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Note

Common Origin of Cubic Binomial Identities; A Generalization of Surányi's Proof on Le Jen Shoo's Formula

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In this paper we prove a binomial identity of five independent variables that is a common generalization of several cubic identities. Our proof is short and elementary, based on a coding process due to Surányi [8], on Surányi's second proof on Le Jen Shoo's formula (see [7]).

Our result contains as a very special case Le Jen Shoo's formula, of course. Some dozen papers have been concerned with this relic of mediaeval Chinese mathematics. Almost all references can be found in the papers of Takács [9] and of Kaucky [4].

The numbering of formulae refers to Gould's book [3].

THEOREM. *Let a, b, c, d, e be natural numbers. Then*

$$\binom{a+c+d+e}{a+c}\binom{b+c+d+e}{c+e}=\sum_k\binom{a+b+c+d+e-k}{a+b+c+d}\binom{a+d}{k+d}\binom{b+c}{k+c}.$$

Proof. We multiply both sides of our formula to $\binom{a+b+c+d}{a+d}$. The theorem is equivalent to

$$\binom{a+b+c+d}{a+d}\binom{a+c+d+e}{a+c}\binom{b+c+d+e}{c+e}$$
$$=\sum_k\frac{(a+b+c+d+e-k)!}{(a-k)!\,(b-k)!\,(c+k)!\,(d+k)!\,(e-k)!} \qquad (*)$$

Let us consider the set X of words containing exactly $a-k$ examples of A, $b-k$ examples of B, $c+k$ examples of C, $d+k$ examples of D, $e-k$ examples of E with an integer k. Let us consider the set Y of ordered triplets of

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$0-1$ vectors (w_1, w_2, w_3), where w_1 contains $a+b+c+d$ bits and $a+d$ of them are equal to 1, w_2 contains $a+c+d+e$ bits and $a+c$ of them are equal to 1, w_3 contains $b+c+d+e$ bits and $c+e$ of them are equal to 1.

Since the right-hand side of $(*)$ is the cardinality of X and the left-hand side is the cardinality of Y, in order to prove the theorem it is enough to prove the existence of a $f: X \to Y$ 1–1 mapping.

Suppose we had a word $w \in X$. We define $f(w) = (w_1, w_2, w_3) \in Y$ in the following way: the ith bit of w_1 is 1 iff after having erased the letters E from w the ith letter is either A or D; the ith bit of w_2 is 1 iff after having erased the letters B from w the ith letter is either A or C; the ith bit of w_3 is 1 iff after having erased the letters A from w the ith letter is either E or C.

We have to prove f to be injective and onto. Suppose we had two words, u and w with $f(u) = f(w)$. There exists a minimum index j such that the jth letter of u differs from the jth letter of w, or u is a beginner segment of w. Having controlled the all possible pairs of the five letters A, B, C, D, E, there exists an index $j \in \{1, 2, 3\}$ with $w_j(u) \neq w_j(w)$ because of the definition of f. So, in the first case we have $f(u) \neq f(w)$, a contradiction. It is easily seen that the second case is impossible, so, f is injection.

We tell the algorithm of computing f^{-1} and will prove its validity by mathematical induction. Let us consider the following table:

	A	B	C	D	E
w_1	1	0	0	1	x
w_2	1	x	1	0	0
w_3	x	0	1	0	1.

Let us make a column of the first bits of w_1, w_2, w_3. Disregarding the positions marked by x, our column will be equal to exactly one of the five columns of the table. We write the letter of the found column, erase those bits of w_i ($i \in \{1, 2, 3\}$) which were equal to the bits of the found column and remain untouched the bit of w_i corresponding to x of the found column. However it happens, that a vector runs out, the algorithm can be applied in this case too, since the (unique!) empty place will coincide with an x. In order to help understand we show an example.

f^{-1} (1001, 0011, 1100) $=$ *EEABBA*						
w_1	1001	1001	1001	001	01	1
w_2	0011	011	11	1	1	1
w_3	1100	100	00	00	0	–
	E	E	A	B	B	A

It is clear, that the algorithm gives $f^{-1}(w_1, w_2, w_3)$ if it finishes recognizing the last letter. The only problem would be a last column of type, e.g., (1, —, —). We prove the validity of the algorithm by induction on the sum of the numbers of bits of w_1, w_2, w_3. If it is 0, 1, 2, 3, the statement can be easily checked. If it is greater than 3, we recognize the first letter by the algorithm (suppose A) and by assumption we know the validity of the f^{-1}-algorithm for $a-1$, b, c, d, e (or a similar statement with an other letter instead of A).

COROLLARY 1 (Le Jen Shoo's identity). *Gould* [3, 6.32].

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{x+2n-k}{2n} = \binom{x+n}{n}^2.$$

Set $a=b=n$, $c=d=0$, $e=x$.

COROLLARY 2 (Surányi's identity [7, 8]). *Gould* [3, 6.19].

$$\sum_{k=0} \binom{n}{k}\binom{r}{k}\binom{x+n+r-k}{n+r} = \binom{x+r}{r}\binom{x+n}{n}.$$

Set $a=n$, $b=r$, $c=d=0$, $e=x$.

COROLLARY 3 (Bizley's identity). *Gould* [3, 6.42].

$$\sum_{K} \binom{B}{K}\binom{C}{K-D}\binom{A+K}{B+C} = \binom{A}{B-D}\binom{A+D}{C+D}.$$

Set $c=0$, $b=B$, $a=B-D$, $d=C-B+D$, $e=A-C$, $k=B-K$.

COROLLARY 4 (Nanjundiah's identity [5]). *Bizley* [1], *Gould* [3, 6.17].

$$\sum_{K} \binom{m-x+y}{K}\binom{n+x-y}{n-K}\binom{x+K}{m+n} = \binom{x}{m}\binom{y}{n}.$$

Set $a=0$, $b=x-y$, $c=n$, $d=m-x+y$, $e=x-m-n$, $k=-K$.

COROLLARY 5 (Stanley's identity [6]). *Gould* [2; 3, 6.52].

$$\sum_{K} \binom{x+y+K}{K}\binom{y}{A-K}\binom{x}{B-K} = \binom{x+A}{B}\binom{y+B}{A}.$$

Set $a=y-A$, $b=x-B$, $c=B$, $d=A$, $e=0$, $k=-K$.

COROLLARY 6 (Gould's identity [3, 6.51]).

$$\sum_K \binom{n}{K}\binom{r}{K}\binom{x+n+r+K}{n+r} = \binom{x+n+r}{n}\binom{x+n+r}{r}.$$

Set $a=b=0$, $c=n$, $d=r$, $k=-K$.

COROLLARY 7 (Takács' identity [9]).

$$\sum_{j=0}^{S} \binom{r}{j}\binom{m-r}{s-j}\binom{t+j}{m} = \binom{t}{m-s}\binom{t-m+s+r}{s}.$$

Set $a=m-s$, $b=r$, $c=0$, $d=s-r$, $e=t-a-d$, $k=r-j$.

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