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ORIGINAL ARTICLE

Unsteady rotational flows of an Oldroyd-B fluid due to tension on the boundary

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Abstract Unsteady Taylor–Couette flows of an Oldroyd-B fluid, which fills a straight circular cylinder of radius R, are studied. Flows are generated by the oscillating azimuthal tension which is given on the cylinder surface. As a novelty, authors used in this paper the governing equation related to the tension field. The closed forms of the shear stress and velocity fields corresponding to the flow problems are obtained by means of the integral transforms method. Expressions for the azimuthal tension and fluid velocity were written as sums between the "permanent component" (the steady-state component) and the transient component. By customizing values of parameters from the mathematical model were obtained the corresponding solutions of other types of fluids, namely, Maxwell fluids. By using numerical simulations and diagrams of the azimuthal stress, the fluid behavior has been analyzed. The necessary time to achieve the "steady-state" was, also, determined.

1. Introduction

The Oldroyd-B fluid model is very important among the fluids of rate type due to its special behavior. Also, this model contains the Newtonian fluid model and Maxwell fluid model as special cases. The Oldroyd-B fluid model [1,2] considers the memory effects and elastic effects exhibited by a large class of fluids, such as the biological and polymeric liquids. Guillope and Saut [3] and Fontelos and Friedman [4] established the stability, existence and uniqueness results for some shearing flows of such fluids. Exact solutions for some simple flows of Oldroyd-B fluids were presented by many authors, See, for example, Rajagopal and Bhatnagar [5], Hayat et al. [6,7]. Recently, various problems regarding flows of Oldroyd-B fluids through cylindrical domains have been studied. Singh and Varshney [8] have considered the unsteady laminar flow of an electrically conducting Oldroyd fluid through a circular cylinder boundary by permeable bed under the influence of an exponentially decreasing pressure gradient in porous medium. Burdujan [9] studied Taylor–Couette flows of the Oldroyd-B fluid with fractional derivatives within the annular region between two infinitely coaxial circular cylinders due to a time-dependent axial tension given on the surface of the inner cylinder. The unsteady unidirectional transient flow of Oldroyd-B fluid with fractional time derivatives, in an annular domain, produced by a constant pressure gradient and a translation with constant velocity of the inner cylinder was studied by Mathur and Khandelwal [10]. Liu et al. [11] studied some helical flows of an Oldroyd-B fluid with time-fractional

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derivatives, between two infinite concentric oscillating cylinders and within an infinite circular oscillating cylinder. Most existing solutions in the literature correspond to problems with boundary conditions on the velocity. There are several practical problems with the specified force on the boundary [12–14]. For example in [12], Renardy has studied the motion of a Maxwell fluid across a strip bounded by parallel plates and proved that, in order to formulate a well posed problem it is necessary to impose the boundary conditions on the stresses at the inflow boundary. In [13], Renardy explained how well posed boundary value problems can be formulated using boundary conditions on stresses. Waters and King [15] were among the first specialists who used the shear stress on the boundary to find exact solutions for motions of rate type fluids. Other interesting problems regarding flows of non-Newtonian fluids, in various geometry or boundary conditions, can be finding in the references [17–23]. Our goal is to investigate unsteady flows of Oldroyd-B fluids in an infinite circular cylinder. In the present paper the governing equation of the flow is related to the azimuthal tension and we considered the boundary conditions on the shear stress. The flow of the fluid is due to rotation of the cylinder around its axis, under the action of oscillating shear stress $fH(t)\sin(\omega t)$ or $fH(t)\cos(\omega t)$ given on the boundary. Finally, solutions of the Maxwell fluid flows are obtained as particular cases of our general results. Also the comparison between models is underlined by graphical illustrations.

2. Problem formulation

The constitutive equations for an Oldroyd-B fluid [1] are

$$\begin{aligned}\mathbf{T} = & \ -p\mathbf{I}+\mathbf{S}, \mathbf{S}+\lambda\left(\frac{d\mathbf{S}}{dt}-\mathbf{LS}-\mathbf{SL}^T\right)\\ = & \ \mu\left\{\mathbf{A}+\lambda_r\left(\frac{d\mathbf{A}}{dt}-\mathbf{LA}-\mathbf{AL}^T\right)\right\},\end{aligned} \tag{1}$$

where $\mathbf{T}$ is Cauchy stress tensor, $-p\mathbf{I}$ is indeterminate spherical stress, $\mathbf{S}$ is extra stress tensor, $\mathbf{L}$ is velocity gradient, μ is the dynamic viscosity, $\mathbf{A}=\mathbf{L}+\mathbf{L}^T$ is first the Rivilin–Erickson tensor, λ and λ_r $(0\leqslant\lambda_r<\lambda)$ are relaxation and retardation time. Assume an infinite circular cylinder of radius R with axis of rotation along z-axis. Cylinder is filled with an Oldroyd-B fluid which is at rest at time $t=0$. After time $t=0^+$ the cylinder applies an oscillating rotational shear stress $fH(t)\sin(\omega t)$ or $fH(t)\cos(\omega t)$ to the fluid, $f>0$ is constant and ω is the angular frequency of oscillations. We assume that, the fluid is incompressible and homogeneous. Furthermore we assume that velocity field and extra-stress tensor are of the form

$$\mathbf{V}=\mathbf{V}(r,t)=w(r,t)\mathbf{e}_\theta,\quad \mathbf{S}=\mathbf{S}(r,t), \tag{2}$$

where $\mathbf{e}_\theta$ is the unit vector in the θ direction of the cylindrical coordinate system. Since the fluid and the cylinder are at rest at time $t=0$, therefore,

$$w(r,0)=0,\ \mathbf{S}(r,0)=\mathbf{0}. \tag{3}$$

Introducing (2) in $(1)_2$ and by using (3) we get $S_{rr}=S_{rz}=S_{z\theta}=S_{zz}=0$, along with the following meaningful partial differential equation

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\tau(r,t)=\mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)w(r,t), \tag{4}$$

where $\tau(r,t)=S_{r\theta}(r,t)$ is one of the nonzero component of extra stress tensor. The balance of linear momentum in the absence of body forces reduces to [8]

$$\rho\frac{\partial w(r,t)}{\partial t}=\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\tau(r,t), \tag{5}$$

$\tau_{st}(r,t)$ ρ being the constant density of the fluid. By eliminating $w(r,t)$ between Eqs. (4) and (5) we get the following governing equation for the shear stress [24]

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial\tau(r,t)}{\partial t}=\nu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{4}{r^2}\right)\tau(r,t), \tag{6}$$

where $\nu=\frac{\mu}{\rho}$ is the kinematic viscosity of the fluid. The appropriate initial and boundary conditions are

$$\tau(r,t)|_{t=0}=\left.\frac{\partial\tau(r,t)}{\partial t}\right|_{t=0}=0, \tag{7}$$

$$\tau(R,t)=fH(t)\sin\omega t\ \ or\ \ \tau(R,t)=fH(t)\cos\omega t, \tag{8}$$

$H(t)$ being the Heaviside unit step function. Converting our problem (6)–(8) into the complex field ($\tau=\tau_c+i\tau_s$ with τ_c and τ_s being solutions for cosine, respectively sine boundary conditions), we have

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial\tau(r,t)}{\partial t}=\nu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{4}{r^2}\right)\tau(r,t) \tag{9}$$

$$\tau(r,t)|_{t=0}=\left.\frac{\partial\tau(r,t)}{\partial t}\right|_{t=0}=0 \tag{10}$$

$$\tau(R,t)=fH(t)e^{i\omega t},\quad f>0. \tag{11}$$

By introducing the following dimensionless quantities

$$t^*=\frac{t}{\lambda},\ r^*=\frac{r}{R},\ w^*=\frac{w}{U_0},\ \tau^*=\frac{\tau}{f},\ U_0=\frac{\lambda f}{\rho R},\beta^*=\frac{\lambda_r}{\lambda},\ \omega^*=\lambda\omega, \tag{12}$$

Eqs. (5), (9)–(11) becomes (dropping the star notation)

$$\frac{\partial w(r,t)}{\partial t}=\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\tau(r,t), \tag{13}$$

$$Re\left(1+\frac{\partial}{\partial t}\right)\frac{\partial\tau(r,t)}{\partial t}=\left(1+\beta\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{4}{r^2}\right)\tau(r,t), \tag{14}$$

$$\tau(r,t)|_{t=0}=\left.\frac{\partial\tau(r,t)}{\partial t}\right|_{t=0}=0,\ \ w(r,0)=0 \tag{15}$$

$$\tau(1,t)=H(t)e^{i\omega t}, \tag{16}$$

where $Re=\frac{R^2}{\lambda\nu}$ is the Reynolds number.

3. Solution of the problem

In order to determine the exact analytical solution, we shall use the Laplace and finite Hankel transforms [25]. By applying the temporal Laplace transform to Eqs. (14) and (16) and using the initial conditions (15) we get the following transformed forms,

$$Re(1+q)q\bar{\tau}(r,q)=(1+\beta q)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{4}{r^2}\right)\bar{\tau}(r,q), \tag{17}$$

$$\bar{\tau}(1,q)=\frac{1}{q-i\omega}, \tag{18}$$

where $\bar{\tau}(r,q)=L\{\tau(r,t)\}$ and q is the Laplace transform parameter. In order to find the solution of the problem (17) and (18), we use the finite Hankel transform of the order two (see (A.1) from Appendix A) along with relation (B.3) from Appendix B. We obtain the following expression for the Hankel transforms of function $\bar{\tau}(r,q)$

$$\bar{\tau}_H(r_n,q)=-r_nJ_1(r_n)\frac{1}{(q-i\omega)}\frac{\beta q+1}{Req^2+(Re+\beta r_n^2)q+r_n^2}. \tag{19}$$

In order to apply the inverse Laplace transform [26] we rewrite Eq. (19) into the suitable equivalent form

$$\bar{\tau}_H(r_n,q)=-r_nJ_1(r_n)\left\{\frac{\beta}{Re}\frac{1}{q-i\omega}\frac{(q+\frac{a_n}{2Re})}{(q+\frac{a_n}{2Re})^2-(\frac{b_n}{2Re})^2}+\frac{2Re-\beta a_n}{Reb_n}\frac{1}{q-i\omega}\frac{(\frac{b_n}{2Re})}{(q+\frac{a_n}{2Re})^2-(\frac{b_n}{2Re})^2}\right\}, \tag{20}$$

where $a_n=Re+\beta r_n^2$, $b_n^2=a_n^2-4Rer_n^2$. Now, applying the inverse Laplace transform [26], using the formulae (B.4) and (B.5) and the convolution theorem [25] we get

$$\tau_H(r_n,t)=-r_nJ_1(r_n)\left\{(A_{1n}+iB_{1n})\sinh\left(\frac{b_nt}{2Re}\right)e^{-\frac{a_nt}{2Re}}+(A_{2n}+iB_{2n})\cosh\left(\frac{b_nt}{2Re}\right)e^{-\frac{a_nt}{2Re}}-(A_{2n}+iB_{2n})e^{i\omega t}\right\}, \tag{21}$$

where

$$A_{1n}=\frac{(r_n^2-Re\omega^2)(\beta r_n^2-Re)+\omega^2a_n(a_n\beta-2Re)}{b_n\{(r_n^2-Re\omega^2)^2+(\omega a_n)^2\}}$$

$$B_{1n}=\frac{-\omega a_n(\beta r_n^2-Re)+\omega(a_n\beta-2Re)(r_n^2-Re\omega^2)}{b_n\{(r_n^2-Re\omega^2)^2+(\omega a_n)^2\}},$$

$$A_{2n}=\frac{-r_n^2+Re\omega^2-\omega^2a_n\beta}{(r_n^2-Re\omega^2)^2+(\omega a_n)^2}, \tag{22}$$

$$B_{2n}=\frac{\omega\beta(-r_n^2+Re\omega^2)+\omega a_n}{(r_n^2-Re\omega^2)^2+(\omega a_n)^2}.$$

Using the identity

$$A_{2n}-\frac{1}{r_n^2}+\frac{1}{r_n^2}=A_{3n}+\frac{1}{r_n^2}, \tag{23}$$

with

$$A_{3n}=\frac{Re\omega^2r_n^2+\beta r_n^2a_n\omega^2-\omega^2(a_n^2+Re^2\omega^2)}{r_n^2\{(r_n^2-Re\omega^2)^2+(\omega a_n)^2\}},$$

we get

$$\begin{aligned}\tau_H(r_n,t)=-r_nJ_1(r_n)\Bigg[&\left\{\left(A_{1n}\sinh\left(\frac{b_nt}{2Re}\right)+A_{2n}\cosh\left(\frac{b_nt}{2Re}\right)\right)e^{-\frac{a_nt}{2Re}}+(A_{3n}\cos\omega t+B_{2n}\sin\omega t)+\frac{1}{r_n^2}\cos\omega t\right\}\\&+i\left\{\left(B_{1n}\sinh\left(\frac{b_nt}{2Re}\right)+B_{2n}\cosh\left(\frac{b_nt}{2Re}\right)\right)e^{-\frac{a_nt}{2Re}}+(A_{3n}\sin\omega t-B_{2n}\cos\omega t)+\frac{1}{r_n^2}\sin\omega t\right\}\Bigg].\end{aligned} \tag{24}$$

Now applying inverse Hankel transform (Eq. A.2) to Eq. (24) and using identity (Eq. (B.6)) with condition $J_2(r_n)=0$ and identity (Eq. (B.7))), [27], separating real and imaginary parts we get the exact expression for shear stresses corresponding to considered problems. For cosine oscillation,

$$\tau_c(r,t)=\text{Real part }\{\tau(r,t)\}=\tau_{cs}(r,t)+\tau_{ct}(r,t),\quad t>0, \tag{25}$$

where

$$\tau_{cs}(r,t)=r^2\cos\omega t-2\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{J_1(r_n)}r_n(A_{3n}\cos\omega t+B_{2n}\sin\omega t), \tag{26}$$

$$\tau_{ct}(r,t)=-2\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{J_1(r_n)}r_n\left(A_{1n}\sinh\left(\frac{b_nt}{2Re}\right)+A_{2n}\cosh\left(\frac{b_nt}{2Re}\right)\right)e^{-\frac{a_nt}{2Re}}. \tag{27}$$

For sine oscillation,

$$\tau_s(r,t)=\text{Imiginary part }\{\tau(r,t)\}=\tau_{ss}(r,t)+\tau_{st}(r,t),\quad t>0, \tag{28}$$

where

$$\tau_{ss}(r,t)=r^2\sin\omega t-2\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{J_1(r_n)}r_n(A_{3n}\sin\omega t-B_{2n}\cos\omega t), \tag{29}$$

$$\tau_{st}(r,t)=-2\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{J_1(r_n)}r_n\left(B_{1n}\sinh\left(\frac{b_nt}{2Re}\right)+B_{2n}\cosh\left(\frac{b_nt}{2Re}\right)\right)e^{-\frac{a_nt}{2Re}}. \tag{30}$$

It is important to point out that solutions (25) and (28) are written as sums between the steady-state and transient solutions. By replacing Eqs. (25) and (28) in Eq. (13), integrating with respect to time t and using the initial condition $(15)_2$ we get velocity corresponding to cosine and sine oscillation of Oldroyd-B fluid respectively,

$$\begin{aligned}w_c(r,t)=&-2\sum_{n=1}^{\infty}\frac{J_1(rr_n)r_n^2}{J_1(r_n)}\left[\frac{e^{-\frac{a_nt}{2Re}}}{2r_n^2}\left\{-(a_nA_{1n}+b_nA_{2n})\sinh\left(\frac{b_nt}{2Re}\right)-(b_nA_{1n}+a_nA_{2n})\cosh\left(\frac{b_nt}{2Re}\right)\right\}+\frac{1}{\omega}(A_{3n}\sin(\omega t)-B_{2n}\cos(\omega t))\right]\\&-2\sum_{n=1}^{\infty}\frac{J_1(rr_n)r_n^2}{J_1(r_n)}\left\{\frac{1}{2r_n^2}(b_nA_{1n}+a_nA_{2n})+\frac{B_{2n}}{\omega}\right\}+\frac{4r}{\omega}\sin(\omega t),\end{aligned} \tag{31}$$

$$\begin{aligned}w_s(r,t)=&-2\sum_{n=1}^{\infty}\frac{J_1(rr_n)r_n^2}{J_1(r_n)}\left[\frac{e^{-\frac{a_nt}{2Re}}}{2r_n^2}\left\{-(a_nB_{1n}+b_nB_{2n})\sinh\left(\frac{b_nt}{2Re}\right)-(b_nB_{1n}+a_nB_{2n})\cosh\left(\frac{b_nt}{2Re}\right)\right\}+\frac{1}{\omega}(-A_{3n}\cos(\omega t)-B_{2n}\sin(\omega t))\right]\\&-2\sum_{n=1}^{\infty}\frac{J_1(rr_n)r_n^2}{J_1(r_n)}\left\{\frac{1}{2r_n^2}(b_nB_{1n}+a_nB_{2n})+\frac{A_{3n}}{\omega}\right\}+\frac{4r}{\omega}(1-\cos(\omega t)).\end{aligned} \tag{32}$$

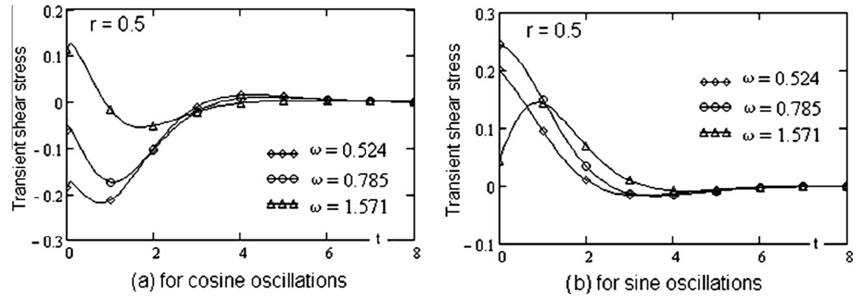

Figure 1 Decay time of transient component for the shear stress $\tau_{ct}(r,t)$ and $\tau_{st}(r,t)$ of Oldroyd-B fluids given by Eqs. (27) and (30) for $\mathrm{Re}=25, \beta=0.5$ and different values of angular frequency ω.

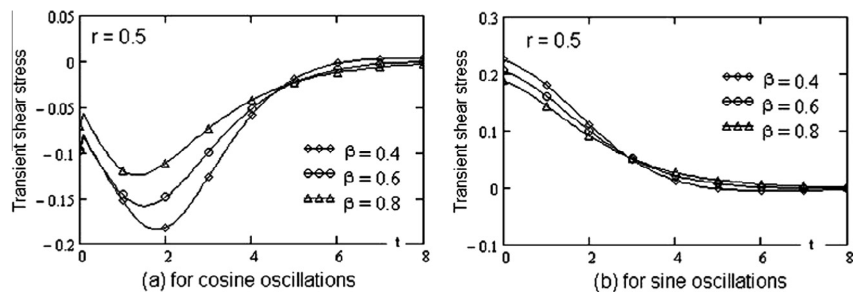

Figure 2 Decay time of transient component for the shear stress $\tau_{ct}(r,t)$ and $\tau_{st}(r,t)$ of Oldroyd-B fluids given by Eqs. (27) and (30) for $\mathrm{Re}=50, \omega=0.393$ and different values of the ratio β.

By now letting $\omega \to 0$ into Eqs. (25) and (31), as it was to be expected, we recover the known results [16, Eqs. (15) and (17)] corresponding to the motion induced by the flat plate that applies a constant shear $fH(t)$ to the fluid.

4. Limiting cases

4.1. The case $\beta \to 0$ (Maxwell fluids)

By taking $\beta \to 0$ into Eqs. (25)–(32) we get exact expressions of the shear stress and velocity for both, cosine and sine cases corresponding to the motion of a Maxwell fluid in an infinite circular cylinder which is rotating under the boundary conditions (16)

$$\tau_{cM}(r,t) = -2\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{J_1(r_n)}r_n\left\{\left(A_{11n}\sinh\left(\frac{\sqrt{(Re^2-4Rer_n^2)}t}{2Re}\right) + A_{22n}\cosh\left(\frac{\sqrt{(Re^2-4Rer_n^2)}t}{2Re}\right)\right)e^{-\frac{t}{2Re}} + (A_{33n}\cos\omega t + B_{22n}\sin\omega t)\right\} + r^2\cos\omega t, \quad t>0, \tag{33}$$

$$\tau_{sM}(r,t) = -2\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{J_1(r_n)}r_n\left\{\left(B_{11n}\sinh\left(\frac{\sqrt{(Re^2-4Rer_n^2)}t}{2Re}\right) + B_{22n}\cosh\left(\frac{\sqrt{(Re^2-4Rer_n^2)}t}{2Re}\right)\right)e^{-\frac{t}{2Re}} + (A_{33n}\sin\omega t - B_{22n}\cos\omega t)\right\} + r^2\sin\omega t, \quad t>0, \tag{34}$$

$$w_{cM}(r,t) = -2\sum_{n=1}^{\infty}\frac{J_1(rr_n)r_n^2}{J_1(r_n)}\left[\frac{e^{-\frac{t}{2}}}{2r_n^2}\left\{-\left(ReA_{11n}+\sqrt{(Re^2-4Rer_n^2)}A_{22n}\right)\sinh\left(\frac{\sqrt{(Re^2-4Rer_n^2)}t}{2Re}\right) - \left(\sqrt{(Re^2-4Rer_n^2)}A_{11n}+ReA_{22n}\right)\cosh\left(\frac{\sqrt{(Re^2-4Rer_n^2)}t}{2Re}\right)\right\} + \frac{1}{\omega}(A_{33n}\sin(\omega t) - B_{22n}\cos(\omega t))\right] - 2\sum_{n=1}^{\infty}\frac{J_1(rr_n)r_n^2}{J_1(r_n)}\left\{\frac{1}{2r_n^2}\left(\sqrt{(Re^2-4Rer_n^2)}A_{11n}+ReA_{22n}\right)+\frac{B_{22n}}{\omega}\right\} + \frac{4r}{\omega}\sin(\omega t), \tag{35}$$

$$w_{sM}(r,t) = -2\sum_{n=1}^{\infty}\frac{J_1(rr_n)r_n^2}{J_1(r_n)}\left[\frac{e^{-\frac{t}{2}}}{2r_n^2}\left\{-\left(ReB_{11n}+\sqrt{(Re^2-4Rer_n^2)}B_{22n}\right)\sinh\left(\frac{\sqrt{(Re^2-4Rer_n^2)}t}{2Re}\right) - \left(\sqrt{(Re^2-4Rer_n^2)}B_{11n}+ReB_{22n}\right)\cosh\left(\frac{\sqrt{(Re^2-4Rer_n^2)}t}{2Re}\right)\right\} + \frac{1}{\omega}\left(-A_{33n}\cos(\omega t) - B_{22n}\sin(\omega t)\right)\right] - 2\sum_{n=1}^{\infty}\frac{J_1(rr_n)r_n^2}{J_1(r_n)}\left\{\frac{1}{2r_n^2}\left(\sqrt{(Re^2-4Rer_n^2)}B_{11n}+ReB_{22n}\right)+\frac{A_{33n}}{\omega}\right\} + \frac{4r}{\omega}(1-\cos(\omega t)), \tag{36}$$

where

$$A_{11n} = -\frac{Rer_n^2 + Re^2\omega^2}{\sqrt{1-4Rer_n^2}\{(r_n^2 - Re\omega^2)^2 + \omega^2\}},$$

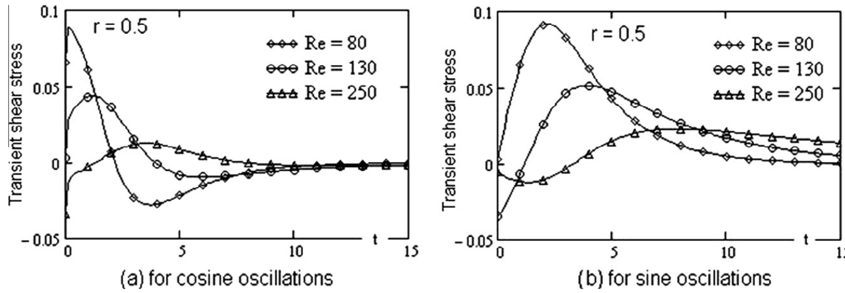

Figure 3 Decay time of transient component for the shear stress $\tau_{ct}(r,t)$ and $\tau_{st}(r,t)$ of Oldroyd-B fluids given by Eqs. (27) and (30) for $\omega = 0.785, \beta = 0.6$ and different values of the Reynolds number Re.

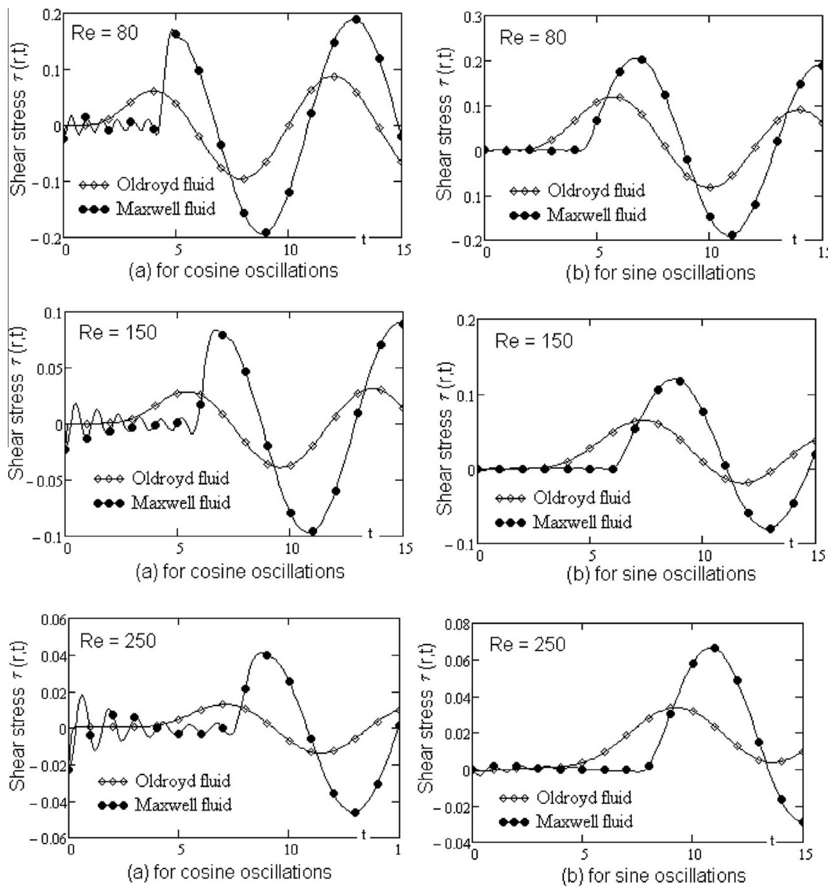

Figure 4 Profiles of both cosine and sine shear stress for Oldroyd-B fluid and Maxwell fluid for $\beta = 0.6$, $\omega = 0.785$, $r = 0.5$ and different values of Reynolds number.

$$
\begin{aligned}
B_{11n} &= \frac{\omega Re\{Re - 2Re(r_n^2 - Re\omega^2)\}}{\sqrt{1 - 4Rer_n^2\{(r_n^2 - Re\omega^2)^2 + \omega^2\}}},\\
A_{22n} &= \frac{-r_n^2 + Re\omega^2}{(r_n^2 - Re\omega^2)^2 + \omega^2},\\
B_{22n} &= \frac{\omega Re}{(r_n^2 - Re\omega^2)^2 + \omega^2},\\
A_{33n} &= \frac{Re\omega^2 r_n^2 - \omega^2 Re^2(1 + \omega^2)}{r_n^2\{(r_n^2 - Re\omega^2)^2 + \omega^2\}}.
\end{aligned}
\tag{37}
$$

5. Numerical results and discussion

Generally, in the rheological measurements, the transient parts of the starting solutions are neglected. Practically, it is of interest to find the required time to reach the large-time state for solutions, consequently, to approximate the time after which the fluid is moving according to the large-time state is an important problem regarding the technical relevance of the solutions. In our problem the required time for the decay of transient part of solutions for both cosine and sine shear stresses depends only of three quantities, namely, ω the angular frequency of oscillations, β the ratio of the retardation time and the relaxation time and Re the Reynolds number. Fig. 1 depicts that the decaying of transient part for the solutions (27) and (30), corresponding to both cosine and sine shear stresses is faster for increasing angular frequency ω. Fig. 2 shows that the decay time of transient part for the solutions (27) and (30), corresponding to both cosine and sine shear stresses decreases with the increase of ratio β. Fig. 3 depicts that the decay time of transient part for the solutions (27) and (30), corresponding to both cosine and sine shear stresses increases with the increase in Reynolds number. In each of the examined cases, it can be determined from the chart, the approximate values of the time t after which the transient solutions can be neglected. Fig. 4 shows the comparison between Oldroyd-B and Maxwell fluids. It shows that the amplitude of the wave cycle for shear stress corresponding to Oldroyd-B and Maxwell fluids decreases with increasing the Reynolds number Re. Moreover, amplitude of wave cycle for Oldroyd-B fluids is always lesser in comparison with that of Maxwell fluids for both sine and cosine shear stresses.

Also, it is important to note that, for a short time-interval, the flow of Maxwell fluid exhibits instability for cosine oscillations. In the case of sine oscillations, for a short time interval, the Maxwell fluid is not moving. The length of this interval increases for increasing of Reynold's number.

6. Conclusions

In this paper we have studied unsteady flows on Oldroyd-B fluids through a circular cylinder on whose boundary was given the azimuthal tension in the form $f\sin(\omega t)$ or $f\cos(\omega t)$. If, usually in literature the governing equation of the flow is related at the velocity field, in the present paper, the basic flow equation is related at the azimuthal tension. Solutions of the initial-boundary value flow problem have been obtained by means of the Laplace and Hankel transforms. Expressions of the shear stress corresponding to both types of oscillations have been written as a sum between "the steady-state" (or permanent solution) and the transient solution which tends to zero for large values of the time t.

Analyzing some specific situations, the decreasing of transient solutions was studied. The approximate values of the time for which the transient solution can be neglected were determined. Corresponding solutions for Maxwell fluids have been obtained as particular case and a comparison between both models was presented. The roots of the equations $J_2(x) = 0$, numerical calculations and graphs were obtained using subroutines from Mathcad15.

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Appendix A. For the function $g(r)$ the finite Hankel transform of order 2 [25] is defined by

$$H_2\{g(r)\} = g_H(r_n) = \int_0^1 r g(r) J_2(r r_n) dr, \tag{A.1}$$

where $J_2(\cdot)$ is the Bessel function of the first kind of order 2 and $0 < r_1 < r_2 < \ldots$ are the positive roots of the equation $J_2(r) = 0$. The inverse finite Hankel transform of order 2 of function $g_H(r_n)$ is defined by

$$H_2^{-1}\{g_H(r_n)\} = g(r) = 2\sum_{n=1}^{\infty} \frac{J_2(r r_n)}{J_3^2(r_n)} g_H(r_n), \tag{A.2}$$

where the summation is defined over all positive roots of $J_2(r) = 0$.

Appendix B. By using the following formulae [27,28] and definition of finite Hankel transform of order 2 (A.1)

$$\frac{dJ_1[m(r)]}{dr} = \left\{\frac{1}{m(r)} J_1[m(r)] - J_2[m(r)]\right\} m'(r), \tag{B.1}$$

$$\frac{dJ_2[m(r)]}{dr} = \left\{-\frac{2}{m(r)} J_2[m(r)] + J_1[m(r)]\right\} m'(r), \tag{B.2}$$

we get

$$\int_0^1 r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial r} - \frac{4}{r^2}\right)\overline{\tau}(r,q) J_2(r r_n) dr = -r_n^2 \overline{\tau}_H(r_n, q) - r_n \overline{\tau}(1,q) J_1(r_n). \tag{B.3}$$

$$\int e^{as}\cosh(bs)ds = \frac{e^{as}}{a^2 - b^2}\{a\cosh(bs) - b \times \sinh(bs)\}, \quad a^2 \neq b^2. \tag{B.4}$$

$$\int e^{as}\sinh(bs)ds = \frac{e^{as}}{a^2 - b^2}\{a\sinh(bs) - b \times \cosh(bs)\}, \quad a^2 \neq b^2. \tag{B.5}$$

$$J_1(r_n) + J_3(r_n) = \frac{4}{r_n} J_2(r_n). \tag{B.6}$$

For finite inverse Hankel transform of order 2 with $J_2(r_n) = 0$ we have the following identity

$$H_2^{-1}\left\{-\frac{1}{r_n} J_1(r_n)\right\} = r^2. \tag{B.7}$$

Proof Since,

$$H_2\{r^2\} = \int_0^1 r^3 J_2(r r_n) dr.$$

Replace $z = r r_n$, then

$$H_2\{r^2\} = \frac{1}{r_n^4}\int_0^{r_n} z^3 J_2(z) dz.$$

Using $\int_0^z t^3 J_2(t) dt = z^3 J_3(z)$ [28] and $J_2(r_n) = 0$ along with $J_3(z) = \frac{4}{z} J_2(z) - J_1(z)$ [28] we get,

$$H_2\{r^2\} = -\frac{1}{r_n} J_1(r_n).$$

Hence,

$$H_2^{-1}\left\{-\frac{1}{r_n} J_1(r_n)\right\} = r^2.$$

References

[1] J.G. Oldroyd, On the formulation of rheological equations of state, Proc. R. Soc. Lond. A. 200 (1950) 523–541.

[2] J.G. Oldroyd, The motion of an elastico-viscous liquid contained between coaxial cylinders, Q. J. Mech. Appl. Math. 4 (1951) 271–282.

[3] C. Guillope, J.C. Saut, Global existence and one-dimensional non-linear stability of shearing motions of viscoelastic fluids of Oldroyd type, RAIRO Model. Math. Anal. Number. 24 (1990) 369–401.

[4] M.A. Fontelos, A. Friedman, Stationary non-Newtonian fluid flows in channellike and pipe-like domains, Arch. Rational Mech. Anal. 151 (2000) 1–43.

[5] K.R. Rajagopal, P.K. Bhatnagar, Exact solutions for some simple flows of an Oldroyd-B fluid, Acta Mech. 113 (1995) 233–239.

[6] T. Hayat, A.M. Siddiqui, S. Asghar, Some simple flows of an Oldroyd-B fluid, Int. J. Eng. Sci. 39 (2001) 135–147.

[7] T. Hayat, M. Khan, M. Ayub, Exact solutions of flow problems of an Oldroyd-B fluid, Appl. Math. Comput. 151 (2004) 105–119.

[8] B. Singh, N.K. Varshney, Effect of MHD visco-elastic fluid (Oldroyd) and porous medium through a circular cylinder bounded by a permeable bed, Int. J. Math. Arch. 3 (8) (2012) 2912–2917.

[9] I. Burdujan, The flow of a particular class of Oldroyd-B fluids, Ann. Acad. Romanian Sci. Ser. Math. Appl. 3 (1) (2011) 23–45.

[10] V. Mathur, K. Khandelwal, Exact solution for the flow of Oldroyd-B fluid between coaxial cylinders, Int. J. Eng. Res. Technol. (IJERT) 3 (1) (2014) 949–954.

[11] Y. Liu, F. Zong, J. Dai, Unsteady helical flow of a generalized Oldroyd-B fluid with fractional derivative, Int. J. Math. Trends Technology. 5 (2014) 66–76.

[12] M. Renardy, Inflow boundary condition forsteady flow of viscoelastic fluids with differential constitutive laws, Rocky Mount. J. Math. 18 (1998) 445–453.

[13] M. Renardy, An alternative approach to inflow boundary conditions for Maxwell fluids in three space dimensions, J. Non-Newtonian Fluid Mech. 36 (1990) 419–425.

[14] R. Talhouk, Unsteady flows of viscoelastic fluids with inflow and outflow boundary conditions, Appl. Math. Lett. 9 (1996) 93–98.

[15] N.D. Waters, M.J. King, Unsteady flow of an elastico-viscous liquid, Rheol. Acta. 93 (1970) 345–355.

[16] Corina Fetecau, Mehwish Rana, Niat Nigar, Constantin Fetecau, First exact solutions for flows of rate type fluids in a circular duct that applies a constant couple to the fluid, Z. Naturforsch. 69a (2014) 232–238.

[17] Muhammad Jamil C. Fetecau, M. Rana, Some exact solutions for Oldroyd-B fluid due to time dependent prescribed shear stress, J. Theor. Appl. Mech. 50 (2) (2012) 549–562.

[18] D. Vieru, I. Siddique, Axial flow of several non-Newtonian fluids through a circular cylinder, Int. J. Appl. Mech. 2 (3) (2010) 543–556.

[19] A. Karami, T. Yousefi, S. Mohebbi, C. Aghanajafi, Prediction of free convection from vertical and inclined rows of horizontal isothermal cylinders using ANFIS, Arab. J. Sci. Eng. 39 (2014) 4201–4209.

[20] H.A. Attia, M.A. Abdeen, W.A. El-Meged, Transient generalized Couette flow of viscoelastic fluid through a porous medium with variable viscosity and pressure gradient, Arab. J. Sci. Eng. 38 (2013) 3451–3458.

[21] G. Nagaraju, J.V. Ramana Murthy, MHD flow of longitudinal and torsional oscillations of a circular cylinder with suction in a couple stress fluid, Int. J. Appl. Mech. Eng. 18 (4) (2013) 1099–1114.

[22] B.J. Gireesha, K.R. Madhura, C.S. Bagewadi, Flow of an unsteady dusty fluid through porous media in a uniform pipe with sector of a circle as cross-section, Int. J. Pure Appl. Math. 76 (1) (2012) 29–47.

[23] S.P. Anjali Devi, B. Ganga, Viscous dissipation effects on nonlinear MHD flow in a porous medium over a stretching porous surface, Ind. J. Appl. Math. 5 (7) (2009) 45–49.

[24] C. Fetecau, Q. Rubbab, S. Akhter, C. Fetecau, New methods to provide exact solutions for some unidirectional motions of rate type fluids, Thermal Sci. (2013), http://dx.doi.org/10.2298/TSCI130225130F.

[25] L. Debnath, D. Bhatta, Integral Transforms and their Applications, Chapman and Hall/CRC Press, Boca Raton, London, New York, 2007.

[26] G.E. Roberts, H. Kaufman, Table of Laplace Transforms, W.B. Saunders Company, Philadelphia and London, 1968.

[27] N.W. McLachlan, Bessel Functions for Engineers, Oxford Univeristy Press, London, 1955.

[28] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, in: NBS, Appl. Math. Series, vol. 55. Washington, DC, 1964.