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Advances in Mathematics 211 (2007) 34–71

ADVANCES IN Mathematics

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Frobenius–Schur indicators and exponents of spherical categories

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Received 27 February 2006; accepted 25 July 2006

Available online 26 September 2006

Communicated by Pavel Etingof

Abstract

We obtain two formulae for the higher Frobenius–Schur indicators: one for a spherical fusion category in terms of the twist of its center and the other one for a modular tensor category in terms of its twist. The first one is a categorical generalization of an analogous result by Kashina, Sommerhäuser, and Zhu for Hopf algebras, and the second one extends Bantay's 2nd indicator formula for a conformal field theory to higher degrees. These formulae imply the sequence of higher indicators of an object in these categories is periodic. We define the notion of Frobenius–Schur (FS-)exponent of a pivotal category to be the global period of all these sequences of higher indicators, and we prove that the FS-exponent of a spherical fusion category is equal to the order of the twist of its center. Consequently, the FS-exponent of a spherical fusion category is a multiple of its exponent, in the sense of Etingof, by a factor not greater than 2. As applications of these results, we prove that the exponent and the dimension of a semisimple quasi-Hopf algebra H have the same prime divisors, which answers two questions of Etingof and Gelaki affirmatively for quasi-Hopf algebras. Moreover, we prove that the FS-exponent of H divides $\dim(H)^4$. In addition, if H is a group-theoretic quasi-Hopf algebra, the FS-exponent of H divides $\dim(H)^2$, and this upper bound is shown to be tight.

Keywords: Frobenius–Schur indicator; Exponent; Fusion category; Modular tensor category; Quasi-Hopf algebra

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1 The author is supported by the NSA grant number H98230-05-1-0020.
2 The author is supported by the *Deutsche Forschungsgemeinschaft* through a Heisenberg fellowship.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2006.07.017

1. Introduction

We continue our investigation, begun in [26,27], of the higher Frobenius–Schur indicators for quasi-Hopf algebras and, more generally, certain fusion categories.

The classical (degree two) Frobenius–Schur indicator introduced a century ago as well as the higher indicators are well-known invariants of an irreducible representation of a finite group. The degree two indicators for simple modules over a semisimple Hopf algebra were studied by Linchenko and Montgomery [19], a version for certain fusion categories by Fuchs, Ganchev, Szlachányi, and Vescernyés [11], and a more general version for simple objects in pivotal categories by Fuchs and Schweigert [10]. Bantay introduced a notion of Frobenius–Schur indicator for a primary field of a conformal field theory via a formula in terms of the modular data. The higher indicators for Hopf algebras were introduced and studied in depth by Kashina, Sommerhäuser, and Zhu [17]. The degree two indicators for simple modules of a semisimple quasi-Hopf algebra were studied by Mason and Ng [20], and given a different treatment by Schauenburg [32]. The higher indicators introduced in [26] are a generalization of all of the above to the case of pivotal fusion categories, with more details and examples for the special case of modules over a semisimple quasi-Hopf algebra worked out in [27].

The definition of the indicator $\nu_n(V)$ of a simple object V in a pivotal fusion category $\mathcal{C}$ in [26] says that $\nu_n(V)$ is the trace of a certain endomorphism $E_V^{(n)}$ of $\mathcal{C}(I, V^{\otimes n})$. If $\mathcal{C} = H\text{-}\mathbf{mod}_{\mathrm{fin}}$ for a semisimple Hopf algebra over $\mathbb{C}$, we can identify $\mathcal{C}(I, V^{\otimes n})$ with the invariant subspace of $V^{\otimes n}$, and $E_V^{(n)}$ corresponds to a cyclic permutation of tensor factors. Thus, our definition corresponds to the "first formula" for the indicator given in [17, Corollary 2.3]. The *definition* of the higher indicators in [17] is in terms of Hopf powers of the integral, namely

$$\nu_n(V) = \chi(\Lambda_{(1)} \cdots \Lambda_{(n)}),$$

where Λ is the normalized integral and χ the character of V. This formula is generalized by the description in [26] of the nth indicator as the pivotal trace of the nth Frobenius–Schur endomorphism of V; that this is indeed a direct generalization of the defining formula in [17] becomes evident from the calculations done in [27] for the quasi-Hopf algebra case.

In the present paper, we prove a generalization of the "third formula" [17, 6.4, Corollary] for the higher indicators, which says that the nth indicator of V is the trace of the Drinfeld element of the double $D(H)$ taken on the induced module $D(H) \otimes_H V$. The appropriate generalization to the categorical setting is the following: For a spherical fusion category $\mathcal{C}$, consider the two-sided adjoint $K : \mathcal{C} \to Z(\mathcal{C})$ of the forgetful functor, where $Z(\mathcal{C})$ is the (left) center of $\mathcal{C}$. Then the nth indicator $\nu_n(V)$ is $(\dim \mathcal{C})^{-1}$ times the pivotal trace on $K(V)$ of the nth power of the twist in the ribbon category $Z(\mathcal{C})$. We prove this in Section 4, making use of results of Müger on the adjoint K and Ocneanu's tube algebra for $\mathcal{C}$. The key ingredient is another formula for the nth indicator in terms of the nth power of a special central element t in the tube algebra of $\mathcal{C}$ obtained in Section 3.

In the Hopf algebra case, the "second formula" [17, 3.2, Proposition], of which the "third formula" is a reformulation using the terminology of the Drinfeld double, connects the theory of indicators to the exponent of a Hopf algebra studied by Kashina [14,15] and Etingof and Gelaki [6]. In more detail, Kashina, Sommerhäuser and Zhu define the exponent of an irreducible representation of a semisimple complex Hopf algebra H, and they characterize it as the order of a certain endomorphism of the induced representation, the traces of whose powers are the higher indicators. As a consequence of their results, the exponent of H is the least common multiple of the exponents of the irreducible representations; the exponent of an irreducible representation in turn is the period of the sequence formed by its higher indicators, or the least number n such that the nth indicator is the representation's dimension.

Theorem 4.1 implies that the sequence $\{\nu_n(V)\}_n$ of the higher indicators of an object V in the spherical fusion category $\mathcal{C}$ is periodic as well. Moreover, the mth term of the sequence is the pivotal dimension $d(V)$ of V whenever m is a multiple of the order of the twist of $Z(\mathcal{C})$.

In Section 5 we define the Frobenius–Schur exponent of an object V in a pivotal category $\mathcal{C}$ to be the least positive integer n such that $\nu_n(V)$ is the pivotal dimension $d(V)$ of V, and we define the Frobenius–Schur exponent of $\mathcal{C}$ to be the least positive integer n such that $\nu_n(V) = d(V)$ for all $V \in \mathcal{C}$. We then prove that if $\mathcal{C}$ is a spherical fusion category, then its Frobenius–Schur exponent is equal to the order of the twist of $Z(\mathcal{C})$. By [5] the exponent is finite and our result implies that the exponent of $\mathcal{C}$ divides the Frobenius–Schur exponent. As it turns out in the first examples of the same section, the Frobenius–Schur exponent is different in general from the exponent as defined by Etingof [5]. In Section 6, however, we show that the Frobenius–Schur exponent can at most be twice the exponent.

Bantay introduced the (degree 2) Frobenius–Schur indicator for a primary field V of a conformal field theory via a formula in terms of the modular data of the CFT and he showed that the indicator of V is 0 if $V \ncong V^*$ and ± 1 if $V^* \cong V$ (cf. [3]). In Section 7, we derive a formula for the nth indicator of a simple object V of a modular tensor category by computing the trace of $E_V^{(n)}$. Our formula for $\nu_n(V)$ contains Bantay's formula for degree two indicators as the special case $n = 2$. An important consequence of this formula is the invariance of Frobenius–Schur exponent of a spherical fusion category under the center construction. Moreover, the Frobenius–Schur exponent of a modular tensor category is equal to the order of the twist.

In Section 8 we obtain several results for complex semisimple quasi-Hopf algebras H, which have been known for Hopf algebras, making use of Frobenius–Schur indicators and exponents. We prove that the dimension of H is even if H admits a self-dual simple module, and we generalize the Hopf algebra version of Cauchy's Theorem from [17] to our setting: The exponent, in the sense of Etingof [5], and the dimension of a quasi-Hopf algebra H have the same prime factors. This result answer two questions of Etingof and Gelaki [6] affirmatively for semisimple complex quasi-Hopf algebras. We have also shown that the exponent and the Frobenius–Schur exponent coincide if the dimension of H is odd.

In Section 9 we prove two bounds for the Frobenius–Schur exponent. In [15] Kashina asked whether the exponent of a semisimple complex Hopf algebra always divides the dimension. Etingof and Gelaki [6, Theorem 4.3] have shown that the exponent divides the third power of the dimension. As an immediate consequence of the results in [5] the Frobenius–Schur exponent of a semisimple complex quasi-Hopf algebra H divides the fifth power of the dimension of H. We improve this bound by lowering the exponent to the fourth power. We also study an important class of quasi-Hopf algebras, namely the group-theoretical quasi-Hopf algebras corresponding to the group-theoretical fusion categories introduced by Ostrik [29]. For a group-theoretical quasi-Hopf algebra H we can show that the (Frobenius–Schur) exponent divides the square of the dimension. More precisely, we can express the Frobenius–Schur exponent of a group-theoretical quasi-Hopf algebra, which is constructed from a finite group G and a three-cocycle ω on G, in terms of the cohomology class of ω and its restrictions to the cyclic subgroups of G. This same description of the exponent and the resulting general bound on the exponent for group-theoretical categories were recently obtained by Natale [24].

2. Preliminaries

We will collect some conventions and facts on monoidal categories and quasi-Hopf algebras. Most of these are well known (and we refer to [26,27] and the literature cited there), the others are easy observations.

In a monoidal category $\mathcal{C}$ with tensor product $\otimes$ we denote the associativity isomorphism by $\Phi:(U\otimes V)\otimes W\to U\otimes(V\otimes W)$. We assume that the unit object $I\in\mathcal{C}$ is strict. If $X,Y\in\mathcal{C}$ are obtained by tensoring together the same sequence of objects with two different arrangements of parentheses, there is an isomorphism between them which is obtained by composing several instances of Φ or Φ^{-1}; it is unique by coherence, and will be denoted by $\Phi^{?}:X\to Y$. A monoidal functor $\mathcal{F}:\mathcal{C}\to\mathcal{D}$ preserves tensor products by way of a coherent isomorphism $\xi:\mathcal{F}(V)\otimes\mathcal{F}(W)\to\mathcal{F}(V\otimes W)$ and $\mathcal{F}(I)=I$. An equivalence of monoidal categories is a monoidal functor that is an equivalence.

A left dual object $(V^\vee,\mathrm{ev},\mathrm{db})$ consists of an object $V^\vee$ and morphisms $\mathrm{ev}:V^\vee\otimes V\to I$ and $\mathrm{db}:I\to V\otimes V^\vee$ such that

$$V\xrightarrow{\mathrm{db}\otimes V}\bigl(V\otimes V^\vee\bigr)\otimes V\xrightarrow{\Phi}V\otimes\bigl(V^\vee\otimes V\bigr)\xrightarrow{V\otimes\mathrm{ev}}V,$$

$$V^\vee\xrightarrow{V^\vee\otimes\mathrm{db}}V^\vee\otimes\bigl(V\otimes V^\vee\bigr)\xrightarrow{\Phi^{-1}}\bigl(V^\vee\otimes V\bigr)\otimes V^\vee\xrightarrow{\mathrm{ev}\otimes V^\vee}V^\vee$$

are identities. Right duals are defined analogously. If every object has a (left) dual, $\mathcal{C}$ is called (left) rigid. If $\mathcal{C}$ is left rigid, taking duals extends naturally to a monoidal functor $(-)^\vee:\mathcal{C}^{\mathrm{op}}\to\mathcal{C}$. Double dualization is consequently a monoidal functor $(-)^{\vee\vee}:\mathcal{C}\to\mathcal{C}$. A pivotal monoidal category is a left rigid monoidal category equipped with an isomorphism $j:V\to V^{\vee\vee}$ of monoidal functors. Let $f:V\to V$ be a morphism in a pivotal category $\mathcal{C}$. The left and right pivotal traces of f are

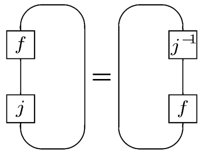
$$\underline{\mathrm{ptr}}^r(f)=\underline{\mathrm{tr}}(j_Vf)=\bigl(I\xrightarrow{\mathrm{db}}V\otimes V^\vee\xrightarrow{f\otimes V^\vee}V\otimes V^\vee\xrightarrow{j_V\otimes V^\vee}V^{\vee\vee}\otimes V^\vee\xrightarrow{\mathrm{ev}}I\bigr),$$

$$\underline{\mathrm{ptr}}^\ell(f)=\bigl(I\xrightarrow{\mathrm{db}}V^\vee\otimes V^{\vee\vee}\xrightarrow{V^\vee\otimes j_V^{-1}}V^\vee\otimes V\xrightarrow{V^\vee\otimes f}V^\vee\otimes V\xrightarrow{\mathrm{ev}}I\bigr).$$

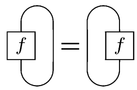
The left and right pivotal dimensions of $V\in\mathcal{C}$ are $d_\ell(V)=\underline{\mathrm{ptr}}^\ell(\mathrm{id}_V)$ and $d_r(V)=\underline{\mathrm{ptr}}^r(\mathrm{id}_V)$. If the left and right traces of every morphism are the same, then $\mathcal{C}$ is called a spherical monoidal category. In this case, traces and pivotal dimensions will be denoted by $\underline{\mathrm{ptr}}(f)$ and $d(V)$. If $\mathcal{C}$ is $\mathbb{C}$-linear, semisimple, and $d_\ell(V)=d_r(V)$ for each simple V, then $\mathcal{C}$ is spherical.

Any monoidal category is equivalent as a monoidal category to a strict monoidal category, that is, one in which the associativity isomorphism Φ is the identity. A pivotal monoidal category $\mathcal{C}$ is, moreover, equivalent as a pivotal monoidal category to a strict pivotal monoidal category $\mathcal{C}_{\mathrm{str}}$, that is, a pivotal monoidal category in which the associativity isomorphism, the pivotal structure j, and the canonical isomorphism $(V\otimes W)^\vee\to W^\vee\otimes V^\vee$ are identities. Equivalence as pivotal monoidal categories means that the monoidal equivalence $\mathcal{C}\to\mathcal{C}_{\mathrm{str}}$ preserves pivotal structures in a suitable sense; we refer to [26] for details. If $\mathcal{C}$ is spherical, then so is $\mathcal{C}_{\mathrm{str}}$.

In a strict monoidal category we make free use of graphical calculus. For instance, the condition on a pivotal category to be spherical is depicted as

$\underline{\mathrm{ptr}}^r(f) =$  $= \underline{\mathrm{ptr}}^{\ell}(f).$

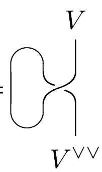
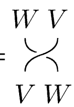
The strict pivotal case allows us to simply drop any instance of j resulting in

$\underline{\mathrm{ptr}}(f) =$ .

Now let $\mathcal{C}$ be a braided monoidal category with braiding c. The Drinfeld isomorphism is the natural isomorphism $u : \mathrm{Id} \to (-)^{\vee\vee}$ given by

$$u_V := \left(\mathrm{ev}_V \otimes V^{\vee\vee}\right) \circ \Phi^{-1}_{V^\vee, V, V^{\vee\vee}} \circ \left(V^\vee \otimes c^{-1}_{V, V^{\vee\vee}}\right) \circ \Phi_{V^\vee, V^{\vee\vee}, V} \circ \left(\mathrm{db}_{V^\vee} \otimes V\right).$$

If $\mathcal{C}$ is strict, then we have the graphical representation

$u_V =$  , where $c_{VW} =$  and $(c_{VW})^{-1} =$  .

We see that $u_{V \otimes W} = (u_V \otimes u_W) c^{-1}_{VW} c^{-1}_{WV}$. In particular, there is a bijective correspondence between pivotal structures j on $\mathcal{C}$ and *twists*, that is, automorphisms θ of the identity endofunctor satisfying

$$\theta_{V \otimes W} = (\theta_V \otimes \theta_W) c_{WV} c_{VW} \quad \text{and} \quad \theta_I = \mathrm{id}_I, \tag{2.1}$$

given by $\theta = u^{-1} j$.

The (left) center $Z(\mathcal{C})$ of a monoidal category $\mathcal{C}$ has objects pairs (V, e_V) with $V \in \mathcal{C}$ and $e_V(-) : V \otimes (-) \to (-) \otimes V$ a natural isomorphism satisfying the properties $e_V(I) = \mathrm{id}_V$ and

$$\left(X \otimes e_V(Y)\right) \circ \Phi_{X,V,Y} \circ \left(e_V(X) \otimes Y\right) = \Phi_{X,Y,V} \circ e_V(X \otimes Y) \circ \Phi_{V,X,Y}$$

for all $X, Y \in \mathcal{C}$. The tensor product of $Z(\mathcal{C})$ is given by $(V, e_V) \otimes (W, e_W) = (V \otimes W, e_{V \otimes W})$, where

$$e_{V \otimes W}(X) = \Phi_{X,V,W} \circ \left(e_V(X) \otimes W\right) \circ \Phi^{-1}_{V,X,W} \circ \left(V \otimes e_W(X)\right) \circ \Phi_{V,W,X}$$

for any $X \in \mathcal{C}$, and the neutral object is (I, e_I) with $e_I(X) = \mathrm{id}_X$. The associativity isomorphism in $Z(\mathcal{C})$ is that of $\mathcal{C}$. In this way $Z(\mathcal{C})$ is a monoidal category, and braided with braiding given by $e_V(W)$.

If $\mathcal{C}$ is left rigid, then $Z(\mathcal{C})$ is rigid; the dual of $(V, e_V) \in Z(\mathcal{C})$, is $(V^\vee, e_{V^\vee})$ with

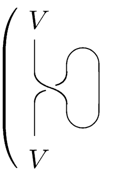
$$e_{V^\vee}(X) = \big(\mathrm{ev}_V \otimes \big(X \otimes V^\vee\big)\big) \circ \Phi^? \circ \big(V^\vee \otimes \big(e_V(X)^{-1} \otimes V\big)\big) \circ \Phi^? \circ \big(V^\vee \otimes (X \otimes \mathrm{db}_V)\big).$$

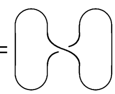
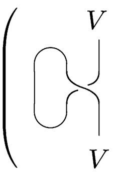
The evaluation and dual basis homomorphisms for the dual in $Z(\mathcal{C})$ are those of the dual $V^\vee$ in $\mathcal{C}$. If $\mathcal{C}$ is pivotal, the pivotal structure $j : \mathrm{Id} \to (-)^{\vee\vee}$ induces a pivotal structure in $Z(\mathcal{C})$. If $\mathcal{C}$ is spherical, so is $Z(\mathcal{C})$.

Any equivalence $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ of monoidal categories induces in a natural way an equivalence $\hat{\mathcal{F}} : Z(\mathcal{C}) \to Z(\mathcal{D})$ of braided monoidal categories. In addition, if $\mathcal{C}$ and $\mathcal{D}$ are pivotal monoidal categories and $\mathcal{F}$ preserves their pivotal structures, then $\hat{\mathcal{F}} : Z(\mathcal{C}) \to Z(\mathcal{D})$ also preserves their pivotal structures. Moreover, $\hat{\mathcal{F}}$ preserves the twists θ associated with their pivotal structures, i.e.

$$\hat{\mathcal{F}}(\theta_{(V,e_V)}) = \theta_{\hat{\mathcal{F}}(V,e_V)}.$$

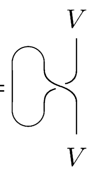
A fusion category over $\mathbb{C}$ is a rigid $\mathbb{C}$-linear monoidal category which is semisimple with finitely many non-isomorphic simple objects, one of them the neutral object I, whose endomorphism rings are isomorphic to $\mathbb{C}$. If $\mathcal{C}$ is a strict spherical fusion category with a braiding c, then by (2.1) the twist θ associated with the pivotal structure of $\mathcal{C}$ is identical to u^{-1} where u is the Drinfeld isomorphism associated with the braiding of $Z(\mathcal{C})$. Moreover, for any simple object V of $\mathcal{C}$,

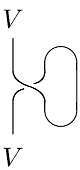
$\underline{\mathrm{ptr}}(\theta_V) = \underline{\mathrm{ptr}}\Bigg($ 

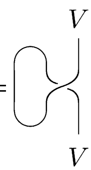
$\Bigg) =$  $= \underline{\mathrm{ptr}}\Bigg($ 

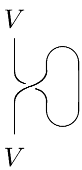
$\Bigg)$

and so

$\theta_V = u_V^{-1} =$ 

$=$ 

, and $\theta_V^{-1} =$ 

$=$ 

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In particular, $Z(\mathcal{C})$ is a ribbon category with respect to the twist θ.

If H is a quasi-Hopf algebra with associator $\phi \in H^{\otimes 3}$ and (quasi-)antipode (S, α, β), then the category H-$\mathbf{mod}_{\mathrm{fin}}$ of finite-dimensional left H-modules is a rigid monoidal category with associativity isomorphism Φ given as left multiplication with ϕ, dual object $V^\vee = V^*$ the vector space dual with module structure the transpose of the action through S, evaluation $\mathrm{ev}(f \otimes v) = f(\alpha v)$, and dual basis morphism $\mathrm{db}(1) = \sum_i \beta v_i \otimes v^i$, where $\{v_i\}_i$ and $\{v^i\}_i$ are dual bases for V and V^* respectively. If H is semisimple, we denote the normalized integral in H by Λ.

By a result of Etingof, Nikshych and Ostrik [8, Section 8], the categories H-$\mathbf{mod}_{\mathrm{fin}}$ for semisimple quasi-Hopf algebras H can be characterized as those fusion categories $\mathcal{C}$ for which the Frobenius–Perron dimension of every simple object is an integer. For such a category, [8] show that $\mathcal{C}$ is pseudo-unitary, and it has a pivotal structure j determined by the condition $d(V) = \mathrm{ev}_V(j_V \otimes \mathrm{id})\,\mathrm{db}_V = \dim(V)\,\mathrm{id}_{\mathbb{C}}$. We will call this the canonical pivotal structure; it is spherical. For $\mathcal{C} = H$-$\mathbf{mod}_{\mathrm{fin}}$ with a quasi-Hopf algebra H we have $j(v) = g^{-1}v$ for a certain element g of H called the trace-element of H, once we identify a vector space and its double dual space in the usual way.

Remark 2.1. If j' is a second pivotal structure on a pivotal fusion category $\mathcal{C}$ over $\mathbb{C}$ with pivotal structure j, we have $j' = j\lambda$ for some monoidal automorphism λ of the identity functor. For a simple V, the component λ_V is a scalar. In particular, $\lambda_I = 1$. If we denote the left and right pivotal dimensions with respect to the pivotal structure j' by d'_ℓ and d'_r, we see $d'_\ell(V) = \lambda_V^{-1} d_\ell(V)$ and $d'_r(V) = \lambda_V d_r(V)$ for simple V. Also, since $V \otimes V^\vee$ contains I, we have $\lambda_{V^\vee}\lambda_V = 1$. By [8, Proposition 2.9],

$$\left|d'_\ell(V)\right|^2 = |V|^2 = \left|d_\ell(V)\right|^2.$$

In particular, the absolute values of the pivotal dimensions of simple objects and the pivotal dimension of $\mathcal{C}$ do not depend on the choice of a pivotal structure. Finally, if both pivotal structures are spherical, then $\lambda_V = \pm 1$ for each simple V.

Remark 2.2. Analogous to the left center construction, the right version of the center $\overline{Z}(\mathcal{C})$ of $\mathcal{C}$ consists of objects pairs $(V, \bar{e}_V)$ with $V \in \mathcal{C}$ and $\bar{e}_V(-) : (-) \otimes V \to V \otimes (-)$ a natural isomorphism satisfying the properties $\bar{e}_V(I) = \mathrm{id}_V$ and

$$\bar{e}_V(X \otimes Y) = \Phi_{V,X,Y} \circ \big(\bar{e}_V(X) \otimes Y\big) \circ \Phi^{-1}_{X,V,Y} \circ \big(X \otimes \bar{e}_V(Y)\big) \circ \Phi_{X,Y,V} \quad \text{for } X, Y \in \mathcal{C}.$$

The tensor product of $\overline{Z}(\mathcal{C})$ is given by $(V, \bar{e}_V) \otimes (W, \bar{e}_W) = (V \otimes W, \bar{e}_{V\otimes W})$, where

$$\bar{e}_{V\otimes W}(X) = \Phi^{-1}_{V,W,X} \circ \big(V \otimes \bar{e}_W(X)\big) \circ \Phi_{V,X,W} \circ \big(\bar{e}_V(X) \otimes W\big) \circ \Phi^{-1}_{X,V,W}$$

for any $X \in \mathcal{C}$, and the neutral object is $(I, \bar{e}_I)$ with $\bar{e}_I(X) = \mathrm{id}_X$. The associativity isomorphism in $\overline{Z}(\mathcal{C})$ is also the same as in $\mathcal{C}$. The right center $\overline{Z}(\mathcal{C})$ is also a braided monoidal category with braiding given by $\bar{e}_V(W) : (W, \bar{e}_W) \otimes (V, \bar{e}_V) \to (V, \bar{e}_V) \otimes (W, \bar{e}_W)$. In addition, if $\mathcal{C}$ is left rigid, then $\overline{Z}(\mathcal{C})$ is rigid; the dual of $(V, \bar{e}_V) \in \overline{Z}(\mathcal{C})$, is $(V^\vee, \bar{e}_{V^\vee})$ with

$$\bar{e}_{V^\vee}(X)^{-1} = \big(\mathrm{ev}_V \otimes \big(X \otimes V^\vee\big)\big) \circ \Phi^{?} \circ \big(V^\vee \otimes \big(\bar{e}_V(X) \otimes V\big)\big) \circ \Phi^{?} \circ \big(V^\vee \otimes (X \otimes \mathrm{db}_V)\big)$$

$$\mathrm{ev}_{(V,\bar{e}_V)} = \mathrm{ev}_V, \quad \text{and} \quad \mathrm{db}_{(V,\bar{e}_V)} = \mathrm{db}_V.$$

By [13], the natural isomorphism $c'_{(V,e_V),(W,e_W)} := e_W(V)^{-1}$ for (V, e_V), $(W, e_W) \in Z(\mathcal{C})$ also defines a braiding on $Z(\mathcal{C})$, and we denote by $Z'(\mathcal{C})$ the braided monoidal category $Z(\mathcal{C})$ with the braiding c'. Then $Z'(\mathcal{C})$ and $\overline{Z}(\mathcal{C})$ are equivalent braided monoidal categories under monoidal equivalence $(T, \xi) : Z'(\mathcal{C}) \to \overline{Z}(\mathcal{C})$ with $\xi : T(V, e_V) \otimes T(W, e_W) \to T(V \otimes W, e_{V\otimes W})$ the identity, $T(V, e_V) = (V, e_V^{-1})$ and $T(f) = f$ for any objects (V, e_V), (W, e_W) and map f of $Z'(\mathcal{C})$. If $\mathcal{C}$ admits a pivotal structure, (T, ξ) preserves the induced pivotal structures of $Z'(\mathcal{C})$ and $\overline{Z}(\mathcal{C})$ as well as their associated twists.

In addition, if $\mathcal{C}$ is a spherical fusion category over $\mathbb{C}$ with the pivotal structure j, then $Z(\mathcal{C})$, $Z'(\mathcal{C})$ and $\overline{Z}(\mathcal{C})$ are pivotal fusion categories with their pivotal structures inherited from $\mathcal{C}$. Let θ, θ' and $\bar{\theta}$ be the associated twists of $Z(\mathcal{C})$, $Z'(\mathcal{C})$ and $\overline{Z}(\mathcal{C})$ respectively and (V, e_V) a simple object of $Z(\mathcal{C})$. Then $\theta_{(V,e_V)} = \omega\, \mathrm{id}_{(V,e_V)}$ for some non-zero scalar ω in $\mathbb{C}$. By considering the strictifications of these pivotal fusion categories, we have $\theta'_{(V,e_V)} = \omega^{-1}\,\mathrm{id}_{(V,e_V)}$ and hence $\bar{\theta}_{T(V,e_V)} = \omega^{-1}\,\mathrm{id}_{T(V,e_V)}$. In particular, we have

$$\theta'_{(V,e_V)} = \theta^{-1}_{(V,e_V)}.$$

3. Tube algebra of a strict spherical fusion category

In this section, we consider Ocneanu's tube algebra Θ_L of a strict spherical fusion category and a special element $t \in \Theta_L$. This element t has been considered in [12,23] for the computations of the Gauss sums of $\mathcal{C}$. We will show in Lemma 3.2 that the nth FS-indicator can be expressed in terms of t^n. This observation is essential to our proofs of Theorem 4.1 and Proposition 4.5.

Let $\mathcal{C}$ be a strict spherical fusion category over $\mathbb{C}$. Let X_i, $i \in \Gamma$, be the set of isomorphism classes of simple objects of $\mathcal{C}$. Since the left dual and right dual of an object in $\mathcal{C}$ are isomorphic, we simply denote the left (or right) dual of any object $V \in \mathcal{C}$ by V^*. For any $i \in \Gamma$, we define $\bar{i}$ by the equation

$$X_i^* = X_{\bar{i}}$$

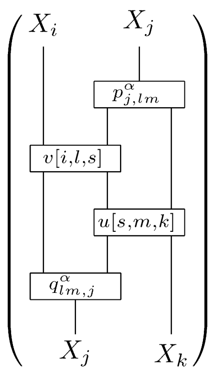
and we define $0 \in \Gamma$ by $X_0 = I$, the neutral object of $\mathcal{C}$.

For any $V \in \mathcal{C}$, we let

$$d(V) = \underline{\mathrm{ptr}}(\mathrm{id}_V), \qquad d_i = d(X_i) \quad \text{for } i \in \Gamma, \qquad \text{and} \qquad \dim\mathcal{C} = \sum_{i\in\Gamma} d_i d_{\bar{i}}.$$

Note that $d_0 = 1$ and $d_i = d_{\bar{i}}$ for $i \in \Gamma$. Following [23], we consider Ocneanu's tube algebra $\Theta_L = \bigoplus_{i,j,k\in\Gamma} \mathcal{C}(X_i \otimes X_j, X_j \otimes X_k)$ with multiplication given by

$$uv[i,j,k] = \frac{1}{d_j\lambda} \sum_{s,l,m\in\Gamma} d_l d_m \sum_{\alpha=1}^{N^j_{lm}}$$



where $\{p^{\alpha}_{j,lm}\}_\alpha$ is a basis for $\mathcal{C}(X_j, X_l \otimes X_m)$ and $\{q^{\alpha}_{lm,j}\}_\alpha$ the basis for $\mathcal{C}(X_l \otimes X_m, X_j)$ dual to $\{p^{\alpha}_{j,lm}\}_\alpha$ and $\lambda = \sqrt{\dim\mathcal{C}}$. The product of Θ_L is independent of the choice of basis for $\mathcal{C}(X_j, X_l \otimes X_m)$ (cf. [9,12,28] for the original definition of Ocneanu's tube algebra). The identity element $\mathbf{1}$ of Θ_L is given by

$$\mathbf{1}[i,j,k] = \lambda\delta_{j,0}\delta_{i,k}\,\mathrm{id}_{X_i}.$$

Let $\Theta_i = \bigoplus_{j\in\Gamma} \mathcal{C}(X_i \otimes X_j, X_j \otimes X_i)$ and $\Theta_{\mathcal{C}} = \bigoplus_{i\in\Gamma} \Theta_i$. For any $u \in \Theta_{\mathcal{C}}$, one can define $\hat{u} \in \Theta_L$ given by

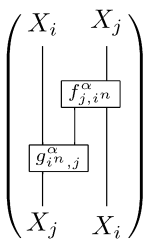
$$\hat{u}[i,j,k] = \delta_{i,k} u[i,j] \quad \text{for all } i,j,k \in \Gamma.$$

We will identify $\Theta_{\mathcal{C}}$ with a subspace of Θ_L under this identification. It is easy to see that $\Theta_{\mathcal{C}}$ is closed under the multiplication on Θ_L, and it also contains the identity element $\mathbf{1}$ of Θ_L. Consider the element $t \in \Theta_{\mathcal{C}}$ given by

$$t[i,j] = \frac{\lambda}{d_i} \delta_{ij}\, \mathrm{id}_{X_i^{\otimes 2}}. \tag{3.1}$$

Lemma 3.1.

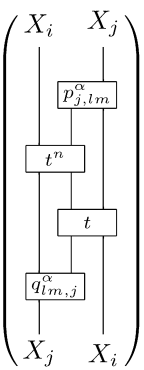
$$t^n[i,j] = \frac{\lambda}{d_j} \sum_{\alpha=1}^{N_{i^n}^j} \tag{3.2}$$



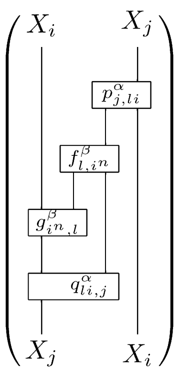
for $n \geqslant 1$ where $N_{i^n}^j = \dim \mathcal{C}(X_j, X_i^{\otimes n})$, $\{f_{j,i^n}^\alpha\}_\alpha$ is a basis for $\mathcal{C}(X_j, X_i^{\otimes n})$, and $\{g_{i^n,j}^\alpha\}_\alpha$ is the dual basis for $\mathcal{C}(X_i^{\otimes n}, X_j)$.

Proof. For $n = 1$, the formula holds by definition of t. Assume that t^n is given by (3.2). Then

$$t^{n+1}[i,j] = \frac{1}{d_j \lambda} \sum_{l,m\in\Gamma} d_l d_m \sum_{\alpha=1}^{N_{lm}^j}$$



$$= \frac{\lambda}{d_j} \sum_{l\in\Gamma} \sum_{\beta=1}^{N_{i^n}^j} \sum_{\alpha=1}^{N_{li}^j}$$



Note that $\{(f_{l,i^n}^\beta \otimes X_i) \circ p_{j,li}^\alpha\}_{\alpha,\beta,l}$ forms a basis for $\mathcal{C}(X_j, X_i^{\otimes(n+1)})$ with dual basis

$$\{q_{li,j}^\alpha \circ (g_{i^n,l}^\beta \otimes X_i)\}_{\alpha,\beta,l}$$

for $\mathcal{C}(X_i^{\otimes(n+1)}, X_j)$. Thus, the result follows by induction. $\square$

Define ϕ and $\phi_i \in \Theta_{\mathcal{C}}^*$ by

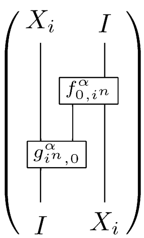
$$\phi(u) = \lambda \sum_{i \in \Gamma} d_i \, \underline{\mathrm{ptr}}_{X_i}\big(u[i,0]\big), \quad \text{and} \quad \phi_i(u) = \phi \circ \pi_i(u)$$

where π_i is the natural projection from $\Theta_{\mathcal{C}}$ to Θ_i.

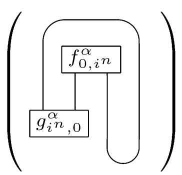
Lemma 3.2. *For any $i \in \Gamma$, $\lambda^2 d_i \overline{\nu_n(X_i)} = \phi_i(t^n)$.*

Proof. By Lemma 3.1,

$$\phi_i(t^n) = \frac{\lambda^2}{d_0} \sum_{\alpha=1}^{N_{i^n}^0} d_i \, \underline{\mathrm{ptr}}_{X_i}$$



$$= \lambda^2 \sum_{\alpha=1}^{N_{i^n}^0} d_i$$



$$= \lambda^2 d_i \nu_{n,n-1}(X_i) = \lambda^2 d_i \overline{\nu_n(X_i)}$$

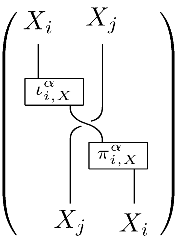
where $N_{i^n}^0 = \dim \mathcal{C}(X_0, X_i^{\otimes n})$. The third and the fourth equalities follow from definitions and results in [26]. $\square$

4. Indicators and the twist of the center

We continue to consider a strict spherical fusion category $\mathcal{C}$ over $\mathbb{C}$. The center $Z(\mathcal{C})$ of $\mathcal{C}$ is a ribbon category with the twist θ associated with the pivotal structure of $\mathcal{C}$. In this section, we obtain a formula, in Theorem 4.1, for the nth Frobenius–Schur indicator $\nu_n(V)$ of an object V in a spherical fusion category $\mathcal{C}$ over $\mathbb{C}$ in terms of the twist θ of the center. The result is a categorical generalization of the formula for higher indicators of Hopf algebras in [17, 6.4, Corollary]. By [23], $Z(\mathcal{C})$ is a modular tensor category and hence θ is of finite order (cf. [2,33]). The formula implies that the sequence $\{\nu_n(V)\}_n$ is periodic for each object V and $\nu_n(V) = d(V)$ if n is a multiple of the order of θ. In addition, if $d(V)$ is positive for simple V, we show in Proposition 4.5 that $d(V) = \nu_n(V)$ if and only if n is a multiple of the order of θ, and for any n the inequality $|\nu_n(V)| \leqslant d(V)$ holds for all $V \in \mathcal{C}$. By a result of [8, Section 8], there is at most one pivotal structure on a fusion category such that the pivotal dimension $d_\ell(V)$ of a simple object V is positive. In this case, the pivotal structure is spherical and $d(V)$ is the Frobenius–Perron dimension of V.

Let $\hat{\Gamma}$ be the set of isomorphism classes of simple objects of $Z(\mathcal{C})$. For any $(X, e_X) \in \hat{\Gamma}$, $\theta_{(X,e_X)} = \omega_{(X,e_X)} \operatorname{id}_{(X,e_X)}$ for some root of unity $\omega_{(X,e_X)} \in \mathbb{C}$. By Müger's results [23, Propositions 5.4 and 5.5], for any $(X, e_X) \in \hat{\Gamma}$, the element $z_{(X,e_X)} \in \Theta_{\mathcal{C}}$ defined by

$$z_{(X,e_X)}[i, j] = \frac{d(X)}{\lambda d_i} \sum_{\alpha=1}^{N_i^X}$$



is a primitive central idempotent of $\Theta_{\mathcal{C}}$ and $\sum_{(X,e_X)\in\hat{\Gamma}} z_{(X,e_X)} = \mathbf{1}_{\Theta_{\mathcal{C}}}$, where $\{\iota_{i,X}^{\alpha}\}_{\alpha}$ is a basis for $\mathcal{C}(X_i, X)$ and $\{\pi_{i,X}^{\alpha}\}_{\alpha}$ its dual basis for $\mathcal{C}(X, X_i)$, and $N_i^X = \dim \mathcal{C}(X_i, X)$. By [23, Lemma 5.17], the element t defined in (3.1) can be written as

$$t = \sum_{(X,e_X)\in\hat{\Gamma}} \omega_{(X,e_X)}^{-1} z_{(X,e_X)}. \tag{4.1}$$

By another result of Müger [23, Proposition 8.1], the forgetful functor $H : Z(\mathcal{C}) \to \mathcal{C}$ has a two-sided adjoint $K : \mathcal{C} \to Z(\mathcal{C})$ such that

$$K(Y) \cong \bigoplus_{(X,e_X)\in\hat{\Gamma}} \dim\big(\mathcal{C}(X, Y)\big)(X, e_X) \tag{4.2}$$

for any $Y \in \mathcal{C}$. Now, we can prove our formula for the Frobenius–Schur indicators of an object in $\mathcal{C}$.

Theorem 4.1. *Let $\mathcal{C}$ be any spherical fusion category over $\mathbb{C}$ with pivotal structure j and let u be the Drinfeld isomorphism of $Z(\mathcal{C})$. For any $V \in \mathcal{C}$,*

$$\nu_n(V) = \frac{1}{\dim \mathcal{C}} \underline{\operatorname{ptr}}\big(\theta_{K(V)}^n\big),$$

where $\theta = u^{-1} j$ is the twist of $Z(\mathcal{C})$ associated with j.

Proof. Since Frobenius–Schur indicators as well as pivotal traces are invariant under tensor equivalences that preserve the pivotal structures, we may assume that $\mathcal{C}$ is strict spherical. In this case $V^{**} = V$ and $j_V = \operatorname{id}_V$ for all $V \in \mathcal{C}$. Moreover, since $\mathcal{C}$ is semisimple and K preserves direct sums, it suffices to prove the case when $V = X_i$ for some $i \in \Gamma$. By (4.1) and Lemma 3.2, we have

$$\lambda^2 d_i \overline{\nu_n(X_i)} = \phi_i\big(t^n\big) = \sum_{(X,e_X)\in\hat{\Gamma}} \omega_{(X,e_X)}^{-n} \phi_i(z_{(X,e_X)}) = \sum_{(X,e_X)\in\hat{\Gamma}} \omega_{(X,e_X)}^{-n} d(X) N_i^X d_i,$$

where $N_i^X = \dim \mathcal{C}(X_i, X)$. By [8, Corollary 2.10], $d(V) = d(V^*)$ is real for all $V \in \mathcal{C}$. Therefore,

$$\begin{aligned}\nu_n(X_i) &= \overline{\frac{1}{\lambda^2}\sum_{(X,e_X)\in\hat{\Gamma}} \omega_{(X,e_X)}^{-n} d(X) N_i^X} \\ &= \frac{1}{\lambda^2}\sum_{(X,e_X)\in\hat{\Gamma}} \omega_{(X,e_X)}^{n} d(X) N_i^X \\ &= \frac{1}{\dim\mathcal{C}}\,\underline{\mathrm{ptr}}\big(\theta^n_{K(X_i)}\big). \qquad \square \end{aligned} \tag{4.3}$$

Remark 4.2. Since

$$\underline{\mathrm{ptr}}(\theta_{K(V^*)}) = \sum_{(X^*,e_{X^*})\in\hat{\Gamma}} N_{V^*}^{X^*}\omega_{(X^*,e_{X^*})} d\big(X^*\big) = \sum_{(X^*,e_{X^*})\in\hat{\Gamma}} N_V^X \omega_{(X,e_X)} d(X) = \underline{\mathrm{ptr}}(\theta_{K(V)})$$

for $V \in \mathcal{C}$, Theorem 4.1 implies that $\nu_n(V) = \nu_n(V^*)$ for all positive integers n, which has been proved in [26] using graphical calculus.

Remark 4.3. By Remark 2.2, $\theta'_{(V,e_V)} = \theta^{-1}_{(V,e_V)}$ for $(V, e_V) \in Z(\mathcal{C})$. Then the nth Frobenius–Schur indicator of V in $\mathcal{C}$ can be rewritten as

$$\nu_n(V) = \frac{1}{\dim\mathcal{C}}\,\underline{\mathrm{ptr}}\big({\theta'}^{-n}_{K(V)}\big).$$

It follows from the equivalence of $Z'(\mathcal{C})$ and $\overline{Z}(\mathcal{C})$ that we also have

$$\nu_n(V) = \frac{1}{\dim\mathcal{C}}\,\underline{\mathrm{ptr}}\big(\bar{\theta}^{-n}_{\overline{K}(V)}\big)$$

where $\overline{K}$ is the two-sided adjoint for the forgetful functor from $\overline{Z}(\mathcal{C})$ to $\mathcal{C}$ and $\bar{\theta}$ is the twist of $\overline{Z}(\mathcal{C})$ associated with the pivotal structure of $\mathcal{C}$.

Remark 4.4. Since θ has finite order, the sequence $\{\nu_n(V)\}_n$ is periodic for any $V \in \mathcal{C}$, with a period that divides the order of θ.

The following Eq. (4.4) can also be obtained easily by [8, Proposition 5.4] using the fact that $\sum_{i\in\Gamma} |d_i|^2 = \dim\mathcal{C}$. In the following proposition, we give another proof for the formula. The special case of the equation for $V = I$ is the class equation in [8, Proposition 5.7].

Proposition 4.5. *Let $\mathcal{C}$ be a spherical fusion category over $\mathbb{C}$ and θ the twist of $Z(\mathcal{C})$ associated with the pivotal structure of $\mathcal{C}$. Then*

$$d(V) = \frac{1}{\dim\mathcal{C}}\sum_{(X,e_X)\in\hat{\Gamma}} \dim\big(\mathcal{C}(X,V)\big) d(X) \tag{4.4}$$

for all $V \in \mathcal{C}$. In particular, if $\theta^n_{K(V)} = \mathrm{id}_{K(V)}$, then $\nu_n(V) = d(V)$. In addition, if $d_i > 0$ for all $i \in \Gamma$, then the converse also holds, and we have

$$\big|\nu_r(V)\big| \leqslant d(V)$$

for all positive integers r and $V \in \mathcal{C}$.

Proof. Let n be a positive integer such that $\theta^n = \mathrm{id}$. Then $\omega^n_{(X,e_X)} = 1$ for all $(X, e_X) \in \hat{\Gamma}$. In particular, $t^n = \mathbf{1}$ by (4.1). By Lemma 3.2, we have

$$\lambda^2 d_i \overline{\nu_n(X_i)} = \phi_i(\mathbf{1}) = \lambda^2 d_i^2.$$

Since d_i is real, by Theorem 4.1, we have

$$d_i = \nu_n(X_i) = \frac{1}{\dim \mathcal{C}} \underline{\mathrm{ptr}}(\mathrm{id}_{K(X_i)}) = \frac{1}{\dim \mathcal{C}} \sum_{(X,e_X)\in\hat{\Gamma}} N_i^X d(X).$$

The first statement follows directly from the additivity of the dimension function. In particular, if $\theta^n_{K(V)} = \mathrm{id}_{K(V)}$, then

$$\nu_n(V) = \frac{1}{\dim \mathcal{C}} \underline{\mathrm{ptr}}(\mathrm{id}_{K(V)}) = \frac{1}{\dim \mathcal{C}} \sum_{(X,e_X)} \dim\bigl(\mathcal{C}(X, V)\bigr) d(X) = d(V).$$

If, in addition, $d_i > 0$ for all $i \in \Gamma$, then $d(V) > 0$ for all non-zero $V \in \mathcal{C}$. Thus, for any positive integer r and for any $V \in \mathcal{C}$, we have

$$\begin{aligned}
\bigl|\nu_r(V)\bigr| &= \left| \frac{1}{\dim \mathcal{C}} \sum_{(X,e_X)\in\hat{\Gamma}} \dim\bigl(\mathcal{C}(X, V)\bigr) d(X) \omega^r_{(X,e_X)} \right| \\
&\leqslant \frac{1}{\dim \mathcal{C}} \sum_{(X,e_X)\in\hat{\Gamma}} \dim\bigl(\mathcal{C}(X, V)\bigr) d(X) = d(V).
\end{aligned}$$

Moreover, if $\nu_n(V) = d(V)$, then $\omega^n_{(X,e_X)} = 1$ for any component (X, e_X) of $K(V)$. Hence, $\theta^n_{K(V)} = \mathrm{id}_{K(V)}$. $\square$

Remark 4.6. Note that

$$\delta_{i,0} = \nu_1(X_i) = \frac{1}{\dim \mathcal{C}} \sum_{(X,e_X)\in\hat{\Gamma}} N_i^X d(X) \omega_{(X,e_X)} \tag{4.5}$$

for $i \in \Gamma$. If $d_i > 0$ for all $i \in \Gamma$, (4.4) implies that $\omega_{(X,e_X)} = 1$ whenever $N_0^X \neq 0$. Thus, the class equation in [8] is the special case of Eq. (4.5) when $i = 0$.

5. Frobenius–Schur exponent

In this section, we define the *Frobenius–Schur exponent* of a pivotal category over any field k, and give an example to demonstrate the difference between the Frobenius–Schur exponent and the (quasi-)exponent, in the sense of Gelaki and Etingof, of a spherical fusion category. We prove in Theorem 5.5 that the Frobenius–Schur exponent of a spherical fusion category $\mathcal{C}$ over $\mathbb{C}$ is equal to the order of the twist θ of $Z(\mathcal{C})$ associated with the pivotal structure of $\mathcal{C}$. It will be shown later in Section 7 that the Frobenius–Schur exponent of a spherical fusion category is invariant under the center construction. We then apply Theorems 4.1 and 5.5 to a semisimple quasi-Hopf algebra H over $\mathbb{C}$ to obtain a formula for the nth indicator of an H-module in terms of the value $\hat{\chi}_V((gu)^{-n})$ of the character $\hat{\chi}_V$ of the induced module $D(H) \otimes_H V$, where g is the trace-element of H (cf. [20, Section 6]) and u is the Drinfeld element of $D(H)$. When H is an ordinary Hopf algebra, then $g = 1$ and the formula reduces to the one in [17, 6.4, Corollary].

As in [26], a pivotal category over a field k is a k-linear pivotal monoidal category with a simple neutral object, finite-dimensional morphism spaces, and $\mathrm{End}(V) = k$ for every simple object V.

Definition 5.1. Let $\mathcal{C}$ be a pivotal category over any field k. The *Frobenius–Schur exponent of an object* V in $\mathcal{C}$, denoted by $\mathrm{FSexp}(V)$, is defined to be the least positive integer n such that

$$\nu_n(V) = d_\ell(V),$$

where $d_\ell(V)$ and $\nu_n(V)$, respectively, denote the left pivotal dimension and the nth Frobenius–Schur indicator of V. If such an integer does not exist, we define $\mathrm{FSexp}(V) = \infty$. We call the least positive integer n such that $\nu_n(V) = d_\ell(V)$ for all $V \in \mathcal{C}$ the *Frobenius–Schur exponent of* $\mathcal{C}$ and denote it by $\mathrm{FSexp}(\mathcal{C})$. If such an integer does not exist, we define $\mathrm{FSexp}(\mathcal{C}) = \infty$.

If H is a finite-dimensional semisimple Hopf algebra over $\mathbb{C}$, $\mathrm{FSexp}(V)$ is identical to the exponent of V for $V \in H\text{-}\mathbf{mod}_{\mathrm{fin}}$ defined in [17]. Moreover, the results of [17] show that $\mathrm{FSexp}(H)$ is the same as the exponent of $H\text{-}\mathbf{mod}_{\mathrm{fin}}$ in the sense of Etingof and Gelaki (cf. [5,7]).

Remark 5.2. Let $\mathcal{C}$ be a pivotal category over a field k with the pivotal structure j. For a simple object $V \in \mathcal{C}$, if j' is another pivotal structure, then j_V and j'_V differ by a scalar factor, say $j'_V = \lambda j_V$ with $\lambda \in k$. From the definition of the left pivotal trace, it is clear that the left pivotal dimension $d'_\ell(V)$ computed with respect to j' is $d'_\ell(V) = \lambda^{-1} d_\ell(V)$. Equally, one can read off from the definition of the Frobenius–Schur indicators that the indicator $\nu'_n(V)$ computed with respect to j' is $\nu'_n(V) = \lambda^{-1}\nu_n(V)$. In particular the Frobenius–Schur exponent of a simple object does not depend on the choice of a pivotal structure, and neither does the Frobenius–Schur exponent of a pivotal fusion category over k.

Recall from [8] that a fusion category $\mathcal{C}$ over $\mathbb{C}$ is pseudo-unitary if it admits a spherical pivotal structure such that the pivotal dimension $d(V)$ for non-zero $V \in \mathcal{C}$ is positive.

Proposition 5.3. *Let $\mathcal{C}$ be a pseudo-unitary fusion category over $\mathbb{C}$. If the object $V \in \mathcal{C}$ contains every simple object of $\mathcal{C}$, then* $\mathrm{FSexp}(V) = \mathrm{FSexp}(\mathcal{C})$. *In particular, if H is a semisimple complex quasi-Hopf algebra, then* $\mathrm{FSexp}(H) = \mathrm{FSexp}(H\text{-}\mathbf{mod}_{\mathrm{fin}})$.

Proof. Let $V \cong \sum_{i \in \Gamma} n_i X_i$ and $n = \mathrm{FSexp}(V)$. Then by additivity of ν_n

$$\sum_{i \in \Gamma} n_i d_i = d(V) = \nu_n(V) = \sum_{i \in \Gamma} n_i \nu_n(X_i).$$

Since $|\nu_n(X_i)| \leqslant d_i$ for $i \in \Gamma$ and $n_i > 0$ by assumption, the equality implies that $d_i = \nu_n(X_i)$ for all $i \in \Gamma$. Now it follows from the additivity of ν_n that the Frobenius–Schur exponent of $\mathcal{C}$ is n.

If H is a semisimple complex quasi-Hopf algebra, $\mathcal{C} = H\text{-}\mathbf{mod}_{\text{fin}}$ is a pseudo-unitary fusion category over $\mathbb{C}$. We have $\sum_{i\in\Gamma} d_i X_i \cong H \in H\text{-}\mathbf{mod}_{\text{fin}}$. Therefore, $\mathrm{FSexp}(H) = \mathrm{FSexp}(\mathcal{C})$. □

It is reasonable to conjecture that the Frobenius–Schur exponent of a semisimple quasi-Hopf algebra is identical to its exponent. However, the following example demonstrates the difference between the two exponents of a quasi-Hopf algebra and hence $\mathrm{FSexp}(\mathcal{C})$ is different from the exponent of $\mathcal{C}$ in general.

Example 5.4. Let $G = \{1, x\}$ be an abelian group of order 2 and ω a 3-cocycle of G given by

$$\omega(a,b,c) = \begin{cases} -1 & \text{if } a = b = c = x, \\ 1 & \text{otherwise.} \end{cases}$$

The dual group algebra $\mathbb{C}[G]^*$ is a well-known Hopf algebra with the usual multiplication, comultiplication Δ, counit ϵ, and antipode S. Let $\{e(1), e(x)\}$ be the dual basis of $\{1, x\}$ for $\mathbb{C}[G]^*$. Define

$$\phi = \sum_{a,b,c\in G} \omega(a,b,c)e(a)\otimes e(b)\otimes e(c), \qquad \alpha = 1_{\mathbb{C}[G]^*},$$

$$\beta = \sum_{a\in G} \omega\big(a, a^{-1}, a\big)e(a) = e(1) - e(x).$$

Then $H = (\mathbb{C}[G]^*, \Delta, \epsilon, \phi, \alpha, \beta, S)$ is a quasi-Hopf algebra and $D(H) = D^{\omega}(G)$. The universal R-matrix is given by

$$R = \sum_{a,h\in G} e(a)\otimes 1\otimes e(h)\otimes a.$$

Then

$$R_{21}R = \sum_{a,b,h,k\in G} \big(e(k)\otimes b\big)\cdot\big(e(a)\otimes 1\big)\otimes\big(e(b)\otimes 1\big)\cdot\big(e(h)\otimes a\big) = \sum_{a,b\in G} e(a)\otimes b\otimes e(b)\otimes a.$$

Since

$$(R_{21}R)^2 = \sum_{a,b\in G} \theta_a(b,b)\theta_b(a,a)e(a)\otimes b^2\otimes e(b)\otimes a^2 = \sum_{a,b\in G} e(a)\otimes 1\otimes e(b)\otimes 1 = 1_{D(H)},$$

the exponent of $H\text{-}\mathbf{mod}_{\text{fin}}$, in the sense of Etingof and Gelaki, is 2. Here, we have used the notation and multiplication for $D^{\omega}(G)$ given in [20, Section 5].

The Frobenius–Schur indicators of the nontrivial representation of H, on the other hand, were already computed in [27]. In fact $H \cong \mathbb{C}[G]_x$ is a special case of the constructions studied in Section 5 of [27], and hence the indicators of the nontrivial simple H-module $\overline{V}$ are

$$\nu_n(\overline{V}) = \cos\left(\frac{(n-1)\pi}{2}\right).$$

In particular, the Frobenius–Schur exponent of H is 4.

Theorem 5.5. *Let $\mathcal{C}$ be a spherical fusion category over $\mathbb{C}$. The Frobenius–Schur exponent of $\mathcal{C}$ is equal to the order of the twist θ of $Z(\mathcal{C})$ associated with the pivotal structure of $\mathcal{C}$. In particular,* $\operatorname{FSexp}(\mathcal{C})$ *is finite.*

Proof. Since the Frobenius–Schur exponent as well as the order of θ are invariant under monoidal equivalences of $\mathcal{C}$ preserving the pivotal structure, it suffices to prove the claim in the case when $\mathcal{C}$ is a strict spherical category. It follows from (4.3) that for any positive integer n

$$\begin{aligned}\sum_{i\in\Gamma}\nu_n(X_i)d_i &= \frac{1}{\dim(\mathcal{C})}\sum_{(X,e_X)\in\hat{\Gamma}}\omega^n_{(X,e_X)}\sum_{i\in\Gamma}N_i^X d_i d(X)\\ &= \frac{1}{\dim(\mathcal{C})}\sum_{(X,e_X)\in\hat{\Gamma}}\omega^n_{(X,e_X)}d(X)^2. \qquad (5.1)\end{aligned}$$

If $\nu_n(V)=d(V)$ for all $V\in\mathcal{C}$, then (5.1) becomes

$$\dim(\mathcal{C})^2=\sum_{(X,e_X)\in\hat{\Gamma}}\omega^n_{(X,e_X)}d(X)^2.$$

Since $\dim(Z(\mathcal{C}))=\dim(\mathcal{C})^2$ (cf. [23]), we have

$$\sum_{(X,e_X)\in\hat{\Gamma}}d(X)^2=\dim\bigl(Z(\mathcal{C})\bigr)=\sum_{(X,e_X)\in\hat{\Gamma}}\omega^n_{(X,e_X)}d(X)^2. \qquad (5.2)$$

By [8], $d(X)$ is real for $(X,e_X)\in\hat{\Gamma}$. Equation (5.2) implies that $\omega^n_{(X,e_X)}=1$ for all $(X,e_X)\in\hat{\Gamma}$. Therefore,

$$\theta^n_{(X,e_X)}=\mathrm{id}_{(X,e_X)}.$$

Conversely, suppose that $\theta^n=\mathrm{id}$. By Theorem 4.1, we obtain

$$\nu_n(V)=\frac{1}{\dim\mathcal{C}}\,\underline{\mathrm{ptr}}(\mathrm{id}_{K(V)})=\frac{1}{\dim\mathcal{C}}\sum_{(X,e_X)\in\hat{\Gamma}}N_V^X d(X)=d(V)$$

for any $V\in\mathcal{C}$. □

Since the Drinfeld isomorphism is clearly invariant under equivalences of braided monoidal categories, we can immediately conclude:

Corollary 5.6. *The Frobenius–Schur exponent of a spherical fusion category $\mathcal{C}$ depends only on the equivalence class of the spherical braided monoidal category $Z(\mathcal{C})$.*

As it will turn out, the Frobenius–Schur exponent of a spherical fusion category over $\mathbb{C}$ is actually invariant under the center construction. This invariance follows immediately from a formula of higher Frobenius–Schur indicators for a modular tensor category which will be derived in Section 7.

If the fusion category in question is given as the representation category of a semisimple quasi-Hopf algebra H, then the characterizations of the Frobenius–Schur indicators and exponent in terms of the Drinfeld isomorphism in the center turn, of course, into descriptions in terms of the Drinfeld element of the double of H. The following corollary spells out the details, and the remark following it gives some more information about the explicit form of the elements of $D(H)$ involved. Here $D(H)$ stands for a version of the Drinfeld double construction for quasi-Hopf algebras matching the left center construction, so that $D(H)$-$\mathbf{mod}_{\mathrm{fin}}$ is equivalent to the center $Z(H\text{-}\mathbf{mod}_{\mathrm{fin}})$ of H-$\mathbf{mod}_{\mathrm{fin}}$ as in [31]. See Remark 5.8 below.

Corollary 5.7. *Let H be a finite-dimensional semisimple quasi-Hopf algebra over $\mathbb{C}$ and u the Drinfeld element of the quantum double $D(H)$ of H and g the trace-element of H. Then for any simple H-module V of H and $n \in \mathbb{N}$,*

$$\nu_n(V) = \frac{1}{\dim(H)} \hat{\chi}_V\big((gu)^{-n}\big)$$

where $\hat{\chi}_V$ is the induced character of the character χ_V of V to $D(H) \otimes_H V$. Moreover, the Frobenius–Schur exponent of H is equal to the order of gu.

Proof. Note that the canonical pivotal structure on H-$\mathbf{mod}_{\mathrm{fin}}$ is given by the formula

$$j_V(x) = g^{-1}x$$

for any $V \in H$-$\mathbf{mod}_{\mathrm{fin}}$ and $x \in V$. Moreover, $d(V) = \dim(V)$ for any $V \in \mathcal{C}$. Therefore, the pivotal trace of any f in $\mathrm{End}_H(V)$ is identical to the ordinary trace of the linear map f and H-$\mathbf{mod}_{\mathrm{fin}}$ is spherical. The Drinfeld isomorphism u_Y of $D(H)$-$\mathbf{mod}_{\mathrm{fin}}$ is given by the action of u on the $D(H)$-module Y and the associated twist θ is given by the action of $(gu)^{-1}$. Since we always have the natural isomorphism

$$\mathrm{Hom}_{D(H)}\big(D(H) \otimes_H V, Y\big) \cong \mathrm{Hom}_H(V, Y),$$

by the uniqueness of adjoint functors, $D(H) \otimes_H -$ is naturally equivalent to K. By Theorem 4.1, we have

$$\nu_n(V) = \frac{1}{\dim(H\text{-}\mathbf{mod}_{\mathrm{fin}})} \mathrm{Tr}_{D(H)\otimes_H V}\big((gu)^{-n}\big) = \frac{1}{\dim(H)} \hat{\chi}_V\big((gu)^{-n}\big).$$

The second statement follows immediately from Theorem 5.5. □

Remark 5.8. The Drinfeld double $\overline{D}(H)$ of a finite-dimensional quasi-Hopf algebra H is usually defined so that $\overline{D}(H)$-$\mathbf{mod}_{\mathrm{fin}} = \overline{Z}(H\text{-}\mathbf{mod}_{\mathrm{fin}})$ (cf. [18,21] for the Drinfeld double of a Hopf algebra). If H is a complex semisimple quasi-Hopf algebra, it follows from Remark 4.3 that

$$\nu_n(V) = \frac{1}{\dim H} \hat{\chi}_V\big((gu)^n\big)$$

for $V \in H$-$\mathbf{mod}_{\mathrm{fin}}$, where u is the Drinfeld element of $\overline{D}(H)$ and g is the trace-element of H. This formula for higher indicators recovers the one in [17, 6.4, Corollary] when H is a Hopf algebra.

Remark 5.9. Let H be a quasi-Hopf algebra. By [1], the Drinfeld element u of $D(H)$ is given by

$$u = S\big(\phi^{(-2)}\beta S\big(\phi^{(-3)}\big)\big)S\big(R^{(2)}\big)\alpha R^{(1)}\phi^{(-1)},$$

where $\phi^{(-1)} \otimes \phi^{(-2)} \otimes \phi^{(-3)}$ is the inverse of the associator of $D(H)$.

If H is a semisimple complex quasi-Hopf algebra, one can find a formula for gu that does not contain the element g corresponding to the pivotal structure explicitly. Using the expressions for g in [20, Corollary 8.5], or using [32, Lemma 3.1], we find that for any $t \in H \otimes H$ satisfying $S(t^{(1)})\alpha t^{(2)} = 1$ we have

$$\begin{aligned} gu &= gS\big(\phi^{(-2)}\beta S\big(\phi^{(-3)}\big)\big)S\big(R^{(2)}\big)\alpha R^{(1)}\phi^{(-1)} \\ &= gS^2\big(\phi^{(-3)}\big)S(\beta)S\big(\phi^{(-2)}\big)S\big(R^{(2)}\big)\alpha R^{(1)}\phi^{(-1)} \\ &= \phi^{(-3)}gS(\beta)S\big(\phi^{(-2)}\big)S\big(R^{(2)}\big)\alpha R^{(1)}\phi^{(-1)} \\ &= \phi^{(-3)}\Lambda_{(2)}t^{(2)}S\big(\Lambda_{(1)}t^{(1)}\big)S\big(\phi^{(-2)}\big)S\big(R^{(1)}\big)\alpha R^{(1)}\phi^{(-1)}. \end{aligned}$$

Possible choices are $t = p_L$ or $t = p_R$.

We have already computed the Frobenius–Schur exponent of the nontrivial two-dimensional quasi-Hopf algebra above, and seen that it differs from the exponent in the sense of Etingof. We shall redo this example now using the new characterization of the Frobenius–Schur exponent as the order of gu:

Example 5.10. Let H be the quasi-Hopf algebra given in Example 5.4. Since $D^{\omega}(G)$ is commutative and the antipode of $D^{\omega}(G)$ is the identity map, the general formula for u from [1] specializes to

$$u = R^{(2)}R^{(1)} = \sum_{a\in G} e(a) \otimes a = e(1) \otimes 1 + e(x) \otimes x.$$

Direct computation shows that $\operatorname{ord}(u) = 4$. By [20], the trace-element g of $D(H)$ is given by

$$g = \sum_{a\in G} \omega\big(a, a^{-1}, a\big)e(a) \otimes 1 = \big(e(1) - e(x)\big) \otimes 1.$$

Therefore, $\operatorname{ord}(g) = 2$. By the commutativity of $D^{\omega}(G)$ again,

$$\operatorname{ord}(gu) = 4.$$

Therefore, as we have already seen in Example 5.4, the Frobenius–Schur exponent of H is 4.

Since the canonical pivotal structure of the module category over a semisimple quasi-Hopf algebra is preserved by every monoidal equivalence, we can deduce some invariance properties of the Frobenius–Schur exponent:

Proposition 5.11. *Let H, H' be complex semisimple quasi-Hopf algebra.*

(i) *The Frobenius–Schur exponent of H depends only on the gauge equivalence class of the double $D(H)$ as a quasi-triangular quasi-Hopf algebra, i.e.* $\mathrm{FSexp}(H) = \mathrm{FSexp}(H')$ *provided $D(H)$ and $D(H')$ are gauge equivalent quasi-triangular quasi-Hopf algebras.*
(ii) $\mathrm{FSexp}(H \otimes H') = \mathrm{lcm}(\mathrm{FSexp}(H), \mathrm{FSexp}(H'))$.
(iii) $\mathrm{FSexp}(H^{\mathrm{op}}) = \mathrm{FSexp}(H^{\mathrm{cop}}) = \mathrm{FSexp}(H)$.
(iv) $\mathrm{FSexp}(D(H)) = \mathrm{FSexp}(H)$.

Proof. Since the pivotal structure is preserved under any monoidal equivalence between module categories of semisimple complex quasi-Hopf algebras, (i) is a direct consequence of Corollary 5.6. Statement (ii) can be verified directly from the definition of the Frobenius–Schur exponent, since one easily sees $\nu_n(V \otimes W) = \nu_n(V)\nu_n(W)$ for $V \in H\text{-}\mathbf{mod}_{\mathrm{fin}}$ and $W \in H'\text{-}\mathbf{mod}_{\mathrm{fin}}$. As for (iii), it suffices to treat H^{cop}, since H^{op} and H^{cop} are gauge equivalent through the antipode. Now $H^{\mathrm{cop}}\text{-}\mathbf{mod}_{\mathrm{fin}} = (H\text{-}\mathbf{mod}_{\mathrm{fin}})^{\mathrm{sym}}$ is the category $H\text{-}\mathbf{mod}_{\mathrm{fin}}$ with the opposite tensor product. For $n = \mathrm{FSexp}(H)$ we have

$$\dim(V) = \overline{\dim(V)} = \overline{\nu_n(V)} = \nu_{n,n-1}(V) = \nu_n\big(V^{\mathrm{sym}}\big)$$

by [26, Theorem 5.1, Lemma 5.2], where V^{sym} denotes the module V considered as an object of $(H\text{-}\mathbf{mod}_{\mathrm{fin}})^{\mathrm{sym}}$. Thus $\mathrm{FSexp}(H)$ divides $\mathrm{FSexp}(H^{\mathrm{cop}})$, and by symmetry we are done. Finally (iv) follows since $D(D(H))$ is gauge equivalent to $D(H \otimes H^{\mathrm{op}})$. This is can be rephrased and proved entirely in categorical terms (see the remark below). For quasi-Hopf algebras we can argue as follows, without using semisimplicity: By [31] the category of modules over the double $D(H)$ is isomorphic to the monoidal category ${}^H\mathcal{C}^H$ of H-H-bicomodules in the monoidal category $\mathcal{C} = H\text{-}\mathbf{mod}_{\mathrm{fin}}\text{-}H$ of H-bimodules. By [30] the center of this bicomodule category is equivalent to the center of the underlying category, and so

$$\begin{aligned} D\big(H \otimes H^{\mathrm{op}}\big)\text{-}\mathbf{mod}_{\mathrm{fin}} &\cong Z\big(H \otimes H^{\mathrm{op}}\text{-}\mathbf{mod}_{\mathrm{fin}}\big) \cong Z(\mathcal{C}) \cong Z\big({}^H\mathcal{C}^H\big) \cong Z\big(D(H)\text{-}\mathbf{mod}_{\mathrm{fin}}\big) \\ &\cong D\big(D(H)\big)\text{-}\mathbf{mod}_{\mathrm{fin}} \end{aligned}$$

as braided monoidal categories. □

Remark 5.12. For any fusion category $\mathcal{C}$, we have a braided monoidal category equivalence $Z(Z(\mathcal{C})) \cong Z(\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{sym}})$ by [23, Section 7]. Thus at least in the semisimple case we need here the result on the double of the double of a quasi-Hopf algebra above has a categorical version using a suitable tensor product of categories. However, we did not verify that the equivalence preserves the pivotal structures in this case, so we cannot draw the desired conclusion on the Frobenius–Schur exponent of the center for general spherical fusion categories. We will arrive at that result with an entirely different proof in Section 7.

We close this section with

Corollary 5.13. *Let $\mathcal{C}$ be a spherical fusion category over $\mathbb{C}$ with pivotal structure j. Then for any simple $V \in \mathcal{C}$, I is a summand of $V^{\otimes n}$, whenever n is a multiple of* $\mathrm{FSexp}(V)$ *or* $\mathrm{FSexp}(\mathcal{C})$. *Moreover, for any pivotal structure j' on $\mathcal{C}$, $j^{-1}j'$ is a finite order monoidal automorphism of the identity functor of $\mathcal{C}$, and* $\mathrm{ord}(j^{-1}j') \mid \mathrm{FSexp}(\mathcal{C})$.

Proof. Clearly we need only treat the case that $n \in \{\mathrm{FSexp}(V), \mathrm{FSexp}(\mathcal{C})\}$. By [26], $\nu_r(V)$ is the ordinary trace of an $\mathbb{C}$-linear automorphism $E_V^{(r)}$ on $\mathcal{C}(I, V^{\otimes r})$. Thus, if $\nu_r(V) \neq 0$, then

$\mathcal{C}(I, V^{\otimes r}) \neq 0$. By [8, Theorem 2.3], $d(V) \neq 0$ for any simple object V, and $\nu_n(V) = d(V)$. Therefore, $\mathcal{C}(I, V^{\otimes n}) \neq 0$. Let $\lambda = j^{-1} j'$. Then the component λ_V is a scalar, and $\lambda_I = 1$ (cf. Remark 2.1). Since I is a summand of $V^{\otimes n}$, we have $\lambda_V^n = 1$ and hence $\operatorname{ord}(\lambda_V) \mid \operatorname{FSexp}(\mathcal{C})$. Since $\operatorname{ord}(\lambda) = \operatorname{lcm} \operatorname{ord}(\lambda_V)$, where V runs through a complete set of non-isomorphic simple objects of $\mathcal{C}$, the divisibility $\operatorname{ord}(\lambda) \mid \operatorname{FSexp}(\mathcal{C})$ follows. □

The fact that I is a direct summand of some tensor power of every simple V generalizes a result from [17, Section 4]; in their terminology (introduced for modules over Hopf algebras), any simple V has finite order.

6. Etingof's exponent vs. Frobenius–Schur exponent

We have discussed already that the Frobenius–Schur exponent of a quasi-Hopf algebra can differ from its exponent as defined by Etingof. In the example, the difference amounts to a factor 2. The main result of this section implies that this is the most general discrepancy between the two notions that can occur. In particular, results about the Frobenius–Schur exponent have implications for the exponent in the sense of Etingof.

Let $\mathcal{C}$ be a ribbon fusion category over $\mathbb{C}$ with the twist θ and braiding c. Let $M_{V,W} = c_{W,V} \circ c_{V,W}$ for any $V, W \in \mathcal{C}$. By [5], θ has finite order, and M has finite order with $\operatorname{ord}(M) \mid \operatorname{ord}(\theta)$.

Proposition 6.1. $\operatorname{ord}(\theta) = \operatorname{ord}(M)$ *or* $2\operatorname{ord}(M)$.

Proof. Let $X_1, \ldots, X_l$ be the complete set of isomorphism classes of simple objects of $\mathcal{C}$, and let $\theta_{X_i} = \omega_i \operatorname{id}_{X_i}$ for some root of unity ω_i. Then

$$X_i^* = X_{\bar{i}}$$

for some $\bar{i}$. The ribbon structure θ implies that $\omega_i = \omega_{\bar{i}}$ and $\mathcal{C}$ is spherical. Hence, $d_i = d_{\bar{i}}$. Let $n = \operatorname{ord}(M)$. Then the equality

$$\theta_{V\otimes W} = (\theta_V \otimes \theta_W) M_{V,W}$$

implies that

$$\theta^n_{V\otimes W} = \left(\theta^n_V \otimes \theta^n_W\right)$$

for all $V, W \in \mathcal{C}$. In particular, we have

$$\theta^n_{X_i\otimes X_j} = \left(\theta^n_{X_i} \otimes \theta^n_{X_j}\right) \tag{6.1}$$

for all $i, j \in \{1, \ldots, l\}$. Let $N_{ij}^k = \dim(\mathcal{C}(X_k, X_i \otimes X_j))$. Taking trace on both sides of (6.1), we have

$$\sum_{k\in\Gamma} \omega_k^n d_k N_{ij}^k = \omega_i^n \omega_j^n d_i d_j$$

where d_i is the pivotal dimension of X_i. Let

$$v_i = \omega_i^n d_i, \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_l \end{bmatrix}, \qquad N_i = \left[N_{ia}^b\right]_{a,b}.$$

Then we have

$$N_i \mathbf{v} = v_i \mathbf{v}, \qquad N_{\bar{i}} = N_i^t.$$

Taking the complex transposition of the first equation, we have

$$\bar{\mathbf{v}}^t N_{\bar{i}} = \bar{v}_i \bar{\mathbf{v}}^t$$

and hence

$$\bar{v}_i \bar{\mathbf{v}}^t \mathbf{v} = \bar{\mathbf{v}}^t N_{\bar{i}} \mathbf{v} = v_{\bar{i}} \bar{\mathbf{v}}^t \mathbf{v}.$$

Since $\mathcal{C}$ is spherical and $\mathbf{v}$ is a non-zero complex vector, we obtain

$$\bar{\omega}_i^n d_i = \bar{v}_i = v_{\bar{i}} = \omega_i^n d_{\bar{i}} = \omega_i^n d_i.$$

Since $d_i \neq 0$, $\bar{\omega}_i^n = \omega_i^n$. Therefore, $\omega_i^n = \pm 1$ and so $\omega_i^{2n} = 1$ for all i. Equivalently, $\theta^{2n} = \mathrm{id}$. $\square$

Corollary 6.2. *For any semisimple quasi-Hopf algebra H over $\mathbb{C}$ we have* $\mathrm{FSexp}(H) = \exp(H)$ *or* $\mathrm{FSexp}(H) = 2\exp(H)$.

We close this section with the following lemma which will be used in Section 8 to prove the Cauchy Theorem for semisimple quasi-Hopf algebras.

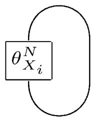
Lemma 6.3. *Let $\mathcal{C}$ be a ribbon fusion category over $\mathbb{C}$ with the braiding c, the pivotal structure j, and the twist θ. Let $M_{V,W} = c_{W,V} c_{V,W}$ for $V, W \in \mathcal{C}$. If* $\mathrm{ord}(\theta) = 2\,\mathrm{ord}(M)$ *and* $\mathrm{ord}(M)$ *is odd, then there exists a spherical pivotal structure $\hat{j}$ on $\mathcal{C}$ such that the order of the twist $\hat{\theta}$ associated with $\hat{j}$ is equal to the order of M, and $\hat{d}(V) = \pm d(V)$ where $\hat{d}(V)$ and $d(V)$ denote the dimensions of V computed with the pivotal structures $\hat{j}$ and j, respectively. In addition, if $\mathcal{C}$ is a MTC, then $\mathcal{C}$ with the twist $\hat{\theta}$ is also a MTC.*

Proof. Without loss of generality, we may assume that $\mathcal{C}$ is strict pivotal. Let $N = \mathrm{ord}(M)$ and $\hat{\theta} = \theta^{N+1}$. Since N is odd, $\gcd(N+1, 2N) = 2$ and so $\mathrm{ord}(\hat{\theta}) = N$. Moreover,

$$\theta_{V\otimes W}^N = \theta_V^N \otimes \theta_W^N$$

for any $V, W \in \mathcal{C}$. Thus, $\hat{\theta}$ is a twist and $\mathcal{C}$ is a ribbon category with respect to $\hat{\theta}$. Let $\hat{j} = u\hat{\theta}$ be the spherical pivotal structure on $\mathcal{C}$ associated with $\hat{\theta}$, where u is the Drinfeld isomorphism of the braiding c. Since $\mathcal{C}$ is strict pivotal, $u = \theta^{-1}$ and so $\hat{j} = \theta^N$. Let X_i, $i \in \Gamma$, be a complete set of non-isomorphic simple objects of $\mathcal{C}$. Then, for $i \in \Gamma$, $\theta^N_{X_i} = \pm \mathrm{id}_{X_i}$ and

$$\hat{d}_i = \hat{d}(X_i) =$$



$$= \pm d(X_i) = \pm d_i.$$

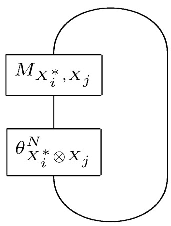
In particular, $\sum_{i\in\Gamma} \hat{d}_i^2 = \sum_{i\in\Gamma} d_i^2 = \dim \mathcal{C}$.

Suppose, in addition, $\mathcal{C}$ is a MTC with respect to the twist θ. Let $\Gamma^{\pm} = \{i \in \Gamma \mid \theta^N_{X_i} = \pm \mathrm{id}_{X_i}\}$. Let $[s_{ij}]_{i,j\in\Gamma}$ be the S-matrix of this MTC. Now, we compute the S-matrix $[\hat{s}_{ij}]$ of the ribbon category $\mathcal{C}$ with respect to the twist $\hat{\theta}$. Note that

$$\theta^N_{X_i^*\otimes X_j} = \theta^N_{X_i^*} \otimes \theta^N_{X_j} = \begin{cases} \mathrm{id}_{X_i^*\otimes X_j} & \text{if } i, j \in \Gamma^+ \text{ or } i, j \in \Gamma^-, \\ -\mathrm{id}_{X_i^*\otimes X_j} & \text{otherwise.} \end{cases}$$

Thus

$$\hat{s}_{ij} = \frac{1}{\sqrt{\dim \mathcal{C}}} \cdot$$



$$= \begin{cases} s_{ij} & \text{if } i, j \in \Gamma^+ \text{ or } i, j \in \Gamma^-, \\ -s_{ij} & \text{otherwise.} \end{cases}$$

Therefore, $[s_{ij}]$ and $[\hat{s}_{ij}]$ are conjugate matrices. Hence the matrix $[\hat{s}_{ij}]$ is non-singular and $\mathcal{C}$ is a MTC with respect to $\hat{\theta}$. $\square$

7. Frobenius–Schur indicator formula for modular tensor categories

In [3], Bantay has defined a scalar, called the (2nd) Frobenius–Schur indicator, by the formula

$$\sum_{i,j\in\Gamma} N_{ij}^k s_{0i} s_{0j} \left(\frac{\omega_i}{\omega_j}\right)^2 \tag{7.1}$$

for each simple object X_k of a modular tensor category $\mathcal{C}$ (MTC), where $[s_{ab}]$ denotes the S-matrix of $\mathcal{C}$ and $N_{ij}^k = \dim(\mathcal{C}(X_i \otimes X_j, X_k))$. It is shown in the paper that the value of the expression can only be 0, 1, or -1.

In this section, we will derive a formula (Theorem 7.5) for the nth indicator $\nu_n(V)$ of a simple object V in a modular tensor category $\mathcal{C}$ using the definition of higher indicators introduced in [26]. Bantay's formula (7.1) is recovered as the special case $n = 2$. This also shows that Bantay's notion of Frobenius–Schur indicator is the 2nd FS-indicator in our sense.

As an immediate consequence of the formula for $\nu_n(V)$, we show that FSexp($\mathcal{C}$) is equal to the order of the twist θ of $\mathcal{C}$ and that the Frobenius–Schur exponent of a spherical fusion category is invariant under the center construction.

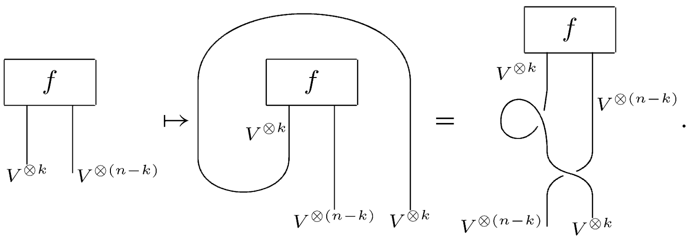
We continue to use the notation introduced in Section 3. Let $\mathcal{C}$ be a *strict* modular tensor category, i.e. a MTC whose underlying spherical category is strict. Suppose that X_i, $i \in \Gamma$, is a complete set of non-isomorphic simple objects of $\mathcal{C}$. We define $\omega_i \in \mathbb{C}$, $i \in \Gamma$, by the equation

$$\theta_{X_i} = \omega_i \operatorname{id}_{X_i}.$$

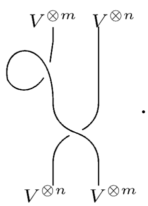
Since the twist θ of $\mathcal{C}$ is of finite order, ω_i is a root of unity.

By [26], the (n,k)th Frobenius–Schur indicator $\nu_{n,k}(V)$ of the object $V \in \mathcal{C}$ is the ordinary trace of the linear operator $(E_V^{(n)})^k : \mathcal{C}(I, V^{\otimes n}) \to \mathcal{C}(I, V^{\otimes n})$ defined by

$\left(E_V^{(n)}\right)^k:$



Let $C_V^{(m,n)} \in \mathcal{C}(V^{\otimes(m+n)}, V^{\otimes(m+n)})$ denote the map

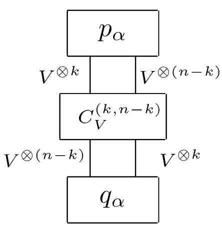


In particular,

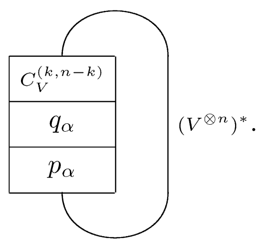
$$C_{X_j}^{(1,n-1)} = \omega_j \cdot c_{X_j, X_j^{\otimes(n-1)}}. \tag{7.2}$$

For any positive integers n, k, let $\{p_\alpha\}$ be a basis for $\mathcal{C}(I, V^{\otimes n})$, and $\{q_\alpha\}$ the dual basis for $\mathcal{C}(V^{\otimes n}, I)$. Then we have

$\operatorname{Tr}\left(\left(E_V^{(n)}\right)^k\right) = \sum_\alpha$

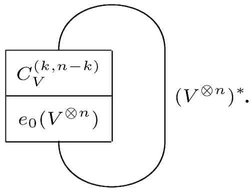


$= \sum_\alpha$



Let $e_0(V^{\otimes n}) = \sum_\alpha p_\alpha q_\alpha$. Then we have

$$\nu_{n,k}(V) =$$



(7.3)

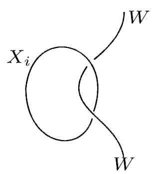
Note that $e_0(V^{\otimes n})$ is independent of the choices of the basis $\{p_\alpha\}$ for $\mathcal{C}(I, V^{\otimes n})$. In general, the map $e_0(W)$, for any object W of $\mathcal{C}$, is the idempotent of $\mathcal{C}(W, W)$ given by

$$e_0(W) = \iota\pi, \tag{7.4}$$

where $\pi : W \to W^{\mathrm{triv}}$ and $\iota : W^{\mathrm{triv}} \to W$ are the epimorphism and the monomorphism associated with the summand W^{triv} of W.

Lemma 7.1. *For any $W \in \mathcal{C}$, we have*

$$e_0(W) = \frac{1}{\dim(\mathcal{C})} \sum_{i\in\Gamma} d_i$$

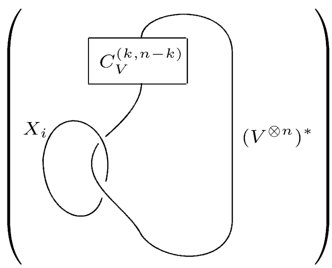


.

Proof. The statement follows directly from (7.4) and [2, Corollary 3.1.11]. □

By Lemma 7.1 and (7.3), we have

$$\nu_{n,k}(V) = \frac{1}{\dim(\mathcal{C})} \sum_{i\in\Gamma} d_i$$

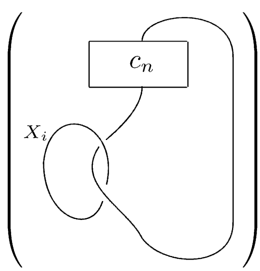


. (7.5)

Since $\nu_n(V) = \nu_{n,1}(V)$, the following lemma follows immediately from (7.2).

Lemma 7.2. *For any $j \in \Gamma$ and integer $n \geqslant 1$, we have*

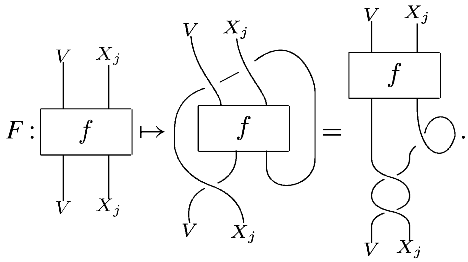
$$\nu_n(X_j) = \frac{1}{\dim(\mathcal{C})} \sum_{i\in\Gamma} d_i \omega_j$$



,

where c_n denotes the map $c_{X_j, X_j^{\otimes(n-1)}}$.

Let $V \in \mathcal{C}$. We define $F : \mathcal{C}(V \otimes X_j, V \otimes X_j) \to \mathcal{C}(V \otimes X_j, V \otimes X_j)$ by



(7.6)

Let $m_n = c_{X_j^{\otimes n}, V} \circ c_{V, X_j^{\otimes n}}$, and $c_n = c_{X_j, X_j^{\otimes(n-1)}}$ for any integer $n \geqslant 1$. Then, (7.6) says that

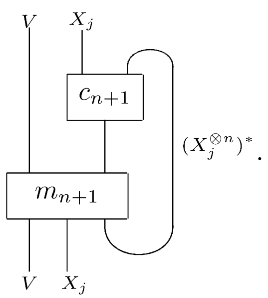
$$F(f) = \omega_j \cdot m_1 \circ f$$

for $f \in \mathcal{C}(V \otimes X_j, V \otimes X_j)$. In particular,

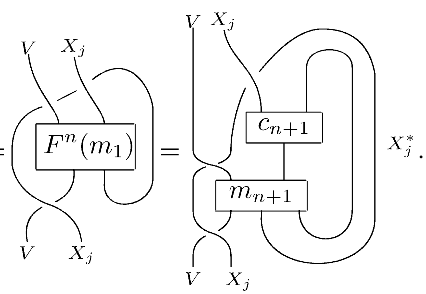
$$F^{n-1}(m_1) = \omega_j^{n-1} \cdot m_1^n, \tag{7.7}$$

where F^0 denotes the identity map.

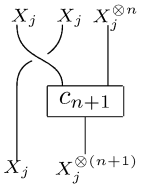
Lemma 7.3. *For any integer $n \geqslant 0$, we have*

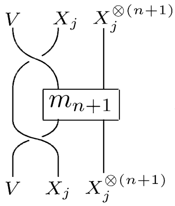
$F^n(m_1) =$ 

Proof. The equality is obviously true for $n = 0$ as $c_1 = \mathrm{id}_{X_j}$. Assume the equation holds for some integer $n \geqslant 0$. Then

$F^{n+1}(m_1) =$ 

Since

$c_{n+2} =$ 

and $m_{n+2} =$ 

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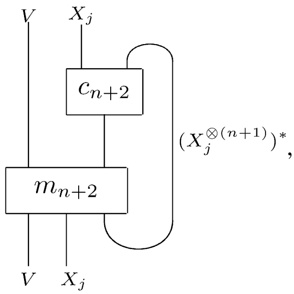
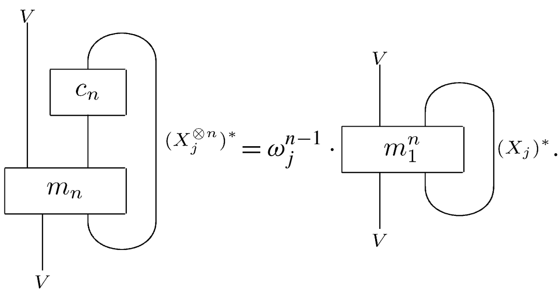
we have

$F^{n+1}(m_1) =$ 

and this completes the proof. □

Lemma 7.4. *For any integer $n \geqslant 1$,*



Proof. The equality follows immediately from Lemma 7.3 and (7.7). □

Theorem 7.5. *For any $j \in \Gamma$ and positive integer n, we have*

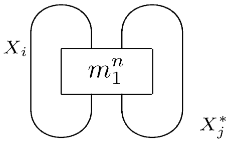
$$\nu_n(X_j) = \frac{1}{\dim(\mathcal{C})} \sum_{i,k\in\Gamma} N_{ik}^j d_i d_k \left(\frac{\omega_i}{\omega_k}\right)^n,$$

where $N_{ik}^j = \dim \mathcal{C}(X_i \otimes X_k, X_j)$. In particular, if $N = \exp(\mathcal{C})$ or $\operatorname{ord}(m_1)$, then

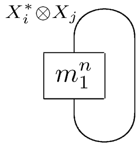
$$\nu_N(X_j) = \omega_j^N d_j = \pm d_j.$$

Proof. By Lemmas 7.2 and 7.4,

$$\nu_n(X_j) = \frac{1}{\dim(\mathcal{C})} \sum_{i\in\Gamma} d_i \omega_j^n \cdot$$



$$= \frac{1}{\dim(\mathcal{C})} \sum_{i\in\Gamma} d_i \omega_j^n \cdot$$



$$. \qquad (7.8)$$

Since $\omega_i \omega_j m_1 = \theta_{X_i^* \otimes X_j}$, we have

$$m_1^n = \frac{1}{(\omega_i\omega_j)^n} \bigoplus_{k\in\Gamma} \left(\theta_{X_k}^n\right)^{\oplus N_{\bar{i}j}^k}.$$

Therefore,

$$\nu_n(X_j) = \frac{1}{\dim(\mathcal{C})} \sum_{i,k\in\Gamma} d_i \frac{\omega_k^n}{\omega_i^n} N_{\bar{i}j}^k d_k = \frac{1}{\dim(\mathcal{C})} \sum_{i,k\in\Gamma} d_i d_k N_{ik}^j \frac{\omega_i^n}{\omega_k^n}.$$

In particular, if $N = \exp(\mathcal{C})$, then $m_1^N = \mathrm{id}$. By Proposition 6.1, $\omega_j^N = \pm 1$. It follows from (7.8) that

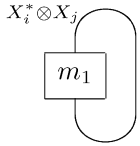
$$\nu_N(X_j) = \frac{1}{\dim(\mathcal{C})} \sum_{i\in\Gamma} d_i d_{\bar{i}} d_j \omega_j^N = \omega_j^N d_j. \qquad \square$$

Remark 7.6. Theorem 7.5 implies that the Frobenius–Schur indicators of the objects in a MTC are real. The formula for the nth indicators can be rewritten in term of the modular data of $\mathcal{C}$ by the Verlinde formula (cf. [2]), namely

$$N_{ik}^j = \sum_{r\in\Gamma} \frac{s_{ir} s_{kr} s_{\bar{j}r}}{s_{0r}},$$

where the S-matrix of $\mathcal{C}$ is given by

$$s_{ij} = \frac{1}{\sqrt{\dim(\mathcal{C})}}$$



In particular, $s_{0i} = d_i/\sqrt{\dim(\mathcal{C})}$ for all $i \in \Gamma$. Hence

$$\nu_n(X_j) = \sum_{i,k\in\Gamma} N_{ik}^j s_{0i} s_{0k} \frac{\omega_i^n}{\omega_k^n}.$$

For $n = 2$, this recovers Bantay's formula for the (degree 2) Frobenius–Schur indicator in conformal field theory (cf. [3]).

Theorem 7.7. *Let $\mathcal{C}$ be a MTC with the twist θ. Then* $\mathrm{FSexp}(\mathcal{C}) = \mathrm{ord}(\theta)$.

Proof. Let n be a positive integer. If $\theta^n = \mathrm{id}$, then

$$\nu_n(X_j) = \frac{1}{\dim(\mathcal{C})} \sum_{i,k} N_{ik}^j d_i d_k = \frac{1}{\dim(\mathcal{C})} \sum_i d_i d_{\bar{i}} d_j = d_j$$

for all $j \in \Gamma$. Conversely, if $\nu_n(X_j) = d_j$ for all $j \in \Gamma$, then

$$\begin{aligned}\dim(\mathcal{C}) = \sum_j d_j^2 &= \frac{1}{\dim(\mathcal{C})} \sum_{i,k,j\in\Gamma} d_i d_k N_{ik}^j d_j \frac{\omega_i^n}{\omega_k^n} \\ &= \frac{1}{\dim(\mathcal{C})} \sum_{i,k\in\Gamma} d_i^2 d_k^2 \frac{\omega_i^n}{\omega_k^n} = \frac{1}{\dim(\mathcal{C})} \left| \sum_{i\in\Gamma} d_i^2 \omega_i^n \right|^2.\end{aligned}$$

Hence, we have

$$\sum_{i\in\Gamma} d_i^2 = \dim(\mathcal{C}) = \left| \sum_{i\in\Gamma} d_i^2 \omega_i^n \right|.$$

The equalities imply that ω_i^n are all identical for $i \in \Gamma$. Since $\omega_0^n = 1$, $\omega_i^n = 1$ for all i and so $\theta^n = \mathrm{id}$. □

Corollary 7.8. *Let $\mathcal{C}$ be a spherical fusion category over $\mathbb{C}$. Then* $\mathrm{FSexp}(\mathcal{C}) = \mathrm{FSexp}(Z(\mathcal{C}))$.

Proof. By [23], $Z(\mathcal{C})$ is a modular tensor category and hence $\mathrm{FSexp}(Z(\mathcal{C})) = \mathrm{ord}(\theta)$ by Theorem 7.7, where θ is the twist of $Z(\mathcal{C})$ associated with the pivotal structure of $\mathcal{C}$. By Theorem 5.5, we also have $\mathrm{FSexp}(\mathcal{C}) = \mathrm{ord}(\theta)$ and so the result follows. □

8. Cauchy theorem for quasi-Hopf algebras

In [6], Etingof and Gelaki asked the following two questions for a complex semisimple Hopf algebra H:

(i) If p is a prime divisor of $\dim(H)$, does p divides $\exp(H)$?
(ii) If $\exp(H)$ is a power of a prime p, is $\dim(H)$ a power of p?

The questions have been recently answered by Kashina, Sommerhäuser and Zhu [17]. They proved that $\exp(H)$ and $\dim(H)$ have the same prime factors.

In this section, we generalize their result and prove an analog of *Cauchy's Theorem* (Theorem 8.4) for a complex semisimple quasi-Hopf algebra H: $\exp(H)$ and $\dim(H)$ have the same prime factors. If H admits a simple self-dual module, then $\dim(H)$ is even. Moreover, if $\dim(H)$ is odd, $\mathrm{FSexp}(H)$ and $\exp(H)$ are the same.

Throughout the section, we consider a spherical fusion category $\mathcal{C}$ over $\mathbb{C}$. Let m be the order of the twist θ of $Z(\mathcal{C})$ associated with the pivotal structure of $\mathcal{C}$; by Theorem 5.5 this is the same as the Frobenius–Schur exponent of $\mathcal{C}$. We let $\zeta_m \in \mathbb{C}$ be a primitive mth root of unity.

Recall that the (n,r)th Frobenius–Schur indicator of any object in $V \in \mathcal{C}$ is a cyclotomic integer in $\mathbb{Q}_n$. Let m be the order of the twist θ of $Z(\mathcal{C})$ associated with the pivotal structure of $\mathcal{C}$

and let $\zeta_m \in \mathbb{C}$ be a primitive mth root of unity. Let $\mathbb{F}$ be the smallest extension over $\mathbb{Q}$ containing d_i for all $i \in \Gamma$; we will call $\mathbb{F}$ the *dimension field* of $\mathcal{C}$. Note that $\mathrm{Gal}(\mathbb{F}(\zeta_m)/\mathbb{F})$ is isomorphic to a subgroup of $U(\mathbb{Z}_m)$. Let r be an integer such that $\sigma : \zeta_m \mapsto \zeta_m^r$ defines an automorphism of $\mathbb{F}(\zeta_m)/\mathbb{F}$; in particular, r is relatively prime to m. By Theorem 4.1, $\nu_n(V) \in \mathbb{F}(\zeta_m)$ for $V \in \mathcal{C}$ and a positive integer n. Moreover,

$$\sigma\big(\nu_n(V)\big) = \sigma\left(\frac{1}{\dim \mathcal{C}}\, \underline{\mathrm{ptr}}\big(\theta^n_{K(V)}\big)\right) = \frac{1}{\dim \mathcal{C}}\, \underline{\mathrm{ptr}}\big(\theta^{nr}_{K(V)}\big) = \nu_{nr}(V).$$

By (4.5), we have

$$\begin{aligned}\delta_{i,0} &= \sigma\left(\frac{1}{\dim \mathcal{C}} \sum_{(X,e_X)\in\hat{\Gamma}} N_i^X d(X)\omega_{(X,e_X)}\right)\\ &= \frac{1}{\dim \mathcal{C}} \sum_{(X,e_X)\in\hat{\Gamma}} N_i^X d(X)\omega^r_{(X,e_X)} = \nu_r(X_i).\end{aligned} \tag{8.1}$$

These equalities imply some congruences when r is a prime.

Proposition 8.1. *Let $\mathcal{C}$ be a spherical fusion category over $\mathbb{C}$ and $m = \mathrm{FSexp}(\mathcal{C})$. For any prime p such that $\zeta_m \mapsto \zeta_m^p$ defines an automorphism of $\mathbb{F}(\zeta_m)/\mathbb{F}$, so in particular $p \nmid m$, we have*

$$N_0^V \equiv N_0^{V^{\otimes p}} \mod p$$

for all $V \in \mathcal{C}$ where N_i^X denotes the integer $\dim \mathcal{C}(X_i, X)$.

Proof. Note that $V = \bigoplus_{i\in\Gamma} N_i^V X_i$, where $N_i^V = \dim \mathcal{C}(X_i, V)$. By (8.1),

$$\nu_p(V) = \sum_{i\in\Gamma} N_i^V \nu_p(X_i) = N_0^V.$$

By the definition of Frobenius–Schur indicators, $\mathrm{Tr}(E_V^{(p)}) = \nu_p(V)$ is an integer. Since $E_V^{(p)}$ is an $\mathbb{C}$-linear automorphism on $\mathcal{C}(I, V^{\otimes p})$ of order 1 or p, by a linear algebra argument in [17],

$$\mathrm{Tr}\big(E_V^{(p)}\big) \equiv \dim \mathrm{Hom}_H\big(I, V^{\otimes p}\big) \mod p.$$

Hence, the result follows. □

As an immediate consequence of Proposition 8.1, we prove the following corollary which is a generalization of a result in [16].

Corollary 8.2. *Let $\mathcal{C}$ be a spherical fusion category over $\mathbb{C}$, $\mathbb{F}$ the dimension field of $\mathcal{C}$ and m the Frobenius–Schur exponent of $\mathcal{C}$. Suppose that there exists a simple object $V \ncong I$ of $\mathcal{C}$ such that $V^* \cong V$.*

(i) *If $\mathbb{F} \cap \mathbb{Q}_m = \mathbb{Q}$, then m is even and $(\dim \mathcal{C})^5/2$ is an algebraic integer.*
(ii) *If $\mathbb{F} = \mathbb{Q}$, then $\dim \mathcal{C}$ is divisible by 2.*

Proof. (i) Suppose that $2 \nmid m$. Then $\zeta_m \mapsto \zeta_m^2$ defines an automorphism of $\mathbb{F}(\zeta_m)/\mathbb{F}$ as the condition $\mathbb{F} \cap \mathbb{Q}(\zeta_m) = \mathbb{Q}$ implies that $\mathrm{Gal}(\mathbb{F}(\zeta_m)/\mathbb{F}) = U(\mathbb{Z}_m)$. It follows from Proposition 8.1 that

$$0 = N_0^V \equiv N_0^{V^{\otimes 2}} \mod 2.$$

This contradicts that $N_0^{V^{\otimes 2}} = \dim \mathcal{C}(V, V^*) = 1$. Therefore, $2 \mid m$. By [5], $\dim Z(\mathcal{C})^{5/2}/m$ is an algebraic integer. Since $\dim Z(\mathcal{C}) = \dim(\mathcal{C})^2$, we obtain that $(\dim \mathcal{C})^5/2$ must be an algebraic integer.

(ii) If $\mathbb{F} = \mathbb{Q}$, then $\dim \mathcal{C}$ is indeed an integer and the assumption in (i) holds obviously. It follows from (i) that $\dim(\mathcal{C})^5/2$ is an algebraic integer in $\mathbb{Q}$ and hence an integer. Therefore, $\dim(\mathcal{C})$ is even. $\square$

Consider a positive integer n and a primitive lth root of unity ζ_l for $l = \mathrm{lcm}(m, n)$. Further let r be a positive integer such that $\sigma : \zeta_l \mapsto \zeta_l^r$ defines an automorphism of $\mathbb{F}(\zeta_l)/\mathbb{F}$; in particular r is relatively prime to l. Then $\sigma(\nu_n(V)) = \nu_{n,r}(V)$ and $\sigma(\nu_n(V)) = \nu_{nr}(V)$. Thus, we have the following proposition which generalizes the corresponding result in [17].

Proposition 8.3. *Let n be any positive integer, $l = \mathrm{lcm}(m, n)$ and r an integer relatively prime to l. If the assignment $\sigma : \zeta_l \mapsto \zeta_l^r$ defines an automorphism of $\mathbb{F}(\zeta_l)/\mathbb{F}$, where ζ_l is a primitive lth root of unity, then*

$$\nu_{n,r}(V) = \nu_{nr}(V) \tag{8.2}$$

for all $V \in \mathcal{C}$.

Proof. The claim follows from the fact that $\sigma|_{\mathbb{F}(\zeta_n)}(x) = \sigma|_{\mathbb{F}(\zeta_m)}(x)$ for all $x \in \mathbb{F}(\zeta_n) \cap \mathbb{F}(\zeta_m)$. $\square$

Now, we turn to the proof of the main theorem in this section.

Theorem 8.4. *Let $\mathcal{C}$ be a spherical fusion category over $\mathbb{C}$ such that the Frobenius–Perron dimension of any simple object is an integer. Then* $\mathrm{FSexp}(\mathcal{C})$, $\exp(\mathcal{C})$ *and* $\dim(\mathcal{C})$ *have the same set of prime factors. Equivalently, if H is a semisimple quasi-Hopf algebra over $\mathbb{C}$, then* $\mathrm{FSexp}(H)$, $\exp(H)$ *and* $\dim(H)$ *have the same set of prime factors.*

Proof. The equivalence of the two statements follows directly from the independence of $\dim(\mathcal{C})$ and $\mathrm{FSexp}(\mathcal{C})$ on the choice of a pivotal structure on $\mathcal{C}$ (cf. Remarks 2.1 and 5.2), and the characterization of H-$\mathbf{mod}_{\mathrm{fin}}$ for some semisimple quasi-Hopf algebra over $\mathbb{C}$ as a fusion category over $\mathbb{C}$ with integer Frobenius–Perron dimension for each simple object by [8, Section 8].

Without loss of generality, we may assume $\mathcal{C}$ is strict pivotal. Since both the Frobenius–Schur exponent and the dimension of $\mathcal{C}$ are independent of the choice of a spherical structure, we can assume that $\mathcal{C}$ is endowed with the canonical spherical structure with $d(V)$ equal to the Frobenius–Perron dimension of $V \in \mathcal{C}$. The center $Z(\mathcal{C})$ is a modular tensor category with the twist θ associated with the canonical spherical structure of $\mathcal{C}$. By [5], the order of θ divides $\dim(Z(\mathcal{C}))^{5/2} = \pm \dim(\mathcal{C})^5$. Thus, the prime factors of $\mathrm{ord}(\theta)$ are also prime factors of $\dim(\mathcal{C})$. Let $\{X_i\}_{i \in \Gamma}$ be a complete set of non-isomorphic simple objects of $\mathcal{C}$. Recall from [5] that $V =$

$\sum_{i\in\Gamma} d(X_i)X_i$ defines a rank one ideal in the Grothendieck ring of $\mathcal{C}$ such that $X_i \otimes V \cong d(X_i)V$ and

$$d(V) = \sum_{i\in\Gamma} d(X_i)^2 = \dim(\mathcal{C}).$$

Suppose p is not a prime factor of $\operatorname{ord}(\theta)$. Since $d(X_i)$ is an integer for all $i \in \Gamma$, $\mathbb{F} = \mathbb{Q}$. By Proposition 8.1, we have

$$N_0^V \equiv N_0^{V^{\otimes p}} \mod p.$$

Since

$$V^{\otimes p} \cong d\big(V^{\otimes(p-1)}\big)V = d(V)^{p-1}V = \dim(\mathcal{C})^{p-1}V,$$

we find

$$\dim \mathcal{C}\big(I, V^{\otimes p}\big) = \dim(\mathcal{C})^{p-1}.$$

Therefore,

$$1 \equiv \dim(\mathcal{C})^{p-1} \mod p$$

and hence $p \nmid \dim(\mathcal{C})$. Thus, by Theorem 5.5, $\operatorname{FSexp}(\mathcal{C})$ and $\dim(\mathcal{C})$ have the same set of prime factors.

By Corollary 6.2,

$$\operatorname{FSexp}(\mathcal{C}) = \exp(\mathcal{C}) \quad \text{or} \quad \operatorname{FSexp}(\mathcal{C}) = 2\exp(\mathcal{C}).$$

To complete the proof, it suffices to show that $\exp(\mathcal{C})$ is even whenever $\operatorname{FSexp}(\mathcal{C})$ is even. Suppose $\operatorname{FSexp}(\mathcal{C})$ is even and $\exp(\mathcal{C})$ is odd. Then we have $\operatorname{FSexp}(\mathcal{C}) = 2\exp(\mathcal{C})$. Recall that $\operatorname{FSexp}(\mathcal{C})$ is the order of θ. By Lemma 6.3, there exists another spherical pivotal structure $\hat{j}$ on $Z(\mathcal{C})$ such that $\operatorname{ord}(\hat{\theta}) = \exp(\mathcal{C})$ for the associated twist $\hat{\theta}$, while $\hat{d}(V) = \pm d(V)$ for any simple object V of $Z(\mathcal{C})$, where $\hat{d}$ is the dimension function associated with $\hat{j}$. Let $\hat{\mathcal{C}}$ denote the spherical fusion category $Z(\mathcal{C})$ endowed with the spherical structure $\hat{j}$. By Lemma 6.3, $\hat{\mathcal{C}}$ is modular since $Z(\mathcal{C})$ is a MTC. By Theorem 7.7, Corollary 7.8, and Remark 5.2, we have

$$\operatorname{FSexp}(\mathcal{C}) = \operatorname{FSexp}\big(Z(\mathcal{C})\big) = \operatorname{FSexp}(\hat{\mathcal{C}}) = \operatorname{ord}(\hat{\theta}) = \exp(\mathcal{C}),$$

a contradiction! $\square$

Corollary 8.5. *Let H be an odd-dimensional semisimple quasi-Hopf algebra over $\mathbb{C}$. Then the exponent of H and Frobenius–Schur exponent of H are identical.*

Proof. By the preceding theorem, $\operatorname{FSexp}(H)$ is odd and so the claim follows from Corollary 6.2. $\square$

9. Bounds on the exponent

A direct application of Etingof's bound on the order of the twist of a modular tensor category [5] shows that the Frobenius–Schur exponent (and hence the exponent) of a spherical fusion category divides the fifth power of its dimension. In this section we will strengthen this bound in important special cases. For a semisimple quasi-Hopf algebra H, we can show that the Frobenius–Schur exponent of H divides $\dim(H)^4$. The techniques in this case are close to those of Etingof and Gelaki in [7]. Special care has to be taken to deal with the nontrivial associativity isomorphisms (which can be avoided in [5] by using categorically defined determinants, at the cost of a higher bound). Our bound is higher than the bound of $\dim(H)^3$ obtained for Hopf algebras by Etingof and Gelaki, which can perhaps be tracked to the fact that we cannot use the dual Hopf algebra H^*. For the special but important class of group-theoretical quasi-Hopf algebras introduced by Ostrik [29], on the other hand, we derive a bound of $\dim(H)^2$. This bound, which is even better than the best previously known bound for general semisimple *ordinary* Hopf algebras, was recently obtained by Natale [24]. Using the general theory of the Frobenius–Schur exponent, we can reduce the problem to the case where the quasi-Hopf algebra is just a dual group algebra with a quasi-bialgebra structure induced by a three-cocycle. In this case, we can compute the Frobenius–Schur exponent directly by considering the indicators of individual representations, or really (after a further reduction using the invariance properties) of just one example.

Theorem 9.1. *Let H be a semisimple complex quasi-Hopf algebra. Then the Frobenius–Schur exponent of H divides* $\dim(H)^4$.

Proof. For $X \in H\text{-}\mathbf{mod}_{\mathrm{fin}}$ and $V \in D(H)\text{-}\mathbf{mod}_{\mathrm{fin}} = Z(H\text{-}\mathbf{mod}_{\mathrm{fin}})$ we write

$$\hat{e}_V(X) = \tau_{XV} e_V(X) : V \otimes X \to V \otimes X,$$

where τ is the ordinary vector space flip. From the hexagon equation

$$e_V(X \otimes Y) = \Phi^{-1}_{XYV}\big(X \otimes e_V(Y)\big)\Phi_{XVY}\big(e_V(X) \otimes Y\big)\Phi^{-1}_{VXY}$$

we deduce

$$\det \hat{e}_V(X \otimes Y) = \det\big(\Phi^{-1}_{XYV}\big)\det\big(\hat{e}_V(Y)\big)^{\dim X}\det(\Phi_{XVY})\det\big(\hat{e}_V(X)\big)^{\dim(Y)}\det\big(\Phi^{-1}_{VXY}\big).$$

Specializing $Y = H$ and using $X \otimes H \cong H^{\dim X}$, we find

$$\begin{aligned}\det\big(\hat{e}_V(H)\big)^{\dim X} &= \det\big(\hat{e}_V(X \otimes H)\big)\\ &= \det\big(\Phi^{-1}_{XHV}\big)\det\big(\hat{e}_V(H)\big)^{\dim X}\det(\Phi_{XVH})\det\big(\hat{e}_V(X)\big)^{\dim H}\det\big(\Phi^{-1}_{VXH}\big)\end{aligned}$$

and thus

$$\det\big(\hat{e}_V(X)\big)^{\dim H} = \det(\Phi_{XHV})\det(\Phi_{VXH})\det\big(\Phi^{-1}_{XVH}\big).$$

For $V, W \in D(H)\text{-}\mathbf{mod}_{\mathrm{fin}}$ we then find

$$\begin{aligned}
&\det\bigl(e_V(W)e_W(V)\bigr)^{\dim H}\\
&\quad=\det\bigl(\hat{e}_V(W)\bigr)^{\dim H}\det\bigl(\hat{e}_W(V)\bigr)^{\dim H}\\
&\quad=\det(\Phi_{WHV})\det(\Phi_{VWH})\det\bigl(\Phi_{WVH}^{-1}\bigr)\det(\Phi_{VHW})\det(\Phi_{WVH})\det\bigl(\Phi_{VWH}^{-1}\bigr)\\
&\quad=\det(\Phi_{WHV})\det(\Phi_{VHW}).
\end{aligned}$$

The other hexagon equation

$$e_{V\otimes W}(X)=\Phi_{XVW}\bigl(e_V(X)\otimes W\bigr)\Phi_{VXW}^{-1}\bigl(V\otimes e_W(X)\bigr)\Phi_{VWX}$$

gives

$$\det\bigl(\hat{e}_{V\otimes W}(X)\bigr)=\det\bigl(\hat{e}_V(X)\bigr)^{\dim W}\det\bigl(\hat{e}_W(X)\bigr)^{\dim V}\det(\Phi_{VWX})\det\bigl(\Phi_{VXW}^{-1}\bigr)\det(\Phi_{XVW}).$$

For $W=D(H)$, using $V\otimes D(H)\cong D(H)^{\dim V}$ as $D(H)$-modules, and abbreviating $D:=D(H)$, we find

$$\det\bigl(\hat{e}_D(X)\bigr)^{\dim V}=\det\bigl(\hat{e}_V(X)\bigr)^{\dim D}\det\bigl(\hat{e}_D(X)\bigr)^{\dim V}\det(\Phi_{VDX})\det\bigl(\Phi_{VXD}^{-1}\bigr)\det(\Phi_{XVD})$$

and hence

$$\begin{aligned}
\det\bigl(\hat{e}_V(X)\bigr)^{-\dim D}&=\det(\Phi_{VDX})\det\bigl(\Phi_{VXD}^{-1}\bigr)\det(\Phi_{XVD})\\
&=\det(\Phi_{VHX})^{\dim H}\det\bigl(\Phi_{VXH}^{-1}\bigr)^{\dim H}\det(\Phi_{XVH})^{\dim H}
\end{aligned}$$

because the associativity isomorphisms depend only on the H-module structures of the objects involved, and $D(H)\cong H^{\dim H}$ as H-module. Comparing with the previous calculation, which gives

$$\begin{aligned}
\det\bigl(\hat{e}_V(X)\bigr)^{\dim D}&=\bigl(\det\bigl(\hat{e}_V(X)\bigr)^{\dim H}\bigr)^{\dim H}\\
&=\det(\Phi_{XHV})^{\dim H}\det(\Phi_{VXH})^{\dim H}\det\bigl(\Phi_{XVH}^{-1}\bigr)^{\dim H}
\end{aligned}$$

we find

$$\det(\Phi_{VHX})^{\dim H}\det(\Phi_{XHV})^{\dim H}=1,$$

and in particular

$$\det\bigl(e_V(D)e_D(V)\bigr)^{\dim H}=\det(\Phi_{DHV})\det(\Phi_{VHD})=\det(\Phi_{HHV})^{\dim H}\det(\Phi_{VHH})^{\dim H}=1.$$

We continue as in [5] and [7]: Since $\theta_{V\otimes D(H)}$ and $\theta_V\otimes\theta_{D(H)}$ agree up to a factor $e_D(V)e_V(D)$, we have

$$\begin{aligned}
\det(\theta_{D(H)})^{\dim V\dim H}&=\det(\theta_{V\otimes D(H)})^{\dim H}=\det(\theta_V\otimes\theta_{D(H)})^{\dim H}\\
&=\det(\theta_V)^{\dim D(H)\dim H}\det(\theta_{D(H)})^{\dim V\dim H}
\end{aligned}$$

and thus $\det(\theta_V)^{\dim D(H)\dim H} = 1$. If V is simple, and thus θ_V is a scalar, we conclude

$$\theta_V^{\dim V \dim H^3} = 1.$$

Since $\dim V$ divides $\dim H$, we are done. □

Now we turn to the announced bound on the (Frobenius–Schur) exponent of a group-theoretical quasi-Hopf algebra.

Let G be a finite group and $\omega : G^3 \to \mathbb{C}^\times$ a three-cocycle. Let $H \subset G$ a subgroup, and ψ a 2-cochain of H with $d\psi = \omega|_H$. The group-theoretical category $\mathcal{C}(G, H, \omega, \psi)$ defined by Ostrik [29] is the category of $\mathbb{C}_\psi[H]$-bimodules in the category $\mathcal{C}(G, \omega)$, which in turn is the category of G-graded vector spaces, made into a monoidal category with the usual tensor product but nontrivial associativity constraint Φ given by ω. More precisely $\Phi_{UVW} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ is given by $\Phi(u \otimes v \otimes w) = \omega(|u|, |v|, |w|) u \otimes v \otimes w$ if u, v, w are homogeneous (and $|x|$ denotes the degree of a homogeneous element x). It makes sense to consider $\mathbb{C}_\psi[H]$-bimodules in $\mathcal{C}(G, \omega)$ since the twisted group algebra $\mathbb{C}_\psi[H]$ is an associative algebra in the monoidal category $\mathcal{C}(G, \omega)$ (thanks to the condition $d\psi = \omega|_H$). By [30], the center of $\mathcal{C}(G, H, \omega, \psi)$ is isomorphic, as a braided monoidal category, to the center of $\mathcal{C}(G, \omega)$. The quasi-Hopf algebras over $\mathbb{C}$, whose representation categories are monoidally equivalent to group-theoretical categories, are called group-theoretical quasi-Hopf algebras. The center of the category $\mathcal{C}(G, \omega)$ is isomorphic to the category of modules over the twisted Drinfeld double $D^\omega(G)$ of Dijkgraaf, Pasquier, and Roche [4]. Thus we see that for any group-theoretical quasi-Hopf algebra K, there exist a group G and a 3-cocycle ω on G such that the module categories over $D(K)$ and $D^\omega(G)$ are equivalent braided monoidal categories, that is, $D(K)$ and $D^\omega(G)$ are gauge equivalent quasi-triangular quasi-Hopf algebras. This argument can be found in [25], where Natale also shows that group-theoretical quasi-Hopf algebras can in fact be characterized as those whose double is gauge equivalent to a twisted double of a finite group.

The Frobenius–Schur indicators of the objects of $\mathcal{C}(G, \omega)$ were already computed in [27]. In the terminology of Frobenius–Schur indicators, [27, Proposition 7.1] says that the Frobenius–Schur exponent of a simple object V_x associated with the element $x \in G$ is equal to $\operatorname{ord}(x) \operatorname{ord}(\operatorname{res}_{\langle x\rangle}[\omega])$. Hence $\operatorname{FSexp}(\mathcal{C}(G, \omega))$ is the least common multiple of the numbers $\operatorname{ord}(x) \operatorname{ord}(\operatorname{res}_{\langle x\rangle}[\omega])$ for all $x \in G$ (see also below). But by Corollary 7.8 we know that the Frobenius–Schur exponent is invariant under the center construction. Hence, we have proved the following result:

Theorem 9.2. *The Frobenius–Schur exponent of the group-theoretical category $\mathcal{C}(G, H, \omega, \psi)$ is the least common multiple of the numbers $|C| \cdot \operatorname{ord}(\operatorname{res}_C[\omega])$ where C runs through the (maximal) cyclic subgroups of G. In particular,* $\operatorname{FSexp}(\mathcal{C}(G, H, \omega, \psi))$ *divides* $\exp(G)^2$ *and* $\exp(G) \operatorname{ord}([\omega])$. *Hence, for each group-theoretical quasi-Hopf algebra K, we have* $\exp(K) \mid \operatorname{FSexp}(K) \mid \dim(K)^2$.

The result on the Frobenius–Schur exponent of $\mathcal{C}(G, \omega)$ cited above relies on the fact that $\mathcal{C}(G, \omega)$ can be described as the category of modules over a certain quasi-Hopf algebra $H(G, \omega)$. The proof of [27, Proposition 7.1] specializes general formulas for indicators over quasi-Hopf algebras to this case. It may be interesting to see a proof that computes the indicators from scratch using their first definition in [26] and the description of $\mathcal{C}(G, \omega)$ above. We will do this in the rest of the section.

A complete set of representatives for the isomorphism classes of simple objects in $\mathcal{C}(G,\omega)$ is given by $V_g = \mathbb{C}$ as a vector space, made into a homogeneous graded vector space of degree g, for each $g \in G$. We will treat the canonical isomorphisms $V_g \otimes V_h \cong V_{gh}$ as identities. As a consequence, every morphism between iterated tensor products of simples is given as multiplication with a scalar, and we will sometimes identify the morphism and the scalar below. It remains to make a suitable choice of dual objects. We will take $(V_g)^\vee := (V_{g^{-1}}, \mathrm{ev}_g, \mathrm{db}_g)$ with $\mathrm{db}_g = 1 : \mathbb{C} \to V_g \otimes V_{g^{-1}}$, which forces us to choose

$$\mathrm{ev}_g = \omega\big(g, g^{-1}, g\big)^{-1} = \omega\big(g^{-1}, g, g^{-1}\big),$$

since we have to ensure that

$$V_g \xrightarrow{\mathrm{db}_g \otimes V_g} (V_g \otimes V_{g^{-1}}) \otimes V_g \xrightarrow{\Phi} V_g \otimes (V_{g^{-1}} \otimes V_g) \xrightarrow{V_g \otimes \mathrm{ev}_g} V_g$$

is the identity, and $\Phi_{V_g, V_{g^{-1}}, V_g} = \omega(g, g^{-1}, g)$.

We proceed to determine the pivotal structure of $\mathcal{C}$. The component $j_g : V_g \to (V_g)^{\vee\vee} = V_g$ is determined by the requirement that the composition

$$\mathbb{C} \xrightarrow{\mathrm{db}_g} V_g \otimes V_{g^{-1}} \xrightarrow{j_g \otimes V_{g^{-1}}} V_g \otimes V_{g^{-1}} \xrightarrow{\mathrm{ev}_{g^{-1}}} \mathbb{C}$$

be the identity. Thus $j_g = \omega(g^{-1}, g, g^{-1})$, since $\mathrm{ev}_{g^{-1}} = \omega(g, g^{-1}, g)$.

Finally, let us investigate the explicit form of the isomorphism

$$D : \mathcal{C}(I, V \otimes W) \to \mathcal{C}\big(I, W \otimes V^{\vee\vee}\big)$$

introduced in [26, Definition 3.3]. We have

$$D(f) = \big(\mathbb{C} \xrightarrow{\mathrm{db}_{V^\vee}} V^\vee \otimes V^{\vee\vee} \xrightarrow{V^\vee \otimes f \otimes V^{\vee\vee}} \big(V^\vee \otimes (V \otimes W)\big) \otimes V^{\vee\vee} \xrightarrow{\Theta \otimes V^{\vee\vee}} W \otimes V^{\vee\vee}\big),$$

where $\Theta = (V^\vee \otimes (V \otimes W) \xrightarrow{\Phi^{-1}} (V^\vee \otimes V) \otimes W \xrightarrow{\mathrm{ev}_V \otimes W} W)$. Note that if $W = V^\vee$, then

$$\big(V^\vee \xrightarrow{V^\vee \otimes \mathrm{db}_V} V^\vee \otimes \big(V \otimes V^\vee\big) \xrightarrow{\Theta} V^\vee\big) = \mathrm{id}_{V^\vee}.$$

Now if $V = V_g$ and $W = V_{g^{-1}}$, then all the morphisms involved can be identified with scalars. We have $\mathrm{db}_{V_g} = 1$, and so $\Theta = 1$, and since also $\mathrm{db}_{V_{g^{-1}}} = 1$, we finally have $D(f) = f$. In particular, we see that the isomorphism

$$E_{V_g, V_{g^{-1}}} : \mathcal{C}(\mathbb{C}, V_g \otimes V_{g^{-1}}) \to \mathcal{C}(\mathbb{C}, V_{g^{-1}} \otimes V_g)$$

is given, under the identification of both sides with $\mathcal{C}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$, by the scalar $j_g^{-1} = \omega(g, g^{-1}, g)$.

Let us now calculate the higher indicators of V_g in the special case $G = \mathbb{Z}/N\mathbb{Z}$ and $g = \bar{1}$. To describe, as in [22], the cocycles on G, we define $\hat{n} \in \{0, \ldots, N-1\}$ for $n \in \mathbb{Z}$ by $\hat{n} \equiv n$ modulo N. Then the class of the cocycle ω_1 defined by

$$\omega_1(\bar{\ell}, \bar{m}, \bar{n}) = \exp\left(\frac{2\pi i}{N^2} \hat{\ell}\big(\hat{m} + \hat{n} - \widehat{m+n}\big)\right)$$

generates the group $H^3(\mathbb{Z}/N\mathbb{Z},\mathbb{C}^\times)\cong C_N$. In particular, every cocycle on $\mathbb{Z}/N\mathbb{Z}$ has the form $\omega_t=\omega_1^t$ for some $0\leqslant t<N$, with

$$\omega_t(\bar\ell,\bar m,\bar n)=\exp\left(\frac{2\pi i t}{N^2}\hat\ell(\hat m+\hat n-\widehat{m+n})\right).$$

We will determine the indicators of $V_{\bar 1}\in\mathcal{C}(\mathbb{Z}/N\mathbb{Z},\omega_t)$. We start by noting that

$$\omega_t(\bar 1,\bar n,\bar 1)=\exp\left(\frac{2\pi i t}{N^2}(\hat n+1-\widehat{n+1})\right)=\begin{cases}\exp(2\pi i t/N)=\zeta^t & \text{if } \bar n=\overline{N-1},\\ 1 & \text{otherwise,}\end{cases}$$

where we have put $\zeta=\exp(2\pi i/N)$. In particular, $E_{V_{\bar 1},V_{\overline{N-1}}}$ is multiplication with ζ^t. We can also compute the scalar $\Phi_n: V_{\bar 1}^{\otimes(n-1)}\otimes V_{\bar 1}\to V_{\bar 1}^{\otimes n}$. We have the recursion formula

$$\Phi_n=(V_{\bar 1}\otimes\Phi_{n-1})\Phi_{V_{\bar 1},V_{\bar 1}^{\otimes(n-2)},V_{\bar 1}}=\Phi_{n-1}\omega_t(\bar 1,\overline{n-2},\bar 1),$$

and hence $\Phi_{\ell N}=\zeta^{t(\ell-1)}$. Finally, by definition of $E^{(n)}_{V_{\bar 1}}$, which is again a scalar or zero, we know that $\nu_n(V_{\bar 1})=0$ if $N\nmid n$, while $\nu_{\ell N}(V_{\bar 1})=\zeta^t\phi_{N\ell}=\zeta^{t\ell}$.

As a consequence of these calculations, we find:

Lemma 9.3. *Let G be a finite group, and $\omega: G^3\to\mathbb{C}^\times$ a three-cocycle representing $[\omega]\in H^3(G,\mathbb{C}^\times)$.*

(i) *If G is a cyclic group of order N, then* $\mathrm{FSexp}(\mathcal{C}(G,\omega))=N\cdot\mathrm{ord}([\omega])$. *In particular we have $N\mid\mathrm{FSexp}(\mathcal{C}(G,\omega))\mid N^2$, and for each $n\mid N$ there is ω with* $\mathrm{FSexp}(\mathcal{C}(G,\omega))=Nn$.
(ii) $\mathrm{FSexp}(\mathcal{C}(G,H,\omega,\psi))$ *is the least common multiple of the numbers* $\mathrm{ord}(C)\,\mathrm{ord}(\mathrm{res}_C[\omega])$ *where C runs through the* (*maximal*) *cyclic subgroups of G.*

Proof. The calculations preceding the statement of the lemma have shown that $\mathrm{FSexp}(V_{\bar 1})=N\,\mathrm{ord}(\zeta^t)=N\,\mathrm{ord}([\omega])$ if $G=\mathbb{Z}/N\mathbb{Z}$ and $\omega=\omega_t$. But since the left-hand side also only depends on the cohomology class of ω, the formula holds for all cocycles. Thus, more generally, $\mathrm{FSexp}(V_g)=N\,\mathrm{ord}([\omega])$ whenever g generates the cyclic group G, because the group isomorphism $f:G\to\mathbb{Z}/N\mathbb{Z}$ mapping g to $\bar 1$ induces a monoidal category equivalence $\mathcal{C}(G,\omega)\to\mathcal{C}(\mathbb{Z}/N\mathbb{Z},\omega')$ mapping V_g to $V_{\bar 1}$, where ω' corresponds to ω under the isomorphism $H^3(\mathbb{Z}/N\mathbb{Z},\mathbb{C}^\times)\to H^3(G,\mathbb{C}^\times)$ induced by f. For general $g\in G$, we may compute the Frobenius–Schur indicators and exponent of $V_g\in\mathcal{C}(G,\omega)$ by restricting ourselves to the full monoidal subcategory $\mathcal{C}(\langle g\rangle,\omega|_{\langle g\rangle^3})$ containing V_g. Thus

$$\mathrm{FSexp}(V_g)\mid\mathrm{ord}(g)\,\mathrm{ord}\big([\omega_{\langle g\rangle^3}]\big)\mid N\,\mathrm{ord}\big([\omega]\big).$$

Summing up, the Frobenius–Schur exponents of all the simples V_g divide $N\,\mathrm{ord}([\omega])$, and equality occurs for any g generating G. In particular $\mathrm{FSexp}(\mathcal{C}(G,\omega))=N\,\mathrm{ord}([\omega])$, and of course there is a cohomology class of order n for any divisor of N since $H^3(G,\mathbb{C}^\times)\cong C_N$.

For a general finite group G and any simple $V_g\in\mathcal{C}(G,\omega)$, we can compute its Frobenius–Schur exponent inside the category $\mathcal{C}(\langle g\rangle,\omega|_{\langle g\rangle^3})$, and in particular inside one of the categories

$\mathcal{C}(C, \omega|_{C^3})$ for C a cyclic subgroup of G, which we may choose to be a maximal cyclic subgroup. The Frobenius–Schur exponent is thus the least common multiple of the Frobenius–Schur exponents of these categories, which we have already computed. □

Acknowledgment

The authors thank Susan Montgomery for her comments on the paper.

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