On the numerical index of Banach spaces

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Abstract

We give a partial answer to the problem of computing the numerical index of \(l_p\)-space \(1 < p < \infty\). Also, we give an estimate of the numerical index of the two-dimensional real space \(l^2_p, 1 < p < \infty\). For the \(l_p\)-space, we show that its numerical index is greater than the \(l_p\)-space one.

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1. Introduction

Given a Banach space \(X\) over \(\mathbb{R}\) or \(\mathbb{C}\), we write \(B_X\) for the closed unit ball and \(S_X\) for the unit sphere of \(X\). The dual space is denoted by \(X^*\) and \(B(X)\) is the Banach algebra of all bounded linear operators on \(X\). The numerical range of an operator \(T \in B(X)\) is the subset \(V(T)\) of the scalar field defined by

\[ V(T) = \{ x^*(Tx) : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1 \}. \]

The numerical radius is then given by

\[ v(T) = \sup \{ |\lambda| : \lambda \in V(T) \}. \]
Clearly, $v$ is a semi-norm on $B(X)$, and $v(T) \leq \|T\|$ for every $T \in B(X)$. It was shown by Glickfeld [6] (and essentially by Bohnenblust and Karlin [3]) that if $X$ is a complex space, then $e^{-1}\|T\| \leq v(T)$ for every $T \in B(X)$ where $e = \exp 1$, so that for complex spaces $v$ is always a norm and it is equivalent to the operator norm $\|\|$.

Thus it is natural to consider the so-called numerical index of the Banach space $X$, namely the constant $n(X)$, defined by

$$n(X) = \inf\{v(T) : T \in S_{B(X)}\}.$$ 

Obviously, $n(X)$ is the greatest constant $k \geq 0$ such that $k\|T\| \leq v(T)$ for every $T \in B(X)$. Note that for any complex Banach space $X$, $e^{-1} \leq n(X) \leq 1$.

The concept of the numerical index was first suggested by Lumer [8] in a lecture to the North British Functional Analysis Seminar in 1968. At that time, it was known that if $X$ is a complex Hilbert space (with dim $X > 1$) then $n(X) = \frac{1}{2}$ and if it is real then $n(X) = 0$ so that for real spaces, $0 \leq n(X) \leq 1$.

Later, Duncan et al. [4] determined the range of values of the numerical index. More precisely they proved that

$$\{n(X) : X \text{ real Banach space}\} = [0, 1].$$

$$\{n(X) : X \text{ complex Banach space}\} = [e^{-1}, 1].$$

Recently, Finet et al. [5] studied the values of the numerical index from the isomorphic point of view and they proved that

$$\mathcal{N}(c_0) = \mathcal{N}(l_1) = \mathcal{N}(l_{\infty}) = \begin{cases} [0, 1] & \text{in the real case,} \\ [e^{-1}, 1] & \text{in the complex case,} \end{cases}$$

where $\mathcal{N}(X) := \{n(X, p) : p \in \mathcal{E}(X)\}$, $\mathcal{E}(X)$ denotes the set of all equivalent norms on the Banach space $X$ and $n(X, p)$ is the numerical index of $X$ which is equipped with the norm $p$. For general information and background on numerical ranges we refer to the books by Bonsall and Duncan [1,2]. Further developments in the Hilbert space case can be found in [7].

The computation of $n(l_p)$ for $1 < p < \infty$, $p \neq 2$ is complicated, in fact it is an open problem for a long time ([4,9,10]).

In this paper we give a partial answer to this problem (Theorem 2.1). Actually, we prove that for $1 \leq p < \infty$, the numerical index of the Banach space $l_p$ is the limit of the sequence of numerical index of finite-dimensional subspaces $l_m^p$, $m = 1, 2, \ldots$. Also, we give an estimate of the numerical index of a two-dimensional real space $l_2^p$, $1 < p < \infty$ (Theorem 2.2). In fact, we prove that for $1 \leq p < \infty$ there exists an explicit constant $M_p$ such that

$$M_p \leq n(l_p^2) \leq M_p.$$ 

(Moreover, for $p = 1$ and $p = 2$, $M_p = n(l_p^2)$.)
2. Main results

**Theorem 2.1.** For $1 < p < \infty$ the numerical index of the Banach space $l_p$ is given by

$$n(l_p) = \lim_{m} n(l^m_p).$$

Here $l_p$ is real or complex.

**Proof.** We proceed by steps. Let $\{e_n\}_{n \geq 1}$ be the canonical basis of $l_p$ and $\{e^*_n\}_{n \geq 1}$ the biorthogonal functionals associated to $\{e_n\}_{n \geq 1}$.

**Step 1:** For every integer $m \geq 1$, the numerical index of $l^m_p$ is greater than the numerical index of $l_p$ and consequently $\lim \inf m n(l^m_p) \geq n(l_p)$.

First, we note that in general if $X$ is a Banach space and $Y$ is a subspace of $X$ there is no comparison between $n(X)$ and $n(Y)$.

**Proof of Step 1.** Let $m \geq 1$ and $T \in B(l^m_p)$. Consider the linear operator $\tilde{T}$ defined on $l_p$ by $\tilde{T}e_k = T e_k, k = 1, 2, \ldots, m$ and $\tilde{T}e_k = 0, k \geq m + 1$. Clearly $\|T\| = \|\tilde{T}\|$.

We also have $v(T) = v(\tilde{T})$. Indeed, if $x = \sum_{i=1}^{\infty} x_i e_i \in S_{l_p}$ and $x^* = \sum_{i=1}^{\infty} x^*_i e^*_i \in S_{l_q}$ such that $x^* (x) = 1 = \sum_{i=1}^{\infty} |x^*_i||x_i|^{p-1}$ where $\varepsilon_i$ is a scalar number such that $\varepsilon_i x_i = |x_i|$. In the following for a given $x \in S_{l_p}$, the unique point $x^* \in S_{l_q}$ such that $x^* (x) = 1$ will be denoted by $x^*_x$. Taking this in consideration, we have

$$v(T) = \sup_{x \in S_{l_p}} |x^*_x (Tx)| \leq \sup_{z \in S_{l_p}} |z^*_z (Tz)| = v(\tilde{T}).$$

On the other hand, for a given $\varepsilon > 0$ there is $z = \sum_{i=1}^{\infty} z_i e_i \in S_{l_p}$ such that

$$v(\tilde{T}) - \varepsilon < \sum_{i=1}^{\infty} |z_i|^{p-1} \varepsilon^*_i e^*_i \left( \tilde{T} \sum_{i=1}^{\infty} z_i e_i \right) = \sum_{i=1}^{m} |z_i|^{p-1} \varepsilon^*_i e^*_i \left( T \sum_{i=1}^{m} z_i e_i \right).$$

Put $r^p := \sum_{i=1}^{m} |z_i|^p \leq 1$. Then we have

$$v(\tilde{T}) - \varepsilon < r^p \sum_{i=1}^{m} \frac{|z_i|^p}{r^p} \varepsilon^*_i e^*_i \left( T \sum_{i=1}^{m} \frac{z_i}{r} e_i \right) \leq v(T).$$

This implies that $v(\tilde{T}) \leq v(T)$ and therefore $v(T) = v(\tilde{T})$. It follows that

$$\{v(T) : \|T\| = 1, T \in B(l^m_p)\} \subset \{v(U) : \|U\| = 1, U \in B(l_p)\}$$

which yields $n(l_p) \leq n(l^m_p)$. Consequently $n(l_p) = \lim \inf m n(l^m_p)$. □

In the following we shall prove that $n(l_p) \geq \lim \sup m n(l^m_p)$. Let $T \in B(l_p)$ represented by the infinite matrix
We define the sequence of linear operators \( \{T_m\}_{m \geq 1} \) as follows. For each \( m \geq 1 \), \( T_m \) is defined on \( l_p^m \) by

\[
T_m = \begin{bmatrix}
    t_{11} & \cdots & t_{1m} & t_{1m+1} & \cdots \\
    \vdots & & \vdots & & \vdots \\
    t_{m1} & \cdots & t_{mm} & t_{mm+1} & \cdots \\
    t_{m+11} & \cdots & t_{m+1m} & t_{m+1m+1} & \cdots \\
    \vdots & & \vdots & & \vdots 
\end{bmatrix};
\]

\[
T e_j = \sum_{k=1}^\infty t_{kj} e_k, \quad j = 1, 2, \ldots
\]

We have seen above that \( \|T_m\| = \|\tilde{T}_m\| \) and \( v(T_m) = v(\tilde{T}_m) \) for \( m = 1, 2, \ldots \) (note that \( T_m \) is not the restriction of \( T \) on the Banach space \( l_p^m \)). According to these notations we have

**Step 2.**

1. \( \tilde{T}_m \) converges strongly to \( T \) and \( \|\tilde{T}_m\| \leq \|T\| \) for all \( m \geq 1 \).
2. \( v(\tilde{T}_m) \) converges to \( v(T) \) and \( v(\tilde{T}_m) \leq v(T) \) for all \( m \geq 1 \).

**Proof of Step 2.**

(i) Let \( m \geq 1 \). We have \( \|\tilde{T}_m\| = \|T_m\| = \|T_m x_0\| \) for some \( x_0 = x_1 e_1 + \cdots + x_m e_m \), \( \|x_0\| = 1 \). Since \( T x_0 = \sum_{i=1}^\infty \left( \sum_{j=1}^m t_{ij} x_j \right) e_i \), then

\[
\|T x_0\|^p = \sum_{i=1}^\infty \left( \sum_{j=1}^m t_{ij} x_j \right)^p \geq \sum_{i=1}^m \sum_{j=1}^m t_{ij}^p \|x_j\|^p = \|T_m x_0\|^p,
\]

which yields \( \|T\| \geq \|\tilde{T}_m\| \).

Now, for any integer \( j \geq 1 \) and \( m \geq j \) we have \( (T - \tilde{T}_m) e_j = \sum_{k=m+1}^\infty t_{kj} e_k \) and \( \|(T - \tilde{T}_m) e_j\|^p = \sum_{k=m+1}^\infty |t_{kj}|^p \) converges to 0 as \( m \) tends to infinity. It follows that for a finite combination \( \sum_{j=1}^r x_j e_j \), \( \|(T - \tilde{T}_m) (\sum_{j=1}^r x_j e_j)\| \to 0 \). Now let \( x = \sum_{k=1}^\infty x_k e_k \in l_p \) and \( \varepsilon > 0 \). Choose \( r \) sufficiently large so that \( \|\sum_{k=r+1}^\infty x_k e_k\| < \varepsilon \). Then
\[
\| (T - \tilde{T}_m) x \| \leq \| (T - \tilde{T}_m) \sum_{k=1}^{r} x_k e_k \| + 2\| T \| \varepsilon.
\]

This asserts obviously that \( \| (T - \tilde{T}_m) x \| \) converges to 0 as \( m \) tends to infinity. (ii) As in (i) we first have \( v(\tilde{T}_m) \leq v(T) \) for all \( m \geq 1 \). Simply because \( v(\tilde{T}_m) = v(T_m) \) and we have \( |x^*_k(T_m x)| = |x^*_k(T x)| \) for all \( x \in S_{lp} \). Now we shall prove that \( v(\tilde{T}_m) \to v(T) \).

Given \( \varepsilon > 0 \), there exists \( x_0 = \sum_{k=1}^{\infty} e_k^*(x_0) e_k \) in \( S_{lp} \) such that

\[
| x^*_n(T x_0) | > v(T) - \varepsilon.
\]

For each \( n \geq 1 \), consider

\[
x^n_0 = \sum_{k=1}^{n-1} e_k^*(x_0) e_k + \lambda_n e_n; \quad x^{*n}_0 = \sum_{k=1}^{n-1} |e_k^*(x_0)|^{p-1} e_k e_k^* + \lambda_n^{p-1} e_n^*,
\]

where \( \lambda_n = \sum_{k=n}^{\infty} |e_k^*(x_0)|^p \).

Then

\[
x^{*n}_0(x^n_0) = \sum_{k=1}^{n-1} |e_k^*(x_0)|^p + \lambda_n^p = 1 = \| x^{*n}_0 \| = \| x^n_0 \|.
\]

Moreover, \( \| x_0 - x^n_0 \| \to 0 \) and \( \| x^*_n - x^{*n}_0 \| \to 0 \). It follows that \( x^*_n(T x^n_0) \to x^*_0(T x_0) \) as \( n \) tends to infinity. Let \( n_0 \geq 1 \) such that

\[
| x^*_n(T x^n_0) | > v(T) - \varepsilon \quad (n \geq n_0).
\]

Since \( \tilde{T}_m \) converges strongly to \( T \), thus for fixed \( n \geq n_0 \), \( x^*_n(\tilde{T}_m x^n_0) \) converges to \( x^*_n(T x_0) \) as \( m \) tends to infinity. Simply because \( |x^*_n(\tilde{T}_m x^n_0) - x^*_n(T x^n_0)| \leq \| (\tilde{T}_m - T) x^n_0 \| \to 0 \) as \( m \to \infty \). So there is \( m_0 \geq n \) such that

\[
| x^*_n(\tilde{T}_m x^n_0) | > v(T) - \varepsilon \quad (m \geq m_0).
\]

This yields \( v(\tilde{T}_m) > v(T) - \varepsilon \) for all \( m \geq m_0 \) and therefore \( v(\tilde{T}_m) \) converges to \( v(T) \) as \( m \) tends to infinity. \( \square \)

The following step achieves the proof of Theorem 2.1.

Step 3. (i) For a given \( T \in B(l_p) \) we have \( \| T \| = \lim_m \| \tilde{T}_m \| \). (ii) \( n(l_p) \geq \lim sup_m n(l_p^m) \).

The proof of (i) is easy and follows from the fact that \( \tilde{T}_m \) converges strongly to \( T \), and \( \| \tilde{T}_m \| \leq \| T \| \) for all \( m \geq 1 \). For (ii), let \( \varepsilon > 0 \). There exists \( T \in B(l_p) \) with \( \| T \| = 1 \) such that

\[
n(l_p) + \varepsilon > v(T).
\]

Following step 2, \( v(T) = \lim_m v(\tilde{T}_m) \), so we can find \( m_0 \) such that

\[
n(l_p) + \varepsilon > v(\tilde{T}_m)
\]
for all \( m \geq m_0 \). But \( v(\tilde{T}_m) = v(T_m) \geq \|T_m\| n(l_m^* \rho) \) thus \( n(l_p) + \epsilon > \|T_m\| n(l_m^* \rho) \) for all \( m \geq m_0 \). Since \( \|T\| = \lim_{m} \|T_m\| \), then there is \( k_0 > m_0 \) such that \( \|T_m\| \geq 1 - \epsilon \) for all \( m \geq k_0 \). We obtain
\[
n(l_p) + \epsilon > (1 - \epsilon)n(l_m^* \rho) \quad (m \geq k_0).
\]

This asserts that \( n(l_p) \geq \limsup_{m} n(l_m^* \rho) \).

\[\Box\]

**Theorem 2.2.** For \( 1 \leq p \leq 2 \), we have \( \frac{M_p}{2} \leq n(l_2^* \rho) \leq M_p \) where \( M_p = \sup_{t \in [0, 1]} \frac{t^{p-1} - 1}{1 + t^p} \). Here \( l_2^* \rho \) is real.

**Proof.** Let \( T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in B(l_2^* \rho) \). According to Lemma 3.2 and Proposition 1.1 [4] we have
\[
v_p(T) = \sup_{|z|=1} \left| \frac{a + zbt + zct^{p-1} + dt^p}{1 + t^p} \right|.
\]

and
\[
\|T\|_p = \sup \left\| \begin{pmatrix} a + zbt & c + zdt \\ (1 + t^p)^{\frac{1}{p}} & (1 + t^p)^{\frac{1}{p}} \end{pmatrix} \right\|_p
\leq \sup \left\| \begin{pmatrix} |a + zbt| & |c + zdt| \\ (1 + t^p)^{\frac{1}{p}} & (1 + t^p)^{\frac{1}{p}} \end{pmatrix} \right\|
\leq |a| + |b| + |c| + |d|.
\]

The sup is taken over \( \{ t \geq 0, |z| = 1 \} \).

Now let \( S_1 = \{(a, b, c, d) \in \mathbb{R}^4 : |a| + |b| + |c| + |d| = 1 \} \). In the following we identify \( T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) with \( (a, b, c, d) \), then we have clearly
\[
n(l_2^* \rho) = \inf_{T \in S_1} \frac{v_p(T)}{\|T\|_p}.
\]

For \( T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) we have \( \|T\|_p = 1 \) and \( v_p(T) = \sup_{t > 0} \frac{|t^{p-1} - 1|}{1 + t^p} = \sup_{t \in [0, 1]} \frac{t^{p-1} - 1}{1 + t^p} = M_p \). Then \( n(l_2^* \rho) \leq M_p \). Let us prove now that \( n(l_2^* \rho) \geq \frac{M_p}{2} \). Fix \( 1 < p < 2 \), the map
\[
(a, b, c, d) \in S_1 \mapsto M(a, b, c, d, p) = \sup_{t > 0, |z|=1} \left| \frac{a + zbt + zct^{p-1} + dt^p}{1 + t^p} \right|
\]
is continuous on $S_1$ and we have

$$M(a,b,c,d,p) = \max \left\{ \sup_{t \in [0,1]} \frac{|a + zbt + zct^{p-1} + dt^{p}|}{1 + t^{p}} ; \sup_{t \in [0,1]} \frac{|at^{p} + zbt^{p-1} + zct + d|}{1 + t^{p}} \right\}$$

$$= \max \left\{ \sup_{t \in [0,1]} \frac{|a + zbt + zct^{p-1} + dt^{p}|}{1 + t^{p}} ; \sup_{t \in [0,1]} \frac{|at^{p} + zbt^{p-1} + zct + d|}{1 + t^{p}} \right\}$$

$$= \max \left\{ \sup_{t \in [0,1]} \frac{|a + zbt + zct^{p-1} + dt^{p}|}{1 + t^{p}} ; \sup_{t \in [0,1]} \frac{|at^{p} + zbt^{p-1} + zct + d|}{1 + t^{p}} \right\}$$

We shall use many times the simple inequality: $\max(A, B) \geq \frac{A+B}{2}$. First, using (1) and (2) we obtain the following two inequalities

$$M(a,b,c,d,p) \geq \frac{1}{2} \left( a + d + (b + c) \frac{t^{p-1} + t}{1 + t^{p}} \right)$$

and

$$M(a,b,c,d,p) \geq \frac{1}{2} \left( a - d \right) \frac{1 - t^{p}}{1 + t^{p}} - \left( b - c \right) \frac{t^{p-1} - t}{1 + t^{p}}.$$  

(5)

This time we use (1) and (4) to obtain the following inequalities

$$M(a,b,c,d,p) \geq \frac{1}{2} \left( a + d \right) - \left( b - c \right) \frac{t^{p-1} - t}{1 + t^{p}}.$$  

(6)

Also, using (2) and (3) we obtain the following two inequalities

$$M(a,b,c,d,p) \geq \frac{1}{2} \left( a + d \right) - \left( b - c \right) \frac{t^{p-1} - t}{1 + t^{p}}.$$  

(7)

and

$$M(a,b,c,d,p) \geq \frac{1}{2} \left( a - d \right) \frac{1 - t^{p}}{1 + t^{p}} + \left( b + c \right) \frac{t^{p-1} + t}{1 + t^{p}}.$$  

(8)

Finally, we use (3) and (4) to obtain the following inequalities

$$M(a,b,c,d,p) \geq \frac{1}{2} \left( a + d \right) - \left( b + c \right) \frac{t^{p-1} + t}{1 + t^{p}}.$$  

(9)

and

$$M(a,b,c,d,p) \geq \frac{1}{2} \left( a - d \right) \frac{1 - t^{p}}{1 + t^{p}} - \left( b - c \right) \frac{t^{p-1} - t}{1 + t^{p}}.$$  

(10)

and

$$M(a,b,c,d,p) \geq \frac{1}{2} \left( a - d \right) \frac{1 - t^{p}}{1 + t^{p}} + \left( b - c \right) \frac{t^{p-1} - t}{1 + t^{p}}.$$  

(11)

and

$$M(a,b,c,d,p) \geq \frac{1}{2} \left( a - d \right) \frac{1 - t^{p}}{1 + t^{p}} + \left( b - c \right) \frac{t^{p-1} - t}{1 + t^{p}}.$$  

(12)
We claim that
\[ M(a, b, c, d, p) \geq \frac{1}{2} \left[ |a| + |b| + |c| + |d| \right] \sup_{t \in [0,1]} \frac{t^{p-1} - t}{1 + t^p}. \]

Indeed, from inequalities (5) and (11) we have
\[ M(a, b, c, d, p) \geq \frac{1}{2} \left[ |a + d| + |b + c| \frac{t^{p-1} + t}{1 + t^p} \right]. \]

Also, from inequalities (7) and (9) we have
\[ M(a, b, c, d, p) \geq \frac{1}{2} \left[ |a + d| + |b - c| \frac{t^{p-1} - t}{1 + t^p} \right]. \]

Clearly, these two last inequalities yield
\[ M(a, b, c, d, p) \geq \frac{1}{2} \left[ |a + d| + |b - c| \sup_{t \in [0,1]} \frac{t^{p-1} - t}{1 + t^p} \right]. \] (\(*\))

Similarly, from inequalities (6) and (12) we have
\[ M(a, b, c, d, p) \geq \frac{1}{2} \left[ |a - d| \frac{1 - t^p}{1 + t^p} + |b - c| \frac{t^{p-1} - t}{1 + t^p} \right]. \]

and from inequalities (8) and (10) we obtain
\[ M(a, b, c, d, p) \geq \frac{1}{2} \left[ |a - d| \frac{1 - t^p}{1 + t^p} + |b + c| \frac{t^{p-1} + t}{1 + t^p} \right]. \]

From those last two inequalities we obtain
\[ M(a, b, c, d, p) \geq \frac{1}{2} \left[ |a - d| \frac{1 - t^p}{1 + t^p} + (|b| + |c|) \frac{t^{p-1} - t}{1 + t^p} \right]. \] (\(^\dagger\))

Now if \( ad \geq 0 \), then (\(*\)) yields
\[ M(a, b, c, d, p) \geq \frac{1}{2} \left[ |a| + |d| + |b| + |c| \sup_{t \in [0,1]} \frac{t^{p-1} - t}{1 + t^p} \right]. \]

Otherwise, \((ad < 0)\). From (\(^\dagger\)) we obtain
\[ M(a, b, c, d, p) \geq \frac{1}{2} \left[ (|a| + |d|) \frac{1 - t^p}{1 + t^p} + (|b| + |c|) \frac{t^{p-1} - t}{1 + t^p} \right]. \]
for every \( t \in [0, 1] \). Since for each \( t \in [0, 1] \), \((1 - t^P) \geq (t^{P-1} - t)\) then

\[
(|a| + |d|)(1 - t^P) + (|b| + |c|)(t^{P-1} - t) \geq t^{P-1} - t
\]

for every \( t \in [0, 1] \). Thus in any case \( M(a, b, c, d, p) \geq \frac{1}{2} \sup_{t \in [0,1]} \frac{t^{p-1}-t}{1+t^p} \). This completes the proof of Theorem 2.2.

\[\Box\]

**Corollary 2.3.** For every \( p \in [1, \infty[, \)

\[
\frac{1}{2} \sup_{t \in [0,1]} \frac{|t^{p-1} - t|}{1+t^p} \leq n(l_p^2) \leq \sup_{t \in [0,1]} \frac{|t^{p-1} - t|}{1+t^p}.
\]

**Proof.** For \( p \in [2, \infty[, \) the general result \( n(X^*) \leq n(X) \) gives \( n(l_p^2) = n(l_p^q) \) where \( \frac{1}{p} + \frac{1}{q} = 1 \). Then \( q \in [1, 2] \) and the substitution \( t^{q-1} = u \) gives \( \sup_{t \in [0,1]} \frac{t^{q-1}-t}{1+t^{q-1}} = \sup_{u \in [0,1]} \frac{u^{-p}-u^{-1}}{1+u^{-p}} \). \[\Box\]

### 3. On the numerical index of \( L_p \)-space, \( 1 < p < \infty \)

We start this section by some notations and definitions. Let \((\Omega, \Sigma)\) be a measurable space and \(\mu\) a positive measure on \(\Omega\). We denote by \(\mathcal{P}\) the collection of all partitions \(\pi\) of \(\Omega\) into finitely many pairwise disjoint members of \(\Sigma\) with finite strictly positive measures. Ordering this collection by \(\pi_1 \leq \pi_2\) whenever each member of \(\pi_1\) is the union of members of \(\pi_2\). So \(\mathcal{P}\) is a directed set.

For each \(\pi = \{E_1, \ldots, E_m\} \in \mathcal{P}\), we associate the subspace \(V_\pi\) of \(L_p(\mu)\) defined by \(V_\pi = \{\sum_{i=1}^m a_i 1_{E_i} : a_i \in \mathbb{K}\}, \mathbb{K} = \mathbb{R} \) or \(\mathbb{C}\). By \(P_\pi\) we denote the projection of \(L_p(\mu)\) onto \(V_\mu\) defined by

\[
P_\pi f = \sum_{i=1}^m \left[ \frac{1}{\mu(E_i)} \int_{E_i} f(t) \, d\mu(t) \right] 1_{E_i}
\]

for all \(f \in L_p(\mu)\). And \(V\) denotes the union of all subspaces \(V_\pi\) of \(L_p(\mu)\). We recall that \(V\) is a dense subspace of \(L_p(\mu)\), that is, for each \(f \in L_p(\mu)\), the sequence \((P_\pi f)_\pi\) converges to \(f\) in \(L_p(\mu)\).

**Theorem 3.1.** For \(1 \leq p < \infty\), the numerical index of \(L_p(\mu)\) is greater than the \(l_p\) one:

\[n(L_p(\mu)) \geq n(l_p) .\]  \(1\)

**Remark 3.2**

(1) If \(\mu\) is the counting measure on \(\Omega = \{1, 2, \ldots, m\}\), then \(L_p(\mu) = l_p^m\) and we get \(n(l_p^m) \geq n(l_p)\).
(2) If $\mu$ is the counting measure on $\Omega = \mathbb{N}$, then $L^p(\mu) = l_p$ and the equality in (1) holds. So (1) is the best estimation of $n(L^p(\mu))$.

(3) For which spaces $L^p(\mu)$ does the equality in (1) holds? In particular does $n(L^p[0,1])$ equal to $n(l_p)$?

Before to prove Theorem 3.1, we first recall the following classical result.

**Lemma 3.3.** For $1 \leq p < \infty$ and for every partition $\pi = \{E_1, \ldots, E_m\} \in \mathcal{P}$, the subspace $V_\pi$ is isometrically isomorphic to $l^m_p$.

**Proof of Theorem 3.1.** Let $T$ be a bounded linear operator on the Banach space $L^p(\mu)$ with $\|T\| = 1$ and let $\varepsilon > 0$. Choose $x_0 \in S_{L^p(\mu)}$ such that

$$\|Tx_0\| > (1 - \varepsilon). \quad (1)$$

Since $V$ is dense in $L^p(\mu)$, then there exists $\pi_0 \in \mathcal{P}$ such that

$$\|x_0 - P_{\pi_0}x_0\| < \varepsilon. \quad (2)$$

So

$$\|T(P_{\pi_0}x_0)\| \geq \|T(x_0)\| - \|T(P_{\pi_0}x_0 - x_0)\| > 1 - 2\varepsilon.$$

We can also find $\pi \in \mathcal{P}$, $\pi_0 \leq \pi$ such that

$$\|P_{\pi}T(P_{\pi_0}x_0)\| > 1 - 2\varepsilon.
$$

Now define the operator $S$ on $V_\pi$ by

$$S(P_{\pi}x) = P_{\pi}T(P_{\pi}x) \quad (x \in L^p(\mu)).$$

Clearly, $\|S\| \leq 1$, and we have

$$\|S\| \geq \frac{\|S(P_{\pi_0}x_0)\|}{\|P_{\pi_0}x_0\|} > \frac{1 - 2\varepsilon}{\|P_{\pi_0}x_0\|} \geq 1 - 2\varepsilon.$$

It follows that

$$v(S) \geq n(V_\pi)\|S\| \geq n(V_\pi)(1 - 2\varepsilon). \quad (3)$$

Since $V_\pi$ is isometrically isomorphic to $l^m_p$ for some $m \geq 1$, we have $n(V_\pi) = n(l^m_p)$, then

$$v(S) \geq n(l^m_p)(1 - 2\varepsilon).$$

It follows from Step 1 in the proof of Theorem 2.1 that

$$v(S) \geq n(l_p)(1 - 2\varepsilon). \quad (4)$$
The operator $S$ is defined on the finite-dimensional space $V_\pi$, its numerical radius is attained on some point $x_1 \in S_{V_\pi}$, that is,

$$v(S) = \sup_{x \in S_{V_\pi}} |x_1^*(Sx)| = |x_1^*(P_\pi T x_1)| = |P_\pi x_1^* (T x_1)|.$$ 

Since

$$P_\pi x_1^* (x_1) = 1 = \|P_\pi x_1^* \| = \|x_1\|,$$

it follows from (4) that

$$v(T) \geq n(l_p)(1 - 2\varepsilon).$$

Letting $\varepsilon \downarrow 0$, we obtain $v(T) \geq n(l_p)$. This completes the proof of Theorem 3.1.

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References