial_{12} G & \xrightarrow{inc} & \partial_2 G \\ \phi_1^G \downarrow & \overset{\phi_1^G}{\swarrow} & \phi_2^G \downarrow & & \downarrow \phi_2^G \\ G_1 & \xleftarrow[\phi_{12}^G]{} & \partial_1 G_2 & \xrightarrow[inc]{} & G_2. \end{array} \tag{4.1}$$

Notice that each object in (4.1) *is an element of the corresponding space in* (3.20)*, and the maps are induced from those in* (3.20).

Proof. For conceptual ease we first prove this result in the codimension two case. Thus suppose that Y is a scattering-fibered manifold with codimension two corners. Near the corner, there are local coordinates (x_1, x_2, y_1, y_2), such that the fibration ϕ_1 on $H_1 = \{x_1 = 0\}$ takes the form $(x_2, y_1, y_2) \mapsto y_1$, while the fibration on the main face $H_2 = \{x_2 = 0\}$ is the identity. The contact form on ${}^{s\Phi}T_{H_2}Y$ is $\chi = d\nu_1 + x_1\,d\nu_2 - \mu_1 \cdot dy_1 - x_1\mu_2 \cdot dy_2$. Let $k_1 = \dim y_1$ and $k_2 = \dim y_2$. This local model applies everywhere except near $\partial_{12}G$, which we treat later.

In the proof we shall need the following consequence of the implicit function theorem: if V is a compact manifold, W is a manifold and $f : V \to W$ is a smooth map of constant rank, then $f(V)$ is an immersed submanifold of Y and $f : V \to f(V)$ is (locally) a fibration.

By assumption, G is transversal to $\{x_1 = 0\}$ and the restriction of χ to G vanishes. Given $p \in \partial_1 G$, let $T_p(\text{fiber})$ denote the tangent space to the fiber of $\tilde{\phi}_1$. Now consider the space

$$T_p\partial_1 G \cap T_p(\text{fiber});$$

we claim that $d\nu_2 + \mu_2 \cdot dy_2 = 0$ restricted to this space vanishes. To prove this, let V be any vector in $T_p\partial_1 G \cap T_p(\text{fiber})$, and let W be a vector tangent to G and transverse to $\{x_1 = 0\}$. Then $d\chi(V, W) = 0$. But $d\chi = -d\mu_1 \wedge dy_1 + dx_1 \wedge (d\nu_2 - \mu_2 \cdot dy_2)$ at $\partial_1 G$. Since the fibers of $\tilde{\phi}_1$ are given by y_1, ν_1, μ_1 constant, it follows that $(d\mu_1 \wedge dy_1)(V, W)$ vanishes. Also, $dx_1(V)$ vanishes, but $dx_1(W)$ does not. This forces $(d\nu_2 - \mu_2 \wedge dy_2)(V)$ to vanish, which proves that the restriction of $d\nu_2 - \mu_2 \cdot dy_2$ to $T_p\partial_1 G \cap T_p(\text{fiber})$ vanishes. Taking the differential, we see also that $d\mu_2 \wedge dy_2 = 0$ vanishes when restricted to $T_p\partial_1 G \cap T_p(\text{fiber})$.

Now recall that coordinates on the fibers of $\tilde{\phi}_1$ are (y_2, ν_2, μ_2). Since $d\mu_2 \wedge dy_2 = 0$ on $T_p\partial G \cap T_p(\text{fiber})$, the dimension of the projection of this space to the span of the variables $\partial_{y_2}, \partial_{\mu_2}$ is at most k_2; since we further have $d\nu_2 - \mu_2 \cdot dy_2 = 0$, we in fact have

$$\dim\big(T_p(\partial G) \cap T_p(\text{fiber})\big) \leqslant k_2 \tag{4.2}$$

for any $p \in \partial G$.

On the other hand, we can look at the projection of ∂G onto ${}^{s\Phi}N^*Z_1$, via $\tilde{\phi}_1$. We show that the rank of this map, restricted to ∂G, is at most k_1. For if not, then let $k > k_1$ be the maximal rank of this map, and $p \in \partial G$ a point where this maximum is attained. Then the rank is exactly k in a neighborhood of p. Using the implicit function theorem as above we see that the image of ∂G is locally a submanifold of dimension $k > k_1$. However, the form $d\nu_1 + \mu_1 \cdot dy_1$ is zero on this image since it vanishes on ∂G. Therefore the dimension of the projection of ∂G is Legendre and can have dimension at most k_1, which contradicts $k > k_1$. It follows that

$$\operatorname{rank}\tilde{\phi}_1|_{\partial G} = \dim\big(T_p(\partial G)\big) - \dim\big(T_p(\partial G) \cap T_p(\text{fiber})\big) \leqslant k_1. \tag{4.3}$$

On the other hand, $\dim \partial G = k_1 + k_2$, so the sum of the LHSs of (4.2) and (4.3) is everywhere $k_1 + k_2$. Consequently, the dimension of $T_p(\partial G) \cap T_p(\text{fiber})$ is exactly k_2 and the rank of $\tilde{\phi}_1|_{\partial G}$ is exactly k_2, and hence $\tilde{\phi}_1 : \partial G \to {}^{s\Phi}N^*Z_1$ has constant rank k_1. By the implicit function theorem, the image of ∂G in ${}^{s\Phi}N^*Z_1$ is an immersed submanifold, which the reasoning above shows is Legendrian; the fibers of this map are Legendre submanifolds with respect to the contact structure on the fibers of $\tilde{\phi}_1$.

Now we treat the codimension three case. The codimension two argument applies locally everywhere except for a neighborhood of the corner $\partial_{12}G$ where we have be more careful. We claim that the manifold G_2 is transverse (and in particular, regular) up to the boundary of

${}^{s\Phi}N^*Z_2$. To prove this, we note that the implicit function statement above remains true if V and W are manifolds with boundary, provided that f pulls back a boundary defining function for W to a boundary defining function for V. The argument above that $\dim(T_p(\partial G)\cap T_p(\text{fiber}(\tilde{\phi}_2)))\leqslant \dim y_3$ is valid uniformly to the corner, but the argument on the base of the fibration does not extend automatically to the corner because the contact form χ_2 on ${}^{s\Phi}N^*Z_2$ degenerates there. Instead, we must further analyze the structure of G at the corner $\partial_{12}G$. Arguing as above, we see that for $p\in\partial_{12}G$,

$$\chi_{23} \text{ vanishes on } T_p(\partial_{12}G)\cap T_p\big(\text{fibre}(\tilde{\phi}_2)\big),$$
$$\chi_{12} \text{ vanishes on } (\tilde{\phi}_2)^*T_p(\partial_{12}G)\cap T_p\big(\text{fibre}(\tilde{\phi}_{12})\big),$$

and

$$\chi_1 \text{ vanishes on } (\tilde{\phi}_1)^*T_p(\partial_{12}G).$$

The dimension counting argument then shows that $\dim T_p(\partial_{12}G)\cap T_p(\text{fibre}(\tilde{\phi}_2)) = \dim y_3$, $\dim(\tilde{\phi}_2)^*T_p(\partial_{12}G)\cap T_p(\text{fibre}(\tilde{\phi}_{12})) = \dim y_2$ and $\dim(\tilde{\phi}_1)^*T_p(\partial_{12}G) = \dim y_1$ are all constant. This establishes the constancy of the rank of $\tilde{\phi}_2:\partial_2G\to {}^{s\Phi}N^*Z_2$ uniformly to the boundary and thus the regularity of G_2, as well as showing that ∂_1G_2 fibers over G_1 with Legendrian fibers. □

Remark. Notice that, because of our assumption that the fibration at the main face mf is the identity, the scattering-fibered structure locally near the interior of the main face is the same as the scattering structure: locally, we have $\mathcal{V}_{s\Phi}(X)=\mathcal{V}_{sc}(X)$ near the interior of the main face. Consequently, the theory coincides with the theory of Legendre distributions as defined by Melrose and Zworski in the interior of mf.

4.2. Parametrization

Before considering the general case let us consider the special case of Legendrians G which are projectable, meaning that the projection from $G\subset {}^{s\Phi}T^*_{\text{mf}}X\to \text{mf}$ is a diffeomorphism. In this case, G is necessarily given by the graph of the differential of a function. We claim that it is necessarily of the form $f/(x_1x_2x_3)$, where $f\in C^\infty_\Phi(X)$. In fact, consider the graph of $d(f/(x_1x_2x_3))$ for a general smooth f. Expanding this in the basis (3.15), we find that the coordinates $\bar{\nu}_i$ and μ_i are given by

$$\bar{\nu}_1 = f - x_3\partial_{x_3}f,\qquad \bar{\nu}_2 = \partial_{x_1}f - \frac{x_3}{x_2}\partial_{x_3}f,\qquad \bar{\nu}_3 = \frac{1}{x_1}\partial_{x_2}f - \frac{x_3}{x_1x_2}\partial_{x_3}f,$$
$$\mu_1 = \partial_{y_1}f,\qquad \mu_2 = \frac{1}{x_1}\partial_{y_2}f,\qquad \mu_3 = \frac{1}{x_1x_2}\partial_{y_3}f.$$

For this to be a smooth submanifold, it follows that $\partial_{x_2}f$ and $\partial_{y_2}f$ are $O(x_1)$ and $\partial_{x_3}f$ and $\partial_{y_3}f$ are $O(x_1x_2)$. Thus f is of the form

$$f = f_1(y_1) + x_1f_2(x_1,y_1,y_2) + x_1x_2f_3(x_1,x_2,x_3,y_1,y_2,y_3),$$

which is to say that $f \in C^\infty_\Phi(X)$. Now consider the general case. We will use the notation $\vec{x}$, $\vec{\nu}$, $\vec{y}$, $\vec{\mu}$, $\vec{v}$ respectively to denote the sets of coordinates (x_1, x_2, x_3), (ν_1, ν_2, ν_3), (y_1, y_2, y_3), (μ_1, μ_2, μ_3), (v_1, v_2, v_3). A (local) non-degenerate parametrization of G near a point $q \in G \cap {}^{s\Phi}T^*_{H_1 \cap H_2 \cap \mathrm{mf}} X$ given in these coordinates as $q = (\vec{x} = 0, \vec{y}^*, \vec{\nu}^*, \vec{\mu}^*)$ is a smooth function $\psi(\vec{x}, \vec{y}, \vec{v})$ such that ψ has the form

$$\phi(\vec{x}, \vec{y}, \vec{v}) = \psi_1(y_1, v_1) + x_1\psi_2(x_1, y_1, y_2, v_1, v_2) + x_1x_2\psi_3 \tag{4.4}$$

such that ψ_1, ψ_2, ψ_3 are defined on neighborhoods of (y_1^*, v_1^*), $(0, y_1^*, y_2^*, v_1^*, v_2^*)$ and $q' = (0, 0, 0, y_1^*, y_2^*, y_3^*, v_1^*, v_2^*, v_3^*)$ respectively with

$$d\left(\frac{\psi}{\mathbf{x}}\right)(q') = q, \quad d_{\vec{v}}\psi(q') = 0, \tag{4.5}$$

ψ is non-degenerate in the sense that

$$d_{(y_1,v_1)}\frac{\partial\psi_1}{\partial v_1^i}, \quad d_{(y_2,v_2)}\frac{\partial\psi_2}{\partial v_2^j}, \quad d_{(y_3,v_3)}\frac{\partial\psi_3}{\partial v_3^k} \tag{4.6}$$

are independent at (y_1^*, v_1^*), $(y_1^*, y_2^*, v_1^*, v_2^*)$ and q' respectively, and locally near q, G is given by

$$G = \left\{ d\left(\frac{\psi}{\mathbf{x}}\right) \,\middle|\, (\vec{x}, \vec{y}, \vec{v}) \in C_\psi \right\} \tag{4.7}$$

where

$$C_\psi = \left\{ (\vec{x}, \vec{y}, \vec{v}) \,\middle|\, d_{\vec{v}}\psi = 0 \right\}. \tag{4.8}$$

Note that the non-degeneracy conditions imply that C_ψ is a smooth submanifold of codimension $k_1 + k_2 + k_3$ of $X \times \mathbb{R}^{k_1+k_2+k_3}$, and that in the interior of mf, the parametrization is non-degenerate in the sense of [23].

Remark. We also have

$$\begin{aligned} &\psi_1 \text{ is a non-degenerate parametrization of } G_1 \\ &\quad \text{and } \psi_1 + x_1\psi_2 \text{ is a non-degenerate parametrization of } G_2. \end{aligned} \tag{4.9}$$

In addition, for fixed (y_1, v_1) with $d_{v_1}\psi_1 = 0$, the phase function

$$\psi_1(y_1, v_1) + x_1\psi_2(x_1, y_1, y_2, v_1, v_2)$$

parametrizes the fibers of the map $\tilde{\phi}_{12}$, while for fixed $(x_1, y_1, y_2, v_1, v_2)$ with $d_{v_1,v_2}\psi_2 = 0$, the phase function ψ parametrizes the fibers of the map $\tilde{\phi}_{23}$.

4.3. Existence of parametrizations

Proposition 4.4. *Let G be a Legendre submanifold. Then for any point $q \in \partial G$ there is a non-degenerate parametrization of G in some neighborhood of q.*

Proof. It is only necessary to do this in the case of a point q lying over $H_1 \cap H_2 \cap \mathrm{mf}$, since the other cases have already been proven in [10]. By definition of a Legendre submanifold, the boundary $\partial_2 G$ of G at $\{x_2 = 0\}$ fibers, via the map $\tilde{\phi}_{23}$, over G_2 with fibers that are Legendre submanifolds of ${}^{sc}T^*_{\partial F}F$, where F denotes a fiber of H_2. Coordinates on ${}^{sc}T^*_{\partial F}F$ are (y_3, ν_3, μ_3) and, as in [23], Proposition 5, we can find coordinates $y_3 = (y_3^\flat, y_3^\sharp)$ near $\phi(q)$ so that $(y_3^\sharp, \mu_3^\flat)$ form coordinates on the fibers of $\partial_2 G \to G_2$. In turn, $\partial_1 G_2$ fibers over G_1 with fibers that are Legendrian with respect to the contact form $\chi_{12} = d\nu_2 - \mu_2 \cdot dy_2$; hence we can find coordinates $y_2 = (y_2^\flat, y_2^\sharp)$ near $\phi(q)$ so that $(y_2^\sharp, \mu_2^\flat)$ form coordinates on the fibers of $\partial_1 G_2 \to G_1$. Lastly, since G_1 is Legendrian, we can find coordinates $y_1 = (y_1^\flat, y_1^\sharp)$ on Z_1 near $\tilde{\phi}_{13}(q)$ so that $(y_1^\sharp, \mu_1^\flat)$ form coordinates on G_1 locally. Using the transversality of G to $\{x_1 = 0\}$ and $\{x_2 = 0\}$ we see that

$$\mathcal{Z} = \bigl(x_1, x_2, y_1^\sharp, y_2^\sharp, y_3^\sharp, \mu_1^\flat, \mu_2^\flat, \mu_3^\flat\bigr)$$

form coordinates on G near q. Consequently, we can write the other coordinates as functions of these coordinates when restricted to G.

We now use the coordinates (3.15) on the scattering cotangent bundle. The reason is that, in terms of a phase function Φ parametrizing a Legendrian G, the value of $\bar{\nu}_1$ on G is given simply by Φ. The contact form is given by

$$d\bar{\nu}_1 - \bar{\nu}_2\,dx_1 - x_1\bar{\nu}_3\,dx_2 - \mu_1 \cdot dy_1 - x_1\mu_2 \cdot dy_2 - x_1x_2\mu_3 \cdot dy_3. \tag{4.10}$$

Writing $\bar{\nu}_i$, $y_i^\flat$ and $\mu_i^\sharp$ in terms of the coordinates $\mathcal{Z}$ on G, we have

$$\begin{aligned}
\bar{\nu}_1 &= N_1(\mathcal{Z}),\\
\bar{\nu}_2 &= N_2(\mathcal{Z}),\\
\bar{\nu}_3 &= N_3(\mathcal{Z}), \qquad\qquad \text{on } G.\\
y_i^\flat &= Y_i^\flat(\mathcal{Z}), \quad i = 1, \dots, 3,\\
\mu_i^\sharp &= M_i^\sharp(\mathcal{Z}), \quad i = 1, \dots, 3,
\end{aligned} \tag{4.11}$$

Since G is Legendrian, we have

$$\begin{aligned}
&dN_1 - N_2\,dx_1 - x_1N_3\,dx_2 - \mu_1^\flat \cdot dY_1^\flat - M_1^\sharp \cdot dy_1^\sharp\\
&\quad - x_1\bigl(\mu_2^\flat \cdot dY_2^\flat - M_2^\sharp \cdot dy_2^\sharp\bigr) - x_1x_2\bigl(\mu_3^\flat \cdot dY_3^\flat - M_3^\sharp \cdot dy_3^\sharp\bigr) = 0.
\end{aligned} \tag{4.12}$$

We claim that the function

$$\Phi = N_1 + \bigl(y_1^\flat - Y_1^\flat\bigr) \cdot \mu_1^\flat + x_1\bigl(\bigl(y_2^\flat - Y_2^\flat\bigr) \cdot \mu_2^\flat\bigr) + x_1x_2\bigl(\bigl(y_3^\flat - Y_3^\flat\bigr) \cdot \mu_3^\flat\bigr) \tag{4.13}$$

is a local parametrization of G. To avoid confusion, let us write v_i instead of $\mu_i^\flat$ for the corresponding arguments of Φ.

First, observe that N_1 has the form $N_1 = N_{1,1}(y_1, v_1) + O(x_1)$ since at $x_1 = 0$, $\partial_1 G$ fibers over G_1 where the value of $\bar{\nu}_1 = \nu_1$ is determined by $(y_1^\sharp, v_1)$ since these are coordinates on G_1. Similarly, N_1 is a function of $(x_2, y_1, y_2, v_1, v_2)$ plus $O(x_1 x_2)$, $Y_1^\flat$ is a function of (y_1, v_1) plus $O(x_1)$, etc. It follows that Φ has the form (4.4).

Second, suppose that $d_{v_1}\Phi = 0$. This means that

$$\begin{aligned} &d_{v_1} N_1 + y_1^\flat - Y_1^\flat - d_{v_1} Y_1^\flat \cdot v_1 \\ &\quad - x_1\big(d_{v_1} Y_2^\flat \cdot \mu_2^\flat\big) - x_1 x_2 \big(d_{v_1} Y_3^\flat \cdot \mu_3^\flat\big) = 0. \end{aligned} \tag{4.14}$$

Using the dv_1 component of (4.12), this is the same thing as saying that $y_1^\flat = Y_1^\flat$. In a similar way, the conditions that $d_{v_i}\Phi = 0$ imply that $y_i^\flat = Y_i^\flat$, $i = 2, 3$. This also shows the non-degeneracy condition, since $d(\partial_{v_j^i}\Phi) = dy_j^i$ at q which are manifestly linearly independent differentials.

To see that the set

$$G' = \left\{ d\left(\frac{\Phi}{x_1 x_2 x_3}\right) \,\middle|\, d_{v_1, v_2, v_3}\Phi = 0 \right\}$$

coincides with G locally near q, first consider the value of $\mu_1^\flat$; this is given by $d_{y_1^\flat}\Phi = v_1$. So we can re-identify $\mu_1^\flat$ with v_1. Similarly we can re-identify $\mu_2^\flat$ with v_2 and $\mu_3^\flat$ with v_3.

Next consider the value of $\bar{\nu}_1$ on G'. It is given by the value of Φ, that is, by (4.13). This simplifies to N_1 when $d_{v_i}\Phi = 0$, since we have $y_i^\flat = Y_i^\flat$ when $d_{v_i}\Phi = 0$. Now consider the value of $\bar{\nu}_2$. This is given by $d_{x_1}\Phi$ which is equal to

$$d_{x_1} N_1 - d_{x_1} Y_1^\flat \cdot \mu_1^\flat - x_1 d_{x_1} Y_2^\flat \cdot \mu_2^\flat - x_1 x_2 d_{x_1} Y_3^\flat \cdot \mu_3^\flat$$

(again using $y_i^\flat = Y_i^\flat$ when $d_{v_i}\Phi = 0$). Since the dx_1 component of (4.10) vanishes, this is equal to N_2. So $\bar{\nu}_2 = N_2$ on G'. In a similar way we deduce that $\bar{\nu}_3 = N_3$, and $\mu_i^\sharp = M_i^\sharp$ on G'. It follows that G' coincides with G. □

4.4. *Equivalence of phase functions*

In this section we shall give a necessary and sufficient condition for equivalence of two phase functions parametrizing a given Legendrian. This is the key step in showing, in the following subsection, that the class of Legendre distributions does not depend on the choice of phase function, which is crucial for deducing that the class of Legendre distributions has a useful symbol calculus.

Two phase functions ϕ, $\tilde{\phi}$ are said to be *equivalent* if they have the same number of phase variables of each type v_1, v_2, v_3 and there exist maps

$$V_1(\vec{x}, \vec{y}, \vec{v}), \quad V_2(\vec{x}, \vec{y}, \vec{v}), \quad V_3(\vec{x}, \vec{y}, \vec{v})$$

such that

$$\tilde{\phi}(\vec{x}, \vec{y}, V_1, V_2, V_3) = \phi.$$

Proposition 4.5. *The phase functions* $\phi = \psi_1 + x_1\psi_2 + x_1x_2\psi_3$ *and* $\tilde{\phi} = \tilde{\psi}_1 + x_1\tilde{\psi}_2 + x_1x_2\tilde{\psi}_3$ *are locally equivalent iff*

(1) *they parametrize the same Legendrian,*
(2) *they have the same number of phase variables of the form* v_1, v_2, *and* v_3 *separately,*
(3)

$$\operatorname{sgn} d^2_{v_1}\psi_1 = \operatorname{sgn} d^2_{v_1}\tilde{\psi}_1,$$
$$\operatorname{sgn} d^2_{v_2}\psi_2 = \operatorname{sgn} d^2_{v_2}\tilde{\psi}_2,$$
$$\operatorname{sgn} d^2_{v_3}\psi_3 = \operatorname{sgn} d^2_{v_3}\tilde{\psi}_3.$$

Proof. The proof follows [14], Theorem 3.1.6 quite closely (and Lemma 4.5 of [10] even more so), hence we will be brief. To begin, we let C and $\tilde{C}$ denote the respective sets where $d_{\vec{v}}\psi = 0$, $d_{\vec{v}}\tilde{\psi} = 0$, near a given point in the codimension-three corner.

We begin by noting that when we restrict to the face H_1 we have a phase function $\psi = \psi_1(y_1, v_1)$ parametrizing G_1. Hence by the usual argument for equivalence of phase functions ([14], as extended to Legendrians in [23]), there exists a fiber diffeomorphism $\tilde{v}_1 = V_1(y_1, v_1)$ such that $\psi_1(y_1, \tilde{v}_1) = \tilde{\psi}_1(y_1, v_1)$. Furthermore on the face H_2, equivalence of phase functions is guaranteed by [10]. Hence we need only extend from H_1 and H_2 to obtain equivalence on H_3 as well.

The manifolds C_ψ and $C_{\tilde{\psi}}$ are diffeomorphic, via their common fiber-preserving diffeomorphism with the Legendrian they parametrize. As they are smooth manifolds, we may extend this diffeomorphism to a fiber-preserving diffeomorphism F of an open neighborhood of C_ψ with an open neighborhood of $C_{\tilde{\psi}}$. Then the phase function $\bar{\psi} := F^*(\tilde{\psi})$ has the property that $C_{\bar{\psi}} = C_\psi =: C$, and $\psi = \bar{\psi}$ to second order along C. Therefore we have reduced by this initial change of variables to the case in which we may take $\psi, \tilde{\psi}$ equal to second order along C.

We now improve this result to exact equivalence of ψ and $\tilde{\psi}$ on H_3, under the assumption that the functions agree to second order on C. As we have equivalence on H_1, H_2 we may write

$$\psi = \tilde{\psi}_1 + x_1\tilde{\psi}_2 + x_1x_2\psi_3.$$

We may expand in a Taylor series on H_3:

$$\tilde{\psi}_3 - \psi_3 = \frac{1}{2}\big(\nabla'_{\vec{v}}\psi\big)^t B\big(\nabla'_{\vec{v}}\psi\big)$$

for some matrix $B = B(\vec{x}, \vec{y}, \vec{v})$, where we define $\nabla'\psi = (\partial_{v_1}\psi, \partial_{v_2}(\psi_2 + x_2\psi_3), \partial_{v_3}\psi_3)$. Observe as in [14] that the non-degeneracy assumptions on $\psi_3, \tilde{\psi}_3$ means precisely that $\det(I + B_{33}\partial^2_{v_3v_3}\psi_3) \neq 0$ where B_{33} is the $(3,3)$ block of the matrix B. We now expand

$$\psi(\vec{x}, \vec{y}, \tilde{\vec{v}}) - \psi(\vec{x}, \vec{y}, \vec{v}) = (\tilde{\vec{v}} - \vec{v}) \cdot \partial_{\vec{v}}\psi + O\big((\tilde{\vec{v}} - \vec{v})^2\big).$$

Set

$$(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) - (v_1, v_2, v_3) = (x_1 x_2 w_1, x_2 w_2, w_3) \cdot \nabla'_{\vec{v}} \psi$$

where $w_i = w_i(\vec{x}, \vec{y}, \vec{v})$ is a matrix for $i = 1, 2, 3$. We thus have

$$\psi(\vec{x}, \vec{y}, \tilde{\vec{v}}) - \psi(\vec{x}, \vec{y}, \vec{v}) = x_1 x_2 \big(\nabla'_{\vec{v}} \psi\big)^t \big(w + O\big(w^2\big)\big)\big(\nabla'_{\vec{v}} \psi\big).$$

We want

$$\begin{aligned}\psi(\vec{x}, \vec{y}, \tilde{\vec{v}}) - \psi(\vec{x}, \vec{y}, \vec{v}) &= \tilde{\psi}(\vec{x}, \vec{y}, \vec{v}) - \psi(\vec{x}, \vec{y}, \vec{v}) \\ &= x_1 x_2 \big(\tilde{\psi}_3(\vec{x}, \vec{y}, \vec{v}) - \psi_3(\vec{x}, \vec{y}, \vec{v})\big).\end{aligned}$$

We thus need to solve

$$x_1 x_2 \big(\nabla'_{\vec{v}} \psi\big)^t \big(w + O\big(w^2\big)\big)\big(\nabla'_{\vec{v}} \psi\big) = \frac{x_1 x_2}{2} \big(\nabla'_{\vec{v}} \psi\big)^t B \big(\nabla'_{\vec{v}} \psi\big)$$

for w. This can be accomplished for B small, i.e. for ψ_3 and $\tilde{\psi}_3$ close, by the inverse function theorem; Lemma 3.1.7 of [14] enables us to extend to the case of arbitrary ψ_3, $\tilde{\psi}_3$ using the hypotheses on the signatures of $\partial^2_{v_3 v_3} \psi_3$ and $\partial^2_{v_3 v_3} \tilde{\psi}_3$. □

4.5. Legendrian distributions

Let m, r_1, r_2 be real numbers, let $N = \dim X$, let $G \subset {}^{s\Phi}T^*_{\mathrm{mf}} X$ be a Legendre submanifold, and let ν be a smooth non-vanishing scattering-fibered half-density. The set of (half-density) Legendre distributions of order $(m; r_1, r_2)$ associated to G, denoted $I^{m, r_1, r_2}(X, G; {}^{s\Phi}\Omega^{\frac{1}{2}})$, is the set of half-density distributions that can be written in the form $u_1 + u_2 + (u_3 + u_4 + u_5)\nu$, such that

- u_1 is a Legendre distribution of order $(m; r_1)$ associated to G and supported away from H_2,
- u_2 is a Legendre distribution of order $(m; r_2)$ associated to G and supported away from H_1 (both of these are defined in [10]),
- u_3 is given by an finite sum of local expressions of the form

$$\begin{aligned}&\iiint e^{i\psi(x_1, x_2, \vec{y}, \vec{v})/\mathbf{x}} a(\vec{x}, \vec{y}, \vec{v}) \\ &\quad \times x_3^{m-(k_1+k_2+k_3)/2+N/4} x_2^{r_2-(k_1+k_2)/2-f_2/2+N/4} x_1^{r_1-k_1/2-f_1/2+N/4} \, dv_1\, dv_2\, dv_3,\end{aligned} \tag{4.15}$$

  with $v_i \in \mathbb{R}^{k_i}$, a smooth and compactly supported, f_i the dimension of the fibers of H_i and $\psi = \psi_1 + x_1 \psi_2 + x_1 x_2 \psi_3$ a phase function locally parametrizing G near a corner point $q \in \partial_{12} G$, as in Section 4.2,
- u_4 is given by a finite sum of terms of the form

$$\iint e^{i(\psi_1+x_1\psi_2)/\mathbf{x}} b(x_1,x_2,x_3,y_1,y_2,y_3,v_1,v_2) \\ \times x_2^{r_2-(k_1+k_2)/2-f_2/2+N/4} x_1^{r_1-k_1/2-f_1/2+N/4}\, dv_1\, dv_2 \tag{4.16}$$

with ψ_1, ψ_2 and f_i as above, b smooth with support compact and $O(x_3^\infty)$ at mf, and

- $u_5 \in \dot{C}^\infty(X)$. (We use the notation $\dot{C}^\infty(X)$ for $\mathbf{x}^\infty \mathcal{C}^\infty(X)$.)

Remark. The convention regarding orders is as follows: the order increases as the distribution gets 'better,' i.e. vanishes more rapidly, and it is 'zeroed' so that $N/4$ is critical for L^2-membership, i.e. for a distribution with positive symbol, u is in L^2 iff all the orders are more than $N/4$. This somewhat peculiar choice is to conform to the order convention for pseudodifferential operators (apart from the change of sign) on a manifold of dimension n, whose kernels are in L^2 provided the order is less than $-n/2 = -N/4$, where $N = 2n$ is the dimension of the space on which the kernel is defined. In any case, the order convention agrees with that of [10,23] and [11].

Proposition 4.6. *Let $u \in I^{m,r_1,r_2}(X, G; {}^{s\Phi}\Omega^{\frac{1}{2}})$ be a Legendre distribution, and let ψ be any local parametrization of some subset $U \subset G$. After localization to U, the u may be expressed as an oscillatory integral with respect to ψ, modulo $\dot{C}^\infty(X)$.*

Proof. We give a brief sketch of this proof, which follows standard lines.

By definition, u can be written with respect to *some* phase function parametrizing G, say ψ'.

One can modify any phase function (without changing the Legendrian parametrized) by adding a non-degenerate quadratic form $Q_1(w_1) + x_1 Q_2(w_2) + x_1 x_2 Q_3(w_3)$ in extra variables $w_i \in \mathbb{R}^{l_i}$. This does not change, modulo $O(\mathbf{x}^\infty)$, the distributions that can be written with respect to the phase function since the extra oscillatory factor only contributes a factor

$$c x_1^{l_1/2} x_2^{(l_1+l_2)/2} x_3^{(l_1+l_2+l_3)/2}$$

which is just an adjustment of the orders. However it allows us to change the number of phase variables of each type, and the corresponding signature. By modifying both ψ and ψ' in this way we may arrange that they satisfy the conditions of Proposition 4.5. (This requires some mod 2 compatibility conditions between $\dim v_i$ and the signature of $d^2_{v_i v_i}\psi_i$ but these are automatically satisfied; see Theorem 3.1.4 of [14].) One can then use the change of variables given by Proposition 4.5 to write u in terms of the modified phase function ψ, and therefore in terms of ψ itself. □

4.6. Symbol calculus

The previous proposition implies that there is a symbol calculus for Legendre distributions. Since this follows standard lines, we omit the proof.

Let X be a scattering-fibered manifold with codimension 3 corners, let $N = \dim X$, and let G be a Legendre submanifold. Let $\mathbf{x}$ denote the distinguished total boundary defining function for X, and x_1, x_2, x_3 be the set of boundary defining functions for each $H_i \in M_1(X) \setminus \{\mathrm{mf}\}$. The Maslov bundle M and the E-bundle are defined via the scattering structure over the interior of G and extend to smooth bundles over the whole of G (that is, they are smooth up to each boundary of G at ${}^{\mathrm{s}\Phi}T_{H_i\cap\mathrm{mf}}X$); see [11]. We define $N^*_{\mathrm{mf}}\partial X$ to be the bundle over mf given by differentials df of smooth functions f on X vanishing at each boundary hypersurface. It is a line bundle with non-zero section $d\mathbf{x}$.

We define the symbol bundle $S^{[m]}(G)$ of order m over G to be the bundle

$$S^{[m]}(G) = M(G) \otimes E \otimes \left|N^*_{\mathrm{mf}}\partial X\right|^{m-N/4}, \tag{4.17}$$

following [11].

Proposition 4.7. *The symbol map for Legendre distributions, defined in the interior of G [23], extends by continuity to give an exact sequence*

$$\begin{aligned} 0 &\to I^{m+1,r_1,r_2}\big(X,G;{}^{\mathrm{s}\Phi}\Omega^{\frac12}\big) \to I^{m,r_1,r_2}\big(X,G;{}^{\mathrm{s}\Phi}\Omega^{\frac12}\big) \\ &\to x_1^{r_1-m}x_2^{r_2-m}\mathcal{C}^\infty\big(G,\Omega_b^{\frac12}\otimes S^{[m]}(G)\big) \to 0. \end{aligned}$$

If $P \in {}^{\mathrm{s}\Phi}\mathrm{Diff}(X;{}^{\mathrm{s}\Phi}\Omega^{\frac12})$ has principal symbol p and $u \in I^{m,r_1,r_2}(X,G;{}^{\mathrm{s}\Phi}\Omega^{\frac12})$, then $Pu \in I^{m,r_1,r_2}(X,G;{}^{\mathrm{s}\Phi}\Omega^{\frac12})$ and

$$\sigma^m(Pu) = (p \restriction G)\sigma^m(u).$$

Thus, if p vanishes on G, then $Pu \in I^{m+1,r_1,r_2}(X,G;{}^{\mathrm{s}\Phi}\Omega^{\frac12})$. The symbol of order $m+1$ of Pu in this case is given by

$$\left(-i\mathcal{L}_{{}^{\mathrm{sc}}H_p} - i\left(\frac12 + m - \frac N4\right)\frac{\partial p}{\partial \nu_1} + p_{\mathrm{sub}}\right)\sigma^m(u) \otimes |d\mathbf{x}|, \tag{4.18}$$

where ${}^{\mathrm{sc}}H_p$ is the scattering Hamilton vector field of p (that is, the Hamilton vector field multiplied by $\mathbf{x}^{-1}$ and restricted to G), ν_1 is the coordinate in the coordinate system (3.14)*, and p_{sub} is the subprincipal symbol of P.*

Remark. The subprincipal symbol of a differential operator has the following properties: (i) for a multiplication operator f, it is the $O(\mathbf{x})$ part of the Taylor series of f at $x_3 = 0$. (ii) The subprincipal symbol of $i(V - V^*)$, where V is a real vector field, is zero. (iii) The subprincipal symbol of the composition of two differential operators P and Q is $\sigma(P)\sigma_{\mathrm{sub}}(Q) + \sigma(Q)\sigma_{\mathrm{sub}}(P) - i/2\{\sigma(P),\sigma(Q)\}$. These properties in fact uniquely determine the subprincipal symbol for any differential operator.

Example 4.8. A very simple example may help to illustrate the symbol calculus. Let P be the differential operator $x_1x_2x_3(x_1D_{x_1})$, $D = -i\partial$, and let u be the Legendre distribution

$$u = x_3^{m+N/4}x_2^{r_2-f_2/2+N/4}x_1^{r_1-f_1/2+N/4}\left|\frac{dx_1\,dx_2\,dx_3\,dy_1\,dy_2\,dy_3}{x_3^{N+1}x_2^{N+1-f_2}x_1^{N+1-f_1}}\right|^{1/2},$$

a distribution of order (m,r_1,r_2) associated to the zero section (which is a Legendrian submanifold). We assume that the half-density factor above, which is a smooth non-vanishing scattering-fibered half-density, is covariant constant. Hence $Pu = -i(r_1 - f_1/2 + N/4)x_1x_2x_3u$.

In terms of the symbol calculus, the symbol of P is ν_1 which vanishes on the Legendrian, so Proposition 4.7 tells us that the result is a Legendre distribution is of order $(m+1, r_1, r_2)$ and the principal symbol is given by (4.18).

The symbol of u at $x_3 = 0$ is the half-density (where for convenience we write u as a b-half-density on the Legendrian)

$$\sigma^m(u) = x_2^{r_2-m} x_1^{r_1-m} \left| \frac{dx_1\, dx_2\, dx_3\, dy_1\, dy_2\, dy_3}{x_3 x_2 x_1} \right|^{1/2} \otimes \left| d(x_1 x_2 x_3) \right|^{m-N/4}.$$

The scattering Hamilton vector field of P is $x_1 \partial_{x_1}$. The subprincipal symbol of P is $-i(N - 1 - f_1)$, which is easily obtained from the fact that $P + P^*$ has vanishing subprincipal symbol. Finally $\partial p / \partial \nu_1 = 1$. Thus, noting that $\mathcal{L}_{{}^{\mathrm{sc}}H_p}$ leaves the b-half-density dx_1/x_1 invariant, (4.18) says that

$$\begin{aligned} \sigma^{m+1}(Pu) &= \left(-i(r_1 - m) - i\left(\frac{1}{2} + m - N/4 \right) + \frac{-i}{2}(N - 1 - f_1) \right) \sigma^m(u) \\ &= -i(r_1 - f_1/2 + N/4)\sigma^m(u) \end{aligned}$$

in agreement with the direct calculation.

4.7. Residual space

The residual space for the spaces of Legendre distributions $I^{m,r_1,r_2}(X, G; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ is, by definition, the intersection of these spaces over all $m \in \mathbb{R}$, and is denoted $I^{\infty,r_1,r_2}(X, G; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$. Let us consider the special case that $X = Y \times [0, \epsilon)$ as in Example 3.5. In that case, for a fixed $x_3 > 0$ an element of $I^{m,r_1,r_2}(X, G; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ is (after division by $|dx_3|^{1/2}$) a Legendre distribution on Y belonging to $I^{r_1-1/4, r_2-1/4}(Y, x_3^{-1}G_2, {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$, in particular associated to the Legendre submanifold $x_3^{-1}G_2$, where $G_2 = \partial_2 G$ is the boundary of G over H_2 and the factor x_3^{-1} scales the cotangent variables (this follows immediately from (4.16)). We may regard the spaces $I^{r_1-1/4, r_2-1/4}(Y, x_3^{-1}G_2, {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ as forming a smooth bundle over $(0, \epsilon)_{x_3}$. The residual space $I^{\infty,r_1,r_2}(X, G; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ can then be described as a smooth, $O(x_3^\infty)$ section of this bundle on $[0, \epsilon)$. We write this (with a minor abuse of notation) as

$$I^{\infty,r_1,r_2}\big(X, G; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}}\big) \equiv x_3^\infty C^\infty\big([0, \epsilon]; I^{r_1-1/4, r_2-1/4}\big(Y, x_3^{-1}G_2; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}}\big)\big) \otimes |dx_3|^{1/2}.$$

We remark that the rather irritating drop of $1/4$ in the orders, when regarding elements of $I^{\infty,r_1,r_2}(X, G; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ as distributions on Y parametrized by x_3, follows from the order convention where a Legendre distribution is order $N/4$ if it is borderline L^2. In terms of (4.15) and (4.16) it can be seen since f_i and N both decrease by 1 when we fix a value of $x_3 > 0$.

5. Intersecting Legendre distributions

For a manifold with boundary, M, intersecting Legendre distributions were defined in [10] as the analogue of the intersecting Lagrangian distributions of [20]. They are related to a pair of

Legendre submanifolds in ${}^{\mathrm{sc}}T^*_{\partial M}M$ that intersect cleanly in codimension 1. Here we define the analogue for a scattering-fibered manifold with codimension two corners.

5.1. Intersecting Legendre submanifolds

Let X be a scattering-fibered manifold with codimension two corners. By Proposition 3.4, locally near the corner, there are local coordinates (x_1, x_2, y_1, y_2) with respect to which the main face is given by $x_2 = 0$, the boundary hypersurface H_1 is given by $x_1 = 0$ and the fibration at H_1 is given by $(x_2, y_1, y_2) \mapsto y_1$. We define a pair of intersecting Legendre submanifolds, $\tilde{L} = (L, \Lambda)$, in ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}}X$, to be a pair consisting of a Legendre submanifold L in the sense of Definition 4.1, thus a manifold with boundary meeting ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}\cap H_1}X$ transversally, together with a submanifold Λ with codimension two corners of ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}}X$ which is Legendre, transversal to ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}\cap H_1}X$, and satisfying the following:

- Λ has two boundary hypersurfaces, $\partial_1\Lambda = \Lambda \cap {}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}\cap H_1}X$, and $\partial_L\Lambda = L \cap \Lambda$;
- the intersection $L \cap \Lambda$ is clean;
- the images $L_1 = \tilde{\phi}_1(\partial_1 L)$ and $\Lambda_1 = \tilde{\phi}_1(\partial_1\Lambda)$ (which are Legendre in ${}^{\mathrm{s}\Phi}N^*Z_1$ by Proposition 4.3) form an intersecting pair of Legendre submanifolds in ${}^{\mathrm{s}\Phi}N^*Z_1$.

5.2. Parametrization

A local parametrization of (L, Λ) near $q \in L \cap \Lambda \cap {}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}\cap H_1}X$ is a function of the form

$$\Phi(x_1, y_1, y_2, v_1, v_2, s) = \phi_{00}(y_1, v_1) + s\phi_{10}(y_1, v_1, s) + x_1\phi_{01}(x_1, y_1, y_2, v_1, v_2) + x_1 s\phi_{11}(x_1, y_1, y_2, v_1, v_2, s), \tag{5.1}$$

defined in a neighborhood of $q' = (0, y_1^*, y_2^*, v_1^*, v_2^*, 0)$ in $\mathrm{mf} \times \mathbb{R}^{k_1+k_2} \times [0, \infty)$ such that $d_{v_1,v_2,s}\Phi = 0$ at q', $q = (0, y_1, y_2, d(\Phi/x_1x_2)(q'))$, Φ satisfies the non-degeneracy hypothesis

$$ds, \quad d\phi_{10}, \quad d\left(\frac{\partial\phi_{00}}{\partial v_1^j}\right), \quad d\left(\frac{\partial\phi_{01}}{\partial v_2^k}\right) \quad \text{are linearly independent at } q',$$

and near q,

$$L = \left\{\left(x_1, y_1, y_2, d\left(\frac{\Phi}{\mathbf{x}}\right)\right) \;\Big|\; s = 0,\ d_{v_1,v_2}(\Phi) = 0\right\},$$

$$\Lambda = \left\{\left(x_1, y_1, y_2, d\left(\frac{\Phi}{\mathbf{x}}\right)\right) \;\Big|\; s \geqslant 0,\ d_s\Phi = 0,\ d_{v_1,v_2}\Phi = 0\right\}.$$

5.3. Existence of parametrizations

For simplicity we shall prove existence of parametrizations only in a special case, which nevertheless suffices for our application. We shall assume that L is a 'conormal bundle' of a submanifold $N \subset \mathrm{mf}$ that meets the boundary $x_1 = 0$ transversally. We shall further assume that the projection from $L \cap \Lambda$ to N everywhere has maximal rank. We need only prove existence of a parametrization locally near a point $q \in L \cap \Lambda \cap {}^{s\Phi}T^*_{\mathrm{mf}\cap H}X$ as above, since existence near other points has been shown in [10] or in the previous section.

By Proposition 4.3, the boundary of N necessarily fibers over a submanifold $N_1 \subset Z_1$. Choose coordinates $y_1 = (y_1', y_1^\sharp)$ on Z_1 so that $N_1 = \{y_1' = 0\}$ locally. We can then find a splitting $y_2 = (y_2^\flat, y_2^\sharp)$ with respect to which N locally takes the form $\{y_1' = 0, y_2^\flat = 0\}$. Our assumption on L reads as follows in local coordinates:

$$L = \{y_1' = 0, y_2^\flat = 0, \mu_1^\sharp = 0, \mu_2^\sharp = 0, \nu_1 = 0, \nu_2 = 0\}.$$

Let us first parametrize the intersecting pair of Legendrians (L_1, Λ_1). We first claim that one can split (after a suitable linear change of y_1 variables) y_1' as $y_1' = (y_1^\flat, y_1^\natural)$, where $\dim y_1^\natural = 1$, in such a way that $(y_1^\natural, y_1^\sharp, \mu_1^\flat)$ form coordinates locally on Λ_1. In fact, we have local coordinates $(y_1^\sharp, \mu_1')$ on L_1. The second assumption above has the consequence that local coordinates on $L_1 \cap \Lambda_1$ are furnished by $y_1^\sharp$ and all but one of the μ_1' variables; after making a linear change of variables, we may split $\mu' = (\mu_1^\flat, \mu_1^\natural)$ dual to the splitting of the y_1' variables so that $y_1^\sharp$ and $\mu_1^\flat$ are coordinates on $L_1 \cap \Lambda_1$. It then follows from the condition that Λ_1 is Legendre with respect to the contact structure $d\nu_1 + \mu_1 \cdot dy_1$ that $(y_1^\natural, y_1^\sharp, \mu_1^\flat)$ furnish coordinates on Λ_1 locally. Thus we can write the other variables $y_1^\flat, \mu_1^\natural, \mu_1^\sharp, \nu_1$, restricted to Λ_1, uniquely as smooth functions of $(y_1^\natural, y_1^\sharp, \mu_1^\flat)$. In particular we have

$$\nu_1 = N_{1,1}(y_1^\natural, y_1^\sharp, \mu_1^\flat), \qquad y_1^\flat = Y_{1,1}^\flat(y_1^\natural, y_1^\sharp, \mu_1^\flat),$$

and each of these functions is $O(y_1^\natural)$ since they vanish at $L_1 \cap \Lambda_1$ which is $\Lambda_1 \cap \{y_1^\natural = 0\}$.

Then a local parametrization of (L_1, Λ_1) is given by

$$(y_1^\natural - s)v_1^\natural + (y_1^\flat - Y_{1,1}^\flat(s, y_1^\sharp, v_1^\flat)) \cdot v_1^\flat + N_{1,1}(s, y_1^\sharp, v_1^\flat);$$

the reasoning is the same as in the proof of Proposition 4.4.

We now parametrize (L, Λ) in a neighborhood of a point on $L \cap \Lambda \cap \{x_1 = 0\}$. In this case $(x_1, y_1^\sharp, y_2^\sharp, \mu_1^\flat, \mu_1^\natural, \mu_2^\flat)$ furnish local coordinates on L and $(x_1, y_1^\natural, y_1^\sharp, y_2^\sharp, \mu_1^\flat, \mu_2^\flat)$ furnish local coordinates on Λ. As before we write

$$\nu_1 = N_1(x_1, y_1^\natural, y_1^\sharp, y_2^\sharp, \mu_1^\flat, \mu_2^\flat), \qquad y_1^\flat = Y_1^\flat(x_1, y_1^\natural, y_1^\sharp, y_2^\sharp, \mu_1^\flat, \mu_2^\sharp),$$
$$Y_2^\flat(x_1, y_1^\natural, y_1^\sharp, y_2^\sharp, \mu_1^\flat, \mu_2^\sharp).$$

Due to the conditions on L and Λ at $x_1 = 0$ we have $N_1 = N_{1,1} + x_1 N_{1,2}$, $Y_1^\flat = Y_{1,1}^\flat + x_1 Y_{1,2}^\flat$ and $Y_2^\flat = x_1 Y_{2,2}^\flat$ for some smooth functions $N_{1,2}$, $Y_{1,2}^\flat$ and $Y_{2,2}^\flat$. Then the function

$$\begin{aligned}
&(y_1^\natural - s)v_1^\natural + (y_1^\flat - Y_1^\flat(x_1, y_1^\sharp, y_2^\sharp, v_1^\flat, s, v_2^\flat)) \cdot v_1^\flat + N_1(x_1, y_1^\sharp, y_2^\sharp, v_1^\flat, s, v_2^\flat) \\
&\quad = (y_1^\natural - s)v_1^\natural + (y_1^\flat - Y_{1,1}^\flat(s, y_1^\sharp, v_1^\flat)) \cdot v_1^\flat + N_{1,1}(s, y_1^\sharp, v_1^\flat) + O(x_1) \\
&\quad = y_1^\natural v_1^\natural + y_1^\flat \cdot v_1^\flat + y_2^\flat \cdot v_2^\flat + O(s)
\end{aligned}$$

has the form (5.1) and parametrizes (L, Λ).

5.4. Equivalence of phase functions

Two phase functions Φ, $\tilde{\Phi}$ are said to be *equivalent* if they have the same number of phase variables of each type v_1, v_2 and there exist maps

$$V_1(x_1, \vec{y}, \vec{v}, s), \quad V_2(x_1, \vec{y}, \vec{v}, s), \quad S(x_1, \vec{y}, \vec{v}, s)$$

such that

$$\tilde{\Phi}(\vec{x}, \vec{y}, V_1, V_2, S) = \Phi.$$

Proposition 5.1. *The phase functions* $\Phi = \phi_{00} + s\phi_{10} + x_1\phi_{01} + x_1 s\phi_{11}$ *and* $\tilde{\Phi} = \tilde{\phi}_{00} + s\tilde{\phi}_{10} + x_1\tilde{\phi}_{01} + x_1 s\tilde{\phi}_{11}$ *are locally equivalent iff*

(1) *they parametrize the same Legendrians,*
(2) *they have the same number of phase variables of the form* v_1, v_2 *separately,*
(3)

$$\operatorname{sgn} d^2_{v_1}(\phi_{00} + s\phi_{10}) = \operatorname{sgn} d^2_{v_1}(\tilde{\phi}_{00} + s\tilde{\phi}_{10}),$$
$$\operatorname{sgn} d^2_{v_2}(\phi_{01} + s\phi_{11}) = \operatorname{sgn} d^2_{v_2}(\tilde{\phi}_{01} + s\tilde{\phi}_{11}).$$

Proof. Using the equivalence of phase functions in the codimension one case from [10] to solve the problem at $x_1 = 0$, and using Proposition 4.5 to solve at $s = 0$, we may assume that we have reduced to the case

$$\tilde{\Phi} = \phi_{00} + s\phi_{10} + x_1\phi_{01} + x_1 s\tilde{\phi}_{11}.$$

As before, we may further reduce by an initial change of variables to the case in which we may take Φ, $\tilde{\Phi}$ equal to second order along $C = \{d_s\Phi = d_{\vec{v}}\Phi = 0\}$.

As the two functions agree to second order on C, we may expand in a Taylor series

$$\tilde{\phi}_{11} - \phi_{11} = \frac{1}{2}\big(\nabla'_{\vec{v},s}\Phi\big)^t B\big(\nabla'_{\vec{v},s}\Phi\big)$$

where we define $\nabla'_{\vec{v},s}\Phi = (\partial_{v_1}\Phi, \partial_{v_2}(\phi_{01} + s\phi_{11}), \partial_s\Phi)$. We further expand

$$\Phi(x_1, \vec{y}, \tilde{\vec{v}}, \tilde{s}) - \Phi(x_1, \vec{y}, \tilde{\vec{v}}, s) = (\tilde{\vec{v}} - \vec{v}) \cdot \partial_{\vec{v}}\Phi + (\tilde{s} - s) \cdot \partial_s\Phi + O\big((\tilde{\vec{v}} - \vec{v})^2 + (\tilde{s} - s)^2\big).$$

Set

$$(\tilde{v}_1, \tilde{v}_2, \tilde{s}) - (v_1, v_2, s) = (x_1 w_1, w_2, x_1 w_3) \cdot \nabla'_{\vec{v}}\Phi$$

for $w_i = w_i(x_1, \vec{y}, \vec{v}, s)$. We thus have

$$\Phi(x_1, \vec{y}, \tilde{\vec{v}}, \tilde{s}) - \Phi(x_1, \vec{y}, \vec{v}, s) = x_1\big(\nabla'_{\vec{v},s}\Phi\big)^t\big(w + O\big(w^2\big)\big)\big(\nabla'_{\vec{v},s}\Phi\big).$$

We want

$$\begin{aligned}\Phi(x_1,\vec{y},\tilde{\vec{v}},\tilde{s})-\Phi(x_1,\vec{y},\vec{v},s)&=\tilde{\Phi}(x_1,\vec{y},\vec{v},s)-\Phi(x_1,\vec{y},s)\\&=x_1s\big(\tilde{\phi}_{11}(x_1,\vec{y},\vec{v},s)-\phi_{11}(x_1,\vec{y},\vec{v},s)\big).\end{aligned}$$

We thus need to solve

$$x_1(\nabla'\Phi)^t\big(w+O\big(w^2\big)\big)(\nabla'\Phi)=\frac{x_1s}{2}(\nabla'\Phi)^tB(\nabla'\Phi)$$

for w. This can always be accomplished for s small by the inverse function theorem. □

5.5. *Intersecting Legendre distributions*

Let ν be a smooth scattering-fibered half-density. The set of Legendre distributions of order (m,r) associated to $\tilde{L}$, denoted $I^{m,r}(X,\tilde{L};{}^{s\Phi}\Omega^{\frac{1}{2}})$, is the set of half-density distributions of the form $u=u_1+u_2+u_3+(u_4+u_5+u_6)\nu$, where

- $u_1\in I^{m,r}(X,\Lambda;{}^{s\Phi}\Omega^{\frac{1}{2}})$ with the microsupport of u_1 disjoint from $\partial\Lambda$,
- $u_2\in I^{m+1/2,r+1/2}(X,L;{}^{s\Phi}\Omega^{\frac{1}{2}})$,
- u_3 has support disjoint from H_1 and is an intersecting Legendre distribution of order (m,r) associated to (L,Λ) as defined in [10],
- u_4 is a finite sum of terms, each supported near $\mathrm{mf}=\{x_2=0\}$, with an expression

$$x_1^{j_1}x_2^{j_2}\int_0^\infty\iint e^{i\Phi(x_1,y_1,y_2,v_1,v_2,s)/x_1x_2}a(x_1,x_2,y_1,y_2,v_1,v_2,s)\,dv_1\,dv_2\,ds,$$

$$j_1=r-\frac{k_1+1}{2}+\frac{N}{4}-\frac{f}{2},\quad j_2=m-\frac{k_1+k_2+1}{2}+\frac{N}{4}\tag{5.2}$$

  where $v_i\in\mathbb{R}^{k_i}$, a is smooth and compactly supported, f is the dimension of the fibers of H_1, and $\Phi=\phi_{00}+s\phi_{01}+x_1\phi_{10}+x_1s\phi_{11}$ locally parametrizes (L,Λ) near a point $q\in L\cap\Lambda\cap{}^{s\Phi}T^*_{\mathrm{mf}\cap H_1}X$, as in (5.1),
- u_5 is a finite sum of terms of the form

$$x_1^{r-\frac{k_1+1}{2}+\frac{N}{4}-\frac{f}{2}}\int_0^\infty\int e^{i(\phi_{00}+s\phi_{01})/x_1x_2}b(x_1,y_1,x_2,y_2,v_1,s)\,dv_1\,ds,\tag{5.3}$$

  where ϕ_{00},ϕ_{01}, f and v_i are as above, and b is smooth and $O(x_2^\infty)$ at mf, and
- $u_6\in\dot{C}^\infty(X)$.

As in Section 4, u_3 can be written with respect to any local parametrization, up to an error in $\dot{C}^\infty(X)$. This follows from the equivalence result above and the argument in Proposition 4.6.

5.6. *Symbol calculus*

The geometry of intersecting Legendre distributions is such that the symbol on L has a $1/\rho_1$ singularity at Λ, where ρ_1 is a boundary defining function for $\partial\Lambda\subset L$, while the symbol on

Λ is smooth up to the boundary at $L \cap \Lambda$. This allows one to symbolically solve away error terms at L in the equation $Pu = f$ where f is Legendrian on L, and the principal symbol of P vanishes simply at Λ; what happens is that the singularities of the solution u propagate from $L \cap \Lambda$ along Λ. The formal symbol calculus for intersecting Legendre distributions on X follows readily from the codimension one case; we follow the description from [11] closely.

Let $\tilde{L} = (L, \Lambda)$ be a pair of intersecting Legendre submanifolds as in Section 5.1. We consider $u \in I^{m,r}(X, \tilde{L}; {}^{s\Phi}\Omega^{\frac{1}{2}})$. The symbol of u takes values in a bundle over $L \cup \Lambda$. To define this bundle, let ρ_1 be a boundary defining function for $\partial\Lambda$ as a submanifold of L, and ρ_0 be a boundary defining function for $\partial\Lambda$ as a submanifold of Λ. Note that the symbol on L is defined by continuity from distributions in $I^{m+1/2,r+1/2}(X, L; {}^{s\Phi}\Omega^{\frac{1}{2}})$ microsupported away from Λ, and takes values in

$$x_1^{r-m}\rho_1^{-1}\mathcal{C}^\infty\big(\Omega_b^{1/2}(L) \otimes S^{[m+1/2]}(L)\big) = x_1^{r-m}\rho_1^{-1/2}\mathcal{C}^\infty\big(\Omega_b^{1/2}(L \setminus \partial\Lambda) \otimes S^{[m+1/2]}(L)\big), \tag{5.4}$$

while the symbol on Λ, defined by continuity from distributions in $I^{m,r}(X, \Lambda; {}^{s\Phi}\Omega^{\frac{1}{2}})$ microsupported away from $\partial\Lambda$, takes values in

$$x_1^{r-m}\rho_0^{1/2}\mathcal{C}^\infty\big(\Omega_b^{1/2}(\Lambda) \otimes S^{[m]}(\Lambda)\big).$$

Melrose and Uhlmann showed that the Maslov factors were canonically isomorphic on $L \cap \Lambda$, so $S^{[m+1/2]}(L)$ is naturally isomorphic to $S^{[m]}(\Lambda) \otimes |N^*_{\mathrm{mf}}\partial X|^{1/2}$ over $L \cap \Lambda$. Canonical restriction of the half-density factors to $L \cap \Lambda$ gives terms in $\mathcal{C}^\infty(\Omega^{\frac{1}{2}}(L \cap \Lambda) \otimes S^{[m]}(\Lambda) \otimes |N^*_L\partial\Lambda|^{-1/2} \otimes |N^*\partial X|^{1/2})$ and $\mathcal{C}^\infty(\Omega^{\frac{1}{2}}(L \cap \Lambda) \otimes S^{[m]}(\Lambda) \otimes |N^*_\Lambda\partial\Lambda|^{1/2})$ respectively. In fact $|N^*_L\partial\Lambda| \otimes |N^*_\Lambda\partial\Lambda| \otimes |N^*_{\mathrm{mf}}\partial X|^{-1}$ is canonically trivial; an explicit trivialization is given by

$$\big(d\rho_0, d\rho_1, (x_1x_2)^{-1}\big) \mapsto (x_1x_2)^{-1}\omega(V_{\rho_0}, V_{\rho_1}) \restriction L \cap \Lambda, \tag{5.5}$$

where V_{ρ_i} are the Hamilton vector fields of the functions ρ_i, and ω is the standard symplectic form. Thus the two bundles are naturally isomorphic over the intersection.

We define the bundle $S^{[m]}(\tilde{L})$ over $\tilde{L} = L \cup \Lambda$ to be that bundle such that smooth sections of $\Omega_b^{1/2}(\tilde{L}) \otimes S^{[m]}(\tilde{L})$ are precisely those pairs (a, b) of sections of $\rho_1^{-1}\mathcal{C}^\infty(\Omega^{1/2}(L) \otimes S^{[m+1/2]}(L))$ and $\rho_0^{1/2}\mathcal{C}^\infty(\Omega_b^{1/2}(\Lambda) \otimes S^{[m]}(\Lambda))$ such that

$$\rho_1^{1/2}b = e^{i\pi/4}(2\pi)^{1/4}\rho_0^{-1/2}a \quad \text{at } L \cap \Lambda \tag{5.6}$$

under the above identification of bundles (cf. Eq. (3.7) of [11]). The symbol maps of order m on Λ and $m + 1/2$ on L then extend in a natural way to a symbol map of order m on $\tilde{L}$ taking values in $\Omega_b^{1/2}(\tilde{L}) \otimes S^{[m]}(\tilde{L})$.

Proposition 5.2. *The symbol map on $\tilde{L}$ yields an exact sequence*

$$0 \to I^{m+1,r}\big(X, \tilde{L}; {}^{s\Phi}\Omega^{\frac{1}{2}}\big) \to I^{m,r}\big(X, \tilde{L}; {}^{s\Phi}\Omega^{\frac{1}{2}}\big) \to x_1^{r-m}\mathcal{C}^\infty\big(\tilde{L}, \Omega_b^{\frac{1}{2}} \otimes S^{[m]}\big) \to 0. \tag{5.7}$$

Moreover, if we consider just the symbol map to Λ, there is an exact sequence

$$0 \to I^{m+1,r}\big(X, \tilde{L}; {}^{s\Phi}\Omega^{\frac{1}{2}}\big) + I^{m+\frac{1}{2},r}\big(X, L; {}^{s\Phi}\Omega^{\frac{1}{2}}\big) \to I^{m,r}\big(X, \tilde{L}; {}^{s\Phi}\Omega^{\frac{1}{2}}\big)$$
$$\to x_1^{r-m}\mathcal{C}^\infty\big(\Lambda, \Omega^{\frac{1}{2}} \otimes S^{[m]}\big) \to 0. \tag{5.8}$$

If $P \in {}^{s\Phi}\operatorname{Diff}(X; {}^{s\Phi}\Omega^{\frac{1}{2}})$ *has principal symbol* p *and* $u \in I^{m,r}(X, \tilde{L}; {}^{s\Phi}\Omega^{\frac{1}{2}})$*, then* $Pu \in I^{m,r}(X, \tilde{L}; {}^{s\Phi}\Omega^{\frac{1}{2}})$ *and*

$$\sigma^m(Pu) = (p \restriction \tilde{L})\sigma^m(u).$$

Thus, if p *vanishes on* Λ*, then* Pu *is an element of* $I^{m+1,r}(X, \tilde{L}; {}^{s\Phi}\Omega^{\frac{1}{2}}) + I^{m,r}(X, L; {}^{s\Phi}\Omega^{\frac{1}{2}})$ *by* (5.8). *The symbol of order* $m+1$ *of* Pu *on* Λ *in this case is given by* (4.18).

5.7. Residual space

The residual space for the spaces of intersecting Legendre distributions $I^{m,r}(X, \tilde{L}; {}^{s\Phi}\Omega^{\frac{1}{2}})$ is

$$I^{\infty,r}\big(X, \tilde{L}; {}^{s\Phi}\Omega^{\frac{1}{2}}\big) = \bigcap_m I^{m,r}\big(X, \tilde{L}; {}^{s\Phi}\Omega^{\frac{1}{2}}\big).$$

If $X = Y \times [0, \epsilon]_{x_2}$ where Y is a manifold with boundary, then the residual space may be identified with

$$x_2^\infty C^\infty\big([0, \epsilon]; I^{r-1/4}\big(X, \big(x_2^{-1}L_1, x_2^{-1}\Lambda_1\big); {}^{s\Phi}\Omega^{\frac{1}{2}}\big)\big) \otimes |dx_2|^{1/2}.$$

6. Legendrian distributions with conic points

Here we shall define a more singular situation in which the Legendrian $G \subset {}^{s\Phi}T^*_{\mathrm{mf}}X$ has conic singularities. We first give a precise description of 'having conic singularities.' We recall the notion of real blowup. Suppose that X is a compact manifold with corners and $S \subset X$ a compact product-type submanifold,[5] which means that locally near any point s of S, there are local coordinates $x_1, \ldots, x_j, y = (y_1, \ldots, y_k)$, $x_i \in [0, \epsilon)$, $y \in B(0, \epsilon) \subset \mathbb{R}^k$, with s corresponding to the origin of coordinates, such that S is given locally by the vanishing of some subset of these coordinates. Then by $[X; S]$ we denote the blow-up of X around S. As a set this is the union of $X \setminus S$ with the inward pointing spherical normal bundle at S, which we denote $\tilde{S}$. $[X; S]$ carries a natural differentiable structure making it a compact manifold with corners, such that $\tilde{S}$ is one of its boundary hypersurfaces.

Definition 6.1. Let X be a manifold with corners and $S \subset X$ a submanifold, and $G \subset X$ a closed set which is a submanifold locally near every point of $G \setminus S$. We say that G has conic singularities at S if the lift of G to $[X; S]$, i.e. the closure of $G \setminus S$ in $[X; S]$, is a smooth product-type submanifold $\hat{G}$ which is transverse to $\tilde{S}$.

Legendre submanifolds with conic singularities have been defined already in two different settings in [23] and [10], and we review these definitions for the convenience of the reader.

[5] All the submanifolds considered in this paper are product-type submanifolds; from here on we refer to them simply as submanifolds for brevity.

The original setting of Melrose–Zworski was that of a Legendre submanifold $G \subset {}^{sc}T^*_{\partial X}X$ in the boundary of the scattering cotangent bundle of a manifold X with boundary, which has conic singularities at a submanifold $J^\sharp$ which is the span of a smooth projectable Legendrian $G^\sharp$. Projectability means that the restriction of the projection $\pi : {}^{sc}T^*_{\partial X}X \to \partial X$ to $G^\sharp$ is a diffeomorphism, or in other words $G^\sharp$ is a graph over ∂X; then $J^\sharp$, which is obtained by replacing each point of $G^\sharp$ by the ray in ${}^{sc}T^*_{\partial X}X$ through this point, is a submanifold with dimension equal to $\dim X$ (one greater than $\dim G^\sharp$). By choosing coordinates judiciously we may arrange that, in local coordinates (y, ν, μ) on ${}^{sc}T^*_{\partial X}X$ given by writing scattering covectors as

$$\nu d\left(\frac{1}{x}\right) + \mu \cdot \frac{dy}{x},$$

we have $G^\sharp = \{\nu = 1, \mu = 0\}$, and $J^\sharp = \{\mu = 0\}$. We say that $(G, G^\sharp)$ are a pair of Legendre submanifolds with conic points, or a Legendrian conic pair for short, if G has conic singularities at $J^\sharp$, i.e. G lifts to $[{}^{sc}T^*_{\partial X}X; J^\sharp]$ to a smooth submanifold $\hat{G}$ transverse to $\tilde{J}^\sharp$.

We recall what it means to locally parametrize $(G, G^\sharp)$. Transversality of $\hat{G}$ to the span of $G^\sharp$ at $q \in \hat{G} \cap \tilde{J}^\sharp$ means that $d|\mu| \neq 0$ at q; we may assume (after making a linear change of coordinates in y) that the first component μ^1 of μ is a local boundary defining function for the blowup of the span of $G^\sharp$ near q. Assuming this, a local parametrization of $(G, G^\sharp)$ near q is given by a phase function of the form

$$1 + s\psi(y, s, v), \quad s \geqslant 0,\ v \in \mathbb{R}^k$$

defined in a neighborhood of $(y^*, 0, v^*)$, satisfying the non-degeneracy condition

$$d_{y^1}\psi \quad \text{and} \quad d_{y,v}\left(\frac{\partial \psi}{\partial v^i}\right) \quad \text{are linearly independent at } \big(y^*, 0, v^*\big), \tag{6.1}$$

such that $\hat{G}$ is given by

$$\hat{G} = \left\{ d\left(\frac{1 + s\psi(y, s, v)}{x}\right) \,\Big|\, d_{s,v}\psi = 0 \right\}. \tag{6.2}$$

Furthermore we require that $d_v\psi(y^*, 0, v^*) = 0$ and that the point on $\hat{G}$ corresponding to $(y^*, 0, v^*)$ is q. To be precise, the meaning of (6.2) is that when the set on the RHS is *lifted* to the space $[{}^{sc}T^*_{\partial X}X; J^\sharp]$ obtained by blowup of $J^\sharp$ it coincides with $\hat{G}$. We remark that the correspondence in (6.2) lifts to a diffeomorphism from $\{(y, s, v) \mid d_{s,v}\psi = 0\}$ to $\hat{G}$, so the blowup is implicit in the parametrization ψ.

Next we recall the definition of Legendre conic pairs in the case of a manifold X with fibered boundary and codimension 2 corners. Let $G^\sharp$ be a smooth projectable Legendrian submanifold of ${}^{s\Phi}T^*_{mf}X$, and G be a Legendrian submanifold of ${}^{s\Phi}T^*_{mf}X$ which is smooth away from $G^\sharp$ and which has conic singularities at $J^\sharp \subset {}^{s\Phi}T^*_{mf}X$, where $J^\sharp$ is the span of $G^\sharp$. Let $\hat{G}$ denote the lift of G to $[{}^{s\Phi}T^*_{mf}X; J^\sharp]$; we assume that it is transverse to both boundary hypersurfaces of $[{}^{s\Phi}T^*_{mf}X; J^\sharp]$ (that is, transverse to both the lift of ${}^{s\Phi}T^*_{mf\cap H_1}X$ and the lift of $J^\sharp$). Let $\partial_1\hat{G}$ and $\partial_1 G^\sharp$ denote the boundary hypersurface of $\hat{G}$, respectively $G^\sharp$, at (the lift of) ${}^{s\Phi}T^*_{mf\cap H_1}X$. We say that $(G, G^\sharp)$ form a conic Legendrian pair if $\partial_1 \hat{G}$ and $\partial_1 G^\sharp$ fiber over the same Legendrian submanifold $G_1 \subset {}^{s\Phi}N^*Z_1$ as base.

Remark. This implies that the fibers of $\partial_1 \hat{G} \to G_1$ and the fibers of $\partial_1 G^\sharp \to G_1$ form an intersecting pair of Legendre submanifolds in ${}^{sc}T^*_{\partial F}F$ for each fiber $F \subset H_1$. The reasoning is analogous to that in Proposition 4.3.

This differs from the structure above only over the codimension two corner of X, so we shall consider a point of ${}^{s\Phi}T^*_{H_1\cap \mathrm{mf}}X$ lying over the codimension two corner. We shall use coordinates (x_1, x_2, y_1, y_2) as in Section 4.1, and associated dual coordinates $(\nu_1, \nu_2, \mu_1, \mu_2)$ defined by writing scattering-fibered covectors in the form

$$\nu_1 d\left(\frac{1}{x_1x_2}\right) + \nu_2 d\left(\frac{1}{x_2}\right) + \mu_1 \cdot \frac{dy_1}{x_1x_2} + \mu_2 \cdot \frac{dy_2}{x_2}.$$

For definiteness we shall assume that $G_2^\sharp$ is the submanifold $\{\nu_1 = 1, \nu_2 = 1, \mu_1 = 0, \mu_2 = 0\}$ which is parametrized by the function $1 + x_1$. This is the form of $G_2^\sharp$ that turns up in our application (and in any case, it can always be arranged by a change of coordinates). Then the span of $G_2^\sharp$ is given by

$$J_2^\sharp = \{x_2 = 0, \nu_1 = \nu_2, \mu_1 = 0, \mu_2 = 0\}. \tag{6.3}$$

The corresponding Legendrian in ${}^{s\Phi}N^*Z_1$ is

$$G_1^\sharp = \{\nu_1 = 1, \mu_1 = 0\}.$$

The condition of being a conic Legendrian pair means that at $\{x_1 = x_2 = 0\}$, if we set $\nu_1 = 1$, $\mu_1 = 0$ and fix y_1, then we have remaining coordinates (y_2, ν_2, μ_2) and these are local coordinates on the fiber ${}^{sc}T^*_{\partial F}F$ which is a contact manifold with contact form $d\nu_2 + \mu_2 \cdot dy_2$; we are then asking that the restriction of G_2 to this fiber have a conic singularity at (and therefore becomes smooth after blowup of) $\{\mu_2 = 0\}$. In particular $d|\mu_2| \neq 0$ on $\hat{G}_2$ at its intersection with $J_2^\sharp$.

We next recall the form of a parametrization of $(G_2, G_2^\sharp)$ near a point $q \in \hat{G}_2$ on the codimension two corner of $\hat{G}_2$, i.e. lying above $x_1 = 0$ and on $\tilde{J}_2^\sharp$. Assume that coordinates have been chosen so that $dy_2^1 \neq 0$ at q. A local parametrization of $(G_2, G_2^\sharp)$ near q is given by a phase function of the form

$$1 + x_1 + sx_1\psi(x_1, y_1, y_2, s, v), \quad s \geqslant 0, \ v \in \mathbb{R}^k$$

defined in a neighborhood of $(0, y_1^*, y_2^*, 0, v^*)$, satisfying the non-degeneracy condition

$$d_{y_2^1}\psi \quad \text{and} \quad d_{y_2,v}\left(\frac{\partial \psi}{\partial v^i}\right) \quad \text{are linearly independent at } \big(0, y_1^*, y_2^*, 0, v^*\big), \tag{6.4}$$

such that $\hat{G}_2$ is given by

$$\hat{G}_2 = \left\{ d\left(\frac{1 + x_1 + s x_1 \psi(x_1, y_1, y_2, s, v)}{x_1 x_2} \right) \,\Big|\, d_{s,v}\psi = 0 \right\}. \tag{6.5}$$

Furthermore we require that $d_v\psi(0, y_1^*, y_2^*, 0, v^*) = 0$ and that the point on $\hat{G}_2$ corresponding to $(0, y_1^*, y_2^*, 0, v^*)$ is q. The precise meaning of (6.5) is that when the set in (6.5) is lifted to the space obtained by blowup of $J_2^\sharp$ it coincides with $\hat{G}_2$.

Remark. As in the case above, the correspondence in (6.5) lifts to a diffeomorphism from $\{(x_1, y_1, y_2, s, v) \mid d_{s,v}\psi = 0\}$ to $\hat{G}$, so the blowup is implicit in the parametrization ψ. Also, if we fix a value of $\bar{y}_1$, or equivalently fix a point in the base $G_1^\sharp$ of the fibration $\tilde{\phi}_{12}|_G$, then the function $\psi(0, \bar{y}_1, y_2, s, v)$ parametrizes the fiber (which is a Legendrian conic pair in ${}^{\mathrm{sc}}T^*_{\partial F}F$).

6.1. Legendre submanifolds with conic points

We now define Legendre submanifolds with conic points in two new situations, although both are closely analogous to the ones reviewed above.

6.1.1. Codimension two corners

Suppose that X is a scattering-fibered manifold with corners of codimension 2. Let x_2 be a boundary defining function for the main face mf and x_1 a boundary defining function for the fibered face H_1. Let $G_1^\sharp$ be a projectable Legendrian in ${}^{s\Phi}N^*Z_1$, and let J be the lift of the span of $G_1^\sharp$ to ${}^{s\Phi}T^*_{H_1\cap \mathrm{mf}}X$ via the fibration $\tilde{\phi}_{12}$. Let $\hat{G}$ be the lift of G to $[{}^{s\Phi}T^*_{\mathrm{mf}}X, J]$. We shall say that $(G, G_1^\sharp)$ form a conic Legendrian pair of submanifolds if $\hat{G}$ has conic singularities at J, i.e. is transverse to both boundary hypersurfaces of $[{}^{s\Phi}T^*_{\mathrm{mf}}X, J]$ (that is, transverse to both the lift of ${}^{s\Phi}T^*_{\mathrm{mf}\cap H_1}X$ and to the lift $\tilde{J}$ of J).

Let $\partial_1\hat{G}$ and $\partial_\sharp\hat{G}$ denote the boundary hypersurfaces of $\hat{G}$. Also, let G_1 denote the projection of $G \cap \{x_1 = 0\}$ to ${}^{s\Phi}N^*Z_1$ via $\tilde{\phi}_{12}$. It follows from the definition that G_1 has conic singularities at $G_1^\sharp$; let $\hat{G}_1$ be the lift of G to $[{}^{s\Phi}N^*Z_1; J_1]$ where J_1 is the span of $G_1^\sharp$. Then, as a consequence of $(G, G_1^\sharp)$ being a conic Legendrian pair, the fibers of the map $\partial_1\hat{G} \to \hat{G}_1$ are Legendrian, while $\partial_\sharp\hat{G}$ is itself Legendrian with respect to a natural contact structure on the lift of $G_1^\sharp$ to $\tilde{J}$ defined by the leading part of χ.

6.1.2. Codimension three corners

Now let us assume that X is a scattering-fibered manifold with corners of codimension 3, and consider a Legendrian submanifold $G \subset {}^{s\Phi}T^*_{\mathrm{mf}}X$ which is singular at the boundary. We use the notation H_1, H_2, H_3 for boundary hypersurfaces of X and x_1, x_2, x_3 for boundary defining functions as in Section 4.1. Let $G_1 = \phi_{13}(G \cap \{x_1 = 0\})$ and $G_2 = \phi_{23}(G \cap \{x_2 = 0\})$. Here we could consider the cases where either G_1 or G_2 have conic singularities at some Legendrian $G_1^\sharp$ or $G_2^\sharp$; however, we shall only consider the case where G_2 has conic singularities since that is the case that occurs in our applications. Thus, we consider a case where G_1 is smooth, but G_2 has conic singularities, and indeed that there is a projectable smooth Legendrian $G_2^\sharp \subset {}^{s\Phi}N^*Z_2$ such that $(G_2, G_2^\sharp)$ form a Legendrian conic pair. Thus, if J_2 is the span of $G_2^\sharp$ in ${}^{s\Phi}N^*Z_2$, then

G_2 lifts to a smooth manifold $\hat{G}_2$ in $[{}^{s\Phi}N^*Z_2; J_2]$ that is transversal to $\tilde{J}_2$. Let J denote the preimage of J_2 inside ${}^{s\Phi}T^*_{H_2\cap\mathrm{mf}}X$ via $\tilde{\phi}_{23}: {}^{s\Phi}T^*_{\mathrm{mf}\cap H_2}X \to {}^{s\Phi} N^*Z_2$. We shall say that $(G, G_2^\sharp)$ form a conic Legendrian pair if G has conic singularities at J, i.e. the lift $\hat{G}$ of G to $[{}^{s\Phi}T^*_{\mathrm{mf}}X; J]$ is smooth and transverse to $\tilde{J}$ as well as to the lifts of ${}^{s\Phi}T^*_{\mathrm{mf}\cap H_1}X$ and ${}^{s\Phi}T^*_{\mathrm{mf}\cap H_2}X$.

The manifold $\hat{G}$ is a manifold with corners of codimension three. The boundary at ${}^{s\Phi}T^*_{\mathrm{mf}\cap H_1}X$ (more precisely, at the lift of this to $[{}^{s\Phi}T^*_{\mathrm{mf}}X; J]$) is denoted $\partial_1\hat{G}$, the boundary at the lift of ${}^{s\Phi}T^*_{\mathrm{mf}\cap H_2}X$ is denoted $\partial_2\hat{G}$ and the boundary at $\tilde{J}$ is denoted $\partial_\sharp\hat{G}$. It follows from the definition that $\partial_1\hat{G}$ fibers over G_1 with Legendrian fibers relative to χ_{13}, that $\partial_2\hat{G}$ fibers over $\hat{G}_2$ via a map ϕ_{23}^G induced from $\tilde{\phi}_{23}$, with fibers that are Legendrian for the contact structure χ_{23}, and $\partial_\sharp\hat{G}$ is Legendrian for the contact structure on the lift of $G_2^\sharp$ to $\tilde{J}$ given by the leading part of χ.

6.2. Parametrization

6.2.1. Codimension two corners

In this situation, the lifted submanifold $\hat{G}$ is a manifold with corners of codimension two. The two boundary hypersurfaces of $\hat{G}$ are denoted $\partial_1\hat{G}$ (at $x_1 = 0$ and away from $\tilde{J}$) and $\partial_\sharp\hat{G}$ (at $\{x_1 = 0\} \cap \tilde{J}$). Locally near a point on the interior of $\partial_1\hat{G}$ the situation is as for a smooth Legendrian distribution, so consider a point q on $\partial_\sharp\hat{G}$. We need to distinguish two cases: the first is that q is on the codimension two corner $\partial_1\hat{G} \cap \partial_\sharp\hat{G}$, and the second is that q is on the interior of $\partial_\sharp\hat{G}$.

To make things concrete we shall assume that coordinates have been chosen so that $G_1^\sharp$ is the Legendrian $\{\nu_1 = 1, \mu_1 = 0\}$, and that μ_1^1 is a local boundary defining function for $\tilde{J}$. Then a local parametrization of $(G, G_1^\sharp)$ near q is a phase function of the form

$$\psi(s, x_1, y_1, y_2, v_1, v_2) = 1 + s\psi_1(y_1, s, v_1) + x_1\psi_2\left(s, \frac{x_1}{s}, y_1, y_2, v_1, v_2\right), \quad s \geqslant 0,\ v_i \in \mathbb{R}^{k_i}, \tag{6.6}$$

defined in a neighborhood of $(0, 0, y_1^*, y_2^*, v_1^*, v_2^*)$, where ψ_1 and ψ_2 are smooth, satisfying the non-degeneracy condition

$$d_{y_1^1}\psi_1, d_{y_1,v_1}\left(\frac{\partial\psi_1}{\partial v_1^i}\right) \quad \text{and} \quad d_{y_2,v_2}\left(\frac{\partial\psi_2}{\partial v_2^j}\right) \quad \text{are linearly independent at } \big(0, 0, y_1^*, y_2^*, v_1^*, v_2^*\big), \tag{6.7}$$

and such that $\hat{G}$ is given by

$$\left\{d\left(\frac{\psi}{x_1x_2}\right) \,\Big|\, d_{s,v_1,v_2}\psi = 0\right\}. \tag{6.8}$$

Furthermore we require that $d_{s,v_1,v_2}\psi(0, 0, y_1^*, y_2^*, v_1^*, v_2^*) = 0$, and that the point on $\hat{G}$ corresponding to $(0, 0, y_1^*, y_2^*, v_1^*, v_2^*)$ is q.

Remark. The non-degeneracy conditions imply that the subset

$$C_\psi = \left\{(s, u, y_1, y_2, v_1, v_2) \,\middle|\, d_s\psi = 0,\ d_{v_1}(\psi_1 + u\psi_2) = 0,\ d_{v_2}\psi_2 = 0,\ u = \frac{x_1}{s}\right\}$$

is a submanifold[6] and that (6.8) defines a diffeomorphism between C_ψ and $\hat{G}$ locally near $(0, 0, y_1^*, y_2^*, v_1^*, v_2^*)$, so this indeed corresponds to the usual notion of non-degenerate parametrization. Notice that under this correspondence s is a boundary defining function for $\partial_\sharp\hat{G}$ and u is a boundary defining function for $\partial_1\hat{G}$.

In the second case, since we are away from the lift of $\{x_1 = 0\}$, given by $x_1/s = 0$, we do not need the special variable $s \geqslant 0$, and we obtain the following: a local parametrization of $(G, G_1^\sharp)$ near q is a phase function of the form

$$\psi(x_1, y_1, y_2, v) = 1 + x_1\psi(x_1, y_1, y_2, v) \tag{6.9}$$

defined in a neighborhood of $(0, y_1^*, y_2^*, v^*)$, satisfying the non-degeneracy condition

$$d_{y_1,y_2,v}\left(\frac{\partial\psi}{\partial v^j}\right) \quad \text{are linearly independent at } \big(0, y_1^*, y_2^*, v^*\big), \tag{6.10}$$

such that $\hat{G}$ is given by

$$\left\{d\left(\frac{\psi}{x_1x_2}\right) \,\middle|\, d_v\psi = 0\right\}. \tag{6.11}$$

Furthermore we require that $d_v\psi(0, y_1^*, y_2^*, v^*) = 0$, and that the point on $\hat{G}$ corresponding to $(0, y_1^*, y_2^*, v^*)$ is q.

Remark. This is very similar to the parametrization of a smooth Legendrian, but with respect to a different fibration on H_1, where the base of the fibration is a point. This can also be seen by noting that blowing up $\{\mu_1 = 0, x_1 = 0\}$ amounts to introducing the variable $M_1 = \mu_1/x_1$ as a smooth coordinate. This is dual to dy_1/x_2 and so corresponds to a coordinate along the fiber of the fibration rather than on the base. This is related to the blowup of the submanifold W in Section 11.

Remark. Notice that, if we localize the phase function in (6.6) to the region $x_1/s \geqslant \epsilon > 0$, then it can be expressed in the form

$$1 + x_1\big(w\psi_1(y_1, x_1w, v_1) + \psi_2(x_1w, 1/w, y_1, y_2, v_1, v_2)\big), \quad w = \frac{s}{x_1},$$

and is therefore of the form (6.9). So these two forms of parametrization are consistent on their overlapping regions of validity.

[6] The partial derivative $d_s\psi$ in the equation above is taken keeping x_1 fixed, not keeping u fixed.

6.2.2. Codimension three corners

Now the lifted submanifold $\hat{G}$ is a manifold with corners of codimension three. The three boundary hypersurfaces are denoted $\partial_1\hat{G}$ (at $x_1 = 0$), $\partial_2\hat{G}$ (at $x_2 = 0$ and away from $G_2^\sharp$), and $\partial_\sharp\hat{G}$ (at $\{x_2 = 0\} \cap G_2^\sharp$). Locally near a point on the interior of $\partial_1\hat{G}$ or $\partial_2\hat{G}$ the situation is as for a smooth Legendrian distribution, so consider a point q on $\partial_\sharp\hat{G}$. If q is not also in $\partial_1\hat{G}$ then the situation is (locally) the codimension two situation described above, so we assume that $q \in \partial_\sharp\hat{G} \cap \partial_1\hat{G}$. We need to distinguish two cases: the first is that q is on the codimension three corner $\partial_1\hat{G} \cap \partial_2\hat{G} \cap \partial_\sharp\hat{G}$ and the second is that q is on the interior of $\partial_\sharp\hat{G} \cap \partial_1\hat{G}$.

To make things concrete we shall assume that coordinates have been chosen so that $G_1^\sharp$ is the Legendrian $\{\nu_1 = 1, \mu_1 = 0\}$, that $G_2^\sharp$ is the Legendrian $\{\nu_1 = \nu_2 = 1, \mu_1 = 0, \mu_2 = 0\}$, so that $1 + x_1$ parametrizes $G_2^\sharp$, and that μ_2^1 is a local boundary defining function for the third boundary hypersurface of $\hat{G}$. Let $q \in \hat{G}$ lie on the codimension three corner. A non-degenerate parametrization of $(G, G_2^\sharp)$ near $q \in \hat{G}$ is then a smooth phase function Ψ of the form

$$\begin{aligned}&\Psi(s, x_1, x_2, y_1, y_2, y_3, v_2, v_3)\\&\quad = 1 + x_1 + s x_1 \psi_2(s, x_1, y_1, y_2, v_2)\\&\qquad + x_1 x_2 \psi_3(s, x_1, x_2/s, y_1, y_2, y_3, v_2, v_3), \quad s \geqslant 0,\ v_i \in \mathbb{R}^{k_i},\end{aligned} \tag{6.12}$$

where ψ_2 and ψ_3 are smooth, with Ψ non-degenerate in the sense that such that

$$d_{y_2}\psi_2, \quad d_{y_2,v_2}\left(\frac{\partial \psi_2}{\partial v_2^i}\right) \quad \text{and} \quad d_{y_3,v_3}\left(\frac{\partial \psi_3}{\partial v_3^i}\right) \quad \text{are linearly independent at } q' \tag{6.13}$$

with

$$\hat{G} = \left\{ d\left(\frac{\Psi}{x_1 x_2 x_3}\right)(q'') \,\middle|\, q'' \in C_\Psi \right\} \quad \big(\text{lifted to } \big[{}^{s\Phi}T^*_{\partial X}X; J\big]\big) \text{ near } q, \tag{6.14}$$

and such that q' corresponds to q under this correspondence.

The non-degeneracy condition implies that there is a local diffeomorphism between the set

$$C_\Psi = \big\{(s, x_1, u, y_1, y_2, y_3, v_2, v_3) \,\big|\, d_s\Psi = d_{v_2}\Psi = d_{v_3}\Psi = 0 \text{ at } (s, x_1, su, y_1, y_2, y_3, v_2, v_3)\big\}$$

and $\hat{G}$.

In the second case, as we are localizing away from the boundary of $\{x_2 = 0\}$, given by $x_2/s = 0$, we do not need the special variable s. In this case, a non-degenerate parametrization of $(G, G^\sharp)$ near $q \in \hat{G}$ is a smooth phase function Ψ of the form

$$1 + x_1 x_2 \psi(x_1, x_2, y_1, y_2, y_3, v) \tag{6.15}$$

defined on a neighborhood of $q' = (0, 0, y_1^*, y_2^*, y_3^*, v^*)$ with Ψ non-degenerate in the sense that such that

$$d_{y_2}\psi, \quad d_{y_2,v}\left(\frac{\partial \psi}{\partial v^i}\right) \quad \text{are linearly independent at } q' \tag{6.16}$$

with

$$\hat{G} = \left\{ d\left(\frac{\Psi}{x_1x_2x_3}\right)(q'') \,\Big|\, q'' \in C_\Psi \right\} \quad \big(\text{lifted to } \big[{}^{s\Phi}T^*_{\partial X}X; J\big]\big) \text{ near } q, \tag{6.17}$$

and such that q' corresponds to q under this correspondence.

6.3. Existence of parametrizations

For brevity we only show the existence of parametrizations in the codimension 3 setting. The construction is analogous (and simpler) in the codimension 2 setting. We use coordinates as in the proof of Proposition 4.4 above, in which we have $G_1 = \{\bar{\nu}_1 = 1, \mu_1 = 0\}$ and $G_2^\sharp = \{\bar{\nu}_1 = (1 + x_1)\bar{\nu}_2, \mu_1 = 0, \mu_2 = 0\}$.

First let $q \in \hat{G}$ lie on the codimension three corner of $\hat{G}$. Recall that $\hat{G}$ fibers over $\hat{G}_2$ with fibers that are Legendrian submanifolds of ${}^{sc}T^*_{\partial F}F$; therefore we can find a splitting of the y_3 coordinates, $y_3 = (y_3^\flat, y_3^\sharp)$, so that $(y_3^\sharp, \mu_3^\flat)$ form coordinates on the fiber over $\pi(q) \in \hat{G}_2$. Also, as in [10], Proposition 3.5, we can find a splitting of the y_2 coordinates, $y_2 = (y_2^1, y_2^\flat, y_2^\sharp)$, where $y_2^\flat = (y_2^2, \dots, y_2^j)$, so that, with $\hat{\mu}^\flat = (\mu_2^2/\mu_2^1, \dots, \mu_2^j/\mu_2^1)$, $\hat{\mu}_2^\sharp = \mu_2^\sharp/\mu_2^1$, the functions $(y_2^\sharp, \mu_2^1, \hat{\mu}^\flat)$ form coordinates $\hat{G}_2$ near $\pi(q)$. It follows that

$$\mathcal{Z} = \big(x_1, \mu_2^1, x_2/\mu_2^1, y_1, y_2^\sharp, \hat{\mu}_2^\flat, y_3^\sharp, \mu_3^\flat\big)$$

form coordinates on $\hat{G}$ near q.

We now follow the proof of Proposition 4.4 as closely as possible. Writing $\bar{\nu}_i$, $y_i^\flat$ and $\mu_i^\sharp$ in terms of the coordinates $\mathcal{Z}$ on $\hat{G}$, we have

$$\begin{aligned}
&\bar{\nu}_1 = N_1(\mathcal{Z}),\\
&\bar{\nu}_2 = N_2(\mathcal{Z}),\\
&\bar{\nu}_3 = N_3(\mathcal{Z}),\\
&y_i^\flat = Y_i^\flat(\mathcal{Z}), \quad i = 2, 3, \text{ on } \hat{G},\\
&\mu_1 = M_1(\mathcal{Z}),\\
&\hat{\mu}_2^\sharp = M_2^\sharp(\mathcal{Z}),\\
&\mu_3^\sharp = M_3^\sharp(\mathcal{Z}).
\end{aligned} \tag{6.18}$$

Since G is Legendrian, we have

$$\begin{aligned}
&dN_1 + N_2\,dx_1 + x_1N_3\,dx_2 - M_1 \cdot dy_1\\
&\quad - x_1\mu_2^1\big(dY_2^1 + \hat{\mu}_2^\flat \cdot dY_2^\flat + M_2^\sharp \cdot dy_2^\sharp\big) - x_1x_2\big(\mu_3^\flat \cdot dY_3^\flat - M_3^\sharp \cdot dy_3^\sharp\big) = 0.
\end{aligned} \tag{6.19}$$

We claim that the function (where we substitute s for μ_2^1, v_2 for $\hat{\mu}_2^\flat$ and v_3 for $\mu_3^\flat$)

$$\Psi(x_1, x_2, s, y_1, y_2, y_3, v_2, v_3) \\ = N_1 + x_1 s\big(\big(y_2^1 - Y_2^1\big) + \big(y_2^\flat - Y_2^\flat\big)\cdot v_2\big) + x_1x_2\big(\big(y_3^\flat - Y_3^\flat\big)\cdot v_3\big) \tag{6.20}$$

is a local parametrization of G. First, observe that N_1 is equal to 1 at $x_1 = 0$ and is equal to $1 + x_1 + O(s)$ at $s = 0$ since the value of $\bar{\nu}_1$ on $G_2^\sharp$ is equal to $1 + x_1$. Hence it has the form (6.12).

Second, suppose that $d_s\Psi = 0$. This means that

$$d_sN_1 + x_1\big(y_2^1 - Y_2^1\big) - x_1 s\, d_s\big(Y_2^1 + Y_2^\flat \cdot v_2\big) - x_1x_2\, d_sY_3^\flat \cdot v_3 = 0.$$

Using the ds component of (6.19) and dividing by an overall factor of x_1 we now obtain $y_2^1 = Y_2^1$. In a similar way, the conditions that $d_{v_i}\Psi = 0$ imply that $y_i^\flat = Y_i^\flat$, $i = 2, 3$. This also shows the non-degeneracy condition, since $d(d_s\psi_2) = dy_2^1$, $d(d_{v_2}\psi_2) = y_2^\sharp$, $d(\partial_{v_3}\psi_3) = dy_3$ at q; these are manifestly linearly independent differentials.

To see that the set

$$G' = \left\{ d\left(\frac{\Psi}{x_1x_2x_3}\right) \,\Big|\, d_{s,v_2,v_3}\Psi = 0 \right\}$$

coincides with G locally near q, consider the value of μ_2^1 on G'; it is given by $d_{y_2^1}\Psi/x_1 = s$. Similarly, the value of $\mu_2^\flat$ is given by sv_2, and the value of $\mu_3^\flat$ is given by v_3. So we can re-identify these values. Next consider the value of $\bar{\nu}_1$ on G'. It is given by the value of Ψ, that is, by (6.20). This simplifies to N_1 when $d_{s,v_i}\Psi = 0$, since we have $y_2^1 = Y_2^1$ when $d_s\Psi = 0$ and $y_i^\flat = Y_i^\flat$ when $d_{v_i}\Psi = 0$. Next consider the value of $\bar{\nu}_2$. This is given by $d_{x_1}\Psi$ which is equal to

$$d_{x_1}N_1 - x_1 s\, d_{x_1}Y_1^\flat \cdot \mu_1^\flat - x_1 s\, d_{x_1}Y_2^\flat \cdot \mu_2^\flat - x_1x_2\, d_{x_1}Y_3^\flat \cdot \mu_3^\flat$$

(again using $y_i^\flat = Y_i^\flat$ when $d_{s,v_i}\Psi = 0$). Since the dx_1 component of (6.19) vanishes, this is equal to N_2. So $\bar{\nu}_2 = N_2$ on G'. In a similar way we deduce that $\bar{\nu}_3 = N_3$, and $\mu_i^\sharp = M_i^\sharp$ on G'. It follows that G' coincides with G.

6.4. *Equivalence of phase functions*

We sketch the proof of equivalence of parametrizations only in the codimension three case.

Two phase functions Ψ, $\tilde{\Psi}$ are said to be *equivalent* if they have the same number of phase variables of each type v_1, v_2 and there exist maps

$$V_1(x_1, \vec{y}, \vec{v}, s),\ V_2(x_1, \vec{y}, \vec{v}, s),\ S(x_1, \vec{y}, \vec{v}, s)$$

such that

$$\tilde{\Psi}(\vec{x}, \vec{y}, V_1, V_2, S) = \Psi.$$

Proposition 6.2. *The phase functions* $\Psi = 1 + x_1 + sx_1\psi_2 + x_1x_2\psi_3$, $\tilde{\Psi} = 1 + x_1 + sx_1\tilde{\psi}_2 + x_1x_2\tilde{\psi}_3$ *are locally equivalent iff*

(1) *they parametrize the same Legendrians,*
(2) *they have the same number of phase variables of the form v_2, v_3 separately,*
(3)

$$\operatorname{sgn} d^2_{v_2}(\psi_2) = \operatorname{sgn} d^2_{v_2}(\tilde{\psi}_2),$$
$$\operatorname{sgn} d^2_{v_3}(\psi_3) = \operatorname{sgn} d^2_{v_3}(\tilde{\psi}_3).$$

By using the codimension two result from [10], we reduce to the case

$$\Psi = 1 + x_1 + s x_1 \psi_2 + x_1 x_2 \psi_3, \qquad \tilde{\Psi} = 1 + x_1 + s x_1 \psi_2 + x_1 x_2 \tilde{\psi}_3.$$

As usual, we can arrange that the two functions agree to first order along $C := \{d_{s,v_2}(s\psi_2 + x_2\psi_3), d_{v_3}\psi_3 = 0\}$. Thus

$$\tilde{\psi}_3 - \psi_3 = \frac{1}{2}\big(\nabla'_{\vec{v},s}\Psi\big)^t B\big(\nabla'_{\vec{v},s}\Psi\big)$$

where we define $\nabla'\Psi = (\partial_{v_2}(s\psi_2 + \psi_3), \partial_{v_3}\psi_3, \partial_s(s\psi_2 + \psi_3))$. We now expand

$$\Psi(x_1, \vec{y}, \tilde{\vec{v}}, \tilde{s}) - \Psi(x_1, \vec{y}, \tilde{\vec{v}}, s) = (\tilde{\vec{v}} - \vec{v}) \cdot \partial_{\vec{v}}\Psi + (\tilde{s} - s) \cdot \partial_s \Psi + O\big((\tilde{\vec{v}} - \vec{v})^2 + (\tilde{s} - s)^2\big).$$

Set

$$(\tilde{v}_1, \tilde{v}_2, \tilde{s}) - (v_1, v_2, s) = (x_2 w_1, w_2, x_2 w_3) \cdot \nabla'_{\vec{v}}\psi$$

for $w_i = w_i(x_1, \vec{y}, \vec{v}, s)$. Thus

$$\Psi(x_1, \vec{y}, \tilde{\vec{v}}, \tilde{s}) - \Psi(x_1, \vec{y}, \vec{v}, s) = x_1 x_2 (\nabla'_{\vec{v},s}\Psi)^t \big(w + O\big(w^2\big)\big)\big(\nabla'_{\vec{v},s}\Psi\big).$$

We want

$$\begin{aligned}\Psi(x_1, \vec{y}, \tilde{\vec{v}}, \tilde{s}) - \Psi(x_1, \vec{y}, \vec{v}, s) &= \tilde{\Psi}(x_1, \vec{y}, \vec{v}, s) - \Psi(x_1, \vec{y}, \vec{v}, s)\\ &= x_1 x_2(\tilde{\psi}_3 - \psi_3).\end{aligned}$$

We thus need to solve

$$x_1 x_2 (\nabla'\Psi)^t \big(w + O\big(w^2\big)\big)(\nabla'\Psi) = \frac{x_1 x_2}{2}(\nabla'\Psi)^t B(\nabla'\Psi)$$

for w. This can always be accomplished for B small by the inverse function theorem, and extended to the general case by using the condition on signatures.

6.5. Legendre distributions associated to a conic pair

6.5.1. Codimension 2 corners

Let X be a scattering fibered manifold with codimension 2 corners, let $N = \dim X$ and let $(G, G_1^\sharp)$ be a conic Legendrian pair. Let m, p and r be real numbers, and let ν be a smooth non-vanishing scattering-fibered half-density. A Legendre distribution of order $(m, p; r)$ associated to $(G, G_1^\sharp)$ is a half-density distribution of the form $u_1 + (u_2 + u_3 + u_4 + u_5)\nu$, where

- u_1 is a Legendre distribution of order $(m; r)$ associated to G and microsupported away from J,
- u_2 is given by an finite sum of local expressions

$$u_2(x_1, x_2, y_1, y_2) = \int_{\mathbb{R}^{k_2}} \int_{\mathbb{R}^{k_1}} \int_0^\infty e^{i\psi(s,x_1,y_1,y_2,v_1,v_2)/x_1x_2} a\left(s, \frac{x_1}{s}, x_2, y_1, y_2, v_1, v_2\right) \times x_2^{m-(1+k_1+k_2)/2+N/4} \left(\frac{x_1}{s}\right)^{r-(1+k_1)/2-f_1/2+N/4} s^{p-1-f_1/2+N/4}\, ds\, dv_1\, dv_2, \tag{6.21}$$

  where a is a smooth compactly supported function of its arguments, f_1 is the dimension of the fibers of H_1, and $\psi = 1 + s\psi_2 + x_1\psi_2$ is a phase function locally parametrizing $(G, G_1^\sharp)$ near a point $q \in \partial_1\hat{G} \cap \partial_\sharp\hat{G}$, as in (6.6),
- u_3 is given by an finite sum of local expressions

$$u_2(x_1, x_2, y_1, y_2) = \int_{\mathbb{R}^k} e^{i\psi(x_1,y_1,y_2,v)/x_1x_2} \tilde{a}(x_1, x_2, y_1, y_2, v) \times x_2^{m-k/2+N/4} x_1^{p-1-f_1/2+N/4}\, dv, \tag{6.22}$$

  where $\tilde{a}$ is smooth and compactly supported, and ψ is a local parametrization of $(G, G_1^\sharp)$ near a point $q \in \partial_\sharp\hat{G} \setminus \partial_1\hat{G}$ as in (6.9),
- u_4 is given by

$$u_4(x_1, y_1, z) = \int e^{i(1+s\psi_1)/x_1x_2} b\left(x_1, s, \frac{x_1}{s}, y_1, v, z\right) \times \left(\frac{x_1}{s}\right)^{r-(1+k_1)/2-f_1/2+N/4} s^{p-1-f_1/2+N/4}\, dv_2 \tag{6.23}$$

  where ψ_1 is as above and b is smooth and $O(x_2^\infty)$ at $\mathrm{mf} = \{x_2 = 0\}$, and
- $u_5 \in x_1^{p-f_1/2+N/4} x_2^\infty e^{i/x_1x_2} \mathcal{C}^\infty(X)$ (which always contains $\dot{C}^\infty(X)$ as a subset).

The set of such distributions is denoted $I^{m,p;r}(X, (G, G_1^\sharp); {}^{s\Phi}\Omega^{\frac{1}{2}})$.

6.5.2. Codimension 3 corners

We now assume that X is a scattering-fibered manifold with codimension 3 corners. Let $N = \dim X$, let m, r_1, r_2 and p be real numbers, and let ν be a smooth non-vanishing scattering-fibered half-density on X. A Legendre distribution of order $(m, p; r_1, r_2)$ associated to $(G, G_2^\sharp)$ is a half-density distribution of the form $u_1 + u_2 + (u_3 + u_4 + u_5 + u_6)\nu$, where

- u_1 is a Legendre distribution of order $(m; r_1, r_2)$ associated to G and microsupported away from J,
- u_2 is a Legendre distribution of order $(m, p; r_2)$ associated to $(G, G_2^\sharp)$ and supported away from H_1, as defined above,
- u_3 is given by an finite sum of local expressions

$$\begin{aligned} &u_2(x_1, x_2, x_3, y_1, y_2, y_3) \\ &\quad = \int_{\mathbb{R}^{k_3}} \int_{\mathbb{R}^{k_2}} \int_0^\infty e^{i\Psi(x_1,x_2,y_1,y_2,y_3,s,v_2,v_3)/\mathbf{x}} a\left(x_1, s, \frac{x_2}{s}, x_3, y_1, y_2, y_3, v_2, v_3\right) \\ &\qquad \times x_3^{m-(1+k_2+k_3)/2+N/4} \left(\frac{x_2}{s}\right)^{r_2-(1+k_2)/2-f_2/2+N/4} \\ &\qquad \times s^{p-1-f_2/2+N/4} x_1^{r_1-f_1/2+N/4}\, ds\, dv_2\, dv_3, \end{aligned} \tag{6.24}$$

  where a is a smooth compactly supported function of its arguments, f_i are the dimension of the fibers on H_i, and $\Psi = 1 + x_1 + sx_1\psi_2 + x_1x_2\psi_3$ is a local parametrization of $(G, G_2^\sharp)$ near a corner point q as in (6.12),
- u_4 is given by an finite sum of local expressions

$$\begin{aligned} u_2(x_1, x_2, x_3, y_1, y_2, y_3) = \int_{\mathbb{R}^k} \int_0^\infty & e^{i\Psi(x_1,x_2,y_1,y_2,y_3,v)/\mathbf{x}} \tilde{a}(x_1, x_2, x_3, y_1, y_2, y_3, v) \\ & \times x_3^{m-k/2+N/4} x_2^{p-1-f_2/2+N/4} x_1^{r_1-f_1/2+N/4}\, dv, \end{aligned} \tag{6.25}$$

  where $\tilde{a}$ is smooth and compactly supported, Ψ is a local parametrization of $(G, G_2^\sharp)$ near a point $q \in \partial_1\hat{G} \cap \partial_\sharp\hat{G} \setminus \partial_2\hat{G}$ as in (6.15),
- u_5 is given by

$$\begin{aligned} u_4(x_1, x_2, y_1, y_2, z_3) = \int & e^{i(1+sx_1\psi_2)/\mathbf{x}} b\left(x_1, s, \frac{x_2}{s}, y_1, y_2, z_3, v_2\right) \\ & \times \left(\frac{x_2}{s}\right)^{r_2-(1+k_2)/2-f_2/2+N/4} s^{p-1-f_2/2+N/4} x_1^{r_1-f_1/2+N/4}\, dv_1\, dv_2 \end{aligned} \tag{6.26}$$

  where ψ_2 is as above, b is smooth and $O(x_3^\infty)$ at mf, and
- $u_6 \in x_1^{r_1-f_1/2+N/4} x_2^{p-f_2/2+N/4} x_3^\infty e^{i(1+x_1)/\mathbf{x}} \mathcal{C}^\infty(X)$ (which includes $\dot{C}^\infty(X)$ as a subset).

The set of such distributions is denoted $I^{m,p;r_1,r_2}(X,(G,G_2^\sharp);{}^{s\Phi}\Omega^{\frac{1}{2}})$.

6.6. Symbol calculus

6.6.1. Codimension 2 corners

For a conic pair of Legendre submanifolds $\tilde{G}=(G,G_1^\sharp)$, with $\hat{G}$ the desingularized submanifold obtained by blowing up $J=\tilde{\phi}_{12}^{-1}(\operatorname{span} G_1^\sharp)$, the symbol calculus takes the form

Proposition 6.3. *Let s be a boundary defining function for $\partial_\sharp\hat{G}\subset\hat{G}$. Then there is an exact sequence*

$$\begin{aligned}0&\to I^{m+1,p;r}\big(X,\tilde{G};{}^{s\Phi}\Omega^{\frac{1}{2}}\big)\to I^{m,p;r}\big(X,\tilde{G};{}^{s\Phi}\Omega^{\frac{1}{2}}\big)\\&\to x_1^{r-m}s^{p-m}\mathcal{C}^\infty\big(\hat{G},\Omega_b^{\frac{1}{2}}\otimes S^{[m]}(\hat{G})\big)\to 0.\end{aligned}\tag{6.27}$$

If $P\in{}^{s\Phi}\operatorname{Diff}(X;{}^{s\Phi}\Omega^{\frac{1}{2}})$ has principal symbol p and $u\in I^{m,p;r}(X,\tilde{G};{}^{s\Phi}\Omega^{\frac{1}{2}})$, then $Pu\in I^{m,p;r}(X,\tilde{G};{}^{s\Phi}\Omega^{\frac{1}{2}})$ and

$$\sigma^m(Pu)=(p\restriction\hat{G})\sigma^m(u).$$

Thus, if p vanishes on $\hat{G}$, then Pu is an element of $I^{m+1,p;r}(X,\tilde{G};{}^{s\Phi}\Omega^{\frac{1}{2}})$ by (6.27). *The symbol of order $m+1$ of Pu in this case is given by* (4.18).

6.6.2. Codimension 3 corners

Let $\tilde{G}=(G,G_2^\sharp)$ now be a conic pair of Legendre submanifolds in the codimension three setting. Then we have

Proposition 6.4. *Let s be a boundary defining function for $\partial_\sharp\hat{G}\subset\hat{G}$, and let ρ be a boundary defining function for $\partial_2\hat{G}$ (for example, $\rho=x_2/s$). Then there is an exact sequence*

$$\begin{aligned}0&\to I^{m+1,p;r_1,r_2}\big(X,\tilde{G};{}^{s\Phi}\Omega^{\frac{1}{2}}\big)\to I^{m,p;r_1,r_2}\big(X,\tilde{G};{}^{s\Phi}\Omega^{\frac{1}{2}}\big)\\&\to x_1^{r_1-m}\rho^{r_2-m}s^{p-m}\mathcal{C}^\infty\big(\hat{G},\Omega_b^{\frac{1}{2}}\otimes S^{[m]}(\hat{G})\big)\to 0.\end{aligned}\tag{6.28}$$

If $P\in{}^{s\Phi}\operatorname{Diff}(X;{}^{s\Phi}\Omega^{\frac{1}{2}})$ has principal symbol p and $u\in I^{m,p;r_1,r_2}(X,\tilde{G};{}^{s\Phi}\Omega^{\frac{1}{2}})$, then $Pu\in I^{m,p;r_1,r_2}(X,\tilde{G};{}^{s\Phi}\Omega^{\frac{1}{2}})$ and

$$\sigma^m(Pu)=(p\restriction\hat{G})\sigma^m(u).$$

Thus, if p vanishes on $\hat{G}$, then Pu is an element of $I^{m+1,p;r_1,r_2}(X,\tilde{G};{}^{s\Phi}\Omega^{\frac{1}{2}})$ by (6.27). *The symbol of order $m+1$ of Pu in this case is given by* (4.18).

6.7. Residual space

In the codimension two case, consider the case where $X = Y \times [0,\epsilon]_{x_2}$ where Y is a manifold with boundary. In this case, the residual space

$$I^{\infty,p;r}\big(X,\big(G,G_1^\sharp\big);{}^{s\Phi}\Omega^{\frac12}\big) = \bigcap_m I^{m,p;r}\big(X,\big(G,G_1^\sharp\big);{}^{s\Phi}\Omega^{\frac12}\big)$$

may be identified with

$$x_2^\infty C^\infty\big([0,\epsilon]; I^{r-1/4,p-1/4}\big(X,\big(x_2^{-1}G_1,x_2^{-1}G_1^\sharp\big),{}^{s\Phi}\Omega^{\frac12}\big)\big).$$

In the codimension three case, if $X = Y \times [0,\epsilon]_{x_3}$ where Y is a scattering-fibered manifold with codimension two corners, then the residual space is

$$I^{\infty,p;r_1,r_2}\big(X,\big(G,G_2^\sharp\big);{}^{s\Phi}\Omega^{\frac12}\big) = \bigcap_m I^{m,p;r_1,r_2}\big(X,\big(G,G_2^\sharp\big);{}^{s\Phi}\Omega^{\frac12}\big)$$

and this may be identified with

$$x_3^\infty C^\infty\big([0,\epsilon]; I^{r_2-1/4,p-1/4;r_1-1/4}\big(X,\big(x_3^{-1}G_2,x_3^{-1}G_2^\sharp\big),{}^{s\Phi}\Omega^{\frac12}\big)\big)\otimes|dx_3|^{1/2}.$$

7. Legendrian–Lagrangian distributions

7.1. Legendrian–Lagrangian submanifolds

The final type of distribution we shall introduce are 'Legendrian–Lagrangian distributions' associated to the scattering cotangent bundle ${}^{sc}T^*X$ of a manifold with boundary X. We shall restrict attention to X of the form $X = Y \times [0,h_0)$. We ignore the non-compactness of $Y \times [0,h_0)$ as $h \to h_0$ since we will only be interested in distributions supported near the boundary at $h = 0$.

Let ${}^{sc}\overline{T}^*X$ be the compactification of ${}^{sc}T^*X$ via radial compactification of each fiber. This is a manifold with corners of codimension two; its boundary hypersurfaces are the fiberwise radial compactification of ${}^{sc}T^*_{\partial X}X$, which we denote scl ('semiclassical limit'), and the new hypersurface at 'fiber-infinity,' which we shall denote fi. Fiber-infinity has a natural contact structure given by $\rho\sum_i \eta_i dy_i$ in local coordinates y on Y (where η are the dual cotangent coordinates), where ρ is a boundary defining function for fi (e.g. $\rho = 1/|\eta|$). If $\eta_1/|\eta| > 0$ locally then we may take $\rho = 1/\eta_1$ and then the contact form takes the form $dy_1 + \sum_{i\geqslant 2}\eta_i/\eta_1\, dy_i$.

There is a natural subbundle S of ${}^{sc}T^*_{\partial X}X$ given by the annihilator of $h^2\partial_h$, or equivalently, spanned by the one-forms dy_i/h. Let $\partial S \subset \mathrm{fi}\cap\mathrm{scl}$ denote the boundary of S after radial compactification.

Definition 7.1. A Legendrian–Lagrangian submanifold on X is a Legendre submanifold with boundary $L \subset \mathrm{scl}$ that meets the corner $\mathrm{scl}\cap\mathrm{fi}$ transversally, and such that $\partial L \subset \partial S$.

Recall that given local coordinates y on Y, we have coordinates h, y, ν, μ on ${}^{sc}T^*X$ near $\{h = 0\}$ given by writing any element of ${}^{sc}T^*X$ relative to the basis $d(1/h)$ and dy_i/h:

$$ {}^{sc}T^*X \ni p = \nu d\left(\frac{1}{h}\right) + \sum_i \mu_i \frac{dy^i}{h}. $$

The coordinates (ν, μ) are linear coordinates on each fiber, and S is given by $\{h = 0, \nu = 0\}$. Now let q be a point on the corner of ∂L after radial compactification of the fibers. Let us assume for a moment that $\mu_1/|\mu| > 0$ at q (which can always be arranged after a linear change of y variables), so that we can use $\rho = 1/\mu_1$ as a boundary defining function for fi near q. Let $\sigma = \nu/\mu_1$ and $M' = \mu'/\mu_1$, where $\mu' = (\mu_2, \dots, \mu_m)$, $m = \dim Y$. Then (h, y, ρ, σ, M') are local coordinates for ${}^{sc}\overline{T}^*X$ near q.

At scl, the contact structure is given by the form $d\sigma - dy_1 - M' \cdot dy' - \sigma\, d\rho/\rho$. This form vanishes on L. Therefore at ∂L, which is contained in $\{\sigma = 0\}$ by assumption, we have $dy_1 + M' \cdot dy' = 0$. Thus, ∂L can be naturally identified with a Legendrian at fiber-infinity on the fiberwise compactification of T^*Y, and hence with a conic Lagrangian Λ in $T^*Y \setminus 0$. We shall soon see that a Legendrian–Lagrangian distribution is, for fixed $h > 0$, a Lagrangian distribution associated to Λ.

7.2. *Parametrization*

Let $q \in \partial L$. We shall use coordinates (h, y, ρ, σ, M') as above. We recall that $\sigma = 0$ at q, indeed everywhere on ∂L.

A local parametrization of L near q is a function $\Phi/\overline{\rho}$, where $\Phi = \Phi(y, \overline{\rho}, v)$ is a smooth function of y, $\overline{\rho}$ and $v \in \mathbb{R}^k$, defined in a neighborhood of $(y_0, 0, v_0)$ so that

$$ d_{\overline{\rho},v}(\Phi/\overline{\rho})|_{y_0,0,v_0} = 0 $$

and

$$ q = \left(0, y_0; d_{h,y}\left(\frac{\Phi}{\overline{\rho}h}\right)\right), $$

such that Φ is non-degenerate in the sense that

$$ d\left(\frac{\partial \Phi}{\partial v_i}\right), \quad d\Phi \quad \text{and} \quad d\overline{\rho} \quad \text{are linearly independent at } (y_0, 0, v_0), $$

and so that

$$ L = \left\{\left(0, y, d_{h,y}\left(\frac{\Phi}{\overline{\rho}h}\right): d_{\overline{\rho},v}\left(\frac{\Phi}{\overline{\rho}}\right) = 0\right)\right\} \tag{7.1} $$

locally near q. (Note that $\overline{\rho}$ is a parameter to be integrated, on the same footing with v.)

This is a parametrization using 'compact coordinates.' We may also use non-compact or homogeneous coordinates by introducing $w \in \mathbb{R}^{k+1}$ given in terms of $(\overline{\rho}, v)$ by $w = (w_1, w')$ with

$w_1 = 1/\bar\rho$ and $w' = v/\bar\rho$. Also write $\Phi = \Phi_1(y, v) + \bar\rho\Phi_0(y, \bar\rho, v)$. Then changing to the w variables we have a parametrization of the form

$$\Psi_1(y, w) + \Psi_0(y, w)$$

where $\Psi_1 = w_1\Phi_1$ is homogeneous of degree 1 in w and $\Psi_0 = \Phi_0$ is a symbol of order zero in w. Then Ψ_1 parametrizes the Lagrangian Λ.

7.3. Existence of parametrizations

This is proved in the usual way. Let $y = (y_1, y', y'')$, $\mu = (\mu_1, \mu', \mu'')$, $\xi' = \mu'/\mu_1$ and $\xi'' = \mu''/\mu_1$. We choose coordinates so that (ρ, ξ', y'') are coordinates on L near q. (Note that ρ always has non-zero differential on L at q since L is assumed transverse to fi.) We can therefore express the other coordinates on L as smooth functions of (ρ, ξ', y''):

$$\begin{aligned} y_1 &= Y_1(\rho, \xi', y''),\\ y' &= Y'(\rho, \xi', y''),\\ \sigma &= \Sigma(\rho, \xi', y''),\\ \xi'' &= \Xi''(\rho, \xi', y''). \end{aligned}$$

We claim that

$$\frac{\Phi(y, \bar\rho, \overline{\xi'})}{\bar\rho} = \frac{\Sigma(y, \bar\rho, \overline{\xi'}) + (y_1 - Y_1(y, \bar\rho, \overline{\xi'})) + (y' - Y'(y, \bar\rho, \overline{\xi'}))\cdot\xi'}{\bar\rho}$$

parametrizes L near q, with $v = \xi'$. In fact, since L is Legendrian, the form

$$-d\Sigma + \Sigma\frac{d\rho}{\rho} + dY_1 + \xi'\cdot dY' + \Xi''\cdot dy''$$

vanishes on L. Setting the coefficients of $d\rho$, $d\xi'$ and dy'' to zero we find that

$$-\frac{\partial\Sigma}{\partial\rho} + \frac{\Sigma}{\rho} + \frac{\partial Y_1}{\partial\rho} + \xi'\cdot\frac{\partial Y'}{\partial\rho} = 0, \tag{7.2}$$

$$\frac{\partial\Sigma}{\partial\xi'} + \frac{\partial Y_1}{\partial\xi'} + \xi'\cdot\frac{\partial Y'}{\partial\xi'} = 0, \tag{7.3}$$

$$\frac{\partial\Sigma}{\partial y''} + \frac{\partial Y_1}{\partial y''} + \xi'\cdot\frac{\partial Y'}{\partial y''} + \Xi'' = 0. \tag{7.4}$$

Using (7.2) and (7.3), one finds that $d_{\bar\rho,v}(\Phi/\bar\rho) = 0$ implies that $Y_1 = y_1$ and $Y' = y'$ on L, while equating $d_{y_1}(\Phi/\bar\rho)$ with $1/\rho$ and $dy'(\Phi/\bar\rho)$ with ξ'/ρ gives $\bar\rho = \rho$ and $\bar\xi' = \xi'$. Finally one obtains (7.1) near q.

7.4. Equivalence of phase functions

Acceptable changes of variables for our phase function are smooth coordinate changes of the form $(\rho, v) \mapsto (\tilde{\rho}, \tilde{v})$ where $\tilde{\rho} = \rho f$, with $f \in \mathcal{C}^\infty$. In the non-compact model described above, this is equivalent to $w \mapsto \tilde{w}$ where $\tilde{w}$ is a polyhomogeneous symbol of order 1 in the w variables. We therefore declare two phase functions to be equivalent if such a transformation maps one to the other. We continue to employ the non-compact phase variable description of parametrization in what follows.

Proposition 7.2. *The phase functions Ψ, $\tilde{\Psi}$ are locally equivalent near q iff*

(1) *they parametrize the same Legendrian,*
(2) *they have the same number of phase variable,*
(3) $\operatorname{sgn} d_w^2 \Psi = \operatorname{sgn} d_w^2 \tilde{\Psi}$ *at q.*

Proof. We begin as usual by arranging to have Ψ and $\tilde{\Psi}$ in agreement to first order along $C = \{d_w \Psi = 0\}$.

We may thus expand in a Taylor series

$$\tilde{\Psi} - \Psi = \frac{1}{2} (\nabla_w \Psi)^t B (\nabla_w \Psi)$$

for some matrix $B = B(y, w)$. As both $\tilde{\Psi}$ and Ψ are symbols of order 1 in w, B is also symbolic of order 1.

The non-degeneracy assumptions on $\Psi, \tilde{\Psi}$ means precisely that $\det(I + B\partial_w^2 \psi_2) \neq 0$. We now expand

$$\Psi(y, \tilde{w}) - \Psi(y, w) = (\tilde{w} - w) \cdot \partial_w \Psi + O\big((\tilde{w} - w)^2\big).$$

Set

$$\tilde{w} - w = z \cdot \nabla_w \Psi,$$

where z is a matrix depending on w; note that this is a change of variables of the required form. We thus have

$$\Psi(y, \tilde{w}) - \Psi(y, w) = (\nabla_w \Psi)^t \big(z + O\big(z^2\big)\big)(\nabla_w \Psi).$$

We want

$$\Psi(y, \tilde{w}) - \Psi(y, w) = \tilde{\Psi}(y, w) - \Psi(y, w);$$

we thus need to solve

$$(\nabla_w \Psi)^t \big(z + O\big(z^2\big)\big)(\nabla_w \Psi) = \frac{1}{2} (\nabla_w \Psi)^t B (\nabla_w \Psi)$$

for z. This can be accomplished for B small by the inverse function theorem, with a result that is symbolic in w. Lemma 3.1.7 of [14] enables us to extend to the case of arbitrary B. □

7.5. Legendrian–Lagrangian distributions

Let L be a Legendrian–Lagrangian submanifold as above. Let $N = \dim X$. A Legendrian–Lagrangian distribution u of order (m, r) associated to L on X, denoted $u \in I^{m,r}(X, L)$, is a half-density $u = u_1 + u_2 + u_3 + u_4$, where

- u_1 is in $h^\infty C^\infty([0, h_0); I^{-r-1/4}(Y, h^{-1}\Lambda; \Omega^{\frac{1}{2}})) \otimes |dh|^{1/2}$,
- u_2 is a Legendrian distribution of order m associated to L and microsupported away from fiber-infinity,
- u_3 is a sum of terms of the form

$$h^{m-(k+1)/2+N/4} \int_0^\infty \int_{\mathbb{R}^k} e^{i\Phi(y,\rho,v)/\rho h} \rho^{r-k/2-1-N/4} a(h, y, \rho, v)\, dv\, d\rho \left| \frac{dy\, dh}{h^{N+1}} \right|^{1/2} \tag{7.5}$$

  where Φ is a local parametrization of L and a is smooth, and
- $u_4 \in \dot{C}^\infty(X)$.

We remark that (7.5) is an oscillatory integral unless r is sufficiently positive.

If we rewrite this using the homogeneous parametrization, we get

$$h^{m-(k+1)/2+N/4} \int e^{i\Psi_1(y,w)/h} e^{i\Psi_0(y,w)/h} \tilde{a}(h, y, w)\, dw \left| \frac{dy\, dh}{h^{N+1}} \right|^{1/2} \tag{7.6}$$

where $\tilde{a}$ is a (classical) polyhomogeneous symbol of order $-r - (k+1)/2 + N/4 - 1/2$ in w. For $h > 0$ the $e^{i\Psi_0/h}$ factor is a symbol of order zero and so $e^{i\Psi_0/h}\tilde{a}$ is a symbol of the same order as $\tilde{a}$. Hence for fixed $h > 0$ this is a Lagrangian distribution, of order $-r - 1/4$, associated to Λ depending smoothly on h for $h > 0$, and whose symbol is itself oscillatory as $h \to 0$.

7.6. Symbol calculus and residual space

Proposition 7.3. *The symbol map for Legendre distributions, defined in the interior of G* [23], *extends by continuity to give an exact sequence*

$$0 \to I^{m+1,r}\big(X, L; {}^{s\Phi}\Omega^{\frac{1}{2}}\big) \to I^{m,r}\big(X, L; {}^{s\Phi}\Omega^{\frac{1}{2}}\big) \to \rho^{r-N/4} C^\infty\big(L, \Omega_b^{\frac{1}{2}} \otimes S^{[m]}(G)\big) \to 0.$$

The residual space $I^{\infty,r}(X, L; {}^{s\Phi}\Omega^{\frac{1}{2}}) \equiv \bigcap_m I^{m,r}(X, L; {}^{s\Phi}\Omega^{\frac{1}{2}})$ *may be identified with*

$$h^\infty C^\infty\big([0, h_0); I^{-r-1/4}\big(Y, h^{-1}\Lambda; \Omega^{\frac{1}{2}}\big)\big) \otimes |dh|^{1/2}.$$

7.7. Distributional limits of Legendrian conic pairs

We now consider a situation which leads to a Legendrian–Lagrangian distribution. Let M be a compact manifold with boundary. We may view $M \times [0, h_0)$ as a scattering-fibered manifold (where we again ignore the non-compactness at $h = h_0$) with main face $H_2 = M \times \{0\}$ and other boundary hypersurface $H_1 = \partial M \times [0, h_0)$ with fibration $H_1 \to \partial M$ given by projection onto the ∂M factor. Suppose that we have a distribution $u \in I(G, G_1^\sharp)$ associated to a Legendrian conic pair $(G, G_1^\sharp)$ as described in the previous section (codimension 2 case). Coordinates in this case can be taken to be x, h, y near H_1, where y is a local coordinate for ∂M, extended to a collar neighborhood of ∂M, and x is a boundary defining function for M. Corresponding scattering-fibered cotangent coordinates are ν_1, ν_2, μ given by expressing covectors

$$q = \nu_1 d\left(\frac{1}{xh}\right) + \nu_2 d\left(\frac{1}{h}\right) + \mu \cdot \frac{dy}{xh}.$$

We suppose as in Section 6.2.1 that $G_1^\sharp$ is given by $\{\mu = 0, \nu_1 = 1\}$.

We consider the problem of restricting $u \in I^{m,p;r}(X, (G, G_1^\sharp))$ to H_1. The first issue is that u is a half-density, so to restrict we must divide by the half-density $|dx/x|^{1/2}$ to obtain a half-density on H_1. The second issue is that u is oscillatory as $x \to 0$, so we must first multiply by $e^{-i/xh}$ to have any hope of being able to restrict to $x = 0$. Thirdly we must divide by a power of x, depending on the order p at $G_1^\sharp$, in order to get a finite, non-zero limit. If we do all this, and if the Legendrian G intersects $\{\mu = 0\}$ only at $G_1^\sharp$, then it turns out that u has a restriction in the distributional sense. In the non-semiclassical case (no h variable) this was proved by Melrose–Zworski [23].

Let $\hat{G}$ be the blowup of the singular Legendrian G at $J = \{x = 0, \mu = 0\} \subset {}^{s\Phi}T^*_{\mathrm{mf}}(M \times [0, h_0))$. We write $\tilde{J}$ for the new boundary hypersurface created by the blowup. Recall that $\hat{G}$ is a manifold with corners of codimension 2, with one boundary hypersurface $\partial_\sharp \hat{G}$ at $\tilde{J}$ and the other, $\partial_1 \hat{G}$, at $x = 0$ but away from $\tilde{J}$.

Lemma 7.4. *The submanifold $\tilde{J} \cap \{\nu_1 = 1\}$ of $\tilde{J}$ is naturally diffeomorphic to the fiberwise compactification of ${}^{\mathrm{sc}}T^*_{\partial M \times \{0\}}(\partial M \times [0, h_0))$, and under this identification, the boundary hypersurface $\partial_\sharp \hat{G}$ (which lies inside $\tilde{J} \cap \{\nu_1 = 1\}$) is a Legendrian–Lagrangian submanifold L.*

Proof. Scattering-fibered covectors in ${}^{s\Phi}T^*(M \times [0, h_0))$ are represented by forms of the form $d((f(y) + xg)/xh)$ where g is smooth. If we restrict to the set $\nu_1 = 1$, $\mu = 0$ then these are of the form $d((1 + xg)/xh) = d(1/xh) + d(g/h)$, and the coordinates are given by y, $\nu_1 = 1$, $\nu_2 = g$, $\mu = x\, dg$. Thus on the interior of $\tilde{J}$, where we may take x as a boundary defining function, the coordinates are given by y, $\nu_1 = 1$, $\nu_2 = g$, $\mu/x = dg$. It is now clear that the map from the point on $\tilde{J} \cap \{\nu_1 = 1\}$ specified by $(y, \nu_2 = g, \mu/x = dg)$ to $d(g/h) \in {}^{\mathrm{sc}}T^*_{\partial M \times \{0\}}(\partial M \times [0, h_0))$ is a natural diffeomorphism. This identifies the interior of $\tilde{J} \cap \{\nu_1 = 1\}$ with ${}^{\mathrm{sc}}T^*_{\partial M \times \{0\}}(\partial M \times [0, h_0))$. The fibers of the blowdown map $\tilde{J} \to J$ are radial compactifications of vector spaces coordinatized by μ/x, since we have $x/|\mu|$ as a boundary defining function for $\partial \tilde{J}$ and $\mu/|\mu|$ as a coordinate along the boundary. Hence the natural diffeomorphism extends from $\tilde{J} \cap \{\nu_1 = 1\}$ to the fiberwise radial compactification of ${}^{\mathrm{sc}}T^*_{\partial M \times \{0\}}(\partial M \times [0, h_0))$.

The contact form on ${}^{s\Phi}T^*_{\mathrm{mf}}(M \times [0, h_0))$ is given by $d\nu_1 + x\, d\nu_2 + \mu \cdot dy$. Let $\eta = \mu/x$. Then this can be written $d\nu_1 + x(d\nu_2 + \eta \cdot dy)$, which vanishes at $\hat{G}$. Taking the differential and restricting to $\partial_\sharp \hat{G}$ we get $dx \wedge (d\nu_2 + \eta \cdot dy) = 0$, and since $dx \neq 0$ at the interior of $\tilde{J}$ we conclude that $d\nu_2 + \eta \cdot dy = 0$ at $\partial_\sharp \hat{G}$. Since $\nu = 1$ on $G_1^\sharp$, we have $\nu_1 = 1$ at $\partial_\sharp \hat{G}$. Using our identification of the interior of $\tilde{J} \cap \{\nu_1 = 1\}$ with ${}^{\mathrm{sc}}T^*(\partial M \times [0, h_0))$, we see that the image of $\partial_\sharp \hat{G}$ under this identification, which we denote L, is Legendrian. Also, by the transversality requirements in the definition of a Legendrian conic pair, L is transverse to the boundary at fiber-infinity. Finally, since $\hat{G}$ is a compact submanifold of $[{}^{s\Phi}T^*_{\mathrm{mf}}(M \times [0,h_0)); J]$, and since ν_2 is a continuous function on this space, the value of ν_2 is bounded on $\hat{G}$. On the other hand, the value of $\eta = \mu/x$ goes to infinity at the boundary of L. Hence $\nu_2/\eta = 0$ at the boundary of L. It follows that $\partial L \subset \partial S$, so L is a Legendrian–Lagrangian submanifold. □

The analytic result corresponding to this geometric lemma is

Proposition 7.5. *Let X be a scattering-fibered manifold with codimension 2 corners, let $\dim X = N$, and let $(G, G_1^\sharp)$ be as above. Suppose that $u \in I^{m,p;r}(X,(G,G_1^\sharp))$, and assume that $G \cap \{\mu = 0\}$ is contained in $G_1^\sharp$. Then*

$$x^{-p+N/4} e^{-i/xh} \left|\frac{dx}{x}\right|^{-1/2} u \tag{7.7}$$

has a distribution limit as $x \to 0$. The limit is an element of $I^{m-1/4,p-r-(N-1)/4}(H_1, L)$, where L is as in Lemma 7.4.

Proof. By definition, u is a sum of terms u_i as in the definition above (6.21). Clearly we can ignore any summands which are rapidly decreasing at $x = 0$. We next note that, if we microlocalize u to any region where $\mu \neq 0$, then (non-)stationary phase (involving repeated integrations by parts in y) shows that the pairing of u with any function of y is rapidly decreasing in x as $x \to 0$. Hence we can restrict attention to the microlocal region where μ is close to zero, which by assumption is near $G_1^\sharp$, i.e. near the conic singularity of G.

In this region, we have seen that u can be written as a sum of terms of the form (6.21) and (6.22) (with y_2 and v_2 absent and x_1 replaced by x). Consider an integral of the form (6.21):

$$\int_{\mathbb{R}^k}\int_0^\infty e^{i(1+s\psi_1(s,y,v)+x\psi_2(s,x/s,y,v))/xh} a\left(s,\frac{x}{s},y,v,h\right) \times h^{m-(1+k)/2+N/4} s^{p-1/2+N/4}\left(\frac{x}{s}\right)^{r-(1+k)/2-1/2+N/4} \frac{ds}{s}\,dv\,\mu$$

where μ is a scattering-fibered half-density. We may take μ to be

$$\mu = \left|\frac{dy}{(xh)^{N-2}}\frac{dx}{x^2h}\frac{dh}{h^2}\right|^{1/2}.$$

We want to express this in terms of a scattering half-density $\nu = |dy\,dh/h^N|^{1/2}$; we see that

$$\mu = h^{-1/2}x^{-(N-1)/2}\left|\frac{dx}{x}\right|^{1/2}\nu.$$

It follows that (7.7) is given by

$$\int_{\mathbb{R}^k}\int_0^\infty e^{i(s\psi_1(s,y,v)+x\psi_2(s,x/s,y,v))/xh} a\left(s, \frac{x}{s}, y, v, h\right) h^{m-(1+k)/2+N/4-1/2}$$
$$\times \left(\frac{x}{s}\right)^{p-1/2+N/4} \left(\frac{x}{s}\right)^{r-(1+k)/2-1/2+N/4} \frac{ds}{s}\, dv\, \nu.$$

Let us introduce the variables $\eta_1 = s/x$ and $\eta' = v\eta_1 \in \mathbb{R}^k$. We can write this

$$\int_{\mathbb{R}^k}\int_0^\infty e^{i(\eta_1\psi_1(x\eta_1,y,\eta'/\eta_1)+\psi_2(x\eta_1,\eta_1^{-1},y,\eta'/\eta_1))/h} a\left(x\eta_1, \frac{1}{\eta_1}, y, \eta'/\eta_1, h\right)$$
$$\times h^{m-(1+k)/2+N/4-1/2} \eta_1^{p-r+(1+k)/2} \eta_1^{-k} \frac{d\eta_1}{\eta_1}\, d\eta'\, \nu. \tag{7.8}$$

If we set $x = 0$ in the integrand then the integral becomes

$$\int_{\mathbb{R}^k}\int_0^\infty e^{i(\eta_1\psi_1(0,y,\eta'/\eta_1)+\psi_2(0,\eta_1^{-1},y,\eta'/\eta_1))/h} a\left(0, \frac{1}{\eta_1}, y, \eta'/\eta_1, h\right)$$
$$\times h^{m-(1+k)/2+N/4-1/2} \eta_1^{p-r-(1+k)/2}\, d\eta_1\, d\eta'\, \nu.$$

It is straightforward to check that $\eta_1\psi_1(0, y, \eta'/\eta_1) + \psi_2(0, \eta_1^{-1}, y, \eta'/\eta_1)$ is a non-degenerate parametrization of L. Therefore this is a Legendrian–Lagrangian distribution of order $(m - 1/4, p - r - (N-1)/4)$ associated to L. It remains to prove that this is indeed the distributional limit of (7.8) as $x \to 0$. This is clear if the exponent of η_1 is sufficiently negative, since then the integral is absolutely convergent, uniformly in x. In general, we can exploit the fact that $d_{y^1}\psi \neq 0$ according to (6.7) (where y^1 is the first component of y) and integrate by parts repeatedly in y^1, using

$$e^{i\eta_1\psi_1/h} = \frac{h}{i\eta_1}\frac{1}{\partial_{y^1}\psi_1}\partial_{y^1} e^{i\eta_1\psi_1/h}.$$

Doing this sufficiently many times eventually reduces the exponent of η_1 to the point of absolute integrability. We can then take the limit $x \to 0$ and perform the integrations by parts in reverse, which gives the desired conclusion.

A similar argument applied to an integral of the form (6.22) gives the same result (in this case, we only get a Legendrian distribution microlocalized to a compact part of the interior of L since this part is away from the corner of $\hat{G}$). □

8. Quadratic scattering-fibered structure

In order to describe precisely the microlocal structure of the Schrödinger propagator, we need to introduce the *quadratic-scattering fibered structure* on manifolds with codimension three corners. This structure is a variant of the scattering-fibered structure, in which we have an extra order of vanishing of the Lie algebra at some of the boundary hypersurfaces. The basic example, on a manifold with boundary, is the quadratic scattering structure, which we now review.

8.1. The basic structure

Recall that the quadratic scattering structure on a manifold with boundary, X, is given by the quadratic scattering Lie algebra $\mathcal{V}_{\mathrm{qsc}}(X) \equiv x\mathcal{V}_{\mathrm{sc}}(X)$. Locally near the boundary, using coordinates (x, y), $x \geqslant 0$ a boundary defining function, $\mathcal{V}_{\mathrm{qsc}}$ is the $\mathcal{C}^\infty(X)$-span of the vector fields

$$x^3\partial_x, \quad x^2\partial_{y_i}.$$

This structure was used to analyze the propagation of singularities at infinity of solutions to the time-dependent Schrödinger equation [28,36].

In the quadratic scattering-fibered structure on a manifold with codimension 3 corners, we start with a manifold X with fibrations ϕ_i, as in Definition 3.3. However, instead of a distinguished total boundary defining function $\mathbf{x}$ we require a distinguished function $\mathbf{x_q}$ which vanishes to second order at the H_1 and H_2 boundary hypersurfaces; in other words $\mathbf{x_q} = x_1^2x_2^2x_3$ for some boundary defining functions x_i of H_i. Correspondingly, we consider a different Lie algebra of vector fields. In place of Definition 3.7, we make

Definition 8.1. The Lie algebra of quadratic scattering-fibered vector fields $\mathcal{V}_{\mathrm{qs}\Phi}$ is defined by

$$V \in \mathcal{V}_{\mathrm{qs}\Phi}(X) \quad \text{iff } V = x_1x_2W, \qquad W(\mathbf{x_q}) \in x_1^3x_2^3x_3^2\mathcal{C}^\infty(X) \text{ and } W \text{ is tangent to } \Phi. \tag{8.1}$$

An analogue of Proposition 3.4 applies, where we replace the last condition $\Pi x_i = \mathbf{x}$ by $x_1^2x_2^2x_3 = \mathbf{x_q}$. In terms of such coordinates, the Lie algebra is given locally by arbitrary linear combinations (over $\mathcal{C}^\infty(X)$) of vector fields of the form

$$\begin{array}{ll} -(x_1^2x_2^2x_3)x_1\partial_{x_1}, & (x_1^2x_2^2x_3)\partial_{y_1}, \\ x_1x_2^2x_3(x_1\partial_{x_1} - x_2\partial_{x_2}), & (x_1x_2^2x_3)\partial_{y_2}, \\ x_1x_2x_3(x_2\partial_{x_2} - 2x_3\partial_{x_3}), & x_1x_2x_3\partial_{y_3}. \end{array} \tag{8.2}$$

It follows, as in Section 3, that $\mathcal{V}_{\mathrm{qs}\Phi}(X)$ is the space of sections of a vector bundle over X. The dual bundle, denoted ${}^{\mathrm{qs}\Phi}T^*(X)$, is spanned by one-forms of the form $d(f/(x_1^2x_2^2x_3))$ where $f \in C^\infty_\Phi(M)$.

The dual basis to the vector fields (8.2) is

$$d\left(\frac{1}{x_1^2x_2^2x_3}\right), \quad d\left(\frac{1}{x_1x_2^2x_3}\right), \quad d\left(\frac{1}{x_1x_2x_3}\right), \quad \frac{dy_1}{x_1^2x_2^2x_3}, \quad \frac{dy_2}{x_1x_2^2x_3}, \quad \frac{dy_3}{x_1x_2x_3}. \tag{8.3}$$

Here dy_i is shorthand for a k_i-vector of 1-forms, if $y_i \in \mathbb{R}^{k_1}$. An alternative basis is given by

$$d\left(\frac{1}{x_1^2x_2^2x_3}\right), \quad \frac{dx_1}{x_1^2x_2^2x_3}, \quad \frac{dx_2}{x_1x_2^2x_3}, \quad \frac{dy_1}{x_1^2x_2^2x_3}, \quad \frac{dy_2}{x_1x_2^2x_3}, \quad \frac{dy_3}{x_1x_2x_3}.$$

Any element of ${}^{\mathrm{s}\Phi}T^*X$ may therefore be written uniquely as

$$\tilde\nu_1 d\left(\frac{1}{x_1^2x_2^2x_3}\right) + \tilde\nu_2 d\left(\frac{1}{x_1x_2^2x_3}\right) + \tilde\nu_3 d\left(\frac{1}{x_1x_2x_3}\right) + \tilde\mu_1 \cdot \frac{dy_1}{x_1^2x_2^2x_3} + \tilde\mu_2 \cdot \frac{dy_2}{x_1x_2^2x_3} + \tilde\mu_3 \cdot \frac{dy_3}{x_1x_2x_3} \tag{8.4}$$

or, alternatively, as

$$\bar{\tilde{\nu}}_1 d\left(\frac{1}{x_1^2 x_2^2 x_3}\right) + \bar{\tilde{\nu}}_2 \frac{dx_1}{x_1^2 x_2^2 x_3} + \bar{\tilde{\nu}}_3 \frac{dx_2}{x_1 x_2^2 x_3} + \tilde{\mu}_1 \cdot \frac{dy_1}{x_1^2 x_2^2 x_3} + \tilde{\mu}_2 \cdot \frac{dy_2}{x_1 x_2^2 x_3} + \tilde{\mu}_3 \cdot \frac{dy_3}{x_1 x_2 x_3}. \tag{8.5}$$

The function $\tilde{\nu}_1$, regarded as a linear form on the fibers of ${}^{s\Phi}T^*X$, can be identified with the vector field $(x_1^2 x_2^2 x_3) x_1 \partial_{x_1}$, and similarly for the other fiber coordinates. The same expression can be viewed as the canonical one-form on ${}^{qs\Phi}T^*X$. Taking d of (3.14) therefore gives the symplectic form on ${}^{qs\Phi}T^*X$.

The same reasoning as in Section 3, but considering differentials of the form $d(f/(x_1^2 x_2^2 x_3))$ where $f \in C^\infty_\Phi(X)$, leads to the definition of the bundles ${}^{qs\Phi}T^*(F_i, H_i)$ and ${}^{qs\Phi}N^*Z_i$ as well as the induced fibrations $\tilde{\phi}_i$.

The contact form on ${}^{qs\Phi}T^*_{\mathrm{mf}}X$ is defined by contracting the symplectic form ω with $(x_1^2 x_2^2 x_3) x_3 \partial_{x_3}$ and restricting to mf. This gives

$$\begin{aligned} \chi &= d\tilde{\nu}_1 + x_1\, d\tilde{\nu}_2 + x_1 x_2\, d\tilde{\nu}_3 - \tilde{\mu}_1 \cdot dy_1 - x_1 \tilde{\mu}_2 \cdot dy_2 - x_1 x_2 \tilde{\mu}_3\, dy_3 \\ &= d\bar{\tilde{\nu}}_1 - \bar{\tilde{\nu}}_2\, dx_1 - x_1 \bar{\tilde{\nu}}_3\, dx_2 - \tilde{\mu}_1 \cdot dy_1 - x_1 \tilde{\mu}_2 \cdot dy_2 - x_1 x_2 \tilde{\mu}_3\, dy_3 \end{aligned} \tag{8.6}$$

which is exactly the same expression as the contact form in the scattering-fibered case. In a similar way we get induced contact forms on ${}^{qs\Phi}N^*Z_i$ and on the fibers of $\tilde{\phi}_i$.

Given a scattering-fibered manifold, Y, with codimension two corners, we can form the product $X_h = Y \times [0, h_0)$ and endow it with the structure of a scattering-fibered manifold with codimension three corners, as in Section 4, or we can form the product $X_t = Y \times [0, t_0]$ and endow it with the structure of a *quadratic* scattering-fibered manifold with codimension three corners. It turns out that there is a contact transformation Q between ${}^{s\Phi}T^* r_{\mathrm{mf}} X_h \setminus \mathcal{N}$ and ${}^{qs\Phi}T^*_{\mathrm{mf}}X_t \setminus \mathcal{N}$, where $\mathcal{N}$ denotes the subbundle of ${}^{s\Phi}T^*_{\mathrm{mf}}X_h$, respectively ${}^{qs\Phi}T^*_{\mathrm{mf}}X_t$, spanned, at $p \in \mathrm{mf}$, by elements of the form $d(f/(x_1 x_2 x_3))$, respectively $d(f/(x_1^2 x_2^2 x_3))$, where $f \in C^\infty_\Phi(X)$ vanishes at p. This contact transformation is very useful in relating the semiclassical resolvent and the propagator (in the case that $Y = M^2_{\mathrm{b}}$). The map Q is defined by

$$Q\left(\frac{df}{x_1 x_2 h}\right) = \left(\frac{d(f^2)}{2x_1^2 x_2^2 t}\right), \quad f \in C^\infty_\Phi(X_h),\ f \neq 0. \tag{8.7}$$

The proof of this is very straightforward if we use the coordinates $\bar{\nu}_i$, $\bar{\mu}_i$ from (3.17) on ${}^{s\Phi}T^*_{\mathrm{mf}}X_h$ and the analogous coordinates on ${}^{qs\Phi}T^*_{\mathrm{mf}}X_t$. Then, with $\tilde{\chi}$ the contact form on ${}^{qs\Phi}T^*_{\mathrm{mf}}X_t$, we find that $Q^*(\tilde{\chi}) = \bar{\nu}_1 \chi$, showing that Q is a contact transformation away from $\bar{\nu}_1 = 0$, which is the set denoted $\mathcal{N}$ above. We remark that such contact transformations can be defined far more generally; the point of the transformation $f \mapsto f^2/2$ is that shows up when we obtain the propagator from the resolvent via an integral over the spectral measure.

8.2. *Legendre distributions*

The theory of Legendre distributions on quadratic scattering-fibered manifolds proceeds in parallel to that of scattering-fibered manifolds.

Definition 8.2. A Legendre submanifold of a quadratic scattering-fibered manifold X of dimension N is a submanifold G of dimension $N-1$ of ${}^{\mathrm{qs}\Phi}T^*_{\mathrm{mf}}X$ on which the contact form χ vanishes, and such that G is transverse to each boundary ${}^{\mathrm{qs}\Phi}T^*_{\mathrm{mf}\cap H_i}X$ of ${}^{\mathrm{qs}\Phi}T^*_{\mathrm{mf}}X$.

A parametrization of a quadratic scattering-fibered Legendre submanifold can be defined much as in the scattering-fibered case. In fact, near a point $q \in G \cap {}^{\mathrm{qs}\Phi}T^*_{\mathrm{mf}\cap H_1\cap H_2}X$, the definition is identical to that of Section 4.2 except that we replace (4.7) by

$$G = \left\{ d\left(\frac{\psi}{(x_1^2 x_2^2 x_3)} \right) \;\middle|\; (\vec{x}, \vec{y}, \vec{v}) \in C_\psi \right\}. \tag{8.8}$$

We also give the definitions for a local parametrization near a point $q \in G$ lying in the interior of G, or in the interior of one of the boundary hypersurfaces of G. In the former case this is just a standard Legendre parametrization locally. In the latter case the definition is analogous to the codimension 3 case except that, near the interior of H_1 we split the coordinates as $(x_1, x_3, Y_1 = y_1, Y_3 = (y_2, x_2, y_3))$, our phase function is of the form

$$\psi(x_1, Y_1, Y_3, v_1, v_3) = \psi_1(Y_1, v_1) + x_1\psi_2(x_1, Y_1, Y_3, v_1, v_3)$$

and we ignore the variables with a '2' subscript. Near H_2 we split the coordinates $(x_2, Y_2 = (x_1, y_1, y_2), x_3, Y_3 = y_3)$, our phase function is of the form

$$\psi(x_2, Y_2, Y_3, v_2, v_3) = \psi_1(Y_2, v_2) + x_2\psi_2(x_2, Y_2, Y_3, v_2, v_3)$$

and we ignore the variables with a '1' subscript.

Let m, r_1, r_2 be real numbers, let $N = \dim X$, let $G \subset {}^{\mathrm{qs}\Phi}T^*_{\mathrm{mf}}X$ be a quadratic Legendre submanifold, and let ν be a smooth non-vanishing quadratic scattering-fibered half-density. The set of quadratic Legendre distributions of order $(m; r_1, r_2)$ associated to G, denoted $I^{m,r_1,r_2}(X, G; {}^{\mathrm{qs}\Phi}\Omega^{\frac{1}{2}})$, is the set of half-density distributions that can be written in the form $(u_1 + u_2 + u_3 + u_4 + u_5)\nu$, such that

- $u_1 \cdot \nu$ is a quadratic Legendre distribution of order $(m; r_1)$ associated to G and supported away from H_2, i.e. u_1 is given by a finite sum of expressions of the form

$$\int e^{i\psi(x_1,x_2,\vec{y},\vec{v})/(x_1^2 x_2^2 x_3)} a(\vec{x}, \vec{y}, \vec{v}) \times x_3^{m-(k_1+k_2)/2+N/4} x_1^{r_1-k_1-k_2/2-f_1/2+3N/4}\, dv_1\, dv_2, \tag{8.9}$$

  where ψ parametrizes G locally and a is smooth and supported away from $x_2 = 0$,
- $u_2 \cdot \nu$ is similarly a quadratic Legendre distribution of order $(m; r_2)$ associated to G and supported away from H_1,
- u_3 is given by an finite sum of local expressions of the form

$$\int e^{i\psi(x_1,x_2,\vec{y},\vec{v})/(x_1^2 x_2^2 x_3)} a(\vec{x}, \vec{y}, \vec{v}) \times x_3^{m-(k_1+k_2+k_3)/2+N/4} x_2^{r_2-(k_1+k_2)-k_3/2-f_2/2+3N/4} \times x_1^{r_1-k_1-(k_2+k_3)/2-f_1/2+3N/4}\, dv_1\, dv_2\, dv_3, \tag{8.10}$$

with $v_i \in \mathbb{R}^{k_i}$, a smooth and compactly supported, f_i the dimension of the fibers of H_i and $\psi = \psi_1 + x_1\psi_2 + x_1x_2\psi_3$ a phase function locally parametrizing G near a corner point $q \in \partial_{12}G$,

- u_4 is given by a finite sum of terms of the form

$$\int e^{i(\psi_1+x_1\psi_2)/(x_1^2x_2^2x_3)} b(x_1, x_2, x_3, y_1, y_2, y_3) \\ \times x_2^{r_2-(k_1+k_2)-f_2/2+3N/4} x_1^{r_1-k_1-k_2/2-f_1/2+3N/4}\, dv_1\, dv_2 \tag{8.11}$$

with ψ_1, ψ_2 and f_i as above, b smooth with support compact and $O(x_3^\infty)$ at mf, and

- $u_5 \in \dot{C}^\infty(X)$.

If $X = X_t = Y \times [0, t_0]$ as in the previous subsection and G is disjoint from $\mathcal{N}$ then we can locally write $G = Q(G')$ for some Legendrian $G' \subset {}^{s\Phi}T^*_{\mathrm{mf}}X_h$; then if ϕ/x_1x_2h is a local parametrization of G', $\phi^2/2(x_1^2x_2^2t)$ is a local parametrization of G.

Proposition 8.3. *Suppose that $u_h \in I^{m,r_1,r_2}(X_h, G; {}^{s\Phi}\Omega^{\frac{1}{2}})$ is a Legendre distribution associated to the Legendrian G which does not intersect $\mathcal{N}$. Also suppose that $\chi(t)$ is a smooth function of $t \in \mathbb{R}$ that vanishes on $[0, R]$ and is identically equal to 1 on $[2R, \infty]$, for some $R > 0$. Then the integral in h*

$$\int_0^\infty e^{-it/2h^2} \chi(\sqrt{t}/h) u_h \frac{|dh\, dt|^{1/2}}{h} \tag{8.12}$$

is in

$$I^{m+1/2, r_1+m+1/2-N/4, r_2+m+1/2-N/4}\big(X_t, Q(G); {}^{\mathrm{qs}\Phi}\Omega^{\frac{1}{2}}\big),$$

i.e. is a quadratic Legendre distribution associated to $Q(G)$, with orders shifted by $1/2$ at mf *and $m + 1/2 - N/4$ at H_1 and H_2.*

Remark. Our interest in this lemma is for the following reason: if u_h is $(2\pi i)^{-1}$ times the difference of the limit of the semiclassical resolvent on the spectrum, taken from above and below,

$$u_h^\pm = \frac{1}{2\pi i}\Big(\big(h^2\Delta + 2h^2V - (1 + i0)\big)^{-1} - \big(h^2\Delta + 2h^2V - (1 - i0)\big)^{-1}\Big) \otimes |dh|^{1/2},$$

then the integral above gives the Schrödinger propagator $e^{-it(\Delta/2+V)}$ (times $|dt|^{1/2}$). Note that the V term is of a higher semiclassical order in this setting than in the usual semiclassical resolvent, hence V does not affect the Legendrian geometry of the Schrödinger propagator.

Proof. Locally u may be written in the form

$$\int e^{i\psi/(x_1x_2h)} a(\vec{x}, \vec{y}, \vec{v}) \times x_3^{m-(k_1+k_2+k_3)/2+N/4} x_2^{r_2-(k_1+k_2)/2-f_2/2+N/4} \times x_1^{r_1-k_1/2-f_1/2+N/4} \, dv_1\, dv_2\, dv_3 \cdot \nu. \tag{8.13}$$

The condition on G means that $|\psi| \geqslant \epsilon > 0$ at G; by cutting off the symbol close to G we may assume that $|\psi| \geqslant \epsilon$ everywhere on the support of the symbol. Then in the integral (8.12) we get a phase function of the form $-t/2h^2 + \psi/x_1x_2h$. Changing variable to $k = tx_1x_2/h$ this becomes $(-k^2/2 + k\psi)/(x_1^2x_2^2t)$, while the symbol becomes a function of tx_1x_2/k. Due to the χ cutoff, the integral in k is supported in $k \geqslant Rx_1x_2\sqrt{t}$.

Let us insert cutoff functions $1 = \chi_1(k) + \chi_2(k)$, where χ_1 is supported in $\{k \leqslant \epsilon/2\}$ and χ_2 is supported in $\{k \geqslant \epsilon/4\}$.

With χ_1 inserted, there are no stationary points in the integral in k since the phase is stationary when $k = \psi$. This term is in $\dot{C}^\infty(X)$, as follows by writing

$$e^{i(-k^2/2+k\psi)/x_1^2x_2^2t} = \left(i\frac{x_1^2x_2^2t}{k-\psi}\partial_k\right)^N e^{i(-k^2/2+k\psi)/x_1^2x_2^2t}$$

and integrating by parts N times, for arbitrary N. We gain at least $\sqrt{t}x_1x_2$ with each integration-by-parts.

With χ_2 inserted, we avoid the singularity caused by the argument tx_1x_2/k in the symbol, and this term is a quadratic Legendre distribution associated to $Q(G)$ since the phase function $\psi k - k^2/2$ parametrizes $Q(G)$. Collecting powers of t, x_1 and x_2 (bearing in mind that the number k_1 of v_1 variables has increased by 1 due to the appearance of k) completes the proof. □

8.3. Conic pairs

We now give an analogous sketch of the theory of Legendre distributions associated to conic Legendrian pairs on quadratic scattering-fibered manifolds.

Definition 8.4. Let $G_2^\sharp$ be a projectable Legendrian in ${}^{\mathrm{qs}\Phi}N^*Z_2$, and let $G \subset {}^{\mathrm{qs}\Phi}T^*_{\mathrm{mf}}X$ be a Legendre submanifold that is singular at the boundary. Let J_2 be the span of $G_2^\sharp$ in ${}^{\mathrm{qs}\Phi}N^*Z_2$, and J the preimage of J_2 in ${}^{\mathrm{qs}\Phi}T^*_{\mathrm{mf}\cap H_2}X$ under the map $\tilde{\phi}_2$. We say that $(G, G_2^\sharp)$ form an conic Legendrian pair if G has conic singularities at J, i.e. lifts to $[{}^{\mathrm{qs}\Phi}T^*_{\mathrm{mf}}X; J]$ to a smooth manifold $\hat{G}$ transverse to $\tilde{J}$ as well as to the lifts of ${}^{\mathrm{qs}\Phi}T^*_{\mathrm{mf}\cap H_1}X$ and ${}^{\mathrm{qs}\Phi}T^*_{\mathrm{mf}\cap H_2}X$.

A parametrization of a quadratic scattering-fibered Legendre submanifold can be defined much as in the scattering-fibered case. In fact, near a point $q \in G \cap {}^{\mathrm{qs}\Phi}T^*_{\mathrm{mf}\cap H_1\cap H_2}X$, the definition is identical to that of Section 6.2.2 except that we replace (6.14) and (6.17) by

$$\hat{G} = \left\{ d\left(\frac{\Psi}{(x_1^2x_2^2x_3)}(q'')\right) \,\Big|\, q'' \in C_\Psi \right\} \quad \left(\text{lifted to } \left[{}^{\mathrm{qs}\Phi}T^*_{\mathrm{mf}}X; J\right]\right) \text{ near } q. \tag{8.14}$$

If $X = X_t = Y \times [0, t_0]$ as in the previous subsection and G is disjoint from $\mathcal{N}$ then we can locally write $G = Q(G')$ for some Legendrian $G' \subset {}^{s\Phi}T^*_{\mathrm{mf}}X_h$; then if ϕ/x_1x_2h is a local parametrization of G', $\phi^2/2(x_1^2x_2^2t)$ is a local parametrization of G. To simplify the definition of a distribution associated to a quadratic conic Legendrian pair, assume that this is the case. Then we can use the parametrizations of G' from Section 6.5.

Let $N = \dim X$, let m, r_1, r_2 and p be real numbers, and let ν be a smooth non-vanishing quadratic scattering-fibered half-density on X. A quadratic Legendre distribution of order $(m, p; r_1, r_2)$ associated to $(G, G_2^\sharp)$ is a half-density distribution of the form $u_1 + u_2 + (u_3 + u_4 + u_5 + u_6)\nu$, where

- u_1 is a Legendre distribution of order $(m; r_1, r_2)$ associated to G and microsupported away from J,
- u_2 is a Legendre distribution of order $(m, p; r_2)$ associated to $(G, G_2^\sharp)$ and supported away from H_1,
- u_3 is given by an finite sum of local expressions

$$u_2(\vec{x}, \vec{y}) = \int_{\mathbb{R}^{k_3}} \int_{\mathbb{R}^{k_2}} \int_0^\infty e^{i\Psi^2(x_1,x_2,\vec{y},s,v_2,v_3)/2\mathbf{x_q}}\, a\left(x_1, s, \frac{x_2}{s}, x_3, \vec{y}, v_2, v_3\right) x_3^{m-(1+k_2+k_3)/2+N/4} \\ \times \left(\frac{x_2}{s}\right)^{r_2-(1+k_2)-k_3/2-f_2/2+3N/4} \\ \times s^{p-1-f_2/2+3N/4} x_1^{r_1-f_1/2-(k_2+k_3)/2+3N/4}\, ds\, dv_2\, dv_3, \tag{8.15}$$

  where a is a smooth compactly supported function of its arguments, f_i are the dimension of the fibers on H_i, and $\Psi = 1 + x_1 + sx_1\psi_2 + x_1x_2\psi_3$ is a local parametrization of $(G', (G')_2^\sharp)$ near a corner point q as in (6.12),
- u_4 is given by an finite sum of local expressions

$$u_2(\vec{x}, \vec{y}) = \int_{\mathbb{R}^k} \int_0^\infty e^{i\Psi^2(x_1,x_2,\vec{y},v)/2\mathbf{x_q}}\, a(\vec{x}, \vec{y}, v) \\ \times x_3^{m-k/2+N/4} x_2^{p-1-k/2-f_2/2+3N/4} x_1^{r_1-k/2-f_1/2+3N/4}\, dv, \tag{8.16}$$

  where $\tilde{a}$ is smooth and compactly supported, Ψ is a local parametrization of $(G', (G')_2^\sharp)$ near a point $q \in \partial_1\hat{G} \cap \partial_\sharp\hat{G} \setminus \partial_2\hat{G}$ as in (6.15),
- u_5 is given by

$$u_3(x_1, x_2, y_1, y_2, z_3) \\ = \int e^{i(1+sx_1\psi_2)^2/2\mathbf{x_q}} b\left(x_1, s, \frac{x_2}{s}, y_1, y_2, z_3, v_2\right) \\ \times \left(\frac{x_2}{s}\right)^{r_2-(1+k_2)/2-f_2/2+3N/4} s^{p-1-f_2/2+3N/4} x_1^{r_1-k_2/2-f_1/2+3N/4}\, dv_1\, dv_2 \tag{8.17}$$

  where ψ_2 is as above, b is smooth and $O(x_3^\infty)$ at mf, and

- $u_6 \in x_1^{r_1-f_1/2+3N/4} x_2^{p-f_2/2+3N/4} x_3^{\infty} e^{i(1+x_1)^2/2\mathbf{x_q}} \mathcal{C}^\infty(X)$ (which includes $\dot{C}^\infty(X)$ as a subset).

The set of such distributions is denoted $I^{m,p;r_1,r_2}(X, (G, G_2^\sharp); {}^{\mathrm{qs}\Phi}\Omega^{\frac{1}{2}})$.

Proposition 8.5. *Suppose that $u_h \in I^{m,p;r_1,r_2}(X_h, G; {}^{\mathrm{s}\Phi}\Omega^{\frac{1}{2}})$ is a Legendre distribution associated to the Legendrian G which does not intersect $\mathcal{N}$. Then*

$$\int e^{-it/2h^2} u_h \frac{|dh\,dt|^{1/2}}{h}$$

is in $I^{m+1/2,p+m+1/2-N/4;r_1+m+1/2-N/4,r_2+m+1/2-N/4}(X_t, Q(G); {}^{\mathrm{qs}\Phi}\Omega^{\frac{1}{2}})$, i.e. is a quadratic Legendre distribution associated to $Q(G)$, with orders shifted by $1/2$ at mf *and $m + 1/2 - N/4$ at J, H_1 and H_2.*

The proof is identical to that of Proposition 8.3.

Part 3. Resolvent

9. The example of Euclidean space

In this section we look at the structure of the resolvent, Poisson operator, scattering matrix and propagator on Euclidean space, with the flat metric and no potential, from a Legendrian point of view, and show explicitly that these kernels obey the claims made in Theorems 1.1, 1.4 and 1.5, and Corollaries 1.2 and 1.3.

We begin with the outgoing resolvent kernel. We may identify functions and half-densities via the Riemannian half-density, and regard our kernel as acting on half-densities. The kernel itself is then a half-density on $\mathbb{R}^{2n}$. In order to fit into the framework here we need to multiply by a half-density in h, so that the kernel becomes a half-density on X. Which power of h to include with this half-density factor is an arbitrary choice. We will adopt the convention that the semiclassical outgoing (+)/incoming (−) resolvent is

$$\big(h^2\Delta - (1 \pm i0)\big)^{-1}|dh|^{1/2}.$$

The difference of these, multiplied by $(2\pi i)^{-1}|dh/h^2|^{1/2}$, is the spectral measure $dE(\lambda^2)$ ($\lambda = h^{-1}$).

The kernel of the outgoing resolvent is then

$$R_+ = h^{-n} e^{i|z-z'|/h} f_n\big(|z-z'|/h\big)|dz\,dz'|^{1/2}|dh|^{1/2} = e^{i|z-z'|/h} f_n\big(|z-z'|/h\big)h\mu \tag{9.1}$$

where μ is a non-vanishing scattering-fibered half-density, and $f_n(t) \sim c_n t^{-(n-1)/2}$ as $t \to \infty$.

Let us compute the orders of this as a Legendrian distribution at $N^*\Delta_b$ (see (11.4)), at the propagating Legendrian L_+, and at the b-face and the left and right boundary. We recall that the order convention is that the order gets larger as the distribution becomes smaller, i.e. more regular, and that the order is $N/4$ if the distribution is borderline L^2. To determine the order at $N^*\Delta_b$, we microlocalize away from L_+ by inserting a cutoff function χ that vanishes in a neighborhood of $\{|\zeta| = 1\}$ and write the kernel as an oscillatory integral:

$$h \int e^{i(z-z')\cdot\zeta/h}\big(|\zeta|^2 - 1\big)^{-1} \chi(\zeta)\, d\zeta\, \mu.$$

On the one hand, this is a semiclassical pseudodifferential operator of order $(-2, 0, 0)$; on the other hand, it is a Legendre distribution associated to $N^*\Delta_b$, of semiclassical order $m = 1 + n/2 - (2n+1)/4 = 3/4$ and order at bf equal to $r_2 = n/2 + 1/2 - (2n+1)/4 = 1/4$ (by (4.15)). To determine the order at L_+ we use the expression (9.1); then (5.2) gives the semiclassical order as $m = (n+1)/2 - (2n+1)/4 = 1/4$ and the order at bf equal to $r_2 = (n-1)/2 - (2n+1)/4 + 1/2 = -1/4$. Note that both these orders are 1/2 less than the corresponding order at $N^*\Delta_b$ as required for an intersecting Legendre distribution (see Section 5.5). The order at H_1, i.e. the left or right boundaries, is calculated from (9.1) to be $r_1 = s_1 + k/2 + f_1/2 - N/4 = (n-1)/2 + (n+1)/2 - (2n+1)/4 = n/2 - 1/4$.

In the case of the free resolvent, the Legendrian L_+ is smooth at $L^\sharp$. However, when this is true, by writing the phase and the symbol in polar coordinates around the intersection $L_+ \cap L^\sharp$ we can regard an element of $I^{m;r_1,r_2}(X, L; {}^{s\Phi}\Omega^{\frac{1}{2}})$ as an element of $I^{m,r_2+d;r_1,r_2}(X, (L, L^\sharp); {}^{s\Phi}\Omega^{\frac{1}{2}})$ where d is the codimension of the intersection; here, $d = (n-1)/2$. (This is explained in more detail in Section 14 of [23].) Thus we see that the free outgoing resolvent kernel is an element of

$$\Psi^{2,0,0}(X) + I^{1/4;-1/4}\big(X, \big(N^*\Delta_b, L_+\big); {}^{s\Phi}\Omega^{\frac{1}{2}}\big) \\ + I^{1/4,n/2-3/4;n/2-1/4,-1/4}\big(X, \big(L_+, L^\sharp\big); {}^{s\Phi}\Omega^{\frac{1}{2}}\big).$$

We recall that for a fixed $h > 0$, the semiclassical order has no meaning while the other orders must be adjusted by adding 1/4, reflecting the fact that the orders are 'zeroed' using $N/4$ where N is the total dimension. We see then that the orders agree with those claimed in [10] for the resolvent at a fixed energy.

The Poisson kernel has a natural normalization: we can ask that the family $\{P(\lambda)\}$, $\lambda = h^{-1} \in (0, \infty)$, form a unitary operator mapping from M to $L^2(\partial M \times \mathbb{R}_+; \lambda^{n-1}\, d\lambda\, d\omega)$ with measure corresponding to the conic metric $d\lambda^2 + \lambda^2\, d\omega$ (i.e. a scattering metric) (see [10, Section 9]). To do this we need to multiply the Poisson operator of [23] and [10] by the half-density $|dh/h^2|^{1/2}$.

To obtain the Poisson kernel at rb we divide R_+ by $|dr'|^{1/2} e^{-i|z'|/h}$, where $r' = |z'|$, and restrict at $r' = \infty$, i.e. at rb, to get a half-density at rb: we get

$$P\big(h^{-1}\big) = \lim_{|z'|\to\infty} e^{-i|z'|h} e^{i|z-z'|/h} f_n\big(|z-z'|/h\big) h\mu |dr'|^{-1/2}.$$

The limit of $e^{-i|z'|h} e^{i|z-z'|/h}$ as $|z'| \to \infty$ is $e^{-iy'\cdot z/h}$, where $y' = z'/|z'|$. Also, $\mu|dr'|^{-1/2}$ is equal to $h^{-1/2} \times (|z'|/|z|)^{(n-1)/2}$ times a non-zero scattering-fibered half-density ν on the Poisson space $M \times \partial M \times [0, h_0)$, since

$$\left|\frac{dh}{h^2}\frac{dx}{x^2 h}\frac{dx'}{(x')^2 h}\frac{dy}{(xh)^{(n-1)}}\frac{dy'}{(x'h)^{(n-1)}}\right|^{\frac{1}{2}} = \left|\frac{dh}{h^2}\frac{dx}{x^2 h}\frac{dy}{(xh)^{(n-1)}}\frac{dy'}{(xh)^{(n-1)}}\right|^{\frac{1}{2}} |dr'|^{\frac{1}{2}} h^{-\frac{1}{2}} \left(\frac{x}{x'}\right)^{\frac{n-1}{2}} \\ = \nu |dr'|^{\frac{1}{2}} h^{-\frac{1}{2}} \left(\frac{x}{x'}\right)^{\frac{n-1}{2}}.$$

Therefore, using the asymptotic $f_n(t) \sim c_n t^{-(n-1)/2}$ as $t \to \infty$, we have

$$P(h^{-1}) = c_n (x')^{-(n-1)/2} h^{(n-1)/2} h e^{-iy'\cdot z/h} \nu h^{-1/2} \left(\frac{x}{x'}\right)^{\frac{n-1}{2}} = c_n x^{(n-1)/2} h^{n/2} e^{-iy'\cdot z/h} \nu.$$

This gives orders $m = n/2 - (2n)/4 = 0$ at the main face and $(n-1)/2 + 1/2 - (2n)/4 = 0$ at the b-face. The zero orders reflect the unitarity of this operator. A possibly more natural way of writing the kernel is

$$P(\lambda) = c_n e^{-i\lambda y'\cdot z} \left|\lambda^{n-1}\, d\lambda\, dy'\, dz\right|^{1/2},$$

in which it is clear that $P(\lambda)$ is essentially the Fourier transform.

To get the scattering matrix we again divide by $|dr|^{-1/2}$ and restrict at $r = \infty$. Let ν' be a scattering-fibered half-density on the scattering matrix space $\partial M \times \partial M \times [0, h_0)$. Then

$$\begin{aligned} \left|\frac{dh}{h^2}\frac{dx}{x^2 h}\frac{dy}{(xh)^{(n-1)}}\frac{dy'}{(xh)^{(n-1)}}\right|^{1/2} &= \left|\frac{dh}{h^2}\frac{dy}{h^{(n-1)}}\frac{dy'}{h^{(n-1)}}\right|^{1/2} |dr|^{1/2} h^{-1/2} |z|^{n-1} \\ &= \nu' |dr|^{1/2} h^{-1/2} |z|^{n-1}. \end{aligned}$$

Thus,

$$S(h^{-1}) = \lim_{|z|\to\infty} c_n e^{-iy'\cdot z/h} |z|^{-(n-1)/2} h^{(n-1)/2} \nu' |z|^{n-1}.$$

This can be written

$$S(h^{-1}) = \lim_{r\to\infty} c_n e^{-iry'\cdot y/h} \left(\frac{r}{h}\right)^{(n-1)/2} \left|\frac{dh}{h^2}\, dy\, dy'\right|^{1/2} = \delta(y - y')\left|\frac{dh}{h^2}\, dy\, dy'\right|^{1/2}$$

which is the scattering matrix times a scattering half-density in h. We may also write

$$S(h^{-1}) = \int e^{i(y-y')\cdot\eta/h}\, d\eta \left|\frac{dh\, dy\, dy'}{h^2 h^{n-1} h^{n-1}}\right|^{1/2}$$

which has semiclassical order $m = 0 + (n-1)/2 - (2n-1)/4 = -1/4$ and Lagrangian order $-1/4$. This implies that the order as a Lagrangian for a fixed positive h is 0 (see Section 7.5), so this again reflects unitarity of $S(h^{-1})$ for a fixed h.

The free propagator is given, using the convention in [13] regarding the half-density factor in t, by

$$(2\pi i t)^{-n/2} e^{i|z-z'|^2/2t} |dz\, dz'\, dt|^{1/2}.$$

We can write $t^{-n/2}|dz\, dz'\, dt|^{1/2}$ in the form $t^{n/2+1}\mu$, where μ is a scattering fibered half-density, and this in turn can be written

$$t^{n/2+1} (\rho_{\mathrm{lb}}\rho_{\mathrm{rb}}\rho_{\mathrm{bf}})^{N/2} \mu_q,$$

where μ_q is a non-vanishing quadratic scattering half-density. It follows from (4.15) that the orders of this distribution are $n/2+1-(2n+1)/4=3/4$ at $Q(L)$, $(2n+1)/2+(n+1)/2-3(2n+1)/4=1/4$ at lb and rb and $(2n+1)/2+1/2-3(2n+1)/4=-n/2+1/4$ at bf. As with the resolvent, we may regard this as an especially simple case of a Legendrian associated to a pair of intersecting Legendre distributions $(Q(L), Q(L_2^\sharp))$ with conic points, where the distribution is in fact 'smooth across $Q(L_2^\sharp)$.' We then find that the free propagator is an element of

$$I^{3/4,n/2+1/4;1/4,-n/2+1/4}\big(X, \big(Q(L), Q\big(L_2^\sharp\big)\big); {}^{s\Phi}\Omega^{\frac{1}{2}}\big).$$

These orders agree with those calculated in [13].

10. Pseudodifferential construction

In this section we show that the inverse $(h^2\Delta+V-\lambda_0^2)^{-1}$ of our operator family lies in the algebra of *semiclassical pseudodifferential operators* when $\operatorname{Re}\lambda_0\neq 0$.

10.1. *h-pseudodifferential calculus*

The scattering calculus as described here was introduced by Melrose [24], although its roots go back a good deal further: in various guises on $\mathbb{R}^n$ it has been examined by Shubin [30], Parenti [25], and Cordes [4]; on manifolds, it has also been considered by Schrohe [29]. It is also the Weyl calculus for the metric

$$\frac{|dz|^2}{1+|z|^2}+\frac{|d\xi|^2}{1+|\xi|^2}.$$

The semiclassical variant has been considered by Vasy–Zworski [35] and by the second author and Zworski [37]. See the appendix of [37] for a summary of the properties of this class of operators. Here we simply recall that the space $\Psi_{\mathrm{sc,h}}^{m,l,k}(X)$ of semiclassical scattering pseudos is indexed by the differential order m, the boundary order l and the semiclassical order k. This space of operators can be expressed in terms of the space $\Psi_{\mathrm{sc,h}}^{m,0,0}(X)$ by

$$\Psi_{\mathrm{sc,h}}^{m,l,k}(X)=x^l h^{-k}\Psi_{\mathrm{sc,h}}^{m,0,0}(X). \tag{10.1}$$

Following [37], we shall restrict to operators with polyhomogeneous symbols. The symbols of such operators are functions a on $(0,h_0)\times {}^{\mathrm{sc}}\overline{T}^*X$, having the property that $h^k x^{-l}\rho^{-m}a\in \mathcal{C}^\infty([0,h_0)\times {}^{\mathrm{sc}}\overline{T}^*X)$, where ρ is a boundary defining function for the boundary hypersurface of ${}^{\mathrm{sc}}\overline{T}^*X$ at fiber-infinity. The (principal) symbol map is given by restriction of $h^k x^{-l}\rho^{-m}a$ to the boundary of $([0,h_0)\times {}^{\mathrm{sc}}\overline{T}^*X)$ and denoted $\sigma_{\mathrm{sc,h}}^{m,l,k}(A)$, where a is the symbol of A. Since the boundary consists of three different faces, one at $\rho=0$, one at $x=0$, and one at $h=0$, the principal symbol corresponding can be decomposed into three parts (subject, of course, to compatibility conditions where the faces intersect). We shall call these parts the interior symbol, the boundary symbol and the h-symbol respectively. These symbols lead to three separate exact sequences, with each symbol being the obstruction to the operator being of lower order in the corresponding sense: if the h-symbol vanishes, our operator is divisible by an additional power of h; if the x symbol vanishes, by a power of x; and if the ρ symbol vanishes, the operator is of lower order in the (usual) sense of differentiation.

10.2. Resolvent (away from the spectrum)

Let λ_0 have non-zero imaginary part. Then the principal symbol of $h^2\Delta + V - \lambda_0^2$ is equal to $g(z,\xi) + V(z) - \lambda_0^2$. This is invertible on each boundary face, so by the symbol calculus there is an operator $G_1(\lambda_0) \in \Psi_{\mathrm{sc,h}}^{-2,0,0}(X)$ such that

$$\big(h^2\Delta + V - \lambda_0^2\big)G_1(\lambda_0) = \mathrm{Id} - E(\lambda_0), \qquad E(\lambda_0) \in \Psi_{\mathrm{sc,h}}^{-1,1,-1}.$$

Let $E_2(\lambda_0)$ be an asymptotic sum of the Neumann series $\mathrm{Id} + E(\lambda_0) + E(\lambda_0)^2 + \cdots$. Then we have, with $G_2(\lambda_0) = G_1(\lambda_0)E_2(\lambda_0)$,

$$\big(h^2\Delta + V - \lambda_0^2\big)G_2(\lambda_0) = \mathrm{Id} - E_\infty(\lambda_0),$$

with $E_\infty(\lambda_0)$ in the 'completely residual space' $\Psi_{\mathrm{sc,h}}^{-\infty,\infty,-\infty}$; equivalently, the kernel of $E_\infty(\lambda_0)$ is in $h^\infty\rho^\infty C^\infty(M^2)$, where ρ is a product of boundary defining functions for M^2. The inverse of $\mathrm{Id} - E_\infty(\lambda_0)$ certainly exists as a bounded operator on $L^2(M)$, for small h, since the operator norm $\|E_\infty(\lambda_0)\|_{L^2\to L^2}$ is $O(h^\infty)$. Let us write $(\mathrm{Id} - E_\infty(\lambda_0))^{-1} = \mathrm{Id} + S(\lambda_0)$. We then have

$$S(\lambda_0) = E_\infty(\lambda_0) + E_\infty(\lambda_0)^2 + E_\infty(\lambda_0)S(\lambda_0)E_\infty(\lambda_0).$$

This identity shows that the kernel of $S(\lambda_0)$ is also in $h^\infty\rho^\infty C^\infty(M^2)$. Thus, we have $S(\lambda_0) \in \Psi_{\mathrm{sc,h}}^{-\infty,\infty,-\infty}$. The resolvent is equal to

$$\big(h^2\Delta + V - \lambda_0^2\big)^{-1} = G_2(\lambda_0)\big(\mathrm{Id} + S(\lambda_0)\big)$$

which is in $\Psi_{\mathrm{sc,h}}^{-2,0,0}$, as claimed.

11. Structure of the propagating Legendrian

We now consider the case where λ_0 is real and positive, i.e. we are on the spectrum. In the case where λ_0 is in the resolvent set, studied in the previous section, the singularities of $(H - \lambda_0^2)^{-1}$ live on the conormal bundle of the diagonal. Here, by contrast, singularities propagate off the diagonal. The reason for this is that the characteristic variety of the operator $H - \lambda_0^2$, in either the left or the right variable, intersects the conormal bundle of the diagonal on mf (as well as bf). Moreover, the Hamilton vector field along the characteristic set is non-zero at this intersection, which allows singularities to move into the characteristic set away from the diagonal. In this section we analyze the geometric structure of this flowout, along bicharacteristics, from the characteristic variety at the diagonal; we shall see that it forms a Legendre submanifold in ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}}(X)$ which becomes smooth after certain blowups are performed.

The first step is to compute the left and right Hamilton vector fields for the operator $H - \lambda_0^2$. First, we do this in the interior of ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}}X$. We may choose coordinates $z, z', \zeta, \zeta', \tau$, corresponding to writing covectors

$$\tau \cdot d\left(\frac{1}{h}\right) + \zeta \cdot \frac{dz}{h} + \zeta' \cdot \frac{dz'}{h}.$$

The left and right Hamilton vector fields take the form (where we divide by a factor of 2 for convenience)

$$V_l = h\left(g^{ij}(z)\zeta_i \frac{\partial}{\partial z_j} - \frac{1}{2}\left(\frac{\partial g^{ij}(z)}{\partial z_k}\zeta_i\zeta_j + \frac{\partial V}{\partial z_k}\right)\frac{\partial}{\partial \zeta_k} + g^{ij}(z)\zeta_i\zeta_j\frac{\partial}{\partial \tau}\right) \tag{11.1}$$

and

$$V_r = h\left(g^{ij}(z')\zeta_i' \frac{\partial}{\partial (z')^j} - \frac{1}{2}\left(\frac{\partial g^{ij}(z')}{\partial (z')^k}\zeta_i'\zeta_j' + \frac{\partial V}{\partial (z')^k}\right)\frac{\partial}{\partial \zeta_k'} + g^{ij}(z')\zeta_i'\zeta_j'\frac{\partial}{\partial \tau}\right). \tag{11.2}$$

Let us write $V_l' = V_l/h$ and $V_r' = V_r/h$, restricted to $\{h = 0\}$. These vector fields commute, and are tangent to the left and right characteristic sets

$$\Sigma_l = \{g^{ij}(z)\zeta_i\zeta_j + V(z) = \lambda_0^2\}, \qquad \Sigma_r = \{g^{ij}(z')\zeta_i'\zeta_j' + V(z') = \lambda_0^2\}. \tag{11.3}$$

Let $\Sigma = \Sigma_L \cap \Sigma_R$ denote the intersection of the characteristic sets for p_L and p_R, and let

$$N^*\Delta_b = \left\{d\left(\frac{f}{\mathbf{x}}\right)(p) \;\middle|\; p \in \Delta_b,\ f \in C^\infty_\Phi(X),\ f\restriction \Delta_b = 0\right\}; \tag{11.4}$$

in coordinates, $N^*\Delta_b = \{h = 0,\ z = z',\ \zeta = -\zeta',\ \tau = 0\}$. Note that on $N^*\Delta_b$, Σ_L and Σ_R coincide; hence $N^*\Delta_b \cap \Sigma_L = N^*\Delta_b \cap \Sigma_R = N^*\Delta_b \cap \Sigma$ and is codimension 1 in $N^*\Delta_b$.

Notice also that V_l' and V_r' are everywhere non-tangential to $N^*\Delta_b$. In fact, for V_l' to be tangential we would need $\zeta = 0$ and $\nabla V = 0$, which means that $V_l' = 0$; but this contradicts the non-trapping hypothesis. Consider the flowout by V_l' from the intersection of $N^*\Delta_b \cap \Sigma_L$. It is at least locally a smooth Legendre manifold (Legendre because the vector fields V_l' and V_r' are contact vector fields and the initial hypersurface is isotropic of dimension $2n - 1$). However, $V_l' - V_r'$ is tangent to $N^*\Delta_b$. Moreover, $[V_l, V_r] = 0$, as follows directly from the commutation of the left and right operators $h^2\Delta_l$ and $h^2\Delta_r$. The two-plane distribution spanned by V_l, V_r (or V_l', V_r') is therefore integrable; as $V_l' - V_r'$ is tangent to $N^*\Delta_b$, the integral manifold consisting of all leaves through $N^*\Delta_b$ is thus $2n$-dimensional (rather than $2n + 1$-dimensional as one would expect without this tangency). It follows that the flowout from $N^*\Delta_b \cap \Sigma$ by V_l' coincides with the flowout by V_r'.

This geometry holds uniformly to the boundary of $N^*\Delta_b$. We now work near $N^*\Delta_b \cap \mathrm{bf}$. Then we use coordinates (where $\theta = x'/x$ is small, so $x' \ll x$)

$$\lambda' d\left(\frac{1}{x\theta h}\right) + \lambda d\left(\frac{1}{xh}\right) + \tau d\left(\frac{1}{h}\right) + \mu' \frac{dy'}{x\theta h} + \mu\frac{dy}{xh}. \tag{11.5}$$

In fact these coordinates are valid in the region $\theta \leqslant C$ for any finite C, say $C = 2$, a region which includes a neighborhood of the corner $\mathrm{bf} \cap \mathrm{rb}$.

The symbols of $h^2\Delta + V - \lambda_0^2$ acting on the left and right factors are respectively

$$p_L = \lambda^2 + h^{ij}(x, y)\mu_i\mu_j + V(x, y) - \lambda_0^2,$$
$$p_R = (\lambda')^2 + h^{ij}(x', y')\mu_i'\mu_j' + V(x', y') - \lambda_0^2.$$

The left and right vector fields thus take the form

$$V_l = xh\biggl(-\lambda x\partial_x + \lambda\theta\partial_\theta + h^{ij}\mu_i\partial_{y_j} + \biggl(h^{ij}\mu_i\mu_j + \frac{1}{2}x\partial_x\bigl(h^{ij}\mu_i\mu_j + V\bigr)\biggr)\partial_\lambda + \biggl(-\mu_k\lambda - \frac{1}{2}\partial_{y_k}\bigl(h^{ij}\mu_i\mu_j + V\bigr)\biggr)\partial_{\mu_k} - \frac{1}{2}\partial_x h^{ij}\mu_i\mu_j\partial_\tau\biggr), \tag{11.6}$$

and

$$V_r = x\theta h\biggl(-\lambda'\theta\partial_\theta + h^{ij}(x\theta, y')\mu_i'\partial_{y_j'} + \biggl(h^{ij}(x\theta, y')\mu_i'\mu_j' + \frac{1}{2}\theta\partial_\theta\bigl(h^{ij}(x\theta, y')\mu_i'\mu_j' + V(x\theta, y')\bigr)\biggr)\partial_{\lambda'} + \biggl(-\mu_k'\lambda' - \frac{1}{2}\partial_{y_k'}\bigl(h^{ij}(x\theta, y')\mu_i'\mu_j' + V(x\theta, y')\bigr)\biggr)\partial_{\mu_k'} - \frac{1}{2}\bigl(\partial_{x'}h^{ij}\bigr)(x\theta, y')\mu_i'\mu_j'\partial_\tau\biggr). \tag{11.7}$$

In this region let us write $V_l' = V_l/xh$ and $V_r' = V_r/x\theta h$. Then we have

$$\frac{V_l}{xh} + \frac{V_r}{x\theta h} = (\lambda - \lambda')\theta\partial_\theta - \frac{1}{2}\bigl(\partial_x|\mu|^2 + \partial_{x'}|\mu'|^2\bigr)\partial_\tau - \lambda x\partial_x + h^{ij}\mu_i\partial_{y_j} + \biggl(h^{ij}\mu_i\mu_j + \frac{1}{2}x\partial_x\bigl(h^{ij}\mu_i\mu_j + V\bigr)\biggr)\partial_\lambda + \biggl(-\mu_k\lambda - \frac{1}{2}\partial_{y_k}\bigl(h^{ij}\mu_i\mu_j + V\bigr)\biggr)\partial_{\mu_k} + h'^{ij}\mu_i'\partial_{y_j'} + \biggl(h'^{ij}\mu_i'\mu_j' + \frac{1}{2}\theta\partial_\theta\bigl(h'^{ij}\mu_i'\mu_j' + V'\bigr)\biggr)\partial_{\lambda'} + \biggl(-\mu_k'\lambda' - \frac{1}{2}\partial_{y_k'}\bigl(h'^{ij}\mu_i'\mu_j' + V'\bigr)\biggr)\partial_{\mu_k'}. \tag{11.8}$$

Note that this vanishes only on $\{\lambda' = \lambda, \mu = \mu' = 0, x = 0\}$.

We define the sets L_+, L_- and L by

$$L_+/L_- \text{ is the forward/backward flowout from } N^*\Delta_b \cap \Sigma \text{ by } V_l'$$
$$L = L_+ \cup L_-. \tag{11.9}$$

Equivalently, we may define L_+/L_- as the forward/backward flowout from $N^*\Delta_b \cap \Sigma$ by V_r'. By the arguments above, $L_\pm$ and L are Legendrian submanifolds; moreover, the pairs $(N^*\Delta_b, L_\pm)$ form intersecting pairs of Legendre submanifolds in the sense of Section 5.

The main goal of this section is to determine the regularity of the Legendrian L, which we call the 'propagating Legendrian,' as we move far from $N^*\Delta_b$. By symmetry it suffices to consider just L_+. It turns out that L_+ is smooth except for a conic singularity at a submanifold $L_2^\sharp$ of

${}^{s\Phi}T^*_{\mathrm{bf}}X$; when J, the span of $L_2^\sharp$, is blown up, L_+ lifts to a smooth manifold with codimension 3 corners.

First consider smoothness at $\mathrm{bf} = \{x = 0\}$. Notice that the flows V_l', V_r', when restricted to bf, are naturally identified with the flows for a fixed positive value of h from [11], so $L_+ \cap \mathrm{bf}$ can be identified with $L_+(\lambda)$ from [11]. It was shown in [11] that $L_+(\lambda)$ was smooth after the space $\{x = 0, \lambda = \lambda', \mu = \mu' = 0\}$ was blown up.[7] Let us then define

$$J = \{x = 0, \lambda = \lambda', \mu = \mu' = 0\} = \operatorname{span} G_2^\sharp \subset {}^{s\Phi}T^*X_{\mathrm{b}}^2$$

with

$$G_2^\sharp = \{x = 0, \lambda = \lambda' = 1, \mu = \mu' = 0\}$$

and consider the space

$$\left[{}^{s\Phi}T^*X; J\right]. \tag{11.10}$$

We denote by $\tilde{J}$ the lift of J to this space, i.e. the new boundary hypersurface created by the blowup.

Proposition 11.1. *The closure of the lift of L_+ to the space* (11.10) *is a smooth manifold with corners of codimension three. Consequently, in a neighborhood of $G_2^\sharp$, the pair $(L_+, G_2^\sharp)$ forms a conic Legendrian pair of submanifolds in the sense of Section* 6.

Proof. It suffices to show that $L = L_+ \cup L_-$ is a smooth manifold with corners of codimension three, since L is transversal to $N^*\Delta_b$, which divides it smoothly into two pieces L_+ and L_-. By standard ODE theory, L is smooth at all points reachable from $N^*\Delta_b$ by the vector field V_l' or V_r' in a finite time. However, we need to check the regularity of the closure of L at the boundary of ${}^{s\Phi}T^*X$.

It has already been observed that the two-plane distribution D determined by V_l' and V_r' is integrable; therefore L is foliated by two-dimensional leaves, each of which intersects $N^*\Delta_b$ in a one-dimensional set (since $V_l' - V_r'$ is tangent to $N^*\Delta_b$). Consider a point $(q, \tilde{q}) \in N^*\Delta_b \cap \Sigma$, where q is a covector in the interior of ${}^{sc}T_z^*X$ with $|q| = \sqrt{\lambda_0^2 - V(z)}$, and the tilde denotes negation of the fiber variables of q. The leaf containing this point is the set of points $(q_1, \tilde{q}_2)$, where q_i lie on the same bicharacteristic γ as q; we shall denote it $\gamma^2 = \gamma_q^2$.

It is convenient to choose a 'section' S of $N^*\Delta_b$, by which we mean a smooth submanifold of $N^*\Delta_b$ of codimension 1 that intersects each γ^2 transversely at a unique point. It is not difficult to see that a section exists, using the following argument: From (11.6), under the flow along $V_l/(xh)$,

$$x' = -\lambda x, \qquad \lambda' = h^{ij}\mu_i\mu_j + \frac{1}{2}x\partial_x h^{ij}\mu_i\mu_j + \frac{1}{2}x\partial_x V(x, y).$$

The form $h^{ij} + \frac{1}{2}x\partial_x h^{ij}$ is positive definite for small x and the potential term vanishes at $x = 0$, hence choosing $x_1 > 0$ and $\delta > 0$ small, for $x \leqslant x_1$ and $|\lambda| < \lambda_0 - \delta$, we have $\lambda' > 0$ on the

[7] The coordinates λ, λ' here correspond to τ', τ'' from [11].

characteristic variety. For small x, we can now take S to be $N^*\Delta_b \cap \{\lambda = 0\}$. The null bicharacteristics corresponding to such points remain in the region where x is small, since on the flowout of $S \cap \Sigma$, it is easy to verify that $x' < 0$ and $\lambda' > 0$ except possibly when $\lambda > \lambda_0 - \delta$. In the region where x is large—say $x \geqslant x_0$ where $x_0 < x_1$—each entering geodesic meets the boundary $\{x = x_0\}$ in exactly two points (by the same argument as above). We can take the section to be that point on the diagonal of γ^2 corresponding to the point on the geodesic which is halfway (with respect to arc length) between the two intersection points with $\{x = x_0\}$. We interpolate between these two prescriptions to obtain a smooth section S. Then each leaf intersects S in a unique point.

The strategy of our proof is to first restrict attention to a single leaf and analyze its closure; we shall show that it is a manifold with codimension 2 corners. We shall then show that the union of these closed leaves is the closure of L, and that this forms a submanifold with codimension 3 corners.

We will, initially, have to work on a larger (i.e. more blown-up) space than (11.10) (an analogous point arises in the proof of Proposition 7.1 of [10]). Let J_- be the submanifold

$$\{x = 0, \mu = 0, \mu' = 0, \lambda = -\lambda'\} \subset {}^{s\Phi}T^*_{\mathrm{bf}}X$$

(the only difference in this definition and that of J being a change of sign in the equation $\lambda = \pm\lambda'$). We shall blow up the submanifold J_- as well as[8] J. Also consider the submanifold

$$W = \{\theta = 0, \mu' = 0\} \subset {}^{s\Phi}T^*_{\mathrm{rb}}(X).$$

Let W' denote the lift of W and let $\tilde{J}$ and $\tilde{J}_-$ denote the lifts of J respectively J_- to the blow-up of $J \cup J_-$. Then W' is transverse to $\tilde{J}$ and $\tilde{J}_-$. Now consider the space

$$\big[\big[{}^{s\Phi}T^*X; J \cup J_-\big]; W'\big]. \tag{11.11}$$

Denote the new boundary hypersurface created by this blowup by $\tilde{W}$. We shall work on the space (11.11) for most of this proof, although eventually we shall see that we can return to the space (11.10).

Consider a leaf γ^2 of the distribution D which intersects S at (q, q), where q lies in the interior of X^2_{sc}. (Later we consider q lying in the boundary, i.e. at $x = 0$.) Let $y_{-\infty}$, respectively y_∞ be the points on ∂X obtained as the initial, respectively final end of the bicharacteristic through q. Consider the intersection of $\overline{\gamma^2}$ with the boundary of ${}^{s\Phi}T^*X$, i.e. with $\{x = 0\} \cup \{x' = 0\}$. To get there we must send either q_1 or q_2 to infinity along the bicharacteristic. If we send q_1 to infinity keeping q_2 fixed, we arrive at the set

$$\big\{(y_{-\infty}, 0, -1, z', \zeta', 0) \bigm| (z', \zeta') \in \gamma\big\} \cup \big\{(y_\infty, 0, 1, z', \zeta', 0) \bigm| (z', \zeta') \in \gamma\big\} \subset {}^{s\Phi}T^*_{\mathrm{lb}}X$$

[8] Note that although these two submanifolds intersect, the intersection is away from the closure of L, since on $J \cap J_-$, $\mu = \mu' = 0$, $\lambda = \lambda' = 0$; on the other hand, L is contained in $\Sigma_l \cap \Sigma_r$, so over bf this is given by $\lambda^2 + |\mu|^2 = (\lambda')^2 + |\mu'|^2 = \lambda_0^2 > 0$. We are only interested in a neighborhood of the closure of L, so J and J_- are disjoint in the region of interest, hence they can be blown up independently.

using coordinates $(y, \mu, \lambda, z', \zeta, \theta^{-1})$. Similarly, if we send q_2 to infinity keeping q_1 fixed, we arrive at the set

$$\bigl\{(z, \zeta, y_{-\infty}, 0, 1, 0) \bigm| (z, \zeta) \in \gamma\bigr\} \cup \bigl\{(z, \zeta, y_{\infty}, 0, -1, 0) \bigm| (z, \zeta) \in \gamma\bigr\} \subset {}^{\mathrm{s}\Phi}T^*_{\mathrm{rb}}X$$

using coordinates $(z, \zeta, y', \mu', \lambda', \theta)$. If q_1 and q_2 are simultaneously sent to infinity, we end up at the set

$$\bigl\{(y_{-\infty}, 0, -1, y_{-\infty}, 0, 1, \theta)\bigr\} \cup \bigl\{(y_{\infty}, 0, 1, y_{\infty}, 0, -1, \theta)\bigr\}$$

if they go to infinity in the same direction, or

$$\bigl\{(y_{-\infty}, 0, -1, y_{\infty}, 0, -1, \theta)\bigr\} \cup \bigl\{(y_{-\infty}, 0, -1, y_{\infty}, 0, -1, \theta)\bigr\}$$

if they go to infinity in opposite directions.

We claim that the closure $\overline{\gamma^2}$ inside the space (11.11) is a surface with corners, with eight edges as above, as in Fig. 2. Our analysis is based on the following lemma. Before stating this we need the following

Definition 11.2. Let X be on a manifold with corners, and let $\mathcal{V}_b(X)$ denote the smooth vector fields on X tangent to each boundary hypersurface. Let ρ be a boundary defining function for a boundary hypersurface H of X. We say that $V \in \mathcal{V}_b(X)$ is b-normal at H if

$$V = c\rho\partial_\rho + \rho W \quad \text{for some } W \in \mathcal{V}_b(X)$$

where the coefficient c is never zero. We say that V is incoming, respectively outgoing b-normal if c is positive, respectively negative. (We note that the 'radial' vector field $\rho\partial_\rho \restriction H$ is non-zero as a b-vector field, and independent of coordinates.)

Notice that if a vector field V is b-normal at H, then V/ρ is smooth and transverse to H.

Lemma 11.3. *On the space* (11.11)*, the vector field V_r' is incoming/outgoing b-normal at $\tilde{W} \cap \Sigma_r$, $V_l' + V_r'$ is incoming/outgoing b-normal at $\tilde{J} \cap \Sigma_l \cap \Sigma_r$ and $V_l' - V_r'$ is incoming/outgoing b-normal at $\tilde{J}_- \cap \Sigma_l \cap \Sigma_r$. In all cases, the sign of $\lambda, \lambda' \in \{\pm\lambda_0\}$ determines whether the vector field is incoming or outgoing.*

Proof. We first look at $V_l' + V_r'$. On $\Sigma_l \cap \Sigma_r$,

$$\lambda^2 - (\lambda')^2 = |\mu'|^2 - |\mu|^2 \quad \text{on } L. \tag{11.12}$$

Thus $\lambda - \lambda' = o(|\mu'| + |\mu|)$ near p, so we can take a boundary defining function for $\tilde{J}$ in $\Sigma_l \cap \Sigma_r$ to be $\rho_{\tilde{J}} = \sqrt{x^2 + |\mu|^2 + |\mu'|^2}$. By (11.12), $\lambda - \lambda'$ is $O(\rho_{\tilde{J}}^2)$, so (11.8) gives

$$\begin{aligned}\bigl(V_l' + V_r'\bigr)\big|_L = {} & -\lambda\bigl(x\partial_x + \mu \cdot \partial_\mu + \mu' \cdot \partial_{\mu'} + (\lambda - \lambda')(\partial_\lambda - \partial_{\lambda'})\bigr) \\ & + \mu \cdot \partial_y + \mu' \cdot \partial_{y'} + O\bigl(|\mu|^2\bigr)(\partial_\tau, \partial_\theta, \partial_\lambda, \partial_\mu, \partial_{\lambda'}, \partial_{\mu'}).\end{aligned}$$

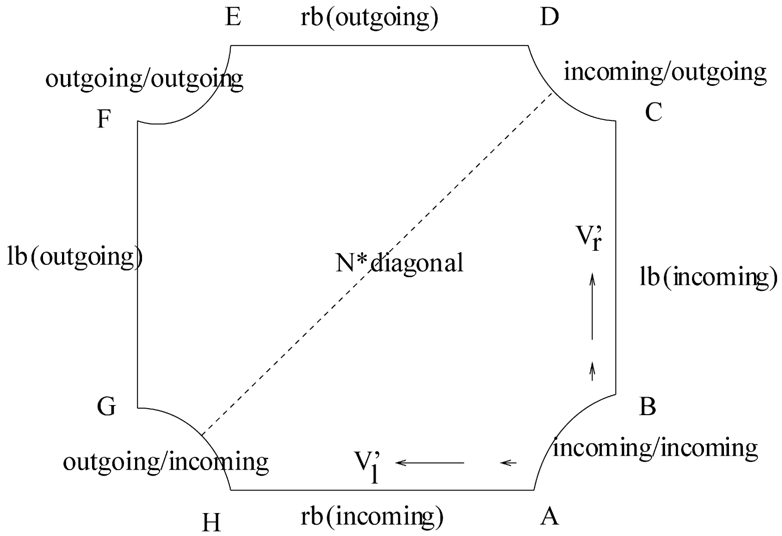

Fig. 2. The closure of a leaf. Here ‘incoming,’ respectively ‘outgoing’ in the left factor means at $\mu = 0$, $\lambda < 0$, respectively $\lambda > 0$, while for the right factor it means $\mu' = 0$, $\lambda' < 0$, respectively $\lambda' > 0$.

This implies that in our local coordinates,

$$V_l' + V_r' = -\lambda \rho_{\tilde{J}} \partial_{\rho_{\tilde{J}}} + \rho_{\tilde{J}} \mathcal{V}_b, \tag{11.13}$$

on the space (11.10). An analogous argument applies to $V_l' - V_r'$ at the blowup of J_-.

We next analyze V_r'. In (11.11), the submanifold $\tilde{W}$ is given by the equations $\{\theta = 0, \mu'/\rho_{\tilde{J}} = 0\}$. Hence in a neighborhood of $\tilde{W}$, $\mu' = o(\sqrt{x^2 + |\mu|^2})$, so we may take $\rho_{\tilde{J}} = \sqrt{x^2 + |\mu|^2}$ in this region, which we shall do from now on. We can switch to local coordinates on (11.10)

$$y, \quad y', \quad \rho_{\tilde{J}} = \sqrt{x^2 + |\mu|^2}, \quad \rho_{\mathrm{bf}} = \frac{x}{x + \rho_{\tilde{J}}}, \quad \theta, \quad M' = \frac{\mu'}{\rho_{\tilde{J}}}, \quad \lambda, \quad \Lambda = \frac{\lambda - \lambda'}{\rho_{\tilde{J}}}, \quad \hat{\mu}. \tag{11.14}$$

From (11.7), in these coordinates,

$$V_r' = -\lambda(\theta \partial_\theta + M' \cdot \partial_{M'}) + \mu' \cdot \partial_{y'} + O\big(|\mu'|^2\big)(\partial_\lambda, \partial_{\mu'}, \partial_\tau),$$

hence

$$V_r' = -\lambda' \rho_{\tilde{W}} \partial_{\rho_{\tilde{W}}} + \rho_{\tilde{W}} \mathcal{V}_b. \qquad \square$$

Continuation of the proof of Proposition 11.1. Now we return to showing that the closure of γ^2 is a smooth 2-manifold with corners. First consider the point A in the figure. This lies on the intersection of $\tilde{J}$ and $\tilde{W}$. We may set $\mathcal{V}_r = V_r'/\theta$ and $\mathcal{V}_c = (V_l' + V_r')/\rho_{\tilde{J}}$; then γ^2 is contained in the flowout from $\gamma^2 \cap S$ by $\mathcal{V}_r$ and $\mathcal{V}_c$. Notice that these vector fields no longer commute, but they still determine an integrable two-plane distribution D. By Lemma 11.3 and the remarks above it, they are both smooth vector fields on (11.11) such that $\mathcal{V}_r$ is transversal to $\tilde{W} \cap \Sigma_r$

and tangent to $\tilde{J}$, while $\mathcal{V}_c$ is transversal to $\tilde{J} \cap \Sigma_l \cap \Sigma_r$ and tangent to $\tilde{W}$. The flowout by these vector fields therefore sweeps out a smooth, closed 2-dimensional manifold with corners meeting the boundary of (11.11) transversally, and it is clear that this is the closure of the leaf.

Since L is invariant under the flow of $\mathcal{V}_r$, which is tangent to the lift of rb and bf, the closure of the leaf is a smooth submanifold which is disjoint from rb and bf (assuming that γ^2 is a leaf through S°, the interior of the section S). It follows that $d\theta \neq 0$ at the intersection of L and $\tilde{W}$, since θ can be taken as a boundary defining function for $\tilde{W}$ away from rb.

Since $\mathcal{V}_c$ and $\mathcal{V}_r$ do not vanish in a neighborhood of A, nearby leaves also have this property, and they vary smoothly with their intersection point $\boldsymbol{\sigma} \in S$ by standard ODE theory. This gives us smooth coordinates on the closure of L near the point A, namely θ, $\rho_{\tilde{J}}$, and a coordinate on S.

Exactly the same argument gives smoothness near the point E. Indeed, a similar argument applies to the corner points D and H in Fig. 2, since there is a symmetry of L coming from the involution $(q, q') \mapsto (q', q)$ on X^2_{sc}. Moreover, essentially the same argument also gives smoothness near the other corner points; the only difference is that we are working near the blowup of J_- rather than J, but this makes no difference at all, because if we replace the minus sign by a plus sign in the left-hand side of (11.12) this makes no difference to the argument.

We also need to check smoothness near a point on an edge. However, we have effectively already done this, because our coordinates are valid for $\theta \leqslant 2$, say, while for $\theta \geqslant 1/2$ we can perform the involution above and use V_l' instead of V_r'.

Notice that the closure of this leaf is disjoint from bf (or more precisely, the lift of bf to (11.11)). In fact the vector fields $\mathcal{V}_c$ and $\mathcal{V}_r$ are everywhere tangent to bf so it is impossible to reach bf after flowing for a finite time along these vector fields. There is another way of seeing this which gives more insight into how these leaves fit together. Notice that a boundary defining function for bf on the space (11.11) can be taken to be

$$\frac{x + x'}{\sqrt{x^2 + (x')^2 + |\mu|^2 + |\mu'|^2}}$$

in a neighborhood of L. For an exactly conic metric, the quantity $x/|\mu|$ is constant along the bicharacteristic and is equal to the maximal value of x that occurs along it. In a general scattering metric, this quantity is approximately equal to the maximal value of x along the bicharacteristic, and this approximation is better and better (in the sense that the ratio of these two quantities tends to 1 uniformly as the bicharacteristic approaches the boundary uniformly); this follows from [9] for example. Hence, for a fixed interior leaf, the limiting value of $x/|\mu|$ is non-zero, which says that the leaf is disjoint from bf. On the other hand, the leaf will approach bf uniformly as the associated bicharacteristic approaches the boundary uniformly.

Now consider a leaf associated to a boundary point of S. In that case, the bicharacteristic is a limiting geodesic contained in the boundary of X, so the leaf is contained in bf. In this case, there is an explicit formula for the leaf. Fix $(y_0, \mu_0) \in T_{y_0}\partial M$ with $|\mu_0| \leqslant 1$. Then the leaf is given by

$$\begin{aligned}\Big\{&(\theta, y, y', \lambda, \lambda', \mu, \mu')\colon \exists (y_0, \hat{\mu}_0) \in S^*\partial X,\ s, s' \in (0, \pi),\ \text{s.t.}\\ &\theta = \frac{\sin s'}{\sin s},\ \lambda = -|\lambda_0| \cos s,\ \lambda' = |\lambda_0| \cos s',\ (y, \mu) = |\lambda_0| \sin s \exp(sH_{\frac{1}{2}h})(y_0, \hat{\mu}_0),\\ &(y', \mu') = -|\lambda_0| \sin s' \exp(s' H_{\frac{1}{2}h})(y_0, \hat{\mu}_0)\Big\}. \end{aligned} \tag{11.15}$$

This corresponds to the interior of the leaf in Fig. 2, which we can think of in this boundary case as the square $(0,\pi)^2$ with the s axis horizontal and the s' axis vertical; V_l' is given by $-\sin s\,\partial_s$ and V_r' is given by $\sin s'\partial_{s'}$ in these coordinates. The closure is given by the closed region in the figure, where the parts over rb (i.e., the boundary lines AH and DE) now lie over the intersection of rb ∩ bf and the parts at $\tilde{J}$ and $\tilde{J}_-$ (the boundary lines AB and EF, respectively CD, GH) now lie in the intersection of those spaces with bf. So the closure of L in the space (11.11) is the disjoint union of the closed leaves, one for every point in S. Since each leaf is contractible, this means that the closure of L on the space (11.11) is diffeomorphic to $S\times\gamma^2$, for some fixed γ^2, and is therefore a smooth manifold with corners of codimension 3.

Now we need to show that the closure of L in the smaller space (11.10) is a smooth manifold of codimension 3. We use the following lemma:

Lemma 11.4. *Assume that Z is a compact manifold with boundary, that $S\subset\partial Z$ is a submanifold and that V is a smooth vector field on Z that lifts to $[Z;S]$ to be b-normal at $\tilde{S}$, the lift of S to $[Z;S]$. Suppose that $L\subset Z^\circ$ is a submanifold of the interior Z° of Z such that V is tangent to L, the closure $\tilde{L}$ of L in $[Z;S]$ is transverse to $\tilde{S}$ and disjoint from $\partial Z\setminus\tilde{S}$. Finally, assume that at each point $s\in\tilde{S}\cap\tilde{L}$,*

$$\text{the intersection of } T_s\tilde{L} \text{ with the tangent space to the fiber of } \tilde{S} \text{ at } s \text{ is trivial.} \tag{11.16}$$

Then the closure $\overline{L}\subset Z$ of L in Z is transverse to ∂Z, and $V|_{\overline{L}}$ is b-normal to $\partial\overline{L}$.

Proof. We can find coordinates (x,y,z) locally near a point of S so that x is a boundary defining function for ∂Z, and S is given by $\{x=y=0\}$. Then coordinates near an interior point $s\in\tilde{S}$ are x, $Y=y/x$ and z, and the fibers of $\tilde{S}$ are parametrized by z. Near s, due to condition (11.16), there is a splitting of the coordinates $z=(z',z'')$ so that (x,z') furnish coordinates on the submanifold $\tilde{L}$. Thus, on $\tilde{L}$, the other coordinates are given by smooth functions of x and z':

$$Y_i=\tilde{Y}_i(x,z'),\qquad z''=\tilde{z}''(x,z').$$

It follows that on L, the coordinates (y,z'') are given by smooth functions of x and z', namely $y=x\tilde{Y}_i(x,z')$, $z''=\tilde{z}''(x,z')$, and hence L is smooth up to, and transverse to, the boundary of Z. Finally, the vector field V, restricted to $\tilde{L}$, has the form

$$x\left(a\partial_x+\sum_i b_i'\partial_{z_i'}\right),$$

where a and b_i' are smooth functions of x and z'. This remains true when viewed as restricted to $\overline{L}\subset Z$, which proves the final statement of the lemma. □

Example 11.5. The following simple examples may help to illustrate the lemma. First consider the vector field $V=-(x\partial_x+y\partial_y+z\partial_z)$ on $Z=\{(x,y,z)\mid z\geqslant 0\}$, let $S\subset Z=\{(0,0,0)\}$ and let L be the flowout from $\{(x,y,z)\mid x^2+y^2=1, z=1\}$ via V. Then condition (11.16) is not satisfied, and L has a genuine conic singularity at S which is resolved by blowup of S.

Second, consider the case where $Z=\{(x,y,z,x',y',z')\mid z\geqslant 0\}$, $V=-(x\partial_x+y\partial_y+z\partial_z)$, $S=\{(0,0,0,x',y',z')\}$ and L is the flowout from $\{(x,y,z,x',y',z')\mid x^2+y^2=1,\ z=1,\ x'=x,\ y'=y,\ z'=z\}$ via V. In this case, (11.16) *is* satisfied, and the closure of L is a smooth manifold with boundary with no blowup required. Indeed, we can take coordinates on L to be $\theta=\tan^{-1}(y'/x')$ and z.

Completion of the proof of Proposition 11.1. We apply the lemma to L, with S equal to W' and V equal to V_r'. Condition (11.16) holds because coordinates on L near $\tilde{W}$ can be taken to be θ, x, y, μ (away from the codimension 3 corner of L). The functions y^i and μ_j have linearly independent differentials since this is true on $N^*\Delta_b$ and since y and μ are invariant under the flow. Near the codimension 3 corner of L, we can take the three boundary defining functions together with y and $\hat{\mu}$ and the same argument goes through. Then the lemma shows that we may blow down $\tilde{W}$ and L is still a manifold with codimension 3 corners, with V_r' still b-normal at $\{\theta = 0\}$.

At $\tilde{J}_-$ a totally different argument is needed. Note the asymmetry between J and J_-: the diagonal $N^*\Delta_b$ intersects J_-, while it is disjoint from J. To understand the structure of L near J_- we can start from $N^*\Delta_b \cap J_-$, which is codimension 1 in L, and flow using either V_l' or V_r'. In the region $\theta \leqslant 2$ it suffices to use V_r'. Then since V_r'/θ is smooth and non-vanishing in this region, we deduce that L is smooth at J_- before blowup. Therefore, (the closure of) L is a smooth manifold with codimension 3 corners on the space (11.10). □

Remark. We emphasize that the blowup at J is essential—it resolves genuine conic singularities of the Legendrian L—while the blowup at J_- resolves no singularities and can be dispensed with. Nevertheless, the blowup at J_- has some good features; in particular, it separates all the leaves. On the space (11.10), the leaves join together at J_- like the pages of a book joined at the binding.

12. Parametrix construction

12.1. Near the h-scattering diagonal

We begin by using a semiclassical scattering pseudodifferential operator to remove the diagonal singularities of the resolvent. Let $P_C = h^2\Delta + V + C$ with $C > -\inf V$. Then in Section 10 we showed that P_C^{-1} is a semiclassical pseudodifferential operator of order $(-2, 0, 0)$. We have

$$\big(h^2\Delta + V - \lambda_0^2\big)P_C^{-1} = \mathrm{Id} - \big(\lambda_0^2 + C\big)P_C^{-1}.$$

Let Q be an asymptotic sum of the Neumann series

$$Q = P_C^{-1}\sum_{j=0}^{\infty}\big(\big(\lambda_0^2 + C\big)P_C^{-1}\big)^j \in \Psi_{\mathrm{sc,h}}^{-2,0,0}(X),$$

which exists since the differential order of P_C^{-1} is negative. Then

$$\big(h^2\Delta + V - \lambda_0^2\big)Q = \mathrm{Id} + E_1, \quad E_1 \in \Psi_{\mathrm{sc,h}}^{-\infty,0,0}(X).$$

Notice that the error term E_1 is trivial except at the boundary of the diagonal $\Delta_b \times [0, h_0)$ on X, i.e., at $\Delta_b \times \{h = 0\}$ and at $\partial\Delta_b \times [0, h_0)$. It remains to solve away the error E_1: we now seek a solution Q' to

$$\big(h^2\Delta + V - \lambda_0^2\big)Q' = -E_1; \tag{12.1}$$

then adding Q' to Q will give the desired parametrix.

12.2. Near the h-b diagonal

We begin by considering the kernel of E_1 on the double space $X = X_{\mathrm{b}}^2 \times [0, h_0)$. The fact that E_1 is an h-pseudodifferential operator of differential order $-\infty$ means that its kernel has an oscillatory integral representation of the form

$$h^{-n} \int e^{i(z-z')\cdot\zeta/h} e(z, \zeta, h)\, d\zeta\, |dz\, dz'|^{1/2}$$

near Δ_b and away from bf, and of the form

$$h^{-n} \int \tilde{e}(x, \theta, y, y', h, \xi, \eta) e^{i\frac{(\theta-1)\xi+(y-y')\cdot\eta}{xh}}\, d\xi\, d\eta\, |dz\, dz'|^{1/2} \quad \left(\theta = \frac{x}{x'}\right)$$

near bf. We multiply this by the half-density $|dh|^{1/2}$, to turn it into a density on X. It may then be regarded as a half-density Legendre distribution of order $(3/4, 1/4)$ associated to the Legendre submanifold $N^*\Delta_b$, where $\Delta_b \subset X \times \{0\}$ is the b-diagonal at $h = 0$. Since we wish to solve the equation (12.1), we need to take into account the (left) characteristic variety $\Sigma_l \subset {}^{s\Phi}T^*(X)$ of the operator $h^2\Delta + V - \lambda_0^2$. The Legendrian $N^*\Delta_b$ is given in the coordinates of (11.5) by

$$\{y = y', \theta = 1, \lambda = -\lambda', \mu = -\mu', \tau = 0\},$$

and the left characteristic variety is given in the same coordinates by

$$\Sigma_l = \{\lambda^2 + h^{ij}(y)\mu_i\mu_j + V(z) = \lambda_0^2\}.$$

These intersect transversely in a submanifold of dimension $2n - 1$, as proved in Section 11. Let $L_\pm$ be defined by (11.9); recall that $L_\pm$ are Legendrian submanifolds with boundary, which intersect $N^*\Delta_b$ cleanly at $\partial L_\pm$, and are both transverse to the boundary bf; hence $(N^*\Delta_b \cap \Sigma, L_\pm)$ have the appropriate geometry for a pair of intersecting Legendre submanifolds, at least in a neighborhood of $N^*\Delta_b$.

We now seek to solve away the error term E_1 near Δ_b using an intersecting Legendrian distribution associated to $(N^*\Delta_b \cap \Sigma, L_+)$; in particular, we would like to find

$$Q_1 \in I^{1/4;-1/4}\big(X; (N^*\Delta_b, L_+)\big)$$

such that $(h^2\Delta + V - \lambda_0^2)Q_1 - E_1$ is microsupported only at L_+, in a region disjoint from $N^*\Delta_b$. (We choose L_+ for the *outgoing* resolvent kernel, and L_- for the *incoming* resolvent kernel; the reason for this is that the coordinate $\bar{\nu}_1$ is positive, respectively negative on L_+, respectively L_- which implies having a positive, respectively negative phase function in the oscillatory integral expression for our kernel.) To do this we solve away the singularity at $N^*\Delta_b$ order by order. (This is a standard construction for intersecting Lagrangian or Legendrian distributions; see [20].)

We begin by choosing a $Q_{1,1}$ to solve away the principal symbol of E_1 at $N^*\Delta_b$. We do this by choosing the symbol of $Q_{1,1}$ at $N^*\Delta_b$ to be

$$\sigma^{3/4}(Q_{1,1}) = \sigma^{3/4}(E_1)/\sigma\big(h^2\Delta + V - \lambda_0^2\big).$$

Of course this has a singularity at $N^*\Delta_b \cap L_+$, but the simple vanishing of $\sigma(h^2\Delta + V - \lambda_0^2)$ at L_+ means this is eligible to be the $N^*\Delta_b$ piece of the symbol of an intersecting Legendrian distribution with respect to $(N^*\Delta_b, L_+)$. The compatibility relation (5.6) then determines the value of the symbol on L_+ at $\partial L_+ = L_+ \cap N^*\Delta_b$; it is essentially given by the residue of the singularity (see Section 5.6). We then specify the symbol at L_+ to be that symbol which solves the transport equation (4.18) along L_+. Since V_l is transverse to $N^*\Delta_b$, this is a regular ODE and there is a unique solution with our initial condition specified above. This gives a $Q_{1,1} \in I^{1/4;-1/4}(X; (N^*\Delta_b, L_+))$ such that

$$\left(h^2\Delta + V - \lambda_0^2\right)Q_{1,1} - E_1 \in I^{5/4,-1/4}\left(X; N^*(\Delta_b), L_+\right)$$

near $N^*\Delta_b$ with principal symbol vanishing at L_+. Using (5.8), we see the error term is actually in

$$I^{7/4,-1/4}\left(X; N^*\Delta_b\right) + I^{9/4,-1/4}\left(X; N^*(\Delta_b), L_+\right). \tag{12.2}$$

The error will thus be more regular at $N^*\Delta_b$ than E_1.

Now we iterate this construction. Assume inductively that we have found $Q_{1,n}$ such that

$$\left(h^2\Delta + V - \lambda_0^2\right)Q_{1,n} - E_1 \in I^{n+3/4,1/4}\left(X; N^*\Delta_b\right) + I^{n+5/4,-1/4}\left(X; N^*(\Delta_b), L_+\right) \tag{12.3}$$

in a neighborhood of $N^*\Delta_b$. We want to improve this by finding $Q_{1,n+1}$ satisfying (12.3) with n replaced by $n+1$. By (5.7) and (5.8) we have to solve away the principal symbol of the first error term $E_{1,n,1}$ in (12.3) at $N^*\Delta_b$, and the principal symbol of the second error term $E_{1,n,2}$ at L_+. We do this as above: we let $Q'_{1,n}$ have symbol at $N^*\Delta_b$ equal to

$$\sigma\left(h^2\Delta + V - \lambda_0^2\right)^{-1}\sigma(E_{1,n,1})$$

and symbol at L_+ given by solving the transport equation on L_+ to remove the principal symbol of $E_{1,n,2}$ there, using the initial condition coming from the compatibility condition (5.6). We cut off this symbol away from $N^*\Delta_b$ to make it supported in a neighborhood of $N^*\Delta_b$. Letting $Q_{1,n+1} = Q_{1,n} + Q'_{1,n}$ completes the inductive step. We can take an asymptotic limit of the $Q_{1,n}$ obtaining a $Q_1 \in I^{1/4,-1/4}(X; (N^*\Delta_b, L_+))$ satisfying

$$\left(h^2\Delta + V - \lambda_0^2\right)Q_1 - E_1 = E_2 + E'_2 \tag{12.4}$$

with $E'_2 \in I^{\infty,1/4}(N^*\Delta_b) + I^{\infty,-1/4}(N^*\Delta_b, L_+)$ and $E_2 \in I^{1/4,-1/4}(X; L_+)$, arising from the cutoff, *microsupported away from* $N^*\Delta_b$. In fact, we can improve this statement to $E'_2 \in I^{\infty,1/4}(N^*\Delta_b) + I^{\infty,3/4}(N^*\Delta_b, L_+)$ since $h^2\Delta + V - \lambda_0^2$ is characteristic at $h^{-1}\partial_{\mathrm{bf}}L_+$ *for every* $h > 0$ which automatically gives us an extra order of vanishing at L_+, hence an improvement by 1 in the order at bf.

12.3. At the propagating Legendrian

As in the finite energy case, we now consider the Legendrian conic pair

$$\tilde{L}(\lambda_0) = \big(L(\lambda_0), L_2^\sharp(\lambda_0)\big),$$

from Proposition 11.1. We set aside the error E_2' until Section 12.4 and seek here to solve away the error E_2 from (12.4) by adding a Legendre distribution $Q_2 \in I^{1/4,p;\mathbf{r}}(X, \tilde{L}(\lambda_0))$, where p is the order at $L_2^\sharp$ and $\mathbf{r}$ represents orders $(r_{\mathrm{bf}}, r_{\mathrm{rb}}, r_{\mathrm{lb}})$ at the other boundary hypersurfaces. We shall see that the orders are $p = n/2 - 3/4$, $r_{\mathrm{bf}} = -1/4$, $r_{\mathrm{rb}} = r_{\mathrm{lb}} = n/2 - 1/4$. Our precise goal in this step in the construction is to find Q_2 so that

$$\big(h^2\Delta + V - 1\big)Q_2 - E_2 \in I^{+\infty,n/2+1/4;(3/4,n/2-1/4,n/2+7/4)}\big(X, \tilde{L}(\lambda_0)\big); \qquad (12.5)$$

that is, the error has been completely solved away at $h = 0$. The space in which the error lies is the same as $h^\infty I^{-1/2,(n-2)/2;(n-1)/2,(n+3)/2}(X_{\mathrm{b}}^2, \partial_{\mathrm{bf}} L, L_2^\sharp)$ (see Section 6.7), that is, a family of Lagrangian distributions associated to the boundary of L at bf and to $L_2^\sharp$, and rapidly decreasing as $h \to 0$. This will reduce the problem to a parametrized version of the problem already studied in [11].

Again we solve away error terms, this time on L_+, order by order. The first step is to find $Q_{2,1}$ solving (12.5) with the order ∞ at $h = 0$ replaced by $9/4$. The order of $Q_{2,1}$ must be $5/4$ at L_+ and $-1/4$ at bf. By (4.18), to solve (12.5) it suffices to obtain q_2 satisfying the ODE

$$\left(\mathcal{L}_{V_l'} - \left(\frac{1}{2} + m - \frac{2n+1}{4}\right)\frac{\partial p_L}{\partial \lambda} + f_l\right)\sigma^{1/4}(q_2) \otimes \big|d(h\sigma x')\big| = e_2, \quad m = \frac{1}{4} \qquad (12.6)$$

and[9] with the 'initial condition' that the symbol q_2 vanishes near $N^*\Delta_b$, reflecting the fact that we do not want to disturb our parametrix near $N^*\Delta_b$. Here we are using coordinates induced from the canonical 1-form

$$\lambda d\left(\frac{1}{x'\sigma h}\right) + \lambda' d\left(\frac{1}{x'h}\right) + \tau d\left(\frac{1}{h}\right) + \mu\frac{dy}{x'\sigma h} + \mu'\frac{dy'}{x'h} \qquad (12.7)$$

which are valid for $\sigma = \theta^{-1} \leqslant 2$, say, thus valid near the corner $\mathrm{lb} \cap \mathrm{bf}$. Also f_l denotes the subprincipal symbol of the (left) operator $h^2\Delta + V - \lambda_0^2$. Since V_l is smooth and non-vanishing in the interior of L_+, this has a unique smooth solution in the interior of L_+. We proceed to analyze the regularity of the symbol at the boundary of L_+. This will be done exploiting the b-normal vector fields from Lemma 11.3. Consider $L_+ \cap \mathrm{lb}$. Here the ODE takes the form

$$\left(\lambda\mathcal{L}_{(\rho_{\mathrm{lb}}\partial_{\rho_{\mathrm{lb}}} + \rho_{\mathrm{lb}}\mathcal{V}_b)} - \left(m - \frac{2n-1}{4}\right)\lambda + f_l\right)q_2 = 0 \qquad (12.8)$$

where $\mathcal{V}_b$ denotes a vector field on L_+ tangent to the boundary at $\rho_{\mathrm{lb}} = 0$. We recall that the sub-principal symbol f_l vanishes at $\mu = 0$, hence is $O(\rho_{\mathrm{lb}})$ at $\rho_{\mathrm{lb}} = 0$. So we may write $f_l =$

[9] The factor $d(h\sigma x')$ in the equation above is a 'formal factor' adjusting for the difference in the symbol bundle (4.17) when the order m changes by 1.

$\rho_{\mathrm{lb}}\tilde{f}_l$. Also recall that q_2 is a half-density and it is convenient to write it as a b-half-density, that is, $q_2 = \tilde{q}_2|d\rho_{\mathrm{lb}}\, d\rho_{\mathrm{bf}}\, d\boldsymbol{\sigma}/\rho_{\mathrm{lb}}|^{1/2}$; note that this half-density is invariant under Lie derivation by $\rho_{\mathrm{lb}}\partial_{\rho_{\mathrm{lb}}}$. We get an equation for $\tilde{q}_2$ of the form

$$\left(\lambda\rho_{\mathrm{lb}}\partial_{\rho_{\mathrm{lb}}} + \rho_{\mathrm{lb}}\mathcal{V}_b - \left(m - \frac{2n-1}{4}\right)\lambda + \rho_{\mathrm{lb}}\tilde{f}_l\right)\tilde{q}_2 = 0$$

hence we obtain

$$\tilde{q}_2 \in \rho_{\mathrm{lb}}^{(2n-1)/4-m}\mathcal{C}^\infty(L)$$

at least locally. Thus, using Proposition 6.3, the order at lb is $(2n-1)/4$.

To show regularity at rb, we use the fact that near rb the symbol q_2 automatically satisfies the right transport equation as well; that is, if we define q_2 using the right transport equation rather than the left, then we get the same result. We shall not give the proof here since it is essentially identical to the proof of the analogous statement in [11, Section 4.4]. Then, reversing the left and right variables in the argument above proves regularity at rb with the order also equal to $(2n-1)/4$.

To show regularity at $L^\sharp(\lambda_0)$, we combine both vector fields. By Lemma 11.3, the vector field $V_l' + V_r'$ is b-normal to $\tilde{J}$, which is the blowup of $L_2^\sharp$; thus we add together the left and the right transport equations. The right transport equation, written with respect to the variables in (11.5), takes the form of (12.6) with the left and right variables switched:

$$\left(\mathcal{L}_{V_r'} - \left(\frac{1}{2} + m - \frac{2n+1}{4}\right)\frac{\partial p_R}{\partial\lambda} + f_r\right)\sigma^{1/4}(q_2) \otimes \left|d(h\theta x)\right| = e_2, \quad m = \frac{1}{4}. \tag{12.9}$$

To compare the two symbols, we must express them with respect to the same total boundary defining function. The total boundary defining function used in (12.6) is hx', while that used in (12.9) is hx. The ratio is θ; in view of the presence of the factor $|d\mathbf{x}|^{m-N/4}$ in the symbol bundle (see (4.17)), the symbol gets multiplied by a factor of $\theta^{m-N/4}$ when we switch (where $N = 2n+1$ here). Thus, with respect to the total boundary defining function hx,

$$\left(\mathcal{L}_{V_r'} - \left(\frac{1}{2} + m - \frac{2n+1}{4}\right)\frac{\partial p_R}{\partial\lambda} + f_r\right)\sigma^{1/4}\left(\theta^{m-N/4}q_2\right) \otimes \left|d(h\theta x)\right| = e_2,$$
$$m = \frac{1}{4}. \tag{12.10}$$

We can multiply this equation by $\theta^{N/4-m}$ and add it to (12.6). The effect of this is that the $-\lambda'\theta\partial_\theta$ term in V_r' gives a contribution of $-(m-N/4)\lambda'$. As a result (taking into account $\lambda = \lambda' + O(\rho_{\tilde{J}})$ at $\tilde{J}$ and $\partial p_L/\partial\lambda = 2\lambda$, $\partial p_R/\partial\lambda' = 2\lambda'$),

$$\left(\mathcal{L}_{V_l'+V_r'} - \left(\frac{1}{2} + m - \frac{2n+1}{4}\right)\lambda + \frac{1}{2}\lambda' + f_l + f_r\right)\sigma^{-1/4}(q_2) \otimes \left|d(h\theta x)\right| = 0,$$
$$m = \frac{1}{4}. \tag{12.11}$$

Since f_l vanishes at $\mu = 0$ and f_r vanishes at $\mu' = 0$, they both vanish at $\tilde{J}$. So we can write $f_l + f_r = \rho_{\tilde{J}} \tilde{f}$. Also, of course $\lambda = \lambda' + O(\rho_{\tilde{J}})$. Thus (12.11) amounts to an equation of the form (again writing $q_2 = \tilde{q}_2$ times a b-half-density)

$$\left(\lambda \rho_{\tilde{J}} \partial_{\rho_{\tilde{J}}} + \rho_{\tilde{J}} \mathcal{V}_b - \left(m - \frac{2n-3}{4}\right)\lambda + \rho_{\tilde{J}} \tilde{f}\right)\tilde{q}_2 = 0 \quad \Rightarrow \quad \tilde{q}_2 \in \rho_{\tilde{J}}^{(2n-3)/4-m} \mathcal{C}^\infty(L)$$

locally. This shows regularity of the symbol at $L^\sharp$, and that the order p at $L^\sharp$ is $n/2 - 3/4$. The error term when we apply the operator is given by (12.5) with $9/4$ replacing ∞. This is because the operator is characteristic at L_+, and at the induced Legendrians at bf and at lb (but not at rb); in addition we have solved the transport equations at L_+ and, trivially, at the left boundary (this because the transport operator is trivial at lb at order $(2n-1)/4$) so we gain two orders in each of these two cases.

Now we iterate the procedure. Assume inductively that we have found $Q_{2,k}$ satisfying

$$\left(h^2 \Delta + V - \lambda_0^2\right) Q_{2,k} - E_2 \in I^{5/4+k, n/2+1/4; (3/4, n/2-1/4, n/2+7/4)}\left(X, \tilde{L}(\lambda_0)\right). \quad (12.12)$$

We want to improve the error term to have order $5/4 + k + 1$ at L_+. To do this, we solve the transport equation at order $5/4 + k$ along L_+, and as above the main point is to determine the regularity of the solution at the boundary of $\hat{L}_+$. Consider the solution of (12.6), with m replaced by $1/4 + k$, and with the right-hand side replaced by the error term in (12.12). Using Proposition 6.3 the right-hand side is $O(\rho_{\mathrm{lb}}^{(2n-1)/4-(1/4+k)+1})$. Therefore the right-hand side avoids the indicial root, in this case $(2n-1)/4 - (1/4 + k)$ which would lead to possible log terms in the solution, and we see that the solution is in $\rho_{\mathrm{lb}}^{(2n-1)/4-(1/4+k)} \mathcal{C}^\infty(L_+)$ locally. Since, as noted above, we get the same parametrix if we solve via the right transport equation instead of the left, the same result is true at rb. Similar reasoning also shows that the symbol is in $\rho_{\tilde{J}}^{(2n-3)/4-(1/4+k)} \mathcal{C}^\infty(\hat{L}_+)$ at $\rho_{\tilde{J}} = 0$; it is essentially the same argument as in [11], Section 4.4, so we omit it. This completes the inductive step. Taking an asymptotic limit of the $Q_{2,k}$ gives a correction term satisfying (12.5).

Remark. If the potential V is replaced by $h^2 V$, then V does not appear in the principal symbol of $H - \lambda_0^2$ and therefore does not affect the bicharacteristic flow or the Legendrian L; on the other hand, it contributes an additional error term on the right-hand side of (12.6). Because of our assumption $V = O(x^2)$, this additional error term is also $O(\rho_{\mathrm{lb}}^{(2n-1)/4-(1/4+k)+1})$, and therefore the construction goes through as above.

12.4. At the boundary for $h > 0$

Our error term is now of the form (using Sections 5.7 and 6.7)

$$E_2' + E_3 \in I^{\infty, 1/4}\left(X, N^* \Delta_b; {}^{s\Phi}\Omega^{\frac{1}{2}}\right) + I^{\infty, 3/4}\left(X, (N^* \Delta_b, L_+), {}^{s\Phi}\Omega^{\frac{1}{2}}\right)$$
$$+ I^{+\infty, n/2+1/4; r_{\mathrm{bf}}+1/4, r_{\mathrm{lb}}+1/4, r_{\mathrm{rb}}+1/4}\left(X, \tilde{L}(\lambda_0), {}^{s\Phi}\Omega^{\frac{1}{2}}\right)$$

where $r_{\mathrm{bf}} = 1/2$, $r_{\mathrm{lb}} = (n+3)/2$ and $r_{\mathrm{rb}} = (n-1)/2$. Equivalently, the error term is a smooth, $O(h^\infty)$ function of h valued in

$$I^0\big(M_{\mathrm{b}}^2, N^*\Delta_b; {}^{\mathrm{s}\Phi}\Omega^{\frac12}\big) + I^{1/2}\big(M_{\mathrm{b}}^2, \big(N^*\Delta_b, h^{-1}\partial_{\mathrm{bf}}L_+\big), {}^{\mathrm{s}\Phi}\Omega^{\frac12}\big)$$
$$+ I^{r_{\mathrm{bf}},n/2;r_{\mathrm{lb}},r_{\mathrm{rb}}}\big(M_{\mathrm{b}}^2, h^{-1}\partial_{\mathrm{bf}}\tilde L(\lambda_0), {}^{\mathrm{s}\Phi}\Omega^{\frac12}\big).$$

We now use the results of [11] to solve away these errors. The main point here is to keep track of powers of h: our error terms are rapidly decreasing in h and we would like to find a correction term that is also rapidly decreasing in h. Examining the construction in [11], we see that the vector fields in the transport equations are linear in $\lambda = h^{-1}$, while λ appears polynomially in the right-hand side due to derivatives bringing down powers of λ from the phase and from the factor λ^2 in front of the potential. It follows that the correction term is $O(h^\infty)$ if the error terms are $O(h^\infty)$. Thus, by [11] we can solve away the error term E_3 above with a term Q_3 in the space

$$Q_3 \in h^\infty\mathcal{C}^\infty\big([0,h_0); I^{-1/2}\big(X_{\mathrm{b}}^2, \partial_{\mathrm{bf}}N^*\Delta_b, h^{-1}\partial_{\mathrm{bf}}L_+\big)\big)$$
$$+ h^\infty\mathcal{C}^\infty\big([0,h_0); I^{-1/2,(n-2)/2;(n-1)/2}\big(h^{-1}\partial_{\mathrm{bf}}L_+, h^{-1}L_2^\sharp\big)\big),$$

or equivalently

$$Q_3 \in I^{-\infty,-1/4}\big(X_{\mathrm{b}}^2, \big(\partial_{\mathrm{bf}}N^*\Delta_b, \partial_{\mathrm{bf}}L_+\big), {}^{\mathrm{s}\Phi}\Omega^{\frac12}\big)$$
$$+ I^{\infty,(2n-3)/4;(2n-1)/4,-1/4}\big(X, \big(\partial_{\mathrm{bf}}L_+, L_2^\sharp\big), {}^{\mathrm{s}\Phi}\Omega^{\frac12}\big)$$

up to a new error term E_4 where the expansions at lb, bf are trivial, but the expansion at rb has not been improved. (We recall that when we act with the operator on the left variable, we can improve our parametrix at lb order by order using the symbol calculus, but to improve at rb we have to solve global problems of the form $(h^2\Delta + V - \lambda_0^2)v = f$, which of course we cannot do until we have constructed the resolvent kernel! So it cannot be expected that we get any improvement at rb.) Thus $E_4 \in I^{\infty,\infty;\infty,\infty,r_{\mathrm{rb}}}(X, (L_+, L_2^\sharp), {}^{\mathrm{s}\Phi}\Omega^{\frac12})$, $r_{\mathrm{rb}} = (n-1)/2$, or more simply,

$$E_4 \in h^\infty x^\infty (x')^{(n-1)/2} e^{i\lambda_0/x'h}\mathcal{C}^\infty\big(X; \Omega_{\mathrm{sf}}^{1/2}\big). \tag{12.13}$$

In summary, we have found a parametrix $G(h)$ in the space

$$\Psi_{\mathrm{sc,h}}^{-2,0,0}(X) \otimes |dh|^{1/2} + I^{1/4;-1/4}\big(X; \big(N^*\Delta_b, L_+\big)\big)$$
$$+ I^{1/4,(2n-3)/4;(2n-1)/4,-1/4}\big(L_+, L_2^\sharp\big) \tag{12.14}$$

such that

$$\big(h^2\Delta + V - \lambda_0^2\big)G(h) - \mathrm{Id} = E_4 \in h^\infty x^\infty (x')^{(n-1)/2} e^{i\lambda_0/x'h}\mathcal{C}^\infty\big(X; \Omega_{\mathrm{sf}}^{1/2}\big). \tag{12.15}$$

13. Resolvent from parametrix

Using the parametrix $G(h)$ constructed in the previous section, which lies in the space (12.14), we can show that the resolvent kernel itself lies in this space for small h. The error term E_4 in the previous section is compact on weighted L^2 spaces $x^sL^2(X)$, for $s > 1/2$. Moreover, the Hilbert–Schmidt norm of E_4, thought of as an operator on $x^sL^2(X)$ parametrized by h, tends to zero. It follows that $\mathrm{Id} + E_4$ is invertible for small h. Let the inverse be $\mathrm{Id} + F(h)$. Then the identity

$$-F = E_4 + E_4^2 + E_4 F E_4 \tag{13.1}$$

shows that F also lies in the space (12.13). Finally, the resolvent kernel is

$$R(h) = G(h) + G(h)F(h).$$

Since $F(h)$ is rapidly decreasing as both $h \to 0$ and as $x \to 0$, it follows from this that $R(h)$ is also in the space (12.14); indeed the rapid decrease of $F(h)$ in x wipes out all expansions of $G(h)$ at bf and at rb in this composition, and the rapid decrease of $F(h)$ as $h \to 0$ wipes out all expansions of $G(h)$ as $h \to 0$. We are left with the expansion of $G(h)$ at lb. This takes the form $e^{i/xh}x^{(n-1)/2}$ times smooth functions of the other variables (ignoring density factors), and the result of the composition is an operator of the form

$$x^{(n-1)/2}(x')^{(n-1)/2}e^{i\lambda_0/xh}e^{i\lambda_0/x'h}h^\infty \mathcal{C}^\infty\big(M^2 \times [0,h_0)\big),$$

rapidly decreasing at bf, at rb, and as $h \to 0$. So $G(h)F(h)$ is a particularly simple example of an operator in (12.14) (corresponding to the term u_6 in Section 6.5.2). This completes the proof of Theorem 1.1.

Part 4. Applications

14. Spectral measure and Schrödinger propagator

In this section we prove Corollary 1.2 and Theorem 1.5. Let H denote $\Delta + V$ in this section, let $R_\pm = (h^2\Delta + h^2V - (1 \pm i0))^{-1}$, and let $\lambda = h^{-1}$. By the remark at the end of Section 12.3, $R_\pm$ has the same structure as the semiclassical resolvent with no potential term. (The term h^2V vanishes to second order at ∂X so it is not present in the principal symbol of the operator, and hence plays no role in determining the Legendrians L or $L_2^\sharp$. It does, of course, affect the *symbol* of the resolvent, but does not change its regularity properties.) Note that as a result, the bicharacteristics arising in the resolvent construction are simply *geodesics* and V is irrelevant to the non-trapping hypothesis.

A direct consequence of Theorem 1.1 is the structure of the spectral measure $dE(\lambda^2)$ for large $\lambda > 0$. By Stone's theorem, we have

$$\begin{aligned} dE\big(\lambda^2\big) &= \frac{1}{2\pi i}\big(\big(H - (\lambda + i0)^2\big)^{-1} - \big(H - (\lambda - i0)^2\big)^{-1}\big)2\lambda\, d\lambda \\ &= \frac{1}{\pi i}\big(R_+(h) - R_-(h)\big) \otimes \left|\frac{dh}{h^2}\right|^{1/2}. \end{aligned}$$

We then have immediately from Theorem 1.1 that $d(E(\lambda^2)) \otimes |dh/h^2|^{-1/2}$ is in the sum of spaces

$$\Psi^{-2,0,0}(X) + I^{-1/4;-1/4}\big((N^*\Delta_b, L_+), X; {}^{s\Phi}\Omega^{\frac{1}{2}}\big)$$
$$+ I^{1/4,n/2-3/4;n/2-1/4,-1/4}\big((L_+, L_2^\sharp), X; {}^{s\Phi}\Omega^{\frac{1}{2}}\big)$$
$$+ I^{-1/4;-1/4}\big((N^*\Delta_b, L_-), X; {}^{s\Phi}\Omega^{\frac{1}{2}}\big) + I^{1/4,n/2-3/4;n/2-1/4,-1/4}\big((L_-, L_2^\sharp), X; {}^{s\Phi}\Omega^{\frac{1}{2}}\big).$$

However, the kernel of $dE(\lambda^2)$ solves an elliptic equation

$$\big(\Delta + V - \lambda^2\big)\, dE\big(\lambda^2\big) = 0.$$

So there can be no singularity of $dE(\lambda^2)$ at $N^*\Delta_b$, except at the characteristic variety $N^*\Delta_b \cap \Sigma_l = N^*\Delta_b \cap L$. along the diagonal. Moreover, $dE(\lambda^2)$ must be Legendrian along $L = L_+ \cup L_-$ at the intersection $L_+ \cap L_- = L \cap N^*\Delta_b$, since it is Legendrian away from $N^*\Delta_b$ and Legendrian regularity propagates along the bicharacteristic flow, which is non-vanishing at $L \cap N^*\Delta_b$. Thus in fact

$$dE\big(\lambda^2\big) \otimes \left|\frac{dh}{h^2}\right|^{-1/2} \in I^{1/4,n/2-3/4;n/2-1/4,-1/4}\big((L, L_2^\sharp), X; {}^{s\Phi}\Omega^{\frac{1}{2}}\big),$$

which is Corollary 1.2.

We now turn to the proof of Theorem 1.5. We begin with some preliminaries on the geometry of the b-double space M_{b}^2. A total boundary defining function for this space can be taken to be $\mathbf{x_b} = (r^2 + (r')^2)^{-1/2}$. We need to consider small neighborhoods of the b-diagonal Δ_b in M_{b}^2. A neighborhood is given, for example, by

$$\big\{(z,z') \bigm| d(z,z') < \epsilon/\mathbf{x_b}\big\} = \big\{(z,z') \bigm| d(z,z') < \epsilon\sqrt{r^2 + (r')^2}\big\}$$

for $\epsilon > 0$. Let ϕ be a smooth function on $[0,\infty)$ equal to 1 on $[0,1]$ and equal to 0 on $[2,\infty)$. Then $\phi(d(z,z')\mathbf{x_b}/\epsilon)$ is a smooth function on M_{b}^2 equal to 1 at Δ_b and supported near Δ_b (for small ϵ). Abusing notation somewhat, we shall denote this function on M_{b}^2 simply by ϕ. The local injectivity radius on M is bounded below by cr for some $c > 0$; we shall assume that $\epsilon > 0$ is chosen so that the local injectivity radius is at least $10\epsilon r$. Then the square of the distance function $d(z,z')^2$ will be smooth on the support of ϕ.

To obtain the kernel of the propagator $e^{-itH/2}$, $H = \Delta + V$, consider the integral over the spectrum:

$$e^{-itH/2} = \int_{-\infty}^{\infty} e^{-it\mu/2}\, dE(\mu). \tag{14.1}$$

To deal with this integral we break it into several pieces. We first use a spectral cutoff. Let us insert $1 = \chi_1(\mu t) + \chi_2(\mu t)$ into the integral, where $\chi_1 \in C_c^\infty(\mathbb{R})$ is equal to 1 on $[-1,1]$ and equal to 0 on $\mathbb{R}\backslash(-2,2)$. The χ_1 term yields the operator $\chi_1(tH)e^{-itH/2}$. Letting $s = \sqrt{t}$, this is a C_c^∞ function of $s^2 H$ and is therefore a semiclassical scattering pseudodifferential operator (with s playing the role of Planck's constant) of order $-\infty$ (cf. [6]). In particular, it is smooth away from the diagonal, and rapidly decreasing as $d(z,z')/s \to \infty$. Let us write this kernel $U_{\mathrm{near},1}$. Notice that $(1-\phi)U_{\mathrm{near},1}$ is residual, i.e. in $\dot{C}^\infty(M_{\mathrm{b}}^2 \times [0,t_0))$, for any function ϕ localized near Δ_b as above (i.e., for any $\epsilon > 0$).

Now consider the integral with $\chi_2(\mu t)$ inserted. For t small enough, the only part of the spectrum of H lying in the support of this term is continuous spectrum in $[0, \infty)$, hence we may change variables, and rewrite this term as

$$e^{-itH/2} = \int_0^\infty \tilde{\chi}_2(\lambda\sqrt{t}) e^{-it\lambda^2/2}\, dE(\lambda^2) \tag{14.2}$$

where

$$\tilde{\chi}_2(\lambda) = \chi_2(\lambda^2).$$

We now localize based on the value of the phase function $\psi/\mathbf{x}$, $\mathbf{x} = \mathbf{x_b} h = \mathbf{x_b}/\lambda$, in the representation of the semiclassical resolvent as a Legendre distribution. Let us write

$$1 = \chi_n(\psi/\epsilon) + \chi_i(\psi/\epsilon) + \chi_f(\psi/\epsilon)$$

where χ_n is supported in $[0, 1/2]$, χ_i is supported in $[1/4, 3]$ and χ_f is supported in $[5/2, \infty)$. We obtain three kernels, denoted $U_{\text{near},2}$, U_{int} and U_{far}, by inserting the cutoffs $\chi_\bullet(\psi/\epsilon)$ into (14.2). Let us also define $U_{\text{near}} = U_{\text{near},1} + U_{\text{near},2}$. Thus we may write the exact propagator

$$e^{-itH/2} = U_{\text{near}} + U_{\text{int}} + U_{\text{far}}.$$

Lemma 14.1.

(i) *The kernel* $d\phi \cdot U_{\text{near}}$ *is in* $\dot{C}^\infty(M_{\text{b}}^2 \times [0, t_0))$.
(ii) *The kernel*

$$(1-\phi)\left(D_t + \frac{1}{2}H\right)U_{\text{int}} + \left(D_t + \frac{1}{2}H\right)U_{\text{far}} \tag{14.3}$$

is in $\dot{C}^\infty(M_{\text{b}}^2 \times [0, t_0))$.
(iii) U_{int} *is a quadratic Legendre distribution associated to* $Q(L)$, *and* U_{far} *is a quadratic Legendre distribution associated to* $(Q(L), Q(L_2^\sharp))$.

Proof. (i) We have already observed that this is true in the case of $U_{\text{near},1}$ so consider $U_{\text{near},2}$. Observe that

$$e^{i\lambda\psi/\mathbf{x_b}} = \frac{-i\mathbf{x_b}}{\lambda}\frac{1}{d_v\psi} d_v e^{i\lambda\psi/\mathbf{x_b}},$$

and that on the support of $1-\phi$ and on the support of $d\chi_n(\psi/\epsilon)$ we have $d_v\psi \neq 0$. (This is because $d_v\psi = 0$ implies that $\psi/\mathbf{x_b} = d(z,z')$, yet $\mathbf{x_b} d(z,z') \geqslant \epsilon$ on the support of $1-\phi$ and $\psi \leqslant \epsilon/2$ on the support of $d\chi_n$.) Thus we can integrate by parts in v as many times as we like.[10] Each integration by parts gains us $\mathbf{x_b}/\lambda$. This allows us to absorb any number of spatial or t-derivatives,

[10] If there are no v variables then we simply use the fact that $\psi/\mathbf{x_b} = d(z,z')$.

as well as any number of negative powers of $\mathbf{x_b}$ or t (remembering that the combination $\lambda^{-2}t^{-1}$ is bounded on the support of $\tilde{\chi}_2$). This proves membership in $\dot{C}^\infty(M_b^2 \times [0, t_0))$.

(ii) Let us start with the first term, $(1-\phi)(D_t + \frac{1}{2}H)U_{\mathrm{int}}$. U_{int} is given by a finite sum of integrals of the form

$$\int\int e^{-it\lambda^2/2} e^{i\lambda\psi(\cdot,v)/\mathbf{x_b}} \tilde{\chi}_2(\lambda\sqrt{t})\chi_i(\psi/\epsilon)a(\lambda,\cdot,v)\,dv\,d\lambda.$$

Here, $\cdot$ refers to the spatial variables on M_b^2. If we apply $(D_t + \frac{1}{2}H)$ to the integral then the result vanishes if none of the derivatives hits one of the cutoffs $\chi_i(\psi/\epsilon)$ or $\tilde{\chi}_2(\lambda\sqrt{t})$, so $(1-\phi)(D_t + \frac{1}{2}H)U_{\mathrm{int}}$ is a sum of terms of the form

$$(1-\phi)\int\int e^{-it\lambda^2/2} e^{i\lambda\psi(\cdot,v)/\mathbf{x_b}} \tilde{\chi}_2'(\lambda\sqrt{t})\chi_i(\psi/\epsilon)\tilde{a}(\lambda,\cdot,v)\,dv\,d\lambda$$

or

$$(1-\phi)\int\int e^{-it\lambda^2/2} e^{i\lambda\psi(\cdot,v)/\mathbf{x_b}} \tilde{\chi}_2(\lambda\sqrt{t})\chi_i^{(k)}(\psi/\epsilon)\tilde{a}(\lambda,\cdot,v)\,dv\,d\lambda$$

where k, the number of derivatives falling on χ_i, is either 1 or 2. In the first case, we can integrate by parts in λ as many times as we like, using the identity

$$e^{i(-t\lambda^2/2+\lambda\psi/\mathbf{x_b})} = \frac{-i\mathbf{x_b}}{-t\lambda\mathbf{x_b}+\psi}\frac{\partial}{\partial\lambda}e^{i(-t\lambda^2/2+\lambda\psi/\mathbf{x_b})}$$

and the fact that the denominator is bounded below since $\psi \geqslant \epsilon/4$ on the support of $\chi_i(\psi/\epsilon)$, $\mathbf{x_b}$ is a bounded function, and it suffices to consider only times $t \ll 1$. This allows us to reduce the order of the symbol in λ, and increase the order in x_1 and x_2, as much as we like. Using the same reasoning as in part (i), the kernel is in $\dot{C}^\infty(M_b^2 \times [0, t_0))$. Exactly the same arguments allows us to dispose of the terms coming from $(D_t + \frac{1}{2}H)U_{\mathrm{far}}$ when a derivative hits $\tilde{\chi}_2$.

In the case of the second integral, we need to further divide into two cases, according as the derivative $\chi_i^{(k)}$ is supported in $[1/4, 1/2]$ or in $[5/2, 3]$. In the first case, supported in $[1/4, 1/2]$, we can integrate by parts in v as many times as we like, as in part (i), and we see that these terms are in $\dot{C}^\infty(M_b^2 \times [0, t_0))$. In the second case, supported in $[5/2, 3]$, we note that modulo $\dot{C}^\infty(M_b^2 \times [0, t_0))$ we can replace the factor $1-\phi$ by 1, for exactly the same reason.

Now we see that these terms, with $\chi_i^{(k)}(\psi/\epsilon)$ supported in $[5/2, 3]$ and with $1-\phi$ replaced by 1, exactly cancel the remaining terms from $(D_t + \frac{1}{2}H)U_{\mathrm{far}}$, since $\chi_i^{(k)}(\psi/\epsilon) = -\chi_f^{(k)}(\psi/\epsilon)$ when restricted to the interval $(\psi/\epsilon) \in [5/2, 3]$. We conclude that (14.3) is in $\dot{C}^\infty(M_b^2 \times [0, t_0))$.

(iii) This follows immediately from Propositions 8.3 and 8.5. □

It appears to be difficult to determine the microlocal nature of U_{near} using the integral (14.2). One reason is that the spectral cutoffs χ_1, χ_2, needed in order to apply Propositions 8.3 and 8.5 in part (iii) of the above lemma, interfere with the microlocal nature of the pieces. In particular, the piece $U_{\mathrm{near},1}$ does *not* lie in the space (1.1). We shall see that this is an artifact of the spectral cutoffs and the sum $U_{\mathrm{near},1} + U_{\mathrm{near},2}$ *does* lie in (1.1). To see this we need to change strategy. What we shall do is construct a parametrix in the near-diagonal region, and show that we can glue it together with the kernel constructed above to obtain the true propagator modulo a $\dot{C}^\infty(M_b^2 \times [0, t_0))$ error.

In the near-diagonal region we use the same ansatz as in Step 1 of [13]. For the reader's convenience we recall that this takes the form

$$(2\pi it)^{-n/2} e^{i\Phi(z,z')/t} \sum_{j=0}^{\infty} t^j a_j(z, z').$$

We want this to be a formal solution, so we apply the operator $t^2 D_t + t^2/2\Delta$ and solve the resulting equations to each order in t. The first is the eikonal equation $-\Phi + g(\nabla_z \Phi, \nabla_z \Phi) = 0$ which has the exact solution $\Phi(z, z') = d(z, z')^2/2$. Thus we see that this is a Legendrian associated to the same Legendre submanifold, namely $Q(L)$, to which U_{int} and U_{far} are associated. The remaining equations are transport equations taking the form (in normal coordinates z about z')

$$\big(z_i + O\big(|z|^2\big)\big)\frac{\partial}{\partial z_i} a_0 = f \cdot a_0,$$

$$\big(z_i + O\big(|z|^2\big)\big)\frac{\partial}{\partial z_i} a_j + j a_j = f \cdot a_j - \frac{i}{2}\Delta_z a_{j-1} \quad (j \geqslant 1), \quad f = \frac{1}{2}\Delta\Phi + \frac{n}{2} = O(z),$$

where all terms are smooth. These equations have unique solutions with a_j smooth and $a_0(0) = 1$. We cut this formal solution off by multiplying by a cutoff function $\phi(d(z, z')/\epsilon r')$ localizing near Δ_b.

This argument only applies away from the front face of M_b^2 since the analysis of [13] was only carried out there. However, the near-diagonal ansatz above holds uniformly up to $\mathrm{bf} \subset M_b^2$, i.e. in a full neighborhood of $\Delta_b \subset M_b^2$. We proceed to show this. We first note that the function $\Psi = d(z, z')^2/2(r')^2$ is a smooth function on M_b^2 in a neighborhood of Δ_b. In fact, if we take coordinates $x', \sigma = x/x', y'$ and y locally near Δ_b, where y' is a local coordinate on ∂M and for a fixed y', y are normal coordinates on ∂_M centered at y' (hence, y is *not* a coordinate lifted from the left factor of ∂M), then Δ_b is defined by $\{\sigma = 1, y = 0\}$ and near Δ_b,

$$\Psi = (\sigma - 1)^2 + \sum y_i^2 + \text{terms vanishing to third order at } \Delta_b.$$

On the other hand, the operator $t^2(D_t + \frac{1}{2}\Delta)$ takes the form

$$t^2 D_t + \big(tx'\sigma^2 D_\sigma\big)^2 + (n-1)t^2 x'\sigma^3 \partial_\sigma + h^{ij}(x)\big((tx'D_{y_i})(tx'D_{y_j}) + \Gamma_{ij}^k(x)\big(t^2(x')^2 D_{y_k}\big)\big)$$

where $\Gamma(x)$ is the Christoffel symbol for the metric $h(x)$. Let us seek a formal solution, as a series in t, near the boundary of Δ_b. It takes the form

$$(2\pi it)^{-n/2} e^{i\Psi/t(x')^2} \sum_{j=0}^{\infty} t^j b_j(x', \sigma, y, y'), \quad \text{with } b_j \text{ smooth.}$$

Since $h^{ij} = \delta_{ij}$ at $y = y'$ and $\Gamma = O(y)$ there, it follows then that we end up with transport equations for the b_j of the form

$$\left(y_i\frac{\partial}{\partial y_i}+(\sigma-1)\frac{\partial}{\partial\sigma}+W\right)b_0=f\cdot b_0,$$
$$\left(y_i\frac{\partial}{\partial y_i}+(\sigma-1)\frac{\partial}{\partial\sigma}+W\right)b_j+jb_j=f\cdot b_j-\frac{i}{2}\Delta b_{j-1}\quad(j\geqslant 1)$$

where all terms are smooth, $f=\frac{1}{2}\Delta\Phi+\frac{n}{2}$ vanishes linearly at Δ_b and W is a vector field vanishing quadratically at Δ_b. These equations have unique smooth solutions b_j in a neighborhood of Δ_b, with $b_0=1$ at Δ_b. An asymptotic sum of this formal series is therefore a solution to the equation to order t^∞, i.e. the error term after applying $t^2(D_t+\frac{1}{2}\Delta)$ is in $t^\infty\mathcal{C}^\infty(M_{\mathrm{b}}^2\times[0,t_0))$ near Δ_b.

We also need our near-diagonal parametrix to be good as $x'\to 0$. To improve the error term at $x'=0$ we expand in a Taylor series in x'. The error term has a Taylor series

$$e^{i\Psi/t(x')^2}\sum_{k=0}^{\infty}(x')^j e_j(t,x',\sigma,y,y'),$$

near Δ_b with each $e_j=O(t^\infty)$ and smooth. We try to solve this away with a series

$$e^{i\Psi/t(x')^2}\sum_{k=0}^{\infty}(x')^j c_j(t,x',\sigma,y,y'). \tag{14.4}$$

This gives us equations of the form

$$\left(y_i\frac{\partial}{\partial y_i}+(\sigma-1)\frac{\partial}{\partial\sigma}+t\frac{\partial}{\partial t}+W\right)c_j=te_j+P(c_0,c_1,\dots,c_{j-1}),$$

where W is as above and P is a differential operator with smooth coefficients. Since $e_j=O(t^\infty)$ there is a unique solution c_j which is $O(t^\infty)$. Adding the correction term (14.4) yields a parametrix with an error term $O(t^\infty(x')^\infty)$ locally near Δ_b. Let us denote this near-diagonal parametrix, defined in a neighborhood of Δ_b by V_{near}.

We now claim that, on the support of $d\phi$, V_{near} is equal to U_{int} up to $\dot{C}^\infty(M_{\mathrm{b}}^2\times[0,t_0))$. In the interior of M_{b}^2, this follows from [13] where we showed that V_{near} is equal, microlocally, to the exact propagator modulo $t^\infty C^\infty(M_{\mathrm{b}}^2\times[0,t_0))$. Our construction is such that U_{near} is in $\dot{C}^\infty(M_{\mathrm{b}}^2\times[0,t_0))$ on the support of $d\phi$ (Lemma 14.1) while U_{far} is microsupported where the phase function is relatively large. (Using the cutoff ψ_f, and the contact transformation Q, we have $\bar{\bar{\nu}}_1\geqslant(5\epsilon/2)^2/2$ on the microsupport of U_{far}, while we have $\bar{\bar{\nu}}_1\leqslant(2\epsilon)^2/2$ on the microsupport of V_{near} and on the support of $d\phi$. Here $\bar{\bar{\nu}}_1$ is the coordinate from (8.5) and Q defined by (8.7).) Therefore, at least away from the boundary of M_{b}^2, V_{near} is equal to U_{int} modulo $t^\infty C^\infty(M_{\mathrm{b}}^2\times[0,t_0))$ on $\operatorname{supp} d\phi$.

However, both V_{near} and U_{int} are Legendre distributions associated to the same Legendrian, and their full symbol expansion at $t=0$ is smooth up to the boundary of M_{b}^2. Since they agree everywhere in the interior of M_{b}^2 on $\operatorname{supp} d\phi$, they agree up to the boundary. Hence V_{near} is equal to U_{int} modulo $t^\infty C^\infty(M_{\mathrm{b}}^2\times[0,t_0))$ globally on the support of $d\phi$. Finally, both V_{near} and U_{int} solve the Schrödinger equation microlocally, and we saw above that the Taylor series of V at

$x' = 0$ was *uniquely* determined by this condition, it follows that $V_{\rm near}$ and $U_{\rm int}$ are equal to all orders in both t and x' microlocally near the Legendrian L and on the support of $d\phi$.

We now construct an accurate global parametrix for the propagator. Define

$$U = \phi V_{\rm near} + (1-\phi)U_{\rm int} + U_{\rm far}.$$

We claim that this parametrix U satisfies the initial condition

$$\lim_{t\to 0} U(t) = \mathrm{Id}$$

distributionally (i.e. the distribution limit of $U(t)$ as $t \to 0$ is equal to the delta function on Δ_b), and satisfies the equation $(D_t + \frac{1}{2}H)U(t) = 0$ up to an error term in $\dot{C}^\infty(M_{\rm b}^2 \times [0,t_0))$ (i.e. smooth and vanishing to infinite order at $t=0$ and all boundary hypersurfaces of $M_{\rm b}^2$). The initial condition follows from the stationary phase lemma applied to Legendre distributions; in particular the delta function on the diagonal comes from $V_{\rm near}$ while $U_{\rm int}$ and $U_{\rm far}$ contribute nothing, since the phase function is always non-zero for all Legendre distributions comprising $U_{\rm int}$ and $U_{\rm far}$.

To prove the claim about U satisfying the equation, we write

$$\begin{aligned}\left(D_t + \frac{1}{2}H\right)U(t) &= \phi\left(D_t + \frac{1}{2}H\right)V_{\rm near} + (1-\phi)\left(D_t + \frac{1}{2}H\right)U_{\rm int} + \left(D_t + \frac{1}{2}H\right)U_{\rm far}\\ &\quad + \nabla\phi\cdot\nabla(V_{\rm near} - U_{\rm int}) + \frac{1}{2}\Delta\phi(V_{\rm near} - U_{\rm int}).\end{aligned} \tag{14.5}$$

We have arranged that $V_{\rm near}$ is an accurate parametrix on the support of ϕ, so the first term is in $\dot{C}^\infty(M_{\rm b}^2 \times [0,t_0))$. Next, Lemma 14.1 shows that the sum of the second and third terms is in $\dot{C}^\infty(M_{\rm b}^2 \times [0,t_0))$. Third, we have seen that $V_{\rm near}$ is equal to $U_{\rm int}$ up to $\dot{C}^\infty(M_{\rm b}^2 \times [0,t_0))$ on the support of $d\phi$. It follows that the last two terms on the right-hand side of (14.5) are in $\dot{C}^\infty(M_{\rm b}^2 \times [0,t_0))$. This completes the proof that U is a parametrix up to $\dot{C}^\infty(M_{\rm b}^2 \times [0,t_0))$ errors.

Finally we correct the error term. It follows from a commutator argument due to Craig [5, Théorème 14], that

$$e^{-itH/2} : \dot{C}^\infty(M) \to \dot{C}^\infty(M) \quad \text{for all } t. \tag{14.6}$$

We can correct our parametrix U to the exact propagator by adding to U the kernel

$$i\int_0^t e^{-i(t-s)H/2}\left(\left(D_s + \frac{1}{2}H\right)U(s)\right)ds \in t^\infty \dot{C}^\infty\big(M_{\rm b}^2 \times [0,t_0)\big).$$

Since $\phi V_{\rm near}$, $(1-\phi)U_{\rm int}$, $U_{\rm far}$ and elements of $\dot{C}^\infty(M_{\rm b}^2 \times [0,t_0))$ are all Legendre distributions associated to the conic pair $(Q(L), Q(L_2^\sharp))$, the proof of the theorem is complete.

Remark. One might wonder why it was necessary to use the cutoff $\chi_1(\lambda\sqrt{t})$, instead of a t-independent cutoff. The reason is that a t-independent cutoff will yield a term that is smooth on

M^2_{sc} down to $t = 0$. This term does not lie in the space (1.1) so it would have to be eliminated by an a posteriori argument. In this respect it is not so different from the term $U_{near,1}$; however $U_{near,1}$ is localized close to the diagonal so it automatically becomes harmless when we glue in our V_{near} parametrix, which is a little more convenient.

Remark. Note that U_{far} need not be supported away from the diagonal. In fact, if there is a geodesic curve on M that self-intersects, then there will be a corresponding part of U_{far} supported over the diagonal, although microlocally it will be away from the zero section. It is for this reason that we introduce U_{int}: we arranged that U_{int} be supported close to, but not at, the diagonal, and this allowed us to piece together V_{near} and U_{int} using the cutoff ϕ in (14.5).

15. Poisson operator and scattering matrix

Having constructed the semiclassical resolvent as a Legendrian distribution, we can now easily determine the structure of the semiclassical Poisson operator and scattering matrix, since the kernels of these operators are related in a simple way to the resolvent kernel.

We recall that the outgoing resolvent kernel was normalized, as a half-density in h, as $(h^2\Delta + V - (\lambda_0^2 + i0))^{-1}|dh|^{1/2}$. The Poisson operator $P(h^{-1})$ may be defined by the restriction of $e^{-i\lambda_0/x'h}|dr'|^{-1/2}$ times the resolvent kernel to the right boundary $\mathrm{rb} = H_1$ of X (see Remark 8.4 of [10]). This may be regarded as the principal symbol of the resolvent kernel at the Legendrian L_1 corresponding to the base of the fibration $\partial_1 L \to L_1$ at $\mathrm{rb} = H_1$ (see Proposition 4.3).

Since the kernel of $P(\lambda)$ is a function on $M \times \partial M \times [0, h_0)$ it is natural to regard $M \times \partial M \times [0, h_0)$ as a scattering-fibered manifold, with the main face being $M \times \partial M \times \{0\}$ and the other boundary hypersurface, $\partial M \times \partial M \times [0, h_0)$ fibered over $\partial M \times \partial M$ by projection off the h variable. To determine the Legendrian structure of $P(h^{-1})$ we start with the geometry of the propagating Legendrian L, defined in (11.9), near the right boundary rb of ${}^{s\Phi}T^*_{mf}X^2_b$. Working near the right boundary, we use coordinates $(x, \theta = x'/x, y, y', h, \lambda, \lambda', \mu, \mu', \tau)$, as defined in (11.5).

Let W be the set $\{\theta = 0, \mu' = 0\} \subset {}^{s\Phi}T^*X^2_b$, and consider the blowup $[{}^{s\Phi}T^*X^2_b; W]$ of ${}^{s\Phi}T^*X^2_b$ at W. Let $\tilde{W}$ denote the new boundary hypersurface created by this blowup, and write $\overline{\mu}' = \mu'/\theta$; this is a smooth function in the interior of $\tilde{W}$.

Lemma 15.1. *$\tilde{W} \cap \{\lambda' = \lambda_0\}$ is diffeomorphic to ${}^{s\Phi}T^*(M \times \partial M \times [0, h_0))$ and hence $\tilde{W} \cap \{\lambda' = \lambda_0, h = 0\}$ has a natural contact structure (degenerating at $x = 0$), contactomorphic to ${}^{s\Phi}T^*_{mf}(M \times \partial M \times [0, h_0))$.*

Proof. The contact form at ${}^{s\Phi}T^*_{mf}X^2_b$ is given in coordinates $(\theta, x, h, y', y; \lambda', \lambda, \tau, \mu', \mu)$ by

$$-d\lambda' - \theta\, d\lambda - x\theta\, d\tau + \mu' \cdot dy' + \theta\mu \cdot dy. \tag{15.1}$$

Performing the blowup of W, i.e. introducing the new coordinate $\overline{\mu}'$, and restricting to $\lambda' = \lambda_0$, we find that this contact form becomes

$$\theta(-d\lambda - x\, d\tau + \overline{\mu}' \cdot dy' + \mu \cdot dy).$$

Dividing by θ, i.e. taking the leading part at $\tilde{W} \cap \{\lambda' = \lambda_0\}$, yields the contact form

$$-d\lambda - x\,d\tau + \overline{\mu}' \cdot dy' + \mu \cdot dy. \tag{15.2}$$

On the other hand, we may write the canonical one-form on ${}^{s\Phi}T^*(M \times \partial M \times [0, h_0))$ as

$$\tilde{\lambda} d\left(\frac{1}{xh}\right) + \tilde{\tau}\, d\left(\frac{1}{h}\right) + \tilde{\mu}\frac{dy}{xh} + \tilde{\mu}'\frac{dy'}{xh};$$

in the induced canonical coordinates, the contact form on this space becomes

$$-d\tilde{\lambda} - x\,d\tilde{\tau} + \tilde{\mu}' \cdot dy' + \tilde{\mu} \cdot dy,$$

hence identifying λ with $\tilde{\lambda}$, τ with $\tilde{\tau}$, μ with $\tilde{\mu}$ and $\overline{\mu}'$ with $\tilde{\mu}'$ exhibits the desired contactomorphism. □

Lemma 15.2. *The propagating Legendrian L intersects $\tilde{W} \cap \{\lambda' = \lambda_0, h = 0\}$ transversally, hence using the identification above we may regard the boundary of L at $\tilde{W}$, which we denote* SR *(for 'sojourn relation'), as a submanifold of ${}^{s\Phi}T^*_{\mathrm{mf}}(M \times \partial M \times [0, h_0))$. Making this identification, then* SR *is a Legendre submanifold of ${}^{s\Phi}T^*_{\mathrm{mf}}(M \times \partial M \times [0, h_0))$ which is smooth after further blowup of $\{x = 0, \mu = \overline{\mu}' = 0\}$.*

Proof. Since L is Legendrian in ${}^{s\Phi}T^*X^2_{\mathrm{b}}$, the form (15.1) vanishes on it. Near rb, since L is contained in Σ_l, the left characteristic variety, we have $(\lambda')^2 = \lambda_0^2 - h^{ij}(x', y')\mu'_i\mu'_j - V(\theta x, y')$. Lemma 11.3 shows that L meets $\{\theta = 0\}$ only in the interior of the blowup of W and does so transversely, so we can use the blow-up variable $\overline{\mu}'_i = \mu'_i/\theta$. In terms of this we have

$$(\lambda')^2 = \lambda_0^2 - \theta^2 h^{ij}\overline{\mu}'_i\overline{\mu}'_j - V(\theta x, y') \quad \Rightarrow \quad d\lambda' = \frac{\theta}{\lambda'}h^{ij}\overline{\mu}'_i\overline{\mu}'_j\,d\theta + O\big(\theta^2\big)$$

(recall that $V(x, y) = O(x^2)$). Thus $d\lambda'/\theta$, which by (15.1) is equal to (15.2) on L, vanishes when restricted to $L \cap \{\theta = 0\}$.

Now we consider the smoothness of SR at the boundary $\{x = 0\}$. By Lemma 11.3, L is desingularized by blowing up first $Z = \{\mu' = \mu = 0, x = 0, \lambda = \lambda'\}$ and then the lift of W. Thus away from Z, the first blowup has no effect and L is desingularized by the blowup of W. We have to analyze the situation further near $L \cap Z$. Here we can take advantage of the explicit formula for $L \cap \{x = 0\}$ given by (11.15). At $x = 0$, we have

$$\theta = \frac{|\mu'|}{|\mu|}. \tag{15.3}$$

It follows that in a neighborhood of $L \cap Z$ we have $|\mu| > |\mu'|$. Similarly, we have

$$\begin{aligned}\lambda' - \lambda &= \sqrt{\lambda_0^2 - |\mu'|^2 - V(x, y)} - \sqrt{\lambda_0^2 - |\mu|^2 - V(x, y)} = O\big(x^2 + |\mu|^2 + |\mu'|^2\big) \\ &= O\big(x^2 + |\mu|^2\big) \quad \text{on } L.\end{aligned} \tag{15.4}$$

It follows that after Z is blown up, we may cover a neighborhood of the intersection of L and the front face by coordinate charts in which either x or $|\mu|$ is a boundary defining function. Thus, in place of $x, \mu, \mu', \lambda' - \lambda$, we may take as coordinates

$$\frac{\mu}{x}, \frac{\mu'}{x}, \frac{\lambda' - \lambda}{x} \text{ and } x,$$

in the region where $dx \neq 0$, and

$$\frac{\mu}{|\mu|}, \frac{\mu'}{|\mu|}, \frac{\lambda' - \lambda}{|\mu|}, \frac{x}{|\mu|} \text{ and } |\mu|$$

in the region where $d|\mu| \neq 0$. As for the second blowup, of $\{\theta = 0, \mu/\rho_{\tilde{Z}} = 0\}$, where $\rho_{\tilde{Z}}$ is a boundary defining function for the face $\tilde{Z}$ created by the Z blowup, (15.3) implies that θ may be taken as a boundary defining function for the new face in a neighborhood of L. Thus coordinates replacing those above become

$$\frac{\mu}{x}, \frac{\mu'}{x\theta} = \frac{\overline{\mu}'}{x}, \frac{\lambda' - \lambda}{x} \text{ and } x; \text{ and } \theta, y, y', h, \lambda, \tau \tag{15.5}$$

in the region where $dx \neq 0$, and

$$\frac{\mu}{|\mu|}, \frac{\mu'}{|\mu|\theta} = \frac{\overline{\mu}'}{|\mu|}, \frac{\lambda' - \lambda}{|\mu|}, \frac{x}{|\mu|} \text{ and } |\mu|; \text{ and } \theta, y, y', h, \lambda, \tau \tag{15.6}$$

in the region where $d|\mu| \neq 0$. It follows from this and from Lemma 11.3 that θ, x, and $2n - 2$ of the remaining coordinates from (15.5) (in the first region), or θ, $x/|\mu|$, $|\mu|$ and $2n - 3$ of the remaining coordinates from (15.6) (in the second region) furnish coordinates on L on this space, and the remaining coordinates (restricted to L) can be written as smooth functions of these coordinates on L. Restricting to $\{\theta = 0\}$, then, we see that SR is desingularized by blowing up $\{x = 0, \mu = 0, \overline{\mu} = 0, \lambda' - \lambda = 0\}$.

We can also observe that $(\lambda' - \lambda)/x$ or $(\lambda' - \lambda)/|\mu|$ cannot serve as a coordinate on L at $\mu = 0$, since we see from (15.4) that this function has vanishing differential there. This implies that SR is also desingularized by blowing up

$$\{x = 0, \mu = 0, \overline{\mu} = 0\},$$

which completes the proof of the lemma. □

Remark. This lemma shows that SR forms a Legendre conic pair with the Legendre submanifold $G^\sharp = \{x = 0, \lambda = \lambda_0, \mu = 0, \overline{\mu}' = 0\}$ which is contained in the contact manifold ${}^{s\Phi}N^*(\partial M \times \partial M)$, $\partial M \times \partial M$ being the base of the fibration at the hypersurface at $x = 0$ of the scattering-fibered manifold $M \times \partial M \times [0, h_0)$.

To interpret the Legendrian SR geometrically, we recall the definition of the sojourn relation from [13] (in fact, we need to generalize it to include the case of a non-zero potential). SR is the graph of a contact transformation S from S^*M° to ${}^{sc}T^*_{\partial M}M$ given as follows: given a unit covector $(z, \hat{\zeta}) \subset S^*M^\circ$, we let $\gamma(s)$ be the bicharacteristic (geodesic, in the case of no potential) emanating from $(z, \hat{\zeta})$. By assumption, $\gamma(s)$ tends to the boundary, and there is a well-defined final 'direction' y. The *action* $A(s)$ accumulated along the bicharacteristic is the integral of $\lambda_0^2 - V$ with respect to s along γ, with initial condition $A(0) = 0$. Since $x = O(s^{-1})$ along γ and $V = O(x^2)$, we see that $A(s) = \lambda_0^2 s + O(1)$. Moreover, it follows from the regularity of the boundary of SR (Lemma 15.2) that $|\mu| = O(x)$, hence $\dot{r} = \lambda_0 + O(s^{-2})$ and so $r(s) = \lambda_0 s + O(1)$. We let ν, the *sojourn time*, be defined by $\nu = \lim_{s\to\infty} A(s) - \lambda_0 r(s)$, which is well defined by the above considerations. We finally define $M = \lim_{s\to\infty} \mu(\gamma(s))/s$. Then the sojourn relation is $S(z, \hat{\zeta}) = (y, \nu, M) \subset {}^{\mathrm{sc}}T^*_{\partial M} M$.[11] If $V \equiv 0$ then $A(s)$ is λ_0 times the geodesic distance along γ.

Lemma 15.3. *The Legendrian* SR *is the* (*twisted*) *graph of the sojourn relation in the interior of* ${}^{\mathrm{s}\Phi}T^*_{\mathrm{mf}}(M \times \partial M \times [0, h_0))$.

Proof. Consider a local parametrization of the Legendrian L near rb and away from $x = 0$. The Legendrian $L_{\mathrm{rb}} = \{\lambda' = \lambda_0, \mu' = 0\}$ is parametrized by the phase function $\lambda_0/\theta x h$, so we can choose our phase function to be of the form $(\lambda_0 + x'\psi)/x'h$, where $\psi = \psi(y', z, v)$ and it is non-degenerate in the sense that

$$d_{z,v}\left(\frac{\partial \psi}{\partial v_i}\right) \quad \text{are linearly independent}, \quad i = 1, \ldots, k \text{ where } v \in \mathbb{R}^k. \tag{15.7}$$

Then L is given locally by

$$L = \left\{(x', y', z, \lambda_0 + x'\psi + (x')^2\psi_{x'}, \psi, d_{y'}\psi, d_z\psi) \;\middle|\; d_v\psi = 0\right\}$$

in coordinates $(x', y', z, \lambda', \tau, M', \zeta)$ given by writing covectors in the form

$$\lambda' d\left(\frac{1}{x'h}\right) + \tau\, d\left(\frac{1}{h}\right) + M'\frac{dy'}{h} + \zeta\frac{dz}{h}.$$

By (11.3), we have $|\zeta|_g^2 + V = \lambda_0^2$, hence under the flow of h^{-1} times the Hamilton vector field, (11.2) gives $\dot{\tau} = \lambda_0^2 - V$. In other words,

$$\frac{\lambda_0}{x'} + \psi = \int \lambda_0^2 - V\, ds = A(s).$$

Thus $\psi(0, y', z, v) = \lim_{s\to\infty} A(s) - \lambda_0/x'$, which is the sojourn time (when $d_v\psi = 0$). Moreover, $d_{y'}\psi = M' = \mu'/x'$ where μ' is the variable dual to $dy'/x'h$. Finally $d_z\psi = d_z(\lambda_0 + x'\psi)/x'$ gives minus the covector $\hat{\zeta}$ at z which is the initial condition $(z, \hat{\zeta})$ for the bicharacteristic.

The boundary of the Legendrian L at $\tilde{W} \cap \{\lambda' = \lambda_0\}$ is given in these coordinates by

$$\mathrm{SR} = \left\{(y', z, \psi, d_{y'}\psi, d_z\psi) \;\middle|\; d_v\psi = 0\right\}$$

[11] The sojourn relation S actually depends on a choice of coordinates; it is invariantly defined on a certain affine bundle identified in [13].

and it is now evident from the interpretations of ψ, $d_{y'}\psi$ and $d_z\psi$ that this is a non-degenerate parametrization of the sojourn relation.[12] □

Proposition 15.4. *The Poisson operator is a Legendrian conic pair associated to the Legendre submanifold* SR *and the submanifold* $G^\sharp$; *in fact,*

$$P\big(h^{-1}\big) \in I^{0,(n-1)/2;0}\big(M \times M \times [0,h_0); \big(\mathrm{SR}, G^\sharp\big)\big).$$

Remark. The fact that the orders of $P(h^{-1})$ at mf and at are equal to zero reflects that the fact that the Poisson operator is a unitary operator mapping between M and the space $\partial M \times \mathbb{R}_+$ with a conic (i.e. scattering) metric, as proved in [10, Section 9].

Proof. The kernel of the resolvent is given by a finite sum of oscillatory integrals, each giving a Legendre distribution associated to the propagating Legendrian L, plus a smooth term vanishing at rapidly at each boundary hypersurface of $X_b^2 \times [0,h_0)$. Consider a single oscillatory integral expression involving a phase function parametrizing some piece of L. There are four different types of such expressions to consider, corresponding to regions of L which are (i) away from $\{x=0\}$, (ii) near $\{x=0\}$ but away from $\{\mu=\overline{\mu}'=0\}$, (iii) near $\{x=0, \mu=\overline{\mu}'=0\}$ and near the codimension three corner of L, (iv) near $\{x=0, \mu=\overline{\mu}'=0\}$ but away from the codimension three corner of L.

In region (i), the result follows directly from the proof of Lemma 15.3. The proof in the other regions follows the same pattern; we need only check that we can choose a non-degenerate phase function of the form $(\lambda_0+\theta\psi)/x\theta h$ for $(L, L^\sharp)$ in each region, such that ψ is a non-degenerate parametrization of $(\mathrm{SR}, G^\sharp)$. This was explicitly noted in (4.9), which covers regions (i) and (ii). In the case of region (iii), we can use a parametrization Ψ as in (6.12); the corresponding function ψ above is $\lambda_0 + s\psi_2 + x_2\psi_3$, using notation from (6.12). Comparison of (6.13) and (6.7) shows that $(\Psi-\lambda_0)/x_1$ is a non-degenerate phase function (where we need to make the transformation $x_2 \to x_1$, $x_3 \to x_2$, $(y_1,y_2) \to y_1$, $y_2 \to \{\ \}$, $v_2 \to v_1$, $v_3 \to v_2$ to make the comparison) in the sense of (6.6). Since we know that it parametrizes SR for $x_2 > 0$, it follows that this is a non-degenerate parametrization of $(\mathrm{SR}, G^\sharp)$. In region (iv) the result follows from the analogous comparison of (6.15) and (6.9).

To determine the orders, notice that we divided by the half-density factor $|dr'|^{1/2}$ to obtain the Poisson kernel. In terms of the boundary defining functions x_1 for rb and x_2 for bf, this is dividing by $|dx_1/x_1^2x_2|^{1/2}$. The semiclassical order is decreased by $-1/4$ accounting for the change in total dimension from N to $N-1$, but the orders at bf *increase* by $1/4$ in view of the power $x_2^{-1/2}$ in $|dx_1/x_1^2x_2|^{1/2}$. This shows that the new orders are as stated in the proposition. □

We now turn to the analysis of the scattering matrix $S(h^{-1})$. This is defined on $f \in \mathcal{C}^\infty(\partial M)$ by distributionally restricting the outgoing part of $x^{-(n-1)/2}e^{-i\lambda_0/xh}P(h^{-1})f$ to ∂M. In terms of kernels, and taking into account the half-density factors, it may be constructed from

[12] It would be more correct to say that we are 'identifying' this with the sojourn relation; it is not exactly the same as the sojourn relation as defined in [13] since it lies in a different bundle, with different scalings as $x' \to 0$. This can be traced to the fact that the bicharacteristics in [13] tend to infinity quadratically, while here they move to infinity linearly, reflecting the different scalings in the two operators (propagator vs. resolvent). The two bundles are related via the identification Q in (8.7).

the Poisson operator by microlocalizing near the intersection of SR and $G^\sharp$, multiplying by $e^{-i\lambda_0/x\lambda}|dx/x^2|^{-1/2}$ and restricting to $x=0$.

Thus the only part of the Legendrian SR of importance for the scattering matrix is the part in a neighborhood of $\mu=0$, i.e. at the blowup of Z. Thus we make a further symplectic reduction and restrict SR to the face Y created by the blowup of $\{x=0, \mu=0, \overline{\mu}'=0\} \subset {}^{s\Phi}T^*(M \times \partial M \times [0,h_0))$; let T denote this set.

Lemma 15.2 tells us that T is a Legendrian–Lagrangian submanifold of ${}^{s\Phi}T^*(\partial M \times \partial M \times [0,h_0))$. Thus, the contact form, which may be written

$$-d\tau + M'' \cdot dy' + M \cdot dy \tag{15.8}$$

in terms of blowup coordinates $M = \mu/x$, $M'' = \overline{\mu}'/x$, vanishes at T.

Let us define the 'total sojourn Legendrian' inside ${}^{sc}T^*_{\partial M \times \partial M \times \{0\}}(\partial M \times \partial M \times [0,h_0))$ as the set consisting of points (y, y', τ, M, M'') such that there a point $(z, \hat{\zeta})$ in the interior of M with $(y, \tau_1, M) = S(z, \hat{\zeta})$, $(y', \tau_2, M'') = S(z, -\hat{\zeta})$ and $\tau = \tau_1 + \tau_2$. We can also express τ as the limit of $A(s_1, s_2) - \lambda_0(1/x_2 + 1/x_1)$ where $s_1 \to -\infty$, $s_2 \to \infty$ and $A(s_1, s_2)$ is the action accumulated along the bicharacteristic determined by $(z, \hat{\zeta})$. If there is no potential then τ is given by the limit of $\lambda_0(d(z_1, z_2) - 1/x(z_1) - 1/x(z_2))$ where z_1 goes to infinity along the geodesic in one direction and z_2 goes to infinity along the geodesic in the opposite direction; this is λ_0 times the original 'sojourn time' defined by Guillemin [8].

Lemma 15.5. *The Legendrian T coincides with the total sojourn Legendrian.*

Proof. The vector field $-V_l'$ is tangent to SR and b-normal to Y. Therefore, every point of T is the endpoint of an integral curve of $-V_l'$ lying inside SR. An arbitrary point of T is therefore obtained from an interior point $(z_0, \hat{\zeta}_0; y', \tau, \mu')$ of SR by flowing along a V_l' integral curve. This does not change the values of y' or M', while $(z, \hat{\zeta})$ moves along the bicharacteristic with initial condition $(z_0, \hat{\zeta}_0)$. Thus when the bicharacteristic arrives at Y the y coordinate is the asymptotic direction of this bicharacteristic, while $M = \mu/x$ is the asymptotic 'angular coordinate.' To work out an interpretation of the τ variable, notice that when we use coordinates on $\tilde{W}$ given by

$$\overline{\tau} d\left(\frac{1}{h}\right) + \frac{\zeta \cdot dz}{h} + M' \frac{dy'}{h}$$

then $\overline{\tau}$ has the interpretation of the sojourn time starting from $(z_0, \hat{\zeta}_0)$ (see the proof of Lemma 15.3). Near Y we change to variables given by

$$\lambda d\left(\frac{1}{xh}\right) + \tau d\left(\frac{1}{h}\right) + \mu \cdot \frac{dy}{xh} + \overline{\mu}' \cdot \frac{dy'}{xh} = \lambda d\left(\frac{1}{xh}\right) + \tau d\left(\frac{1}{h}\right) g + M \cdot \frac{dy}{h} + M' \cdot \frac{dy'}{h}.$$

Comparing the two sets of coordinates gives $\tau = \overline{\tau} - \lambda/x$. Since $\lambda = \lambda_0$ at T, this gives $\tau = \lim_{x\to 0} \overline{\tau} - \lambda_0/x$ on T. Since $\overline{\tau}$ is the sojourn time starting from $(z, \hat{\zeta})$, i.e. the limit of $A - \lambda_0/x'$, this shows that at Y, $\tau = \lim_{x,x'\to 0}(A - \lambda_0/x' - \lambda_0/x)$ is the total sojourn time along the bicharacteristic determined by (y', M'), or equivalently by (y, M). This completes the proof that T is the total sojourn relation. □

Proposition 15.6. *The set T is a Legendrian–Lagrangian submanifold of ${}^{s\Phi}T^*(\partial M \times \partial M \times [0, h_0))$, and the scattering matrix $S(h)$ is a Legendrian–Lagrangian distribution on $\partial M \times \partial M \times [0, h_0)$ associated to T; indeed $S(h^{-1}) \in I^{-1/4,-1/4}(\partial M \times \partial M \times [0, h_0), T; {}^{sc}\Omega^{\frac{1}{2}})$.*

Proof. This follows directly from Proposition 7.5. □

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