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A new approach to the reconstruction of images from Radon projections ☆

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Abstract

A new approach is proposed for reconstruction of images from Radon projections. Based on Fourier expansions in orthogonal polynomials of two and three variables, instead of Fourier transforms, the approach provides a new algorithm for the computed tomography. The convergence of the algorithm is established under mild assumptions.
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1. Introduction

The fundamental problem of the computed tomography (CT) is to reconstruct an image from its Radon projections (X-rays). To be more precise, let us consider the case of a two dimensional image, described by a density function $f(x,y)$ defined on the unit disk $B^2 = \{(x,y)\colon\ x^2+y^2 \leqslant 1\}$ of the $\mathbb{R}^2$ plane. A Radon projection of f is a line integral,

$$\mathcal{R}_\theta(f;t) := \int_{I(\theta,t)} f(x,y)\,dx\,dy, \quad 0 \leqslant \theta \leqslant 2\pi,\ -1 \leqslant t \leqslant 1,$$

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where $I(\theta, t) = \{(x, y)\colon\ x\cos\theta + y\sin\theta = t\} \cap B^2$ is a line segment inside B^2. The reconstruction problem of CT requires finding f from its Radon projections. This question is completely solved if the complete data is given, that is, if projections for all t and θ are given. In practice, however, only a finite number of projections can be measured. Hence, the essential problem of CT is to find an effective algorithm that produces a good approximation to f based on a finite number of Radon projections. The arrangement of the projections is referred as scanning geometry since it is determined by the design of the scanner.

Currently, the most important method in CT is the filtered backprojection (FBP) method which, like several other methods, is based on techniques of Fourier transforms. In principle, the FBP method works for banded limited functions. It requires choosing a function W_b (a low-pass filter with cut-off frequency b) that approximates the δ-distribution, and it includes steps of linear interpolation and a discrete convolution that approximates a continuous convolution. Each of these steps could introduce serious errors in the algorithm. See the discussion in [5,6].

The purpose of the present paper is to develop a direct approach for reconstruction of images in CT. Instead of using Fourier transforms, we will use orthogonal expansions based on orthogonal polynomials on B^2. Let $S_n f$ denote the nth partial sum of such an expansion. It turns out that $S_{2m} f$ satisfies the following remarkable formula

$$S_{2m}(f; x, y) = \sum_{\nu=0}^{2m} \frac{1}{\pi} \int_{-1}^{1} \mathcal{R}_{\phi_\nu}(f; t)\Phi_\nu(t; x, y)\, dt, \tag{1.1}$$

where $\phi_\nu = 2\nu\pi/(2m+1)$ are equally spaced angles along the circumference of the disk and Φ_ν are polynomials of two variables given by explicit formulas (see Theorem 3.3). Applying appropriate quadrature formulas to the integrals in (1.1) leads to an approximation, $\mathcal{A}_{2m} f$, of f that uses discrete Radon data. For example, using an appropriate Gaussian quadrature formula leads to an $\mathcal{A}_{2m} f$ that uses the Radon data

$$\left\{\mathcal{R}_{\phi_\nu}\left(f; \cos\tfrac{j\pi}{2m+1}\right)\colon\ 0 \leqslant \nu \leqslant 2m,\ 1 \leqslant j \leqslant 2m\right\}, \tag{1.2}$$

of the scanning geometry of parallel beams, and the polynomial $\mathcal{A}_{2m} f$ is given by an explicit formula

$$\mathcal{A}_{2m}(f; x, y) = \sum_{\nu=0}^{2m} \sum_{j=1}^{2m} \mathcal{R}_{\phi_\nu}\left(f; \cos\tfrac{j\pi}{2m+1}\right) T_{j,\nu}(x, y), \tag{1.3}$$

where $T_{j,\nu}(x, y)$ are polynomials that are given by simple formulas. The operator $\mathcal{A}_{2m} f$ provides a direct approximation to f, which is the essence of our reconstruction algorithm. The new algorithm can be implemented easily as the $T_{j,\nu}$ are fixed polynomials that can be stored in a table beforehand. Furthermore, the construction allows us to add a multiplier factor and there is a natural extension of the algorithm to a cylindrical domain in $\mathbb{R}^3$.

The operator $\mathcal{A}_{2m}$ in (1.3) reproduces polynomials of degree $2m-1$. Furthermore, we will prove that the operator norm of $\mathcal{A}_{2m}$ in the uniform norm is $\mathcal{O}(m\log(m+1))$, only slightly worse than the norm of S_{2m}, which is $\mathcal{O}(m)$. As a consequence, it follows that the algorithm converges uniformly on B^2 if f is a C^2 function. There is no need to assume that f is banded limited.

At the moment the author is working with Dr. Christoph Hoeschen and Dr. Oleg Tischenko, Medical Physics Group of the National Research Center for Environment and Health, Germany, to implement the algorithm numerically. The results that we have obtained so far are very promising and will be reported elsewhere.

The paper is organized as follows. The background of Radon transforms and orthogonal polynomials is discussed in the next section, where an identity that will play a fundamental role in the analysis is also proved. The Fourier orthogonal expansions are studied in Section 3, where the identity (1.1) will be proved. The reconstruction algorithms are presented in Section 4. The convergence of the basic algorithm for the 2-D reconstruction is proved in Section 5.

2. Radon transform and polynomials

Let $B^2 = \{(x,y)\colon\ x^2+y^2 \leqslant 1\}$ denote the unit disk on the plane. It is often convenient to use the polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \quad r \geqslant 0,\ 0 \leqslant \theta \leqslant 2\pi.$$

Let θ be an angle as in the polar coordinates; that is, θ is measured counterclockwise from the positive x-axis. Let ℓ denote the line $\ell(\theta,t) = \{(x,y)\colon\ x\cos\theta + y\sin\theta = t\}$ for $-1 \leqslant t \leqslant 1$. Clearly the line is perpendicular to the direction $(\cos\theta, \sin\theta)$ and $|t|$ is the distance between the line and the origin. We will use the notation

$$I(\theta,t) = \ell(\theta,t) \cap B^2, \quad 0 \leqslant \theta < 2\pi,\ -1 \leqslant t \leqslant 1, \tag{2.1}$$

to denote the line segment of ℓ inside B^2. The points on $I(\theta,t)$ can be represented as follows:

$$x = t\cos\theta - s\sin\theta, \qquad y = t\sin\theta + s\cos\theta,$$

for $s \in [-\sqrt{1-t^2}, \sqrt{1-t^2}]$.

The Radon projection of a function f in the direction θ with parameter $t \in [-1,1]$ is denoted by $\mathcal{R}_\theta(f;t)$,

$$\begin{aligned}\mathcal{R}_\theta(f;t) &:= \int_{I(\theta,t)} f(x,y)\,dx\,dy \\ &= \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\,ds.\end{aligned} \tag{2.2}$$

In the literature a Radon projection of a bivariate function is also called an X-ray. The definition shows that $\mathcal{R}_\theta(f;t) = \mathcal{R}_{\pi+\theta}(f;-t)$.

Let Π^2 denote the space of polynomials of two variables and let Π_n^2 denote the subspace of polynomials of total degree n in Π^2, which has dimension $\dim \Pi_n^2 = (n+1)(n+2)/2$. If $P \in \Pi_n^2$ then

$$P(x,y) = \sum_{k=0}^{n}\sum_{j=0}^{k} c_{k,j}x^j y^{k-j}.$$

Let p be a polynomial in one variable and $\theta \in [0, \pi]$. For $\xi = (\cos\theta, \sin\theta)$ and $\mathbf{x} = (x, y)$, the ridge polynomial $p(\theta; \cdot)$ is defined by

$$p(\theta; x, y) := p\big(\langle \mathbf{x}, \xi \rangle\big) = p(x\cos\theta + y\sin\theta).$$

This is a polynomial of two variables and is in Π_n^2 if p is of degree n. It is constant on lines that are perpendicular to the direction $(\cos\theta, \sin\theta)$. Especially important to us are the polynomials $U_k(\theta; x, y)$, where U_k denotes the Chebyshev polynomial of the second kind,

$$U_k(x) = \frac{\sin(k+1)\theta}{\sin\theta}, \quad x = \cos\theta.$$

The polynomials U_k, $k = 0, 1, 2, \ldots,$ are orthogonal polynomials with respect to the weight function $\sqrt{1-x^2}$ on $[-1, 1]$:

$$\frac{2}{\pi}\int_{-1}^{1} U_k(x)U_j(x)\sqrt{1-x^2}\,dx = \delta_{k,j}, \quad k, j \geqslant 0.$$

Ridge polynomials play an important role in our study of the Radon transform, as can be seen from the following simple fact.

Lemma 2.1. *For $f \in L^1(B^2)$,*

$$\int_{B^2} f(x, y)U_k(\phi; x, y)\,dx\,dy = \int_{-1}^{1} \mathcal{R}_\phi(f; t)U_k(t)\,dt. \tag{2.3}$$

Proof. The change of variables $t = x\cos\phi + y\sin\phi$ and $s = -x\sin\phi + y\cos\phi$ amounts to a rotation, which leads to

$$\int_{B^2} f(x, y)U_k(\phi; x, y)\,dx\,dy = \int_{B^2} f(t\cos\phi - s\sin\phi, t\sin\phi + s\cos\phi)U_k(t)\,dt\,ds$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\phi - s\sin\phi, t\sin\phi + s\cos\phi)\,ds U_k(t)\,dt,$$

the inner integral is exactly $\mathcal{R}_\phi(f; t)$ by (2.2). □

Let $\mathcal{V}_k(B^2)$ denote the space of orthogonal polynomials of degree k on B^2 with respect to the unit weight function; that is, $P \in \mathcal{V}_k(B^2)$ if P is of degree k and

$$\int_{B^2} P(x, y)Q(x, y)\,dx\,dy = 0, \quad \text{for all } Q \in \Pi_{k-1}^2.$$

A set of polynomials $\{P_{j,k}\colon 0 \leqslant j \leqslant k\}$ in $\mathcal{V}_k(B^2)$ is an orthonormal basis of $\mathcal{V}_k(B^2)$ if

$$\frac{1}{\pi}\int_{B^2} P_{i,k}(x,y)P_{j,k}(x,y)\,dx\,dy = \delta_{i,j}, \quad 0 \leqslant i, j \leqslant k.$$

It is known that the polynomials in $\mathcal{V}_k(B^2)$ are eigenfunctions of a second order differential operator $\mathcal{D}$ (see, for example, [2]),

$$\mathcal{D}P = -(k+2)(k-1)P, \quad \text{for all } P \in \mathcal{V}(B^2);$$

the operator $\mathcal{D}$ is defined by

$$\mathcal{D} := \Delta^2 - \langle \mathbf{x}, \nabla\rangle^2 - 2\langle \mathbf{x}, \nabla\rangle, \tag{2.4}$$

where $\mathbf{x} = (x, y)$, $\nabla = (\partial_1, \partial_2)$ is the gradient operator with $\partial_1 = \partial/\partial x$ and $\partial_2 = \partial/\partial y$, and $\Delta = \partial_1^2 + \partial_2^2$ is the usual Laplacian. It turns out that the Radon projections of orthogonal polynomials in $\mathcal{V}_n(B^2)$ can be computed explicitly.

Lemma 2.2. *If $P \in \mathcal{V}_k(B^2)$ then for each $t \in (-1, 1)$, $0 \leqslant \theta \leqslant 2\pi$,*

$$\mathcal{R}_\theta(P; t) = \frac{2}{k+1}\sqrt{1-t^2}\,U_k(t)P(\cos\theta, \sin\theta). \tag{2.5}$$

Proof. A change of variable in (2.2) shows that

$$\mathcal{R}_\theta(P; t) = \sqrt{1-t^2}\int_{-1}^{1} P\bigl(t\cos\theta - s\sqrt{1-t^2}\sin\theta, t\sin\theta + s\sqrt{1-t^2}\cos\theta\bigr)\,ds.$$

The integral is a polynomial in t since an odd power of $\sqrt{1-t^2}$ in the integrand is always accompanied by an odd power of s, which has integral zero. Therefore, $Q(t) := \mathcal{R}_\theta(P; t)/\sqrt{1-t^2}$ is a polynomial of degree k in t for every θ. Furthermore, the integral also shows that $Q(1) = P(\cos\theta, \sin\theta)$. By (2.3),

$$\int_{-1}^{1} \frac{\mathcal{R}_\theta(P; t)}{\sqrt{1-t^2}} U_j(t)\sqrt{1-t^2}\,dt = \int_{B^2} P(x, y)U_j(\theta; x, y)\,dx\,dy = 0,$$

for $j = 0, 1, \ldots, k-1$, since $P \in \mathcal{V}_k(B^2)$. Since Q is of degree k, we conclude that $Q(t) = cU_k(t)$ for some constant independent of t. Setting $t = 1$ and using the fact that $U_k(1) = k+1$, we have $c = P(\cos\theta, \sin\theta)/(k+1)$. $\square$

This lemma was proved in [4]. It plays an important role in our development below. We have included its short proof in order to make the paper self-contained.

There are several orthogonal or orthonormal bases that are known explicitly for $\mathcal{V}_k(B^2)$ (see [2]). We will work with a special orthonormal basis that is given in terms of the ridge polynomials defined above. Setting $f(x, y) = U_k(\theta; x, y)$ in (2.3) and using (2.5), we derive from the orthogonality of the Chebyshev polynomials that

$$\frac{1}{\pi}\int_{B^2} U_k(\theta; x, y)U_k(\phi; x, y)\,dx\,dy = \frac{1}{k+1}U_k\bigl(\cos(\phi-\theta)\bigr). \tag{2.6}$$

Recall that the zeros of U_k are $\cos\theta_{j,k}$, $1 \leqslant j \leqslant k$, where $\theta_{j,k} = j\pi/(k+1)$. As a consequence of (2.6), we have the following result ([3]; see also [8]):

Proposition 2.3. *An orthonormal basis of $\mathcal{V}_k(B^2)$ is given by*

$$\mathbb{P}_k := \bigl\{U_k(\theta_{j,k}; x, y)\colon\ 0 \leqslant j \leqslant k\bigr\}, \quad \theta_{j,k} = \frac{j\pi}{k+1}.$$

In particular, the set $\{\mathbb{P}_k\colon\ 0 \leqslant k \leqslant n\}$ is an orthonormal basis for Π_n^2.

The polynomials $U_k(\phi; x, y)$ also satisfy a discrete orthogonality. Let

$$\phi_\nu := \frac{2\pi\nu}{2m+1}, \quad 0 \leqslant \nu \leqslant 2m.$$

The discrete orthogonality will follow from the following identity:

Proposition 2.4. *For $k \geqslant 0$ and $\theta \in [0, 2\pi]$, $\phi_\nu = 2\pi\nu/(n+1)$,*

$$\frac{1}{2m+1}\sum_{\nu=0}^{2m} U_k(\phi_\nu; \cos\theta, \sin\theta)U_k(\phi_\nu; x, y) = U_k(\theta; x, y), \quad x, y \in B^2. \tag{2.7}$$

Proof. In order to prove (2.7) we will need the elementary identities

$$\sum_{\nu=0}^{n} \sin k\phi_\nu = 0 \quad \text{and} \quad \sum_{\nu=0}^{n} \cos k\phi_\nu = \begin{cases} 0, & \text{if } k \neq 0 \bmod n+1, \\ n+1, & \text{if } k = 0 \bmod n+1, \end{cases} \tag{2.8}$$

that hold for all nonnegative integers k.

Let us denote by I_k the left-hand side of (2.7). First we consider the case of $k = 2l$. Since U_{2l} is an even polynomial, we can write it as $U_{2l}(t) = \sum_{j=0}^{l} b_j t^{2j}$. Using the polar coordinates $x = r\cos\phi$ and $y = r\sin\phi$, and the fact that $(\cos\psi)^{2j}$ can be written as a sum of $\cos 2i\psi$, we can rearrange the sum to get

$$\begin{aligned} U_{2l}(\theta; x, y) &= U_{2l}\bigl(r\cos(\theta-\phi)\bigr) = \sum_{j=0}^{l} b_j r^{2j}\bigl(\cos(\theta-\phi)\bigr)^{2j} \\ &= \sum_{j=0}^{l} b_j(r)\cos 2j(\theta-\phi), \end{aligned} \tag{2.9}$$

where $b_j(r)$ is a polynomial of degree $2j$ in r. Furthermore, we have

$$U_{2l}(\phi_\nu; \cos\theta, \sin\theta) = U_{2l}\big(\cos(\phi_\nu - \theta)\big) = 1 + 2\sum_{j=1}^{l} \cos 2j(\theta - \phi_\nu). \tag{2.10}$$

These two expressions allow us to write

$$I_{2l} = \sum_{i=0}^{l} d_i \sum_{j=0}^{l} b_j(r) \frac{1}{2m+1} \sum_{\nu=0}^{2m} \cos 2i(\theta - \phi_\nu) \cos 2j(\phi - \phi_\nu)$$

where $d_0 = 1$ and $d_i = 2$ for $1 \leqslant i \leqslant l$. The addition formula of the cosine function shows that

$$\begin{aligned}
&2\cos 2i(\theta - \phi_\nu)\cos 2j(\phi - \phi_\nu) \\
&\quad = \cos 2\big((i+j)\phi_\nu - i\theta - j\phi\big) + \cos 2\big((i-j)\phi_\nu - i\theta + j\phi\big) \\
&\quad = \cos 2(i+j)\phi_\nu \cos 2(i\theta + j\phi) + \sin 2(i+j)\phi_\nu \sin 2(i\theta + j\phi) \\
&\qquad + \cos 2(i-j)\phi_\nu \cos 2(i\theta - j\phi) + \sin 2(i-j)\phi_\nu \sin 2(i\theta - j\phi).
\end{aligned}$$

Since $n + 1 = 2m + 1$ is odd and $\cos 2(i \pm j)\phi_\nu$ have even indices, it follows from the above elementary trigonometric identity and (2.8) that

$$\frac{1}{2m+1} \sum_{\nu=0}^{2m} \cos 2i(\theta - \phi_\nu)\cos 2j(\phi - \phi_\nu) = \begin{cases} 0, & \text{if } i \neq j, \\ \frac{1}{2}\cos 2j(\theta - \phi), & \text{if } i = j \neq 0, \\ 1, & \text{if } i = j = 0. \end{cases}$$

Consequently, using (2.9) again, we conclude that

$$I_{2l} = \sum_{j=0}^{l} b_j(r) \cos 2j(\theta - \phi) = U_{2l}\big(r\cos(\theta - \phi)\big) = U_{2l}(\theta; x, y).$$

This completes the proof for the case $k = 2l$. For the case $k = 2l - 1$, we need to use the identities

$$U_{2l-1}(\phi_\nu; x, y) = U_{2l}\big(r\cos(\phi_\nu - \phi)\big) = \sum_{j=1}^{l} b_j(r)\cos(2j-1)(\phi_\nu - \phi),$$

where $b_j(r)$ is a polynomial of degree $2j - 1$ in r, which is derived using the fact that $(\cos\theta)^{2j-1}$ can be written as a sum of $\cos(2i-1)\theta$. Furthermore, we have

$$U_{2l-1}(\phi_\nu; \cos\theta, \sin\theta) = U_{2l-1}\big(\cos(\phi_\nu - \theta)\big) = 2\sum_{j=1}^{l} \cos(2j-1)(\theta - \phi_\nu).$$

The rest of the proof will follow as in the case $k = 2l$. □

Corollary 2.5. *For $k \geqslant 0$ and $0 \leqslant i, j \leqslant k$,*

$$\frac{1}{2m+1}\sum_{\nu=0}^{2m} U_k(\phi_\nu; \cos\theta_{i,k}, \sin\theta_{i,k}) U_k(\phi_\nu; \cos\theta_{j,k}, \sin\theta_{j,k}) = (k+1)\delta_{i,j}. \tag{2.11}$$

Proof. This follows from setting $\theta = \theta_{i,k}$, $x = \cos\theta_{j,k}$ and $y = \sin\theta_{j,k}$ in the identity (2.7), and using the fact that

$$U_k(\theta_{i,k}; \cos\theta_{j,k}, \sin\theta_{j,k}) = U_k(\cos\theta_{i-j,k}) = (k+1)\delta_{i,j},$$

where in the last step we have used the fact that $U_k(0) = (k+1)$. □

Remark 2.1. The identity (2.7) is established for the case $n = 2m$. It should be pointed out that the result does not hold for $n = 2m-1$. In fact, if $n = 2m-1$, then following the above proof leads to an identity analogous to (2.7) only for $0 \leqslant k \leqslant m-1$. More precisely, we have

$$\frac{1}{2m}\sum_{\nu=0}^{2m-1} U_k(\phi_\nu; \cos\theta, \sin\theta) U_k(\phi_\nu; x, y) = U_k(\theta; x, y), \quad x, y \in B^2,$$

holds for $0 \leqslant k \leqslant m-1$. Indeed, in the case $m = 2p$ it can be proved that

$$\frac{1}{2m}\sum_{\nu=0}^{2m-1} U_m(\phi_\nu; \cos\theta, \sin\theta) U_m(\phi_\nu; \cos\phi, \sin\phi) = U_m\big(\cos(\theta-\phi)\big) + 2T_m\big(\cos(\theta-\phi)\big),$$

where $T_m(x) = \cos m\psi$, $x = \cos\psi$, is the Chebyshev polynomial of the first kind.

There is yet another remarkable identity for $U_k(\cdot; x, y)$, proved in [8], which takes the summation over $\theta_{j,k}$ instead of taking over ϕ_ν:

Lemma 2.6. *For $k \geqslant 0$,*

$$\sum_{j=0}^{k} U_k(\theta_{j,k}; x, y) U_k(\theta_{j,k}; \cos\phi, \sin\phi) = (k+1) U_k(\phi; x, y). \tag{2.12}$$

This identity is a consequence of the compact formula for the reproducing kernel of $\mathcal{V}_n(B^2)$ in [7]. It will also play an important role in our development below.

The two remarkable identities (2.7) and (2.12) hold the key for our new algorithms. The orthogonal polynomials in $\mathcal{V}_n(B^2)$ are usually studied as a special case of the orthogonal polynomials with respect to the weight functions $W_\mu(x, y) := (1 - x^2 - y^2)^{\mu-1/2}$, $\mu \geqslant 0$. Most of their properties are shared by orthogonal families associated with W_μ for all $\mu \geqslant 0$, and furthermore, many properties can be extended to higher dimensions (see [2]). The orthogonality in (2.11), however, is very special; it is not shared by any other orthogonal families associated with W_μ for $\mu \neq 1/2$ and there is no direct extension to higher dimensions; see, for example, the discussion in [8].

3. Fourier orthogonal expansions

3.1. Orthogonal expansions on the disk

The standard Hilbert space theory shows that any function in $L^2(B^2)$ can be expanded as a Fourier orthogonal series in terms of $\mathcal{V}_n$. More precisely,

$$L^2(B^2) = \sum_{k=1}^{\infty} \bigoplus \mathcal{V}_k(B^2): \quad f = \sum_{k=1}^{\infty} \operatorname{proj}_k f, \tag{3.1}$$

where $\operatorname{proj}_k f$ is the orthogonal projection of f onto the subspace $\mathcal{V}_k(B^2)$. Our reconstruction algorithm will be based on this Fourier orthogonal expansion. It is well known that $\operatorname{proj}_n f$ can be written as an integral operator in terms of the reproducing kernel of $\mathcal{V}_k(B^2)$ in $L^2(B^2)$. A compact formula of the reproducing kernel is given in [7]. For our purpose, we seek a formula for $\operatorname{proj}_n f$ that will relate it to the Radon transform.

In terms of the special orthonormal basis $\{\mathbb{P}_k\colon k \geqslant 0\}$ given in Proposition 2.3, we can write $\operatorname{proj}_k f$ as

$$\operatorname{proj}_k f = \sum_{j=0}^{k} \hat{f}_{j,k} U_k(\theta_{j,k}; \cdot), \quad \hat{f}_{j,k} = \frac{1}{\pi} \int_{B^2} f(x,y) U_k(\theta_{j,k}; x, y)\, dx\, dy. \tag{3.2}$$

The remarkable identity (2.7) allows us to express the Fourier coefficients in terms of Radon projections:

Proposition 3.1. *Let m be a nonnegative integer. For $0 \leqslant k \leqslant 2m$, the Fourier coefficient $\hat{f}_{j,k}$ satisfies*

$$\hat{f}_{j,k} = \frac{1}{2m+1} \sum_{\nu=0}^{2m} \frac{1}{\pi} \int_{-1}^{1} \mathcal{R}_{\phi_\nu}(f; t) U_k(t)\, dt\, U_k\big(\cos(\theta_{j,k} - \phi_\nu)\big). \tag{3.3}$$

Proof. Let us denote the right-hand side of (3.3) by $g_{j,k}$. By Eq. (2.3),

$$\begin{aligned} g_{j,k} &= \frac{1}{2m+1} \sum_{\nu=0}^{2m} \frac{1}{\pi} \int_{B^2} f(x,y) U_k(\phi_\nu; x, y)\, dx\, dy U_k\big(\cos(\theta_{j,k} - \phi_\nu)\big) \\ &= \frac{1}{\pi} \int_{B^2} f(x,y) \left[\frac{1}{2m+1} \sum_{\nu=0}^{2m} U_k(\phi_\nu; x, y) U_k\big(\cos(\theta_{j,k} - \phi_\nu)\big) \right] dx\, dy \\ &= \frac{1}{\pi} \int_{B^2} f(x,y) U_k(\theta_{j,k}; x, y)\, dx\, dy \end{aligned}$$

by the identity (2.7). Hence, by the definition, $\hat{f}_{j,k}$, $g_{j,k} = \hat{f}_{j,k}$. □

A further application of the identity (2.12) leads us to the expression of the Fourier projection operator in terms of Radon projections:

Theorem 3.2. *For $m \geqslant 0$ and $k \leqslant 2m$, the operator* $\operatorname{proj}_k f$ *can be written as*

$$\operatorname{proj}_k f(x, y) = \frac{1}{2m+1} \sum_{\nu=0}^{2m} \frac{1}{\pi} \int_{-1}^{1} \mathcal{R}_{\phi_\nu}(f; t) U_k(t)\, dt\, (k+1) U_k(\phi_\nu; x, y). \tag{3.4}$$

Proof. By (3.3) and the identity (2.12) in Lemma 2.6,

$$\begin{aligned} \operatorname{proj}_k f(x, y) &= \frac{1}{2m+1} \sum_{\nu=0}^{2m} \frac{1}{\pi} \int_{-1}^{1} \mathcal{R}_{\phi_\nu}(f; t) U_k(t)\, dt \\ &\quad \times \sum_{j=0}^{k} U_k\big(\cos(\theta_{j,k} - \phi_\nu)\big) U_k(\theta_{j,k}; x, y) \\ &= \frac{1}{2m+1} \sum_{\nu=0}^{2m} \frac{1}{\pi} \int_{-1}^{1} \mathcal{R}_{\phi_\nu}(f; t) U_k(t)\, dt\, (k+1) U(\phi_\nu; x, y), \end{aligned}$$

which is what we need to prove. □

We denote the nth partial sum of the expansion (3.1) by $S_n f$; that is,

$$S_n f(x, y) = \sum_{k=0}^{n} \operatorname{proj}_k f(x, y).$$

The operator S_n is a projection operator from $L^2(B^2)$ onto Π_n^2. An immediate consequence of Theorem 3.2 is the following result that will play an essential role in deriving the new algorithm:

Corollary 3.3. *For $m \geqslant 0$, the partial sum operator $S_{2m} f$ can be written as*

$$S_{2m}(f; x, y) = \sum_{\nu=0}^{2m} \frac{1}{\pi} \int_{-1}^{1} \mathcal{R}_{\phi_\nu}(f; t) \Phi_\nu(t; x, y)\, dt \tag{3.5}$$

where

$$\Phi_\nu(t; x, y) = \frac{1}{2m+1} \sum_{k=0}^{2m} (k+1) U_k(t) U_k(\phi_\nu; x, y). \tag{3.6}$$

The identity (3.5) shows that $S_{2m} f$ can be expressed in terms of Radon projections in $2m+1$ directions. This result, previously unnoticed, holds the key for our new algorithm given in Section 4. It should be pointed out (recall Remark 2.1) that such an identity does not hold for $S_{2m-1}f$.

3.2. Summability of orthogonal expansions

Let $L^p(B^2)$ denote the usual L^p space on B^2 with norm $\|\cdot\|_p$ for $1 \leqslant p < \infty$ and identify it with $C(B^2)$ of continuous functions with uniform norm for $p = \infty$. If $f \in L^p(B^2)$, the error of best approximation by polynomials of degree at most n is defined by

$$E_n(f)_p := \inf\{\|f - P\|_p \colon P \in \Pi_n^2\}. \tag{3.7}$$

It is well-known that $E_n(f)_p \to 0$ for $f \in L^p(B^2)$ as $n \to \infty$. The partial sum $S_n f$ of the orthogonal expansion is the best approximation to f in the L^2 norm; that is,

$$\|f - S_n f\|_2 = E_n(f)_2, \quad f \in L^2(B^2).$$

However, the partial sum $S_n f$ does not converge to f point-wisely if f is merely continuous; see [9] and the discussion below in Section 5.

To study the convergence of our orthogonal expansions we will introduce some summability methods. Such a method takes the form

$$\sum_{j=0}^{\infty} a_{j,n} S_j(f), \quad a_{j,n} \in \mathbb{R} \quad \text{and} \quad \sum_{j=0}^{\infty} a_{j,n} = 1.$$

Many summability methods, for example, the Poisson means and the Cesàro means, have better convergence behavior (see [7]). For our purpose we will use methods for which the operators are polynomials and we would still want to retain the property that polynomials up to certain degree are preserved.

It turns out that this can be done quite easily using a multiplier function.

Definition 3.4. Let r be a positive integer, and let $\eta \in C^r[0,\infty)$. Then η is called a *multiplier function* if

$$\eta(t) = 1, \quad 0 \leqslant t \leqslant 1, \quad \text{and} \quad \operatorname{supp} \eta \subset [0,2].$$

Let η be a multiplier function. We define an operator S_{2m}^{η} by

$$S_{2m}^{\eta}(f; x, y) = \sum_{k=0}^{2m} \eta\left(\frac{k}{m}\right) \operatorname{proj}_k f(x,y).$$

Such an operator was used in the literature for approximation by spherical polynomials on the unit sphere, and it was been used for various other orthogonal expansions in [10], including the expansions on the unit disk. The following theorem is essentially contained in [10].

Proposition 3.5. *Let $\eta \in C^3[0,\infty)$ be a multiplier function. Let $f \in L^p(B^2)$, $1 \leqslant p \leqslant \infty$. Then*

(1) $S_{2m}^{\eta} f \in \Pi_{2m}^2$ *and* $S_{2m}^{\eta} P = P$ *for* $P \in \Pi_m^2$;
(2) *For* $m \geqslant 0$ *there is a constant c such that*

$$\left\| S_{2m}^{\eta} f \right\|_p \leqslant c \|f\|_p \quad \text{and} \quad \left\| f - S_{2m}^{\eta} f \right\|_p \leqslant c E_m(f)_p.$$

This means that the operator S_{2m}^{η} is very well behaved: it preserves polynomials of degree up to m and it approximates f as accurate as any polynomial of degree at most m. Using Theorem 3.2 we also have the following:

Theorem 3.6. *For $m \geqslant 0$, the operator S_{2m}^{η} can be written as*

$$S_{2m}^{\eta}(f; x, y) = \sum_{\nu=0}^{2m} \frac{1}{\pi} \int_{-1}^{1} \mathcal{R}_{\phi_\nu}(f; t) \Phi_\nu^{\eta}(t; x, y)\, dt \tag{3.8}$$

where

$$\Phi_\nu^{\eta}(t; x, y) = \frac{1}{2m+1} \sum_{k=0}^{2m} \eta\left(\frac{k}{m}\right)(k+1) U_k(t_j) U_k(\phi_\nu; x, y). \tag{3.9}$$

Remark 3.1. We can also use other summability methods, not necessarily prescribed by the multiplier function. The essence of Proposition 3.5, however, should be preserved.

3.3. Fourier orthogonal expansion on a cylinder domain

Let $L > 0$ and let B_L be the cylinder region

$$B_L := B^2 \times [0, L] = \{(x, y, z):\ (x, y) \in B^2,\ z \in [0, L]\}.$$

Using the result in Section 3.1 we can also get an expression for the partial sum operator on B_L, which will lead us to a 3D reconstruction algorithm. We consider orthogonal polynomials with respect to the inner product

$$\langle f, g \rangle_{B_L} = \frac{1}{\pi} \int_{B_L} f(x, y, z) g(x, y, z) W_L(z)\, dx\, dy\, dz, \tag{3.10}$$

where W_L is a nonnegative function defined on $[0, L]$ with all its moments on $[0, L]$ assumed finite and normalized so that $\int_0^L W_L(z)\, dz = 1$.

Let Π_n^3 denote the space of polynomials of total degree at most n in three variables. Let $\mathcal{V}_n(B_L)$ denote the subspace of orthogonal polynomials of degree n on B_L with respect to the inner product (3.10); that is, $P \in \mathcal{V}_n(B_L)$ if $\langle P, Q \rangle_{B_L} = 0$ for all polynomial $Q \in \Pi_{n-1}^3$.

Let p_k be the orthonormal polynomials with respect to W_L on $[0, L]$. Let $U_k(\theta_{j,k}; x, y)$ be defined as in the previous subsection.

Proposition 3.7. *An orthonormal basis for $\mathcal{V}_n(B_L)$ is given by*

$$\mathbb{P}_n := \{P_{n,k,j}\colon\ 0 \leqslant j \leqslant k \leqslant n\}, \qquad P_{n,k,j}(x,y,z) = p_{n-k}(z) U_k(\theta_{j,k}; x, y).$$

In particular, the set $\{\mathbb{P}_l\colon\ 0 \leqslant l \leqslant n\}$ is an orthonormal basis for Π_n^3.

This is an easy consequence of the fact that B_L is a product of B^2 and $[0, L]$. For $f \in L^2(B_L)$, the Fourier coefficients of f with respect to the orthonormal system $\{\mathbb{P}_l\colon\ l \geqslant 0\}$ are given by

$$\hat{f}_{l,k,j} = \frac{1}{\pi} \int\limits_{B_L} f(x,y,z) P_{l,k,j}(x,y,z)\, dx\, dy\, W_L(z)\, dz, \quad 0 \leqslant j \leqslant k \leqslant l.$$

Let $S_n f$ denote the Fourier partial sum operator,

$$S_n f(x,y,z) = \sum_{l=0}^{n} \sum_{k=0}^{l} \sum_{j=0}^{k} \hat{f}_{l,k,j} P_{l,k,j}(x,y,z).$$

Just like its counterpart in two variables, this is a projection operator and is independent of the particular choice of the bases of $\mathcal{V}_n(B_L)$.

We retain the notation $\mathcal{R}_\phi(g;t)$ for the Radon projection of a function $g : B^2 \mapsto \mathbb{R}$. For a fixed z in $[0, L]$, we define

$$\mathcal{R}_\phi\big(f(\cdot,\cdot,z); t\big) := \int\limits_{I(\phi,t)} f(x,y,z)\, dx\, dy, \tag{3.11}$$

which is the Radon projection of f in a disk that is perpendicular to the z-axis.

The following is an analogue of Theorem 3.3 for the cylinder B_L.

Theorem 3.8. *For $m \geqslant 0$,*

$$S_{2m} f(x,y,z) = \frac{1}{2m+1} \sum_{\nu=0}^{2m} \frac{1}{\pi} \int_{-1}^{1} \int_{0}^{L} \mathcal{R}_{\phi_\nu}\big(f(\cdot,\cdot,w); t\big) \Phi_\nu(w,t;x,y,z) W_L(w)\, dw\, dt \tag{3.12}$$

where

$$\Phi_\nu(w,t;x,y,z) = \sum_{k=0}^{2m} (k+1) U_k(t) U_k(\phi_\nu; x, y) \sum_{l=0}^{2m-k} p_l(w) p_l(z). \tag{3.13}$$

Proof. Using Proposition 3.1 and the product nature of the region,

$$\hat{f}_{l,k,j} = \frac{1}{2m+1} \sum_{\nu=0}^{2m} \frac{1}{\pi} \int_{-1}^{1} \mathcal{R}_{\phi_\nu}(f_{l-k}; t) U_k(t)\, dt\, U_k\big(\cos(\theta_{j,k} - \phi_\nu)\big),$$

where

$$f_l(x, y) = \frac{1}{\pi} \int_0^L f(x, y, w) p_l(w) W_L(w)\, dw, \quad l \geqslant 0.$$

Substituting this expression of $\hat{f}_{l,k,j}$ into the formula of $S_{2m} f$ and using the identity (2.12) in Lemma 2.6, we then obtain

$$\begin{aligned} S_{2m} f(x, y, z) &= \frac{1}{2m+1} \sum_{\nu=0}^{2m} \frac{1}{\pi} \int_{-1}^{1} \sum_{l=0}^{2m} \sum_{k=0}^{l} \mathcal{R}_{\phi_\nu}(f_{l-k}; t) U_k(t)\, dt \\ &\quad \times (k+1) U_k(\phi_\nu; x, y) p_{l-k}(z) \\ &= \frac{1}{2m+1} \sum_{\nu=0}^{2m} \frac{1}{\pi} \int_{-1}^{1} \sum_{k=0}^{2m} \sum_{l=0}^{2m-k} \mathcal{R}_{\phi_\nu}(f_l; t) p_l(z)(k+1) U_k(\phi_\nu; x, y) U_k(t)\, dt, \end{aligned}$$

where in the second equality we have exchanged the two inner summations. From (3.11) and the definition of f_l it is easy to see that

$$\mathcal{R}_{\phi_\nu}(f_l; t) = \frac{1}{\pi} \int_0^L \mathcal{R}_{\phi_\nu}\big(f(\cdot, \cdot, w); t\big) p_l(w) W_L(w)\, dw,$$

from which the proof follows upon rearranging terms in the summation. □

Remark 3.2. Clearly one can also consider summability of orthogonal expansions on B_L. For example, one can define the operator with multiplier factors just as in the case of orthogonal expansion on B^2. We shall not elaborate.

4. New reconstruction algorithms

4.1. Reconstruction algorithm for 2D images

The identity (3.5) expresses $S_{2m} f$ in terms of the Radon projections $\mathcal{R}_{\phi_\nu}(f; t)$ of $2m+1$ equally spaced angles ϕ_ν, $0 \leqslant \nu \leqslant 2m$, along the circumference of the disk. These Radon projections are defined for all parameters t. In order to make use of the Radon data from the parallel geometry, we will use a quadrature rule to get a discrete approximation of the integrals

$$\int_{-1}^{1} \mathcal{R}_{\phi_\nu}(f; t) U_k(t)\, dt = \int_{-1}^{1} \frac{\mathcal{R}_{\phi_\nu}(f; t)}{\sqrt{1-t^2}} U_k(t) \sqrt{1-t^2}\, dt$$

in (3.5). The result will be our algorithm.

If f is a polynomial then the equation (2.5) shows that $\mathcal{R}_\phi(f;t)/\sqrt{1-t^2}$ is also a polynomial. Hence, we choose a quadrature rule for the integral with respect to $\sqrt{1-x^2}$ on $[-1,1]$. Let us denote such a quadrature rule by $\mathcal{I}_n g$; then

$$\frac{2}{\pi}\int_{-1}^{1} g(t)\sqrt{1-t^2}\,dt \approx \sum_{j=1}^{n}\lambda_j g(t_j) := \mathcal{I}_n(g), \tag{4.1}$$

where $t_1,\ldots,t_n$ are distinct points in $(-1,1)$ and λ_j are real numbers such that $\sum_{j=1}^{n}\lambda_j = 1$. If equality holds in (4.1) whenever g is a polynomial of degree at most ρ, then the quadrature rule is said to have the degree of exactness ρ.

Among all quadrature rules with a fixed number of nodes, the Gaussian quadrature has the highest degree of exactness. It is given by

$$\frac{1}{\pi}\int_{-1}^{1} g(t)\sqrt{1-t^2}\,dt = \frac{1}{n+1}\sum_{j=1}^{n}\sin^2\frac{j\pi}{n+1}\,g\left(\cos\frac{j\pi}{n+1}\right) := \mathcal{I}_n^G(g) \tag{4.2}$$

for all polynomials g of degree at most $2n-1$; that is, its degree of exactness is $2n-1$. Note that $j\pi/(n+1)$ are zeros of the Chebyshev polynomial U_n.

Using quadrature formula in (3.5) gives our reconstruction algorithm, which produces a polynomial $\mathcal{A}_{2m}f$ defined below.

Algorithm 4.1. Let the quadrature rule be given by (4.1). For $m \geqslant 0$ and $(x,y) \in B^2$,

$$\mathcal{A}_{2m}(f;x,y) = \sum_{\nu=0}^{2m}\sum_{j=1}^{n}\mathcal{R}_{\phi_\nu}(f;t_j)T_{j,\nu}(x,y), \tag{4.3}$$

where

$$T_{j,\nu}(x,y) = \frac{\lambda_j}{2(2m+1)\sqrt{1-t_j^2}}\Phi_\nu(t_j;x,y).$$

For a given f, the approximation process $\mathcal{A}_{2m}f$ uses the Radon data

$$\left\{\mathcal{R}_{\phi_\nu}(f;t_j):\ 0 \leqslant \nu \leqslant 2m,\ 1 \leqslant j \leqslant n\right\}$$

of f. The data consists of Radon projections on $2m+1$ equally spaced directions along the circumference of the disk (specified by ϕ_ν) and there are n parallel lines (specified by t_j) in each direction. If these parallel Radon projections are taken from an image f, then the algorithm produces a polynomial $\mathcal{A}_{2m}f$ which gives an approximation to the original image.

The polynomial $\mathcal{A}_{2m}$ is particularly handy for numerical implementation, since one could save $T_{j,\nu}$ in a table beforehand. This provides a very simple algorithm: given the Radon data, one only has to perform addition and multiplication to evaluate $\mathcal{A}_{2m}$ in (4.3) to get a reconstruction of image.

A good choice of the quadrature rule is Gaussian quadrature. We choose in particular $n = 2m$ so that the nodes of the quadrature rule (4.2) becomes $t_j = \cos\theta_{j,2m} = \cos j\pi/(2m+1)$. In this case, our algorithm takes a particular simple form.

Algorithm 4.2. For $m \geqslant 0$, $(x, y) \in B^2$,

$$\mathcal{A}_{2m}(f; x, y) = \sum_{\nu=0}^{2m} \sum_{j=1}^{2m} \mathcal{R}_{\phi_\nu}(f; \cos\theta_{j,2m}) T_{j,\nu}(x, y), \tag{4.4}$$

where

$$T_{j,\nu}(x, y) = \frac{1}{(2m+1)^2} \sum_{k=0}^{2m} (k+1) \sin\big((k+1)\theta_{j,2m}\big) U_k(\phi_\nu; x, y). \tag{4.5}$$

Remark 4.1. It is tempting to choose $n = 2m + 1$ in the Gaussian quadrature. In fact, such a choice has one advantage: the operator $\mathcal{A}_{2m}$ will be a projection operator onto Π_{2m}^2. We choose $n = 2m$ so that $\phi_\nu = \frac{2\nu\pi}{2m+1}$ and $\theta_{j,2m} = \frac{j\pi}{2m+1}$ have common denominator. It also turns out that this choice works perfectly with the fan beam geometry of the projections, which will be reported elsewhere.

The convergence of the algorithm will be discussed in the following section. Here we state one result that is a simple consequence of the definition.

Theorem 4.3. *The operator $\mathcal{A}_{2m}$ in Algorithm* 4.1 *preserves polynomials of degree σ. More precisely, $\mathcal{A}_{2m}(f) = f$ whenever f is a polynomial of degree at most σ. In particular, the operator $\mathcal{A}_{2m}$ in Algorithm* 4.2 *preserves polynomials of degree at most* $2m - 1$.

Proof. The polynomial $\mathcal{A}_{2m} f$ is obtained by applying quadrature rule (4.1) to (3.5). If f is a polynomial of degree at most σ, then so is $\mathcal{R}_{\phi_\nu}(f; t)/\sqrt{1-t^2}$ for every ν. Furthermore, the polynomial $\Phi_\nu(t; x, y)$ is a polynomial of degree $2m$ in t. Hence, when we apply the quadrature rule (4.1) to the integral (3.5), the result is exact. Therefore, $\mathcal{A}_{2m} f = S_{2m} f = f$. □

If the quadrature rule is the Gaussian quadrature in (4.2), then the polynomials preserved by the operator $\mathcal{A}_{2m}$ have the highest degrees among all quadrature rules that use the same number of nodes. Such a choice will ensure better approximation behavior of $\mathcal{A}_{2m}$.

Remark 4.2. Using the angles ϕ_ν, another projection operator, call it $\mathcal{J}_{2m} f$, has been constructed in [1] based on the parallel geometry. For almost all choices of $\{t_j \in (-1, 1)\colon 0 \leqslant j \leqslant m\}$, the operator $\mathcal{J}_{2m}$ is the unique polynomial of degree $2m$ determined by the conditions

$$\mathcal{R}_{\phi_\mu}(\mathcal{J}_{2m} f; t_i) = \mathcal{R}_{\phi_\mu}(f; t_i), \quad 0 \leqslant \nu \leqslant 2m,\ 0 \leqslant j \leqslant m.$$

However, the construction of $\mathcal{I}_{2m}$ requires solving a family of linear system of equations whose coefficient matrices, depending on the choice of t_j, appear to be badly ill-conditioned.

4.2. Reconstruction algorithm for 2D images with a multiplier function

Instead of using (3.5), we can also start from the summability with the multiplier factor (3.8). The result is another reconstruction algorithm. Here we state the resulting algorithm only for the Gaussian quadrature (4.2) with $n = 2m$.

Algorithm 4.4. For $m \geqslant 0$, $(x, y) \in B^2$,

$$\mathcal{A}_{2m}^{\eta}(f; x, y) = \sum_{\nu=0}^{2m} \sum_{j=1}^{2m} \mathcal{R}_{\phi_\nu}(f; \cos\theta_{j,2m}) T_{j,\nu}^{\eta}(x, y), \tag{4.6}$$

where

$$T_{j,\nu}^{\eta}(x, y) = \frac{1}{(2m+1)^2} \sum_{k=0}^{2m} \eta\left(\frac{k}{m}\right)(k+1)\sin\big((k+1)\theta_{j,2m}\big) U_k(\phi_\nu; x, y).$$

For a given f, the approximation process $\mathcal{A}_{2m}^{\eta} f$ uses the same Radon data of f as $\mathcal{A}_{2m} f$. It also has the same simple structure for numerical implementation. Its approximation behavior appears to be better than that of $\mathcal{A}_{2m} f$. We conclude this subsection with the following analogous of Theorem 4.3:

Theorem 4.5. *The operator $\mathcal{A}_{2m}^{\eta}$ preserves polynomials of degree m. More precisely, $\mathcal{A}_{2m}^{\eta}(f) = f$ whenever f is a polynomial of degree at most m.*

We can obtain other reconstruction algorithms using different summability methods; see, however, Remark 3.1.

4.3. Reconstruction algorithm for 3D images

In order to get a reconstruction algorithm for 3D images on the cylinder region B_L, we choose the weight function to be

$$W_L(z) = \frac{1}{\pi} \frac{1}{\sqrt{z(L-z)}}, \quad z \in [0, L],$$

which is the Chebyshev weight function on the interval $[0, L]$, normalized so that its integral over $[0, L]$ is 1. Let T_k be the Chebyshev polynomial of the first kind. Define $\widetilde{T}_k$ by

$$\widetilde{T}_0(z) = 1, \qquad \widetilde{T}_k(z) = \sqrt{2} T_k(2z/L - 1), \quad k \geqslant 1.$$

The polynomials $\widetilde{T}_k$ are orthonormal polynomials with respect to W_L on $[0, L]$.

To obtain an algorithm using parallel Radon projections, we start from (3.12) and apply quadrature rules on its integrals. For the integral in z, we use the Gaussian quadrature for W_L. Set

$$\xi_{i,n} = \frac{(2i+1)\pi}{2n} \quad \text{and} \quad z_i = \frac{1 + \cos\xi_{i,n}}{2}, \quad 0 \leqslant i \leqslant n-1,$$

where z_i are zeros of $T_n(z)$; then the Gaussian quadrature on $[0, L]$ takes the form,

$$\frac{1}{\pi}\int_0^L g(z)\frac{dz}{\sqrt{z(L-z)}} = \frac{1}{n}\sum_{i=0}^{n-1} g(z_i), \tag{4.7}$$

which holds whenever g is a polynomial of degree at most $2n-1$. For the integral in t, we could use the quadrature rule (4.1). For simplicity, however, we will only use the Gaussian quadrature (4.1).

This way we get an algorithm for reconstruction of images on B_L. The algorithm produces a polynomial $\mathcal{B}_{2m}$ of three variables as follows:

Algorithm 4.6. Let $\gamma_{\nu,j,i} = \mathcal{R}_{\phi_\nu}(f(\cdot,\cdot,z_i); \cos\theta_{j,2m})$. For $m \geqslant 0$,

$$\mathcal{B}_{2m} f(x,y,z) := \sum_{\nu=0}^{2m}\sum_{j=1}^{2m}\sum_{i=0}^{n-1} \gamma_{\nu,j,i} T_{\nu,j,i}(x,y,z),$$

where

$$T_{\nu,j,i}(x,y,z) = \frac{1}{n(2m+1)}\Phi_\nu(z_i, \cos\theta_{j,2m}; x, y, z).$$

For a given function f, the approximation process $\mathcal{B}_{2m}$ uses the Radon data

$$\big\{\mathcal{R}_{\phi_\nu}\big(f(\cdot,\cdot,z_i;\cos\theta_{j,2m})\big)\colon\ 0 \leqslant \nu \leqslant 2m,\ 1 \leqslant j \leqslant 2m,\ 0 \leqslant i \leqslant n-1\big\}$$

of f. The data consists of Radon projections on n disks that are perpendicular to the z-axis (specified by z_i); on each disk the Radon projections are taken in $2m+1$ equally spaced directions along the circumference of the disk (specified by ϕ_ν) and $2m$ parallel lines (specified by $\cos\psi_j$) in each direction. We can use this approximation for the reconstruction of the 3D images from parallel X-ray data. In practice, the integer n of z direction should be chosen so that the resolution in the z direction is comparable to the resolution on each disk to achieve isotropic result.

The following theorem is an analogous of Theorem 4.3 for B_L.

Theorem 4.7. *If $n \geqslant 2m$, then the operator $\mathcal{B}_{2m}$ in Algorithm* 4.6 *preserves polynomials of degree* $2m-1$. *More precisely, $\mathcal{B}_{2m}(f) = f$ whenever f is a polynomial of degree at most* $2m-1$.

Proof. If f is a polynomial of degree at most $2m-1$, then $\mathcal{R}_\phi(f(\cdot,\cdot,w); t)/\sqrt{1-t^2}$ is a polynomial of degree at most $2m-1$ both in the t variable and in the w variable. The function $\Phi_\nu(w,t;x,y,z)$ is of degree $2m$ in both the t variable and the w variable. Hence, when we use the quadrature rules for $S_{2m} f$ in (3.12), the result is exact if the quadrature rules are exact for polynomials of degree $4m-1$. For the quadrature rule (4.7) this holds if $n \geqslant 2m$. □

Remark 4.3. In the z direction, we choose the weight function $(z(1-z))^{-1/2}$ instead of constant weight function. The reason lies in the fact that the Chebyshev polynomials of the first kind are simple to work with and the corresponding Gaussian quadrature (4.7) is explicit. If we were to use constant weight functions, we would have to work with Legendre polynomials, whose zeros (the nodes of Gaussian quadrature) can be given only numerically.

5. Convergence of the reconstruction algorithm

In this section we study the convergence behavior of the reconstruction algorithm for 2D images and we work with $\mathcal{A}_{2m}$ in (4.4), for which the quadrature rule is chosen to be Gaussian quadrature.

5.1. Convergence of 2D reconstruction algorithm

Let us consider the uniform norm $\|\cdot\|_\infty$ of continuous functions on B^2. Convergence in the uniform norm guarantees the point-wise convergence of the reconstruction. First we give a formula for the operator norm. Let us denote by

$$\psi_j := \theta_{j,2m} = j\pi/(2m+1), \quad 0 \leqslant j \leqslant 2m.$$

Proposition 5.1. *Let $\|\mathcal{A}_{2m}\|_\infty$ denote the operator norm of $\mathcal{A}_{2m}$ as an operator from $C(B^2)$ into $C(B^2)$. Then*

$$\|\mathcal{A}_{2m}\|_\infty = \max_{(x,y)\in B^2} \Lambda_m(x,y), \quad \Lambda_m(x,y) := \sum_{\nu=0}^{2m}\sum_{j=1}^{2m} \sin\psi_j \left|T_{j,\nu}(x,y)\right|. \tag{5.1}$$

Proof. By the definition of $\mathcal{R}_\phi(f;t)$ we evidently have

$$\left|\mathcal{R}_\phi(f;t)\right| \leqslant \sqrt{1-t^2}\|f\|_\infty, \quad 0 \leqslant \phi \leqslant 2\pi,\ -1 < t < 1,$$

as seen from the second equality of (2.2). It follows immediately that

$$\left|\mathcal{A}_{2m}(f;x,y)\right| \leqslant \|f\|_\infty \Lambda_m(x,y), \quad \text{where } (x,y) \in B^2. \tag{5.2}$$

Taking maximum in both side proves that $\|\mathcal{A}_{2m}\|_\infty \leqslant \max_{(x,y)\in B^2} |\Lambda_m(x,y)|$. To show that the equality holds, let (x_0,y_0) be a point in B^2 at which $\Lambda_m(x,y)$ attains its maximum over B^2. Recall that $I(\theta,t)$ denote a line segment (2.1) inside B^2. Let Σ denote the set of intersection points of any two line segments $I(\phi_\nu, \cos\phi_j)$ and $I(\phi_\mu, \cos\psi_i)$,

$$\Sigma := \left\{(x,y)\colon\ I(\phi_\nu, \cos\psi_j) \cap I(\phi_\mu, \cos\psi_i),\ (i,\mu) \neq (j,\nu)\right\}.$$

The set contains only finitely many points. Let $\varepsilon > 0$ be small enough so that a disk centered at a point in Σ of radius ε contains no other points in Σ. Let Σ_ε denote the union of all such ε disks. We construct a function $f_\varepsilon \in C(B^2)$ as follows:

$$f(x,y) = \sin\psi_j \operatorname{sign} T_{j,\nu}(x_0,y_0), \qquad (x,y) \in I(\phi_\nu, \cos\phi_j) \setminus \left(I(\phi_\nu, \cos\phi_j) \cap \Sigma\right)$$

for all j,ν and $\|f_\varepsilon\|_\infty = 1$. Then $\mathcal{R}_{\phi_\nu}(f_\varepsilon, \cos\psi_j) = \operatorname{sign} T_{j,\nu}(x_0,y_0) + c_{j,\nu}\varepsilon$ for some constant $c_{j,\nu}$. Since there are only finitely many points in Σ, this shows that

$$\|\mathcal{A}_{2m}\|_\infty \geqslant \left|\mathcal{A}_{2m} f_\varepsilon(x_0,y_0)\right| = \Lambda_m(x_0,y_0) - c\,\varepsilon = \max_{(x,y)\in B^2} \Lambda_m(x,y) - c\,\varepsilon,$$

where c is a constant that depends on m. Taking $\varepsilon \to 0$ completes the proof. □

In the following we shall use the notation $A \approx B$ to mean that there exist two positive constants c_1 and c_2 such that $c_1 A \leqslant B \leqslant c_2 A$.

Recall that $\mathcal{A}_{2m} f$ is obtained from the partial sum $S_{2m} f$ of the Fourier orthogonal expansion, upon using Gaussian quadrature. It is proved in [9] that the operator norm for the partial sum operator S_{2m} is

$$\|S_{2m}\|_\infty \approx m. \tag{5.3}$$

One would expect that the operator norm of $\mathcal{A}_{2m}$ is worse than $\mathcal{O}(m)$ due to the additional step of using Gaussian quadrature. Our main result in this section shows that the norm of $\mathcal{A}_{2m}$ does not grow much worse.

Theorem 5.2. *For $\mathcal{A}_{2m}$ defined in* (4.4)*,*

$$\|\mathcal{A}_{2m}\|_\infty \approx m \log(m+1).$$

This shows that the price we pay for using Gaussian quadrature to get to $\mathcal{A}_{2m}$ is just a $\log(m+1)$ factor. This theorem will be proved in the following subsection.

For $f \in C(B^2)$, the quantity $E_n(f)_\infty$ defined in (3.7) denotes the error of best approximation of f by polynomials of degree at most n. It is proved in [10] that if $f \in C^{2r}(B^2)$, $r \in \mathbb{N}$, then

$$E_n(f) \leqslant c n^{-2r} \left\|\mathcal{D}^r f\right\|_\infty, \quad n \geqslant 0, \tag{5.4}$$

where $\mathcal{D}$ is a second order partial differential operator defined in (2.4).

As a consequence of Theorem 5.2, we have the following result:

Theorem 5.3. *If $f \in C^2(B^2)$, then $\mathcal{A}_{2m} f$ converges to f uniformly. In fact, let r be a positive integer*; *then for $f \in C^{2r}(B^2)$,*

$$\|\mathcal{A}_m f - f\|_\infty \leqslant c \frac{\log(m+1)}{m^{2r-1}} \left\|\mathcal{D}^r f\right\|_\infty.$$

Proof. Let p be the best approximation for f from Π_{2m}^2. By the definition of the operator norm, the fact that $\mathcal{A}_{2m}$ preserves polynomials of degree up to $2m-1$, and the triangle inequality we see that

$$\begin{aligned}\|\mathcal{A}_{2m} f - f\|_\infty &\leqslant \left\|C A_{2m}(f-p)\right\|_\infty + \|f-p\|_\infty \\ &\leqslant \left(1 + \|\mathcal{A}_{2m}\|_\infty\right) E_{2m-1}(f)_\infty \leqslant cm \log(m+1) E_{2m-1}(f)_\infty\end{aligned}$$

from which the stated inequality follows from (5.4). □

This theorem shows that the Algorithm 4.2 does converge whenever f is a $C^2(B^2)$ function. In other words, if the original image is C^2 smooth then the reconstruction algorithm 4.2 converges to the image point-wisely and uniformly. The speed of the convergence depends on the smoothness of the function.

The algorithm with multiplier function will likely have better convergence behavior (recall Proposition 3.5), but the estimate is more difficult to establish. We will report results along this line in future communications.

5.2. *Proof of Theorem 5.2, lower bound*

First we need a compact formula for the functions $T_{j,\nu}$ in (4.5). The notation

$$\cos\theta_\nu(x,y) = x\cos\phi_\nu + y\sin\phi_\nu, \quad \phi_\nu = 2\pi\nu/(2m+1)$$

will be used throughout the rest of this paper.

Proposition 5.4. *For* $0 \leqslant \nu \leqslant 2m$ *and* $1 \leqslant j \leqslant 2m$,

$$\begin{aligned}(2m+1)^2 T_{j,\nu}(x,y) &= \frac{-\sin\psi_j[1-(-1)^j T_{2m+1}(\cos\theta_\nu(x,y))]}{2(\cos\theta_\nu(x,y)-\cos\psi_j)^2} \\ &\quad - (-1)^j \sin\psi_j \frac{(2m+1)U_{2m}(\cos\theta_\nu(x,y))}{2(\cos\theta_\nu(x,y)-\cos\psi_j)}. \end{aligned} \tag{5.5}$$

Proof. To derive the formula, we start with the elementary trigonometric identity:

$$\sum_{k=0}^{2m}(k+1)\cos\bigl((k+1)\theta\bigr) = \frac{-1+(2m+2)\cos((2m+1)\theta)-(2m+1)\cos((2m+2)\theta)}{4\sin^2(\theta/2)}.$$

Let $\theta = \theta_\nu(x,y)$ in this proof. We apply the above identity to

$$\begin{aligned}\sin\theta(2m+1)^2 T_{j,\nu}(x,y) &= \sum_{k=0}^{2m}(k+1)\sin\bigl((k+1)\psi_j\bigr)\sin\bigl((k+1)\theta\bigr) \\ &= \frac{1}{2}\sum_{k=0}^{2m}(k+1)\bigl[\cos\bigl((k+1)(\theta-\psi_j)\bigr)-\cos\bigl((k+1)(\theta+\psi_j)\bigr)\bigr]\end{aligned}$$

and combine the two terms together as one fraction. The denominator of the fraction is

$$2\cdot 4\sin^2\frac{\psi_j+\theta}{2}\sin^2\frac{\psi_j-\theta}{2} = 2(\cos\theta-\cos\psi_j)^2$$

and the numerator of the fraction is

$$N(\theta) := \sin^2\frac{\theta+\psi_j}{2}h_{j,m}(\theta-\psi_j) - \sin^2\frac{\theta-\psi_j}{2}h_{j,m}(\theta+\psi_j)$$

where

$$h_{j,m}(\theta) = -1+(2m+2)\cos\bigl((2m+1)\theta\bigr)-(2m+1)\cos\bigl((2m+2)\theta\bigr).$$

We write $N(\theta)$ as a sum of two terms,

$$N(\theta) = N_1(\theta) - (2m+1)N_2(\theta).$$

For the first term we use the identities

$$\cos\big((2m+1)(\theta \pm \psi_j)\big) = (-1)^j \cos\big((2m+1)\theta\big),$$

and $2\sin^2(\theta/2) = 1 - \cos\theta$ to get

$$\begin{aligned}
N_1(\theta) :&= \sin^2\frac{\theta+\psi_j}{2}\big(-1 + (2m+2)\cos\big[(2m+1)(\theta-\psi_j)\big]\big) \\
&\quad - \sin^2\frac{\theta-\psi_j}{2}\big(-1 + (2m+2)\cos\big[(2m+1)(\theta+\psi_j)\big]\big) \\
&= \big(-1 + (2m+2)(-1)^j\cos\big[(2m+1)\theta\big]\big)\big(\cos(\theta-\psi_j) - \cos(\theta+\psi_j)\big)/2 \\
&= \big(-1 + (2m+2)(-1)^j\cos\big[(2m+1)\theta\big]\big)\sin\theta\sin\psi_j.
\end{aligned}$$

For the second term, we use

$$\cos\big((2m+2)(\theta \pm \psi_j)\big) = (-1)^j \cos\big((2m+2)\theta \pm \psi_j\big),$$

$2\sin^2(\theta/2) = 1 - \cos\theta$ and the addition formula of the cosine function to obtain

$$\begin{aligned}
&N_2(\theta) := \sin^2\frac{\theta+\psi_j}{2}\cos\big((2m+2)(\theta-\psi_j)\big) - \sin^2\frac{\theta-\psi_j}{2}\cos\big((2m+2)(\theta+\psi_j)\big) \\
&\quad = \frac{(-1)^j}{2}\big[\cos\big((2m+2)\theta-\psi_j\big) - \cos\big((2m+2)\theta+\psi_j\big)\big] \\
&\qquad + \frac{(-1)^j}{2}\big[\cos(\theta-\psi_j)\cos\big((2m+2)\theta-\psi_j\big) - \cos(\theta+\psi_j)\cos\big((2m+2)\theta+\psi_j\big)\big] \\
&\quad = (-1)^j\sin\psi_j\sin\big((2m+1)\theta\big) + \frac{(-1)^j}{2}\big[-\cos(\theta-\psi_j)\sin(\theta+\psi_i)\sin\big((2m+1)\theta\big) \\
&\qquad + \cos(\theta+\psi_j)\sin(\theta-\psi_i)\sin\big((2m+1)\theta\big)\big] \\
&\quad = (-1)^j\sin\psi_j\sin\big((2m+2)\theta\big) + (-1)^{j+1}\sin\big((2m+1)\theta\big)\sin\psi_j\cos\psi_j,
\end{aligned}$$

where we have used the double angle formula for sine in the last step. Using the addition formula $\sin((2m+2)\theta) = \sin((2m+1)\theta)\cos\theta + \cos((2m+1)\theta)\sin\theta$, we obtain

$$N_2(\theta) = (-1)^j\sin\psi_j\big[\sin\big((2m+1)\theta\big)(\cos\theta - \cos\psi_j) + \cos\big((2m+1)\theta\big)\sin\theta\big].$$

Putting the two terms together we obtain

$$\begin{aligned}
N(\theta) &= -\sin\theta\sin\psi_j + (-1)^j\sin\psi_j\big[(2m+2)\sin\theta\cos\big((2m+1)(\theta)\big) \\
&\quad - (2m+1)\big(\sin\big((2m+1)\theta\big)(\cos\theta - \cos\psi_j) + \cos\big((2m+1)\theta\big)\sin\theta\big)\big] \\
&= -\sin\theta\sin\psi_j\big[1 - (-1)^j\cos\big((2m+1)\theta\big)\big] \\
&\quad - (2m+1)(-1)^j\sin\psi_j\sin\big((2m+1)\theta\big)(\cos\theta - \cos\psi_j).
\end{aligned}$$

Consequently, we have proved that

$$\sin\theta(2m+1)^2 T_{j,\nu}(x,y) = \frac{-\sin\theta\sin\psi_j[1-(-1)^j\cos((2m+1)\theta)]}{2(\cos\theta-\cos\psi_j)^2} - \frac{(2m+1)(-1)^j\sin\psi_j\sin((2m+1)\theta)}{2(\cos\theta-\cos\psi_j)}.$$

Hence, using the fact that $T_n(\cos\theta)=\cos n\theta$ and $U_n(\cos\theta)=\sin(n+1)\theta/\sin\theta$, we conclude that

$$(2m+1)^2 T_{j,\nu}(x,y) = \frac{-\sin\psi_j[1-(-1)^j T_{2m+1}(\cos\theta)]}{2(\cos\theta-\cos\psi_j)^2} - \frac{(2m+1)(-1)^j\sin\psi_j U_{2m}(\cos\theta)}{2(\cos\theta-\cos\psi_j)},$$

which completes the proof. □

Throughout the rest of this section and in the following section, we will use the convention that c denotes a generic constant, independent of f and m, its value may change from line to line. The elementary facts

$$\frac{2}{\pi}t \leqslant \sin t \leqslant t, \quad \text{for } 0 \leqslant t \leqslant \frac{\pi}{2}, \quad \text{and} \quad \cos\alpha - \cos\beta = 2\sin\left(\tfrac{\beta-\alpha}{2}\right)\sin\left(\tfrac{\alpha+\beta}{2}\right)$$

will be used repeatedly without further mention.

We now use the expression (5.5) to derive the lower bound for the estimate.

Proposition 5.5.

$$\|\mathcal{A}_{2m}\|_\infty \geqslant \Lambda_m\left(\cos\frac{\pi}{4m+2}, \sin\frac{\pi}{4m+2}\right) \geqslant c\,m\log(m+1).$$

Proof. Let $x=\cos\frac{\pi}{4m+2}$ and $y=\sin\frac{\pi}{4m+2}$. Then we have $\cos\theta_\nu(x,y)=\cos\frac{(2\nu-1/2)\pi}{2m+1}$, so that $\sin(2m+1)\theta_\nu(x,y)=1$ and $\cos(2m+1)\theta_\nu(x,y)=0$. Let $\theta_\nu=\frac{(2\nu-1/2)\pi}{2m+1}$. Then by (5.5),

$$(2m+1)^2 T_{j,\nu}(x,y) = -\frac{\sin\psi_j}{2}\left[\frac{1}{(\cos\theta_\nu-\cos\psi_j)^2} + \frac{2m+1}{\sin\theta_\nu(\cos\theta_\nu-\cos\psi_j)}\right].$$

We will use the fact that $0<\theta_\nu<\pi$ and $0<\psi_j<\pi/2$ for $0\leqslant\nu,j\leqslant m$. Furthermore, for $0<\theta_\nu\leqslant\pi/2$ and $0<\psi<\pi/2$ we have

$$\frac{\sin\psi_j}{\sin\frac{\psi_j+\theta_\nu}{2}} = \frac{2\sin\frac{\psi_j}{2}\cos\frac{\psi}{2}}{\sin\frac{\psi_j+\theta_\nu}{2}} \leqslant 2\cos\frac{\psi_j}{2} \leqslant 2,$$

which also holds for $\pi/2\leqslant\theta_\nu\leqslant\pi$ and $0<\psi<\pi/2$, since then $\pi/4\leqslant\frac{\psi_j+\theta_\nu}{2}\leqslant 3\pi/4$ and $\sin\frac{\psi_j+\theta_\nu}{2}\geqslant\sqrt{2}/2$. Hence, we have

$$\frac{1}{(2m+1)^2}\sum_{\nu=0}^{m}\sum_{j=1}^{m}\frac{\sin^2\psi_j}{(\cos\theta_\nu-\cos\psi_j)^2}\leqslant\frac{1}{(2m+1)^2}\sum_{\nu=0}^{m}\sum_{j=1}^{m}\frac{1}{\sin^2\frac{\theta_\nu-\psi_j}{2}}$$
$$\leqslant\sum_{\nu=0}^{m}\sum_{j=1}^{m}\frac{1}{(2\nu-j-1/2)^2}\leqslant c\,m.$$

Consequently, it follows that

$$\Lambda_m(x,y)\geqslant\frac{1}{(2m+1)^2}\sum_{\nu=0}^{m}\sum_{j=1}^{m}\sin\psi_j\left|T_{j,\nu}(x,y)\right|$$
$$\geqslant\frac{1}{2m+1}\sum_{\nu=0}^{m}\sum_{j=1}^{m}\frac{\sin^2\psi_j}{|\sin\theta_\nu(\cos\theta_\nu-\cos\psi_j)|}-c\,m.$$

The sum in the last expression is bounded below by

$$\frac{8}{\pi^2}\frac{1}{2m+1}\sum_{\nu=1}^{m}\sum_{j=1}^{m}\frac{\psi_j^2}{\theta_\nu|\theta_\nu-\psi_j|(\theta_\nu+\psi_j)}$$
$$=\frac{8}{\pi^3}\sum_{\nu=1}^{m}\sum_{j=1}^{m}\frac{j^2}{(2\nu-1/2)|2\nu-j-1/2|(2\nu+j-1/2)}$$
$$\geqslant\frac{1}{\pi^3}\sum_{\nu=1}^{m/2}\sum_{j=\nu}^{2\nu-1}\frac{1}{2\nu-j}=\frac{1}{\pi^3}\sum_{\nu=1}^{m/2}\sum_{j=1}^{\nu}\frac{1}{j}=\frac{1}{\pi^3}\sum_{j=1}^{m/2}\frac{m/2-j+1}{j}$$
$$\geqslant\frac{1}{\pi^3}\left(\frac{m}{2}+1\right)\sum_{j=1}^{m/2}\frac{1}{j}-\frac{m}{2\pi^3}\geqslant c\,m\log(m+1).$$

This completes the proof. □

5.3. Proof of Theorem 5.2, upper bound

We will also use expression (5.5) to estimate $\Lambda_m(x,y)$ in (5.1) from above for all $(x,y)\in B^2$. In the following we write $\Lambda(x,y)=\Lambda_m(x,y)$.

It is easy to see that the function $\Lambda(x,y)$ is invariant under the dihedral group I_{2m+1}; that is, it is invariant under the rotation of a angle ϕ_ν for $\nu=0,1,\dots,2m$. Hence, it suffices if we establish the estimate assuming (x,y) is in the wedge

$$\Gamma_m:=\bigl\{(x,y)\colon\ x=r\cos\phi,\ y=r\sin\phi,\ r\geqslant 0,\ |\phi|\leqslant\varepsilon_m\bigr\},\quad \varepsilon_m=\frac{\pi/2}{2m+1}.$$

Note that the set Γ_m is symmetric with respect to the y axis.

We start with a number of reductions. The fact that $\phi_{2m+1-\nu}=2\pi-\phi_\nu$ shows $\cos\phi_{2m+1-\nu}=\cos\phi_\nu$ and $\sin\phi_{2m+1-\nu}=-\sin\phi_\nu$, which implies that $\theta_{2m+1-\nu}(x,y)=\theta_\nu(x,-y)$. Hence, using that $\theta_0(x,y)=x$, we can write

$$\begin{aligned}\Lambda(x,y) &= \sum_{j=1}^{2m} \sin\psi_j\left[\left|T_{0,\nu}(x,y)\right| + \sum_{\nu=1}^{m}\left(\left|T_{j,\nu}(x,y)\right| + \left|T_{j,\nu}(x,-y)\right|\right)\right] \\ &:= H_0(x,y) + \sum_{\nu=1}^{m}\left(H_\nu(x,y) + H_\nu(x,-y)\right).\end{aligned}$$

Since $(x, y) \in \Gamma_m$, we only need to consider the sum over $H_\nu(x, y)$.

Equation (5.5) shows that $T_{j,\nu}$ is naturally split as a sum of two functions,

$$T^{(1)}_{j,\nu} := \frac{-\sin\psi_j[1-(-1)^j T_{2m+1}(\cos\theta_\nu(x,y))]}{2(\cos\theta_\nu(x,y)-\cos\psi_j)^2}, \tag{5.6}$$

$$T^{(2)}_{j,\nu} := -(-1)^j \sin\psi_j \frac{(2m+1)U_{2m}(\cos\theta_\nu(x,y))}{2(\cos\theta_\nu(x,y)-\cos\psi_j)}. \tag{5.7}$$

We shall denote the corresponding splitting of $H_\nu(x, y)$ by $H^{(i)}_\nu(x, y)$ and the splitting of $\Lambda(x, y)$ (only the sum over $H_\nu(x, y)$ by $\Lambda^{(i)}(x, y)$). Thus,

$$\Lambda^{(i)}(x,y) := \sum_{\nu=1}^{m} H^{(i)}_\nu(x,y) \quad \text{and} \quad H^{(i)}_\nu(x,y) := \sum_{j=1}^{2m} \sin\psi_j\left|T^{(i)}_{j,\nu}(x,y)\right|, \quad i = 1, 2.$$

Next we split $H^{(i)}_\nu$ as two sums; the first one is over $j = 1, 2, \ldots, m$ and the second one is over $j = m + 1, m + 2, \ldots, 2m$. Since $\psi_{2m+1-j} = \pi - \psi_j$, we have $\cos\psi_{2m+1-j} = -\cos\psi_j$ so that we can write

$$H^{(i)}_\nu(x,y) = H^{(i)}_{\nu,1}(x,y) + H^{(i)}_{\nu,2}(x,y), \tag{5.8}$$

where, using that $1 - (-1)^j \cos(2m + 1)\theta = 1 - \cos(2m + 1)(\theta - \psi_j)$,

$$\begin{aligned}H^{(1)}_{\nu,1} &= \frac{1}{(2m+1)^2}\sum_{j=1}^{m}\sin^2\psi_j\left|\frac{1-\cos(2m+1)(\theta-\psi_j)}{2(\cos\theta_\nu(x,y)-\cos\psi_j)^2}\right|, \\ H^{(1)}_{\nu,1} &= \frac{1}{(2m+1)^2}\sum_{j=1}^{m}\sin^2\psi_j\left|\frac{1-\cos(2m+1)(\theta-\psi_j)}{2(\cos\theta_\nu(x,y)+\cos\psi_j)^2}\right|,\end{aligned} \tag{5.9}$$

and, using the fact that $(-1)^j \sin(2m + 1)\theta = \sin(2m + 1)(\theta - \psi_j)$,

$$\begin{aligned}H^{(2)}_{\nu,1}(x,y) &= \frac{1}{2m+1}\sum_{j=1}^{m}\sin^2\psi_j\left|\frac{\sin(2m+1)(\theta_\nu(x,y)-\psi_j)}{2\sin\theta_\nu(x,y)(\cos\theta_\nu(x,y)-\cos\psi_j)}\right|, \\ H^{(2)}_{\nu,2}(x,y) &= \frac{1}{2m+1}\sum_{j=1}^{m}\sin^2\psi_j\left|\frac{\sin(2m+1)(\theta_\nu(x,y)-\psi_j)}{2\sin\theta_\nu(x,y)(\cos\theta_\nu(x,y)+\cos\psi_j)}\right|.\end{aligned} \tag{5.10}$$

Let us denote the corresponding split of $\Lambda^{(i)}$ by $\Lambda_j^{(i)}$; that is,

$$\Lambda^{(i)} := \Lambda_1^{(i)} + \Lambda_2^{(i)}, \quad \text{where } \Lambda_j^{(i)} := \sum_{\nu=1}^{m} H_{\nu,j}^{(i)}, \ i, j = 1, 2.$$

We only need to estimate one of the two terms. To see this, let us define

$$\tilde{\phi}_\nu := (2\nu - 1)\pi/(2m+1), \quad 1 \leqslant \nu \leqslant m.$$

Then $\cos\phi_{m+1-\nu} = \cos(\pi - \tilde{\phi}_\nu) = -\cos\tilde{\phi}_\nu$ and $\sin\phi_{m-\nu+1} = \sin\tilde{\phi}_\nu$. Consequently,

$$\cos\theta_{m-\nu+1}(x, y) = -x\cos\tilde{\phi}_\nu + y\sin\tilde{\phi}_\nu = -\cos\tilde{\theta}_\nu(x, -y),$$

where $\tilde{\theta}_\nu(x, y) = x\cos\tilde{\phi}_\nu + y\sin\tilde{\phi}_\nu$. We will also use the notation $\widetilde{H}_{\nu,1}^{(i)}$ and $\widetilde{\Lambda}_j^{(i)}$ when $\theta_\nu(x, y)$ is replaced by $\tilde{\theta}_\nu(x, y)$ in $H_{\nu,1}^{(i)}$. It then follows from (5.10) that

$$H_{m-\nu+1,2}^{(i)}(x, y) = \widetilde{H}_{\nu,1}^{(i)}(x, -y), \quad 1 \leqslant \nu \leqslant m,$$

and, consequently, $\Lambda_2^{(i)}(x, y) = \widetilde{\Lambda}_1^{(i)}(x, -y)$. Hence, the estimate for $\Lambda_2^{(i)}$ will be similar to the estimate for $\Lambda_1^{(2)}$. In fact, set

$$\widetilde{\Gamma}_m := \{(x, y)\colon\ x = r\cos\phi,\ y = r\sin\phi,\ r \geqslant 0,\ |\pi - \phi| \leqslant \varepsilon_m\}, \quad \varepsilon_m = \frac{\pi/2}{2m+1};$$

then the estimate of $\Lambda_2^{(i)}(x, y)$ over $\widetilde{\Gamma}_m$ will be exactly the same as the estimate of $\Lambda_1^{(i)}(x, y)$ over Γ_m. Thus, we only need to estimate one sum, which we choose to be $\Lambda_1^{(i)}$.

We use Eqs. (5.9) and (5.10) to carry out the estimate. The inequality

$$\bigl|U_n(\cos t)\bigr| = \left|\frac{\sin(n+1)t}{\sin t}\right| \leqslant n + 1, \quad 0 \leqslant t \leqslant \pi, \tag{5.11}$$

will be used several times. The estimate is divided into several cases.

Lemma 5.6. *There exist constants c_1 and c_2 such that*

$$H_0^{(1)}(x, y) \leqslant c_1 \quad \text{and} \quad H_0^{(2)}(x, y) \leqslant c_2\left(m + \frac{1}{2}\right)\log(m+1)$$

for $(x, y) \in \Gamma_m$, where $c_1 < \pi^2(2 + \pi^2/12)$ and $c_2 < \pi^2/2 + 1$.

Proof. Since $\cos\theta_0(x, y) = x$ and $(x, y) \in \Gamma_m$, we can write $\theta = \theta_\nu(x, y)$ with $0 \leqslant \theta \leqslant \pi/2$.

We estimate the term $H_\nu^{(2)}$ first. Since $\cos\psi_j > 0$ for $1 \leqslant j \leqslant m$ and $\cos\theta \geqslant 0$, it follows from (5.11) that

$$H_{0,2}^{(2)}(x,y) \leqslant \sum_{j=1}^{m} \frac{\sin^2\psi_j}{\cos\psi_j} \leqslant \sum_{j=1}^{m} \frac{1}{\cos\psi_{m-j}} = \sum_{j=1}^{m} \frac{1}{\sin\frac{(j+1/2)\pi}{2m+1}}$$

$$\leqslant \frac{\pi}{2}\sum_{j=1}^{m} \frac{2m+1}{(j+1/2)\pi} \leqslant \left(m+\frac{1}{2}\right)\log(m+1).$$

For the term $H_{0,1}^{(2)}$, we need to consider the position of θ relative to that of ψ_j, which is divided into several further cases.

(A) If $0 \leqslant \theta \leqslant \varepsilon_m = \pi/(4m+2)$, then by (5.11)

$$H_{0,1}^{(2)}(x,y) \leqslant \sum_{j=1}^{m} \frac{\sin^2\psi_j}{4\sin\frac{\psi_j-\theta}{2}\sin\frac{\psi_j+\theta}{2}} \leqslant \pi^2 \sum_{j=1}^{m} \frac{\psi_j^2}{\psi_j^2-\theta^2} \leqslant \pi^2 \sum_{j=1}^{m} \frac{\psi_j^2}{\psi_j^2-\varepsilon_m^2}$$

$$\leqslant \pi^2 \sum_{j=1}^{m} \frac{j^2}{(j^2-1/4)} \leqslant \pi^2\left(m+\frac{1}{4}\sum_{j=1}^{m}\frac{1}{(j^2-1/4)}\right) \leqslant \pi^2\left(m+\frac{1}{2}\right).$$

(B) If $\psi_l - \varepsilon_m \leqslant \theta \leqslant \psi_l + \varepsilon_m$, where $1 \leqslant l \leqslant m$, then using the fact that

$$\left|\frac{\sin(2m+1)(\theta-\psi)}{\sin\frac{\theta-\psi_l}{2}}\right| \leqslant 2\left|\frac{\sin(2m+1)\frac{\theta-\psi}{2}}{\sin\frac{\theta-\psi_l}{2}}\right| \leqslant 2m+1$$

we obtain that

$$H_{0,1}^{(2)}(x,y) \leqslant \frac{\sin^2\psi_l}{4\sin\theta\sin\frac{\theta+\psi_l}{2}} + \left(\sum_{j=1}^{l-1} + \sum_{j=l+1}^{m}\right)\frac{\sin^2\psi_j}{4\sin\theta\sin\frac{|\psi_j-\theta|}{2}\sin\frac{\psi_j+\theta}{2}}$$

$$\leqslant \frac{\pi^2}{8}\frac{\psi_l^2}{\theta(\psi_l+\theta)} + \frac{\pi^3}{8(2m+1)}\left(\sum_{j=1}^{l-1} + \sum_{j=l+1}^{m}\right)\frac{\psi_j^2}{\theta|\theta-\psi_j|(\theta+\psi_j)}$$

$$\leqslant \frac{\pi^2}{8}\frac{l}{l-1/2} + \frac{\pi^3}{8(2m+1)\theta}\left(\sum_{j=1}^{l-1}\frac{j}{l-j-1/2} + \sum_{j=l+1}^{m}\frac{j}{j-l-1/2}\right).$$

The first sum in the last expression is bounded by

$$\frac{\pi^3}{8(2m+1)\theta}\sum_{j=1}^{l-1}\frac{j}{l-j-1/2} \leqslant \frac{\pi^2(l-1)}{8(l-1/2)}\sum_{j=1}^{l-1}\frac{1}{l-j-1/2} \leqslant \frac{\pi^2}{4}(\log m+2),$$

and the second sum is bounded by

$$\frac{\pi^3}{8(2m+1)\theta}\sum_{j=l+1}^{m}\frac{j}{j-l-1/2}\leqslant\frac{\pi^2}{8(l-1/2)}\sum_{j=1}^{m-l}\frac{j+l}{j-1/2}$$
$$\leqslant\frac{\pi^2 m}{4}\sum_{j=1}^{m-l}\frac{1}{j-1/2}\leqslant\frac{\pi^2 m}{4}(\log m+2).$$

Putting these estimates together and use (5.8), we complete the proof for $H_0^{(2)}$.

The proof for $H_0^{(1)}$ does not need require the splitting argument. By definition and (5.6),

$$H_0^{(1)}(x,y)=\frac{1}{(2m+1)^2}\sum_{j=1}^{2m}\sin^2\psi_j\frac{1-\cos((2m+1)(\theta-\psi_j))}{8\sin^2\frac{\theta-\psi_j}{2}\sin^2\frac{\theta+\psi_j}{2}}$$
$$=\frac{1}{(2m+1)^2}\sum_{j=1}^{2m}\sin^2\psi_j\frac{\sin^2\big((2m+1)\frac{\theta-\psi_j}{2}\big)}{4\sin^2\frac{\theta-\psi_j}{2}\sin^2\frac{\theta+\psi_j}{2}}.$$

Let θ be fixed and $|\theta-\psi_k|\leqslant\varepsilon_m$ for some k. Then by (5.11),

$$H_0^{(1)}(x,y)\leqslant\frac{\sin^2\psi_k}{4\sin^2\frac{\theta+\psi_k}{2}}+\frac{1}{(2m+1)^2}\sum_{j\neq k}\frac{\sin^2\psi_j}{4\sin^2\frac{\theta-\psi_j}{2}\sin^2\frac{\theta+\psi_j}{2}}$$
$$\leqslant\frac{\pi^2}{4}\left(1+\sum_{j=1}^{k-1}\frac{1}{(k-j-1/2)^2}+\sum_{j=k+1}^{m}\frac{1}{(j-k-1/2)^2}\right)$$
$$\leqslant\frac{\pi^2}{2}\left(4+\frac{\pi^2}{6}\right).$$

This completes the proof. □

Putting all pieces together, it is readily seen that the proof of the Theorem 5.2 follows from the conclusion of the following lemma.

Lemma 5.7. *There exist constants c_1 and c_2 such that*

$$\Lambda_1^{(1)}(x,y)\leqslant c_1\,(m+1)\quad\textit{and}\quad\Lambda_1^{(2)}(x,y)\leqslant c_2\,(m+1)\log(m+1)$$

for $(x,y)\in\Gamma_m$, where $c_1<\pi^2/96+7\pi^2/24+1/2$ and $c_2<(3/2)\pi^2+\pi+1$.

Proof. Again we consider the estimate for the sum $\Lambda_1^{(2)}$ first, which is more difficult than the estimate for $\Lambda_1^{(1)}$. Using the polar coordinates $x=r\cos\phi$ and $y=r\sin\phi$, we can write $\cos\theta_\nu(x,y)=r\cos(\phi-\phi_\nu)$. For $(x,y)\in\Gamma_m$, $|\phi|\leqslant\varepsilon_m$, so that

$$\frac{(2\nu-1/2)\pi}{2m+1}\leqslant\phi_\nu-\phi\leqslant\frac{(2\nu+1/2)\pi}{2m+1},$$

from which we conclude that $\cos\theta_\nu(x,y) \geqslant 0$ for $1 \leqslant \nu \leqslant m/2$ and that $\cos\theta_\nu(x,y) \leqslant 0$ for $m/2+1 \leqslant \nu \leqslant m$.

We shall break the sum in $\Lambda_1^{(i)}$ into two parts,

$$\Lambda_1^{(i)}(x,y) = L_1^{(i)}(x,y) + L_2^{(i)}(x,y),$$

where the first one has the sum taken over $1 \leqslant \nu \leqslant m/2$ and the second one has the sum taken over $m/2+1 \leqslant \nu \leqslant m$.

Case 1. We consider the estimate of $L_2^{(i)}(x,y)$. Note that $\cos\theta_\nu(x,y) \leqslant 0$ and $\cos\psi_j \geqslant 0$ for the terms in $L_2^{(i)}(x,y)$. If $\sin\theta_\nu(x,y) \geqslant \sqrt{2}/2$, then

$$\begin{aligned} L_2^{(2)}(x,y) &= \frac{1}{2m+1}\sum_{\nu=m/2+1}^{m}\sum_{j=1}^{m}\sin^2\psi_j\left|\frac{\sin((2m+1)\theta_\nu(x,y))}{2\sin\theta_\nu(x,y)(\cos\theta_\nu(x,y)-\cos\psi_j)}\right| \\ &\leqslant \frac{1}{2m+1}\sum_{\nu=m/2+1}^{m}\sum_{j=1}^{m}\frac{1}{\sqrt{2}}\frac{\sin^2\psi_j}{\cos\psi_j} \leqslant \frac{1}{4\sqrt{2}}\sum_{j=1}^{m}\frac{1}{\sin\frac{(j+1/2)\pi}{2m+1}} \\ &\leqslant \frac{2m+1}{8\sqrt{2}}\sum_{j=1}^{m}\frac{1}{j+1/2} \leqslant \frac{1}{4\sqrt{2}}\left(m+\frac{1}{2}\right)\log(m+1). \end{aligned}$$

If $\sin\theta_\nu(x,y) \leqslant \sqrt{2}/2$, then $-\cos\theta_\nu(x,y) = \cos(\pi-\theta_\nu(x,y)) \geqslant \sqrt{2}/2$. Furthermore, since $-\cos\theta_\nu(x,y) = -r\cos(\phi_\nu-\phi) \leqslant -\cos(\phi_\nu-\phi)$ for $m/2+1 \leqslant \nu \leqslant m$, we have $\theta_\nu(x,y) \leqslant \phi_\nu - \phi$ and $|\sin\theta_\nu(x,y)| = \sin(\pi-\theta_\nu(x,y)) \geqslant (2/\pi)(\pi-\phi_\nu+\phi)$. Hence, for $\sin\theta_\nu(x,y) \leqslant \sqrt{2}/2$,

$$\begin{aligned} L_2^{(2)}(x,y) &\leqslant \frac{1}{2m+1}\sum_{\nu=m/2+1}^{m}\sum_{j=1}^{m}\frac{1}{\sqrt{2}}\frac{2\sin^2\psi_j}{|\sin\theta_\nu(x,y)|} \\ &\leqslant \frac{\pi}{2\sqrt{2}}\sum_{\nu=m/2+1}^{m}\frac{1}{\pi-\phi_\nu+\phi} = \frac{1}{2\sqrt{2}}\sum_{\nu=1}^{m/2}\frac{2m+1}{2\nu+1} \\ &\leqslant \frac{1}{2\sqrt{2}}\left(m+\frac{1}{2}\right)\log(m+1). \end{aligned}$$

The case $L_2^{(1)}(x,y)$ is again easier. We have

$$\begin{aligned} L_2^{(1)}(x,y) &= \frac{1}{(2m+1)^2}\sum_{\nu=m/2+1}^{m}\sum_{j=1}^{m}\sin^2\psi_j\left|\frac{\sin^2\left((2m+1)\frac{\theta_\nu(x,y)-\phi_j}{2}\right)}{2(\cos\theta_\nu(x,y)-\cos\psi_j)^2}\right| \\ &\leqslant \frac{m}{4(2m+1)^2}\sum_{j=1}^{m}\frac{\sin^2\psi_j}{\cos^2\psi_j} \leqslant \frac{m}{16}\sum_{j=1}^{m}\frac{1}{(j+1/2)^2} \leqslant \frac{\pi^2}{96}m. \end{aligned}$$

Case 2. The estimate of $L_1^{(i)}(x, y)$ under the assumption that

$$0 \leqslant r \leqslant \cos(\psi_m - \varepsilon_m) = \sin \pi/(2m+1),$$

where $\cos\theta_\nu(x, y) = r\cos(\phi - \phi_\nu)$ in the polar coordinates for (x, y). The fact that $\cos\theta_\nu(x, y) \leqslant r$ implies that $\theta_\nu(x, y) \geqslant \psi_m - \varepsilon_m = \pi/2 - 2\varepsilon_m$, so that $\sin\theta_\nu(x, y) > 1/2$. Furthermore, $\pi/4 \leqslant \frac{\theta_\nu(x,y)+\psi_j}{2} \leqslant 3\pi/4$, which implies that $\sin\frac{\theta_\nu(x,y)+\psi_j}{2} \geqslant \sqrt{2}/2 > 1/2$. Consequently,

$$\begin{aligned}
L_1^{(2)}(x, y) &\leqslant \frac{1}{2m+1}\sum_{\nu=1}^{m/2}\sum_{j=1}^{m}\sin^2\psi_j\left|\frac{\sin((2m+1)(\theta_\nu(x,y)-\psi_j))}{2\sin\frac{\theta_\nu(x,y)-\psi_j}{2}\sin\frac{\theta_\nu(x,y)+\psi_j}{2}}\right| \\
&\leqslant \frac{1}{2m+1}\sum_{\nu=1}^{m/2}\sum_{j=1}^{m-1}\frac{\sin^2\psi_j}{\sin\frac{|\theta_\nu(x,y)-\psi_j|}{2}} + \sum_{\nu=1}^{m}\frac{\sin^2\psi_m}{\sin\frac{|\theta_\nu(x,y)+\psi_m|}{2}} \\
&\leqslant \frac{m\pi}{2(2m+1)}\sum_{j=1}^{m-1}\frac{1}{\pi/2-\psi_j-2\varepsilon_m} + 2m \\
&\leqslant \frac{m}{2}\sum_{j=1}^{m-1}\frac{1}{m-j-1/2} + 2m \leqslant m(\log m + 2).
\end{aligned}$$

Similarly, we have the estimate

$$\begin{aligned}
L_1^{(1)}(x, y) &= \frac{1}{(2m+1)^2}\sum_{\nu=1}^{m/2}\sum_{j=1}^{m}\sin^2\psi_j\left|\frac{\sin^2((2m+1)\frac{\theta_\nu(x,y)-\psi_j}{2})}{4\sin^2\frac{\theta_\nu(x,y)-\psi_j}{2}\sin^2\frac{\theta_\nu(x,y)+\psi_j}{2}}\right| \\
&\leqslant \frac{\pi^2}{4(2m+1)^2}\sum_{\nu=1}^{m/2}\sum_{j=1}^{m-1}\frac{1}{(\theta_\nu(x,y)-\psi_j)^2} + \sum_{\nu=1}^{m/2}\frac{\sin^2\psi_m}{4\sin^2\frac{\theta_\nu(x,y)+\psi_m}{2}} \\
&\leqslant \frac{m}{2}\sum_{j=1}^{m-1}\frac{1}{(m-j-1/2)^2} + \frac{m}{2} \leqslant \frac{m}{2}\left(\frac{\pi^2}{6}+5\right).
\end{aligned}$$

Case 3. The estimate of $L_1^{(i)}(x, y)$ under the assumption that

$$r \geqslant \cos(\psi_m - \varepsilon_m) = \sin\pi/(2m+1),$$

where $\cos\theta_\nu(x, y) = r\cos(\phi - \phi_\nu)$ in the polar coordinates for (x, y). Note that $\cos\theta_\nu(x, y) \geqslant 0$ for the terms in $L_1^{(i)}(x, y)$. In this case, $\cos\theta_\nu(x, y) - \cos\psi_j$ can be zero. For a fixed r let l be the index such that

$$r = \cos\theta, \qquad |\theta - \psi_l| \leqslant \varepsilon_m, \quad 1 \leqslant l \leqslant m.$$

We split the summation over j as two sums, one over $1 \leqslant j \leqslant l$ and the other over $l+1 \leqslant j \leqslant m$. This leads to a split of the sums in $L_1^{(i)}(x,y)$,

$$L_1^{(i)}(x,y) = \sum_{\nu=1}^{m/2}\sum_{j=1}^{l} + \sum_{\nu=1}^{m/2}\sum_{j=l+1}^{m} := L_{1,1}^{(i)}(x,y) + L_{1,2}^{(i)}(x,y).$$

We consider the two cases separately.

(A) The estimate for $L_{1,1}^{(i)}$. For $j \leqslant l$, $r = \cos\theta \leqslant \cos(\psi_l - \varepsilon) < \cos\psi_j$, so that $\cos\theta_\nu(x,y) \leqslant r < \cos\psi_j$, and consequently,

$$\left|\cos\theta_\nu(x,y) - \cos\psi_j\right| \geqslant \cos\psi_j - \cos\theta_\nu(x,y) \geqslant \cos\psi_j - \cos(\psi_l - \varepsilon_m) > 0.$$

Furthermore, the fact that $\cos\theta_\nu(x,y) = r\cos(\phi - \phi_\nu) \leqslant \cos(\psi_l - \varepsilon_m)$ implies that $\theta_\nu(x,y) \geqslant \psi_l - \varepsilon_m$. Therefore

$$\begin{aligned} L_{1,1}^{(2)}(x,y) &\leqslant \frac{1}{2m+1}\sum_{\nu=1}^{m/2}\sum_{j=1}^{l} \frac{\sin^2\psi_j}{4\left|\sin\theta_\nu(x,y)\sin\frac{\theta_\nu(x,y)-\psi_j}{2}\sin\frac{\theta_\nu(x,y)+\psi_j}{2}\right|} \\ &\leqslant \frac{\pi^3 m}{8(2m+1)}\frac{1}{\psi_l - \varepsilon_m}\sum_{j=1}^{l}\frac{\psi_j^2}{(\psi_l + \psi_j - \varepsilon_m)(\psi_l - \psi_j - \varepsilon_m)} \\ &\leqslant \frac{\pi^2 m}{4(2l-1)}\sum_{j=1}^{l}\frac{j}{l-j-1/2} \leqslant \frac{\pi^2 m}{2}\log m. \end{aligned}$$

The estimate for $L_{1,1}^{(1)}$ is similar:

$$\begin{aligned} L_{1,1}^{(1)}(x,y) &\leqslant \frac{1}{(2m+1)^2}\sum_{\nu=1}^{m/2}\sum_{j=1}^{l} \frac{\sin^2\psi_j}{8\left|\sin^2\frac{\theta_\nu(x,y)-\psi_j}{2}\sin^2\frac{\theta_\nu(x,y)+\psi_j}{2}\right|} \\ &\leqslant \frac{\pi^2 m}{4}\sum_{j=1}^{l}\frac{1}{(l-j-1/2)^2} \leqslant \frac{\pi^2 m}{16}\left(\frac{\pi^2}{6}+4\right). \end{aligned}$$

(B) The estimate for $L_{1,2}^{(i)}$. For $l < j \leqslant m$, $r = \cos\theta \geqslant \cos(\psi_l + \varepsilon) > \cos\psi_j$; hence, $\theta_\nu(x,y) - \cos\psi_j$ can be zero. Since $\cos\theta_\nu(x,y) = r\cos(\phi_\nu - \phi)$ is bounded by both $r \leqslant \cos(\psi_l - \varepsilon_m)$ and $\cos(\phi_\nu - \varepsilon_m)$ as $|\phi| \leqslant \varepsilon_m$, it follows that

$$\theta_\nu(x,y) \geqslant \max\{\psi_l, \phi_\nu\} - \varepsilon_m := z_{l,\nu} - \varepsilon_m > 0.$$

For each ν, we choose an index j_ν such that

$$\left|\theta_\nu(x,y) - \psi_{j_\nu}\right| \leqslant \varepsilon_m.$$

We need an estimate on j_ν. Since

$$2\sin^2\frac{\psi_{j_\nu}-\varepsilon_m}{2} = 1-\cos(\psi_{j_\nu}-\varepsilon_m) \leqslant 1-\cos\bigl(\theta_\nu(x,y)\bigr) = 1-r\cos(\phi_\nu-\phi)$$
$$= 1-r+r\bigl(1-\cos(\phi_\nu-\phi)\bigr) \leqslant 2\sin^2\frac{\psi_l}{2}+2\sin^2\frac{\phi_\nu-\phi}{2},$$

we obtain $(\phi_{j_\nu}-\varepsilon_m)^2 \leqslant \frac{\pi^2}{4}(\psi_l^2+(\phi_\nu-\phi)^2)$, from which it follows that

$$j_\nu - \frac{1}{2} \leqslant \frac{\pi}{2}\sqrt{l^2+(2\nu)^2} \leqslant \frac{\pi}{2}(l+2\nu). \tag{5.12}$$

Using the fact that for $j \neq j_\nu$,

$$\bigl|\theta_\nu(x,y)-\psi_j\bigr| \geqslant |\psi_{j_\nu}-\psi_j| - \bigl|\theta_\nu(x,y)-\psi_{j_\nu}\bigr| \geqslant |\psi_{j_\nu}-\psi_j| - \varepsilon_m,$$

we obtain the estimate

$$L_{1,2}^{(2)}(x,y) = \frac{1}{2m+1}\sum_{\nu=1}^{m/2}\sum_{j=l}^{m}\sin^2\psi_j\left|\frac{\sin((2m+1)(\theta_\nu(x,y)-\psi_j))}{4\sin\theta_\nu(x,y)\sin\frac{\theta_\nu(x,y)-\psi_j}{2}\sin\frac{\theta_\nu(x,y)+\psi_j}{2}}\right|$$
$$\leqslant \sum_{\nu=1}^{m/2}\frac{\sin^2\psi_{j_\nu}}{4\sin\theta_\nu(x,y)\sin\frac{\theta_\nu(x,y)+\psi_{j_\nu}}{2}}$$
$$+ \frac{1}{2m+1}\sum_{\nu=1}^{m/2}\sum_{j\neq j_\nu}\frac{\sin^2\psi_j}{4\sin\theta_\nu(x,y)\sin\frac{|\theta_\nu(x,y)-\psi_j|}{2}\sin\frac{\theta_\nu(x,y)+\psi_j}{2}}$$
$$\leqslant \frac{\pi^2}{8}\sum_{\nu=1}^{m/2}\frac{\psi_{j_\nu}}{z_{l,\nu}-\varepsilon_m} + \frac{\pi^2}{2(2m+1)}\sum_{\nu=1}^{m/2}\frac{1}{z_{l,\nu}-\varepsilon_m}\sum_{j\neq j_\nu}\frac{\psi_j}{|\psi_{j_\nu}-\psi_j|-\varepsilon_m}.$$

Using the inequality (5.12), the first sum in the last expression is bounded by

$$\frac{\pi^3}{16}\sum_{\nu=1}^{m/2}\frac{l+2\nu}{\max\{l,2\nu\}-1/2} \leqslant \frac{\pi^3}{8}m,$$

while the second sum is bounded by, again using (5.12),

$$\frac{\pi^2}{2(2m+1)}\sum_{\nu=1}^{m/2}\frac{1}{z_{l,\nu}-\varepsilon_m}\left(\sum_{j=l}^{j_\nu-1}\frac{j}{j_\nu-j-1/2} + \sum_{j=j_\nu+1}^{m}\frac{j}{j-j_\nu-1/2}\right)$$
$$\leqslant \frac{\pi}{2}\sum_{\nu=1}^{m/2}\frac{1}{\max\{l,2\nu\}-1/2}\left(j_\nu\log(j_\nu+1) + \sum_{j=1}^{m-j_\nu}\frac{j+j_\nu}{j-1/2}\right)$$

$$\leqslant \frac{\pi}{2}\sum_{\nu=1}^{m/2}\frac{1}{\max\{l,2\nu\}-1/2}\big(j_\nu\log(j_\nu+1)+(m-j_\nu)+(j_\nu+1/2)\log(m-j_\nu+1)\big)$$

$$\leqslant \frac{\pi}{2}(\pi+1)\log(m+1)\sum_{\nu=1}^{m/2}\frac{l+2\nu}{\max\{l,2\nu\}-1/2}+m\sum_{\nu=1}^{m/2}\frac{1}{\max\{l,2\nu\}-1/2}$$

$$\leqslant (\pi^2+\pi+1)m\log(m+1).$$

This completes the estimate for $L_{1,2}^{(2)}$. Similarly, but more easily, we have by (5.12),

$$L_{1,2}^{(1)}(x,y)\leqslant \sum_{\nu=1}^{m/2}\frac{\sin^2\psi_{j_\nu}}{4\sin^2\frac{\theta_\nu(x,y)+\psi_{j_\nu}}{2}}+\frac{1}{4(2m+1)^2}\sum_{\nu=1}^{m/2}\sum_{j\neq j_\nu}\frac{1}{\sin^2\frac{\theta_\nu(x,y)-\psi_j}{2}}$$

$$\leqslant \frac{m}{2}+\frac{m}{8}\sum_{j\neq j_\nu}\frac{1}{(j_\nu-j)^2}\leqslant \frac{m}{2}+\frac{m}{4}\sum_{j=1}^{\infty}\frac{1}{j^2}=\frac{m}{2}\left(1+\frac{\pi^2}{12}\right).$$

This competes the estimate of $L_{1,2}^{(i)}$.

Putting all cases together completes the proof of the lemma. It is easy to see that the largest constants in front of $m\log(m+1)$ or m in the three cases come from Case 3. □

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