JOURNAL OF ALGEBRA 139, 134–154 (1991)

State Spaces, Finite Algebras, and Skew Group Rings

RICARDO ALFARO*

Mathematics Department, University of Cincinnati, Cincinnati, Ohio 45221-0025

Communicated by Susan Montgomery

Received July 26, 1989

K. Goodearl [G1] considered the State Space, St(R), of a noetherian ring R; and, in joint work with R. B. Warfield [GW], described the structure of the sate space of commutative rings, hereditary noetherian prime rings, and orders over Dedekind Domains.

In this work, we give some properties of the state space of a ring in relation to the ideas of localization on noncommutative rings and of ramification in generically Galois actions. We show how the structure of St(R) depends on the property of localization at prime ideals. Some general results are given for the state space of a finite algebra over a commutative noetherian ring; then we apply these results to St($S * G$) where S is a commutative noetherian domain, G is a finite group acting faithfully as automorphisms of the ring S, and $|G|$ is invertible in S. We show that the state space St($S * G$) is the affine continuous image of an amalgamation of the state spaces of the ring localized at non-trivial clans, in particular that those arising from the ramified primes of S suffice. We show that St($S * G$) is a singleton if and only if the action is G-Galois on S. Finally, for S a Dedekind domain, we give a structure theorem for the state space of the skew group ring $S * G$.

1. PRELIMINARIES

Let $K_0(R)$, as defined in [B], be Grothendieck group of the category of f.g. projective left R-modules. Let $K_0^+(R)$ denote the *positive cone*, i.e., the set of all isomorphism classes of f.g. projective left R-modules. A pre-order relation is defined in $K_0(R)$ by $X \leqslant Y$ if and only if $Y - X$ is in the positive

* Research partially supported by a Charles P. Taft Fellowship from the University of Cincinnati. Current address: Department of Mathematics, University of Michigan-Flint, Flint, MI 48502.

0021-8693/91 $3.00

cone. An *order-unit* in $K_0(R)$ is an element U of $K_0^+(R)$ such that for every $X \in K_0(R)$, there is a positive integer n with $X \leqslant nU$.

Let U be an order-unit in $K_0(R)$, a *state* on the pre-ordered abelian group $(K_0(R), U)$ is a map $s: K_0(R) \to \mathbb{R}$, where $\mathbb{R}$ is the group of real numbers under addition, satisfying: (1) $(U)\,s = 1$; (2) $(X)\,s \geqslant 0$ for all X in $K_0^+(R)$ and; (3) s is a group homomorphism. The *state space* $\mathrm{St}(K_0(R), U)$ is the set of all the states. The *State Space* $\mathrm{St}(R)$ *of the ring* R is defined to be $\mathrm{St}(K_0(R), [R])$. For this and other background materials on states, see [G1].

The state space is determined by the set of its extreme points, denoted by $\partial_e \mathrm{St}(K_0(R), U)$, and their affine relations, see [GW]. Stafford [St] shows that for commutative rings and for noetherian rings of finite Krull dimension, the extreme points are given by the "generalized rank functions." Let P be a prime ideal of a neotherian ring R and let A be a f.g. projective left R-module, the "generalized rank function" r_P is defined by

$$(A)\,r_P = \frac{\text{length}\,(\mathbf{Q}(R/P) \otimes_R A)}{\text{length}\,(\mathbf{Q}(R/P))},$$

where $\mathbf{Q}(R/P)$ is the simple Artinian quotient ring of the prime noetherian ring R/P. They induce canonically the states s_P on R given by $(X)\,s_P = ([M]\text{-}[N])\,s_P = (M)\,r_P\text{-}(N)\,r_P$. If R is a commutative ring, the generalized rank functions r_P are defined by $(A)\,r_P = \dim_{Q(R/P)} (\mathbf{Q}(R/P) \otimes_R A)$, where $\mathbf{Q}(R/P)$ is the quotient field of the domain R/P. Similarly r_P induces a state s_P on R. Stafford [St, Theorem 6.4] proved that if R is an noetherian ring of finite Krull dimension, then $\partial_e \mathrm{St}(R) \subseteq \{s_P/P \text{ is a } J\text{-prime ideal of } R\}$. For commutative rings, hereditary noetherian prime rings, and simple orders over Dedekind domains, the structure of the state space has been given by K. Goodearl and R. B. Warfield in [GW]. We will use this result to find structure theorems for other state spaces.

We will call a ring R, left regular, if every f.g. left R-module has a finite resolution of projective left R-modules. In this way, this implies both definitions of regular rings, the one in [B] and the one in [Q]. Also, by a noetherian ring we mean left and right noetherian.

In all this work the rings will have 1; when linearity is obvious, the proofs will be restricted to the positive cone of the respective Grothendieck groups; and we will denote a homeomorphism by $\sim$ and an affine homeomorphism by $\approx$. In order to be consistent with the notation, functions act on the left. The skew group ring is defined as in [Mo1].

We now present some results that we will use later in the paper.

PROPOSITION 1.1. *Let* R *and* S *be Morita equivalent rings. Then* $\mathrm{St}(R) \sim \mathrm{St}(S)$. *Moreover, the homeomorphism preserve faces.*

Proof. There exists a f.g. projective generator A for left R-modules with $S \cong \mathrm{End}_R A$ and a group isomorphism $\varphi : K_0(R) \to K_0(S)$ with $([M])\,\varphi = ([A^* \otimes_R M])$ for all f.g. projective left R-modules M. We have $([A])\,\varphi = [S]$ and $(K_0^+(R))\,\varphi = K_0^+(S)$. These conditions allow us to define a map $\vartheta : \mathrm{St}(K_0(R), [A]) \to \mathrm{St}(K_0(S), [S])$ by $(s)\,\vartheta = \varphi s$. It is easy to check that ϑ is an affine homeomorphism. On the other hand, the "normalization" $\theta : \mathrm{St}(K_0(R), [R]) \to \mathrm{St}(K_0(R), [A])$ given by $(r)\,\theta = r\,(1/([A])\,r)$ is a well defined homeomorphism since $[A]$ is an order-unit in $K_0(R)$. Then the product $\theta\vartheta : \mathrm{St}(R) \to \mathrm{St}(S)$ is a homeomorphism. To prove that $\theta\vartheta$ preserves faces, it is enough to show that θ does since ϑ is an affine map. Let $h = f\alpha + g(1-\alpha)$, with $0 < \alpha < 1$ and $f, g \in \mathrm{St}(R)$. Then $(h)\,\theta = (f\alpha + g(1-\alpha))(1/([A])\,h) = (f)\,\theta(([A])f\alpha/([A])\,h) + (g)\,\theta(([A])\,g(1-\alpha)/([A])\,h) = (f)\,\theta\beta + (g)\,\theta(1-\beta)$ with $0 < \beta = ([A])f\alpha/([A])\,h < 1$; thus θ preserves faces.

COROLLARY 1.2. *Let R and S be Morita equivalent rings via the progenerator A of R-*Mod. *Then* $\mathrm{St}(R) \approx \mathrm{St}(S)$ *if and only if* $([A])f = ([A])\,g$ *for all f, g in* $\mathrm{St}(R)$.

Proof. We just need to show that θ is an affine map in the proof of the proposition. But θ is affine if and only if $\beta = \alpha$, which means $([A])\,h = ([A])f$ with $h = f\alpha + g(1-\alpha)$. Hence θ is affine if and only if $([A])f(1-\alpha) = ([A])\,g(1-\alpha)$, but $\alpha < 1$, so $([A])f = ([A])\,g$ for all f, g in $\mathrm{St}(R)$.

EXAMPLE 1.3. Let $M_n(R)$ be the ring of $n \times n$ matrices over a ring R, then $\mathrm{St}(M_n(R)) \approx \mathrm{St}(R)$. Since $M_n(R)$ is Morita equivalent to R via $A = R^n$ we have $([R^n])\,f = n = ([R^n])\,g$ for all $f, g \in \mathrm{St}(R)$.

PROPOSITION 1.4. *Let $R = \oplus\, R_i$ $i \geqslant 0$, be a graded left regular ring. Then* $\mathrm{St}(R) \approx \mathrm{St}(R_0)$.

Proof. The inclusion map $\iota : R_o \hookrightarrow R$ induces a group isomorphism $K_0(R_0) \cong K_0(R)$ by [B, Theorem XII. 3.2] sending $[R_0]$ to $[R]$ and preserving positive cones, thus the induced map $\mathrm{St}(\iota) : \mathrm{St}(R) \to \mathrm{St}(R_0)$ is an injective affine continuous map. On the other hand we have a natural projection $\rho : R \to R/R_+$ $(\cong R_0)$ with $\iota\rho = id_{R_0}$. Hence, since St is a contravariant functor, $\mathrm{St}(\rho)\ \mathrm{St}(\iota) = \mathrm{id}_{\mathrm{St}(R_0)}$ and then $\mathrm{St}(\iota)$ is surjective, thus an affine homeomorphism.

EXAMPLE 1.5. Let S be a graded left regular ring, $S = \oplus S_i$, $i \geqslant 0$. Let G be a finite group acting as automorphisms of S. We say that G "respects the grading" if $(S_i)^g = S_i$ for all $i \geqslant 0$ and for all $g \in G$. If this is the case, then the skew group ring $S * G$ is also a graded left regular ring with $(S * G)_i = S_i * G$ and we obtain, by Proposition 1.4, $\mathrm{St}(S * G) \approx \mathrm{St}(S_0 * G)$.

EXAMPLE 1.6. Let $S = k[x, y]$ be the polynomial ring in two variables where $\mathrm{char}(k) \neq 2$. Thus S is a graded left regular ring where the grading is given by the subgroups S_i generated by monomials of total degree i. Let G be the group generated by the automorphisms α and β given by $(x)\alpha = -x$, $(y)\alpha = y$, and $(x)\beta = x$, $(y)\beta = -y$; so G respects the grading of S. Hence $\mathrm{St}(k[x, y] * G) \approx \mathrm{St}(kG)$. But, since G is abelian, the group algebra kG is isomorphic to the semisimple artinian ring $k \oplus k \oplus k \oplus k$, so $\mathrm{St}(k[x, y] * G)$ is affinely homeomorphic to a 3-dimensional simplex.

Let R be a filtered ring. Quillen's theorem [Q] says that if the associated graded ring S is left noetherian, left regular, and flat as left R_0-module, then the inclusion map $\imath : R_0 \to R$ induces an isomorphism $K_0(R_0) \cong K_0(R)$. This ismomorphism sends $[R_0]$ to $[R]$ and preserves positive cones. Hence the induced map $\mathrm{St}(R) \to \mathrm{St}(R_0)$ is an injective affine continuous map. We now show that it is not necessarily a homeomorphism.

EXAMPLE 1.7. Consider the Weyl algebra $A_1(k)$, where k is a field of characteristic zero. Let G be the cyclic group generated by the automorphism σ acting by $(x)\sigma = \varepsilon x$ and $(y)\sigma = \varepsilon^{-1} y$, where ε is a primitive nth root of unity. It is well known that the skew group ring $A_1(k) * G$ is simple noetherian because the action of G is outer, and hence the state space $\mathrm{St}(A_1(k) * G)$ is a point [St, Theorem 6.4]. On the other hand $A_1(k)$ is a filtered ring, with filtration given by the k-subspaces F_j generated by the monomials $x^m y^n$ with $m + n \leqslant j$. So the group G respects the filtration and hence $A_1(k) * G$ is a filtered ring with $\mathbb{F}_j = F_j * G$. The associated graded ring is $S = \mathrm{gr}(A_1(k)) * G = k[x, y] * G$, where x and y are commuting indeterminants. Then S is a left regular, noetherian ring and flat as a left kG-module. Hence there is an injective map $\mathrm{St}(A_1(k) * G) \subsetneq \mathrm{St}(kG)$; but since G is abelian, the group algebra kG is isomorphic to a direct sum of n copies of the field k and thus $\mathrm{St}(kG)$ is affinely homeomorphic to an $(n-1)$-dimensional simplex.

2. GALOIS ACTIONS AND RAMIFICATION

For any ring S in which a finite group G acts as automorphisms and $|G|$ is invertible in S, there are one-to-one correspondences between the G-orbits of prime ideals of S, certain finite subsets of prime ideals of $S * G$, and certain finite subsets of prime ideals of S^G, for details see [LP1, Pa1, Mo2]. Let P be a prime ideal of S. Define $Lu(P)$ to be the number of prime ideal minimal over $P^0 * G$ where $P^0 = \bigcap P^g$, so $Lu(P) \leqslant |G|$.

Similarly define $Lo(P)$ to be the number of prime ideals minimal over $P \cap S^G$ in S^G, so $Lo(P) \leqslant |G|$. Let $\pi = \sum_{g \in G} g$, then $e = (1/|G|)\,\pi$ is an idempotent in $S * G$ and $eS * Ge \cong S^G$. The map ϕ: Ideals$(S * G) \to$ Ideals(S^G) given by $I \mapsto eIe$ induces a one-to-one correspondence between prime ideals of the skew group ring that do not contain e and prime ideals of the fixed ring. Moreover, if q is a prime ideal of S^G and $\mathcal{Q} = (q)\,\phi^{-1}$, then there exists a prime ideal Q of S such that $\mathcal{Q}$ is minimal over $Q^0 * G$ and q is minimal over $Q^0 \cap S^G$. Hence $Lo(P) \leqslant Lu(P)$. For a detailed example of this correspondence, see [Mo2, Example 4.4].

Now regard S as a left S^G-module and right $S * G$-module with action given by $s \cdot rg = (sr)^g$ for all s, $r \in S$ and $g \in G$. There is a natural map $\varphi : S * G \to \mathrm{End}_{S^G}(S)$ given by right multiplication. The group G is said to be *Galois* (or the action is G-Galois on S) if any of the two following equivalent conditions are satisfied (see [DM]):

(1) S is a finitely generated projective left S^G-module and φ is an ring isomorphism.

(2) $\exists a_1, ..., a_n, b_1, ..., b_n$ in S such that $\sum_{i=1}^{n} a_i b_i^{g^{-1}} = \begin{cases} 1 & \text{if } g = 1 \\ 0 & \text{if } g \neq 1 \end{cases}$.

Note that condition (2) can be written as $S\pi S = S * G$. If it exists, the set $\{a_1, ..., a_n, b_1, ..., b_n\}$ is called a *Galois basis* for S.

EXAMPLE 2.1. *State Spaces for G-Galois actions.* If G is Galois and, in addition, there exists an element of trace 1 in S, then the trace map is a surjective left S^G-module homomomorphism and, by Morita's theorem, S is a f.g. projective generator for left S^G-Mod with $S * G \cong \mathrm{End}_{S^G}(S)$. Hence $S * G$ and S^G are Morita equivalent and then by Proposition 1.1, $\mathrm{St}(S * G) \sim \mathrm{St}(S^G)$. This would be an affine homeomorphism if and only if $([S])\,r = ([S])\,t$ for all r, t in $\mathrm{St}(S^G)$, for example, if S is a free left S^G-module which is the case when S is simple Artinian and G is outer, see [Mo1].

Now assume S is a semiprime noetherian ring, and $|G|^{-1} \in S$. Let $\mathbf{Q}(S)$ be the semisimple Artinian quotient ring of S. It follows from [Mo1] that $S * G$ and S^G are also semiprime noetherian rings and S is a f.g. noetherian left S^G-module. Thus there is an induced action of G on $\mathbf{Q}(S) = S_T$ where T is the set of regular elements of S^G. Hodges [H1] defines G to be *generically Galois* on S if the induced action of G on $\mathbf{Q}(S)$ is G-Galois. The following lemma describes the equivalent notions of generically Galois actions:

LEMMA 2.2 [H1, Lemma 2.1]. *Let S, $\mathbf{Q}(S)$, G, and π be as above. The following are equivalent*:

(1) *G is generically Galois on S.*

(2) $\mathbf{Q}(S)\,\pi\mathbf{Q}(S)=\mathbf{Q}(S)*G$.

(3) *$S\pi S$ is an essential ideal of $S*G$.*

(4) *$S\pi S\cap S$ is an essential ideal of S.*

We now describe the notion of ramification as in [H1]. Let S be a noetherian ring and G be a finite group of automorphisms of S with $|G|^{-1}\in S$. A prime ideal P of S is said to be *unramified* if the induced action of G on S/P^0 is generically Galois. Otherwise, the prime P is said to be *ramified*. The ring S is said to be unramified if and only if all primes are unramified; otherwise, it is called ramified. The following theorem does not seem to appear in the literature:

THEOREM 2.3. *Let S be a noetherian ring and G a finite group of automorphisms of S with $|G|^{-1}\in S$. A prime ideal P of S is unramified if and only if $Lu(P)=Lo(P)$.*

Proof. ($\Rightarrow$) Let $Lu(P)=n$, so there are n minimal primes in $S*G/P^0*G=S/P^0*G$ (see [LP2]). Since S/P^0 is semiprime noetherian, $\mathbf{Q}(S/P^0*G)=\mathbf{Q}(S/P^0)*G$ with n minimal prime ideals $\mathscr{P}$. But since P is unramified, G is Galois on $\mathbf{Q}(S/P^0)$, so $\mathbf{Q}(S/P^0)\,\pi\mathbf{Q}(S/P^0)=\mathbf{Q}(S/P^0)*G$ and thus $e\notin\mathscr{P}$ for any minimal prime ideal $\mathscr{P}$. Hence there are n minimal primes $\rho=e\mathscr{P}e$ of $\mathbf{Q}(S/P^0)^G$. But $\mathbf{Q}(S/P^0)^G=\mathbf{Q}((S/P^0)^G)=\mathbf{Q}(S^G/P^0\cap S^G)$ [Mo 1, Theorem 5.3], thus there are n minimal primes in $S^G/P^0\cap S^G$. Therefore, there are n primes p of S^G minimal over $P^0\cap S^G$.

($\Leftarrow$) After passing to S/P^0, we may assume that S is semiprime and $P^0=0$; thus we need to prove that G is generically Galois on S. Let $\mathscr{P}_1,\dots,\mathscr{P}_n$ be the minimal prime ideals of $S*G$, then $p_1=e\mathscr{P}_1e,\dots,p_n=e\mathscr{P}_ne$ are all the minimal primes of S^G and $e\notin\mathscr{P}_i$ for each i. Thus $\pi\notin\mathscr{P}_i$ for each i, and so $S\pi S\not\subset\mathscr{P}_i$ for each i. But $l(S\pi S)$. $S\pi S=0\subset\mathscr{P}_i$ for each i, hence $l(S\pi S)\subset\mathscr{P}_i$ for each i, and so $l(S\pi S)\subset\bigcap\mathscr{P}_i=0$. Similarly the right annihilator of $S\pi S$ is 0, thus the ideal $S\pi S$ is essential in $S*G$. Then by Lemma 2.2, G is generically Galois on S.

As a consequence of this theorem we can define, for a prime ideal P of S with S and G as above, the *ramification number of P* by $\mathrm{ram}(P)=Lu(P)/Lo(P)$ and so we have that P is an unramified prime ideal if and only if $\mathrm{ram}(P)=1$.

EXAMPLE 2.4. If S is a Dedekind domain and G acts faithfully on S, the fixed ring S^G is also a Dedekind domain and, if p is a maximal ideal of S^G, the ramification index of p, $e(p)$, is defined as the natural number ε such that $(P^0)^\varepsilon=pS$, where $P\cap S^G=p$, see [Re]. Since $|G|^{-1}\in S$, the extension is tamely ramified and hence (see [AR]) $\mathrm{char}(S^G/p)=0$ or it does not divide $|G|$. Wilson [W] proves in this situation that the simple $S/P^{\circ} * G$-modules are $\mu_i(p) = (P^0)^i/(P^0)^{i+1}$ with $\mu_i(p) = \mu_j(p)$ if and only if $i = j$ mod $e(p)$. But $S/P^0 * G$ is semisimple by Maschke's theorem, so there are $e(p)$ simple $S/P^0 * G$-modules, and so $e(p) = Lu(P)$. Since S is commutative, $Lo(P) = 1$ and so $\text{ram}(P) = e(p)$. Thus in this case the ramification number of P coincides with the ramification index of $p = P \cap S^G$.

COROLLARY 2.5. *Let S and G be as in Theorem 2.3. Then:*

(1) *S is unramified if and only if G is Galois on S.*

(2) *If S is unramified then $S * G$ and S^G are Morita equivalent rings.*

(3) *The set of ramified prime ideals of S is Zariski-closed in* $\text{Spec}(S)$.

Proof. Immediate from Theorem 2.3 and [H1, Proposition 2.3].

EXAMPLE 2.6. Let $S = M_n(k)$, the ring of $n \times n$ matrices over a field of characteristic $\neq 2$. Let G be the group generated by the automorphism g of S, where g is conjugation by $x = \text{diag}\{-1, 1, ..., 1\}$. Then $S^G = k \oplus M_{n-1}(k)$ and the skew group ring $M_n(k) * \mathbb{Z}_2$ can be identified with a subring of the ring of $M_{2n}(k)$ as follows: $M_n(k) * \mathbb{Z}_2 \cong \{\left(\begin{smallmatrix} a & bx \\ bx & a \end{smallmatrix}\right)/a, b \in M_n(k)\}$, via $a1 + bg \to \left(\begin{smallmatrix} a & bx \\ bx & a \end{smallmatrix}\right)$. Here there are two prime ideals, $I = \{\left(\begin{smallmatrix} a & a \\ a & a \end{smallmatrix}\right)/a \in M_n(k)\}$ and $J = \{\left(\begin{smallmatrix} a & -a \\ -a & a \end{smallmatrix}\right)/a \in M_n(k)\}$. A calculation using traces of matrices shows that $e = \frac{1}{2}\left(\begin{smallmatrix} I_n & x \\ x & I_n \end{smallmatrix}\right)$ and we note that $eIe \cong M_{n-1}(k)$ and $eJe \cong k$. Thus we have $\text{ram}((0)) = 1$, so the prime ideal (0) is unramified, S is unramified, and the action is G-Galois on S. In particular, if $n = 2$, the following is a Galois basis for S:

$$\left\{\frac{1}{2}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -1 \\ -1 & \frac{1}{2} \end{pmatrix}\right\}.$$

Furthermore, since the action is G-Galois, then $\text{St}(M_n(k) * \mathbb{Z}_2) \approx \text{St}(k \oplus M_{n-1}(k))$ which is affinely homeomorphic to 1-dimensional simplex. The extreme points correspond to the states s_I and s_J.

3. G-ACTIONS ON $K_0(S)$ AND ON $\text{St}(S)$

Let S be a ring with 1, and let G be a finite group acting as automorphisms of S. We describe here an induced action of the group G as group automorphisms of the Grothendieck group $K_0(S)$ and an induced action of G as affine homeomorphisms of the state space $\text{St}(S)$.

$K_0(S)$

Let M be a f.g. projective left S-module. Since $S * G$ is a free left S-module we have $S * G \otimes_S M \cong \sum_{g \in G} \oplus (g \otimes_S M)$ as f.g. projective left

S-modules, whence each left S-module $g \otimes_s M$ is f.g. projective. The action of S on $g \otimes_s M$ is given by $s(g \otimes_s m) = g \otimes_s s^g m$ for all $s \in S$, $m \in M$. Using the notation of [MR], except that in this work the functions act on the left, we define the *skew-module* ${}^g M$ as the f.g. projective left S-module $g \otimes_s M$. For every g in G, ${}^g S \cong S$ as left S-modules. We also have for all M, N f.g. projective left S-modules, ${}^g(M \oplus N) \cong {}^g M \oplus {}^g N$, and then we can define for every isomorphism class $[M] \in K_0(S)$, ${}^g[M] = [{}^g M]$ and extend by linearity to all $K_0(S)$. Moreover we have ${}^h({}^g M) = h \otimes_s g \otimes_s M = Shg \otimes_s M = hg \otimes_s M = {}^{hg}(M)$. Because of this associativity, each action g has an inverse action g^{-1}, thus the group G acts as group automorphisms of $K_0(S)$ which preserve the order unit $[S]$ and the positive cone.

St(S)

The above action of G on $K_0(S)$ induces a natural action of G on St(S) given by

$$(X)\, {}^g s = ({}^g X)\, s \qquad \text{for all } X \in K_0(S), \text{ all } s \in \mathrm{St}(S), \text{ and every } g \in G.$$

Since g, as a group homomorphism, preserves the order-unit $[S]$ and the positive cone, then ${}^g s = gs$ (as a product of group homomorphisms) is a well defined state in S. Moreover, for any g, h in G and s in St(S), we obtain ${}^h({}^g s) = h({}^g s) = h(gs) = hgs = {}^{hg}s$. Is also easy to see that g, as a group homomorphism, preserves affine linear combinations and has an inverse homomorphism g^{-1}, hence the group G acts as affine homeomorphisms of the state space of S.

In particular, if S is a noetherian ring, we obtain:

PROPOSITION 3.1. *Let S be a noetherian ring and let G be a finite group acting as automorphisms of S. Let P be a prime ideal of S. Then* ${}^g(s_P) = s_{P^g}$.

Proof. It is enough to show equality for M a f.g. projective left S-module and the generalized rank function r_P. First, since $S/P \cong S/P^g$ then $\mathbf{Q}(S/P) \cong \mathbf{Q}(S/P^g)$ as rings for every $g \in G$. This induces a right S-module isomorphism $\mathbf{Q}(S/P) \otimes_s g \cong \mathbf{Q}(S/P^g)$, by $([x+P]^{-1}[y+P]) \otimes_s g \mapsto [x^g + P^g]^{-1}[y^g + P^g]$. Hence

$$(M)\, {}^g(r_p) = \frac{\text{length } (\mathbf{Q}(S/P) \otimes_s {}^g M)}{\text{length } (\mathbf{Q}(S/P))}$$

$$= \frac{\text{length } (\mathbf{Q}(S/P^g) \otimes_s M)}{\text{length } (\mathbf{Q}(S/P^g))} = (M)\, r_{P^g}.$$

EXAMPLE 3.2. Let S be an hereditary noetherian prime (HNP) ring, let $\{M_1, ..., M_n\}$ be a cycle of idempotent maximal ideals, i.e., $O_r(M_1) =$

$O_l(M_2), \ldots, O_r(M_n) = O_l(M_1)$, where O_r and O_l denote respectively the right and left orders of the ideals, see [ER]. Then M_i^g is also an idempotent maximal ideal of S for all $i = 1, \ldots, n$ and all $g \in G$. Also $\{M_1^g, \ldots, M_n^g\}$ form a cycle of idempotent maximal ideals in S because, if $O_r(M_1) = O_l(M_2)$, then $O_r({}^gM_1) = O_r(M_1^{g^{-1}}) = (O_r(M_1))^g = (O_l(M_2))^g = O_l(M_2^{g^{-1}}) = O_l({}^gM_2)$. Hence the action of G on $\mathrm{St}(S)$ is given by permuting the m-dimensional simplices, for each m that occurs in the structure theorem given in [GW].

4. Localizations and State Spaces

The classical theory of localization says that we can localize a ring at a set satisfying the Ore condition. Necessary and sufficient conditions to be able to localize a ring at prime or semiprime ideals have been studied by Jategaonkar [J], Mueller [Mu], P. F. Smith [Sm], K. Goodearl [G2], and others, and a decisive point is the notion of cliques, clans, and links; for which definitions we refer to [J]. We now give some results about localized rings and their state spaces.

THEOREM 4.1. *Let R be a noetherian ring and let $\mathscr{C}$ be an Ore set in R. The induced affine continuous map $\Delta : \mathrm{St}(R_{\mathscr{C}}) \to \mathrm{St}(R)$ sends $s_{P_{\mathscr{C}}}$ to s_P for all primes P of R such that $P \cap \mathscr{C} = \varnothing$. Furthermore, if the map $K_0(R) \to K_0(R_{\mathscr{C}})$ is surjective, then $\mathrm{St}(R_{\mathscr{C}})$ embeds into $\mathrm{St}(R)$.*

Proof. Since St is a contravariant functor, the natural map $R \hookrightarrow R_{\mathscr{C}}$ induces a map $\Delta : \mathrm{St}(R_{\mathscr{C}}) \to \mathrm{St}(R)$. Recall that all the prime ideals of $R_{\mathscr{C}}$ are of the form $P_{\mathscr{C}}$ for some prime ideal P of R such that $P \cap \mathscr{C} = \varnothing$, and $\mathbf{Q}(R_{\mathscr{C}}/P_{\mathscr{C}}) \cong \mathbf{Q}(R/P)$ as simple rings (see [J]). Thus we have

$$\begin{aligned}([M])(s_{P_{\mathscr{C}}})\,\Delta &= ([R_{\mathscr{C}} \otimes_R M])\, s_{P_{\mathscr{C}}} \\ &= \frac{\mathrm{length}(\mathbf{Q}(R_{\mathscr{C}}/P_{\mathscr{C}}) \otimes_{R_{\mathscr{C}}} R_{\mathscr{C}} \otimes_R M)}{\mathrm{length}(\mathbf{Q}(R_{\mathscr{C}}/P_{\mathscr{C}}))} \\ &= \frac{\mathrm{length}(\mathbf{Q}(R/P) \otimes_R M)}{\mathrm{length}(\mathbf{Q}(R/P))} = ([M])\, s_P\end{aligned}$$

for all f.g. projective left R-module M. The second part is a direct calculation.

COROLLARY 4.2. *If R is left regular, noetherian and $\mathscr{C}$ is an Ore set, then $\mathrm{St}(R_{\mathscr{C}}) \to \mathrm{St}(R)$ is an embedding.*

Proof. Since R is left regular, [MR, Proposition 12.1.12] gives us the surjectivity of the map $K_0(R) \to K_0(R_{\mathscr{C}})$. Then apply the theorem.

A standard result in localization theory shows that if a prime (resp. semi-prime) ideal of a ring R is localizable, then the resulting localized ring is quasi-local (resp. semi-local) (see [J]). We are interested in the state spaces of this ring.

PROPOSITION 4.3. *If R is a quasi-local ring, then* $\mathrm{St}(R)$ *is a point.*

Proof. Let J be the Jacobson radical of R. Then by [B, IX.1.3], the map $R \to R/J$ induces a monomorphism $K_0(R) \to K_0(R/J)$. But since R is quasi-local, R/J is simple artinian and hence $K_0(R/J) \cong \mathbb{Z}$. Moreover, any subgroup of the cyclic group $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$, so $K_0(R) \cong \mathbb{Z}$. Hence there exists only one state of R and $\mathrm{St}(R)$ is a singleton.

As a consequence of this result we now obtain some information about $\mathrm{St}(R)$ from the localizable prime ideals. Denote by $\mathscr{C}_R(P)$ the set of elements of R which are regular modulo the prime ideal P, i.e., $\mathscr{C}_R(P) = \{x \in R/[x+P]$ is a regular element in $R/P\}$, and drop the subscript R when the ring is understood.

THEOREM 4.4. *Let R be a noetherian ring and let P be a prime ideal if R. If P is localizable, then $s_Q = s_T$ in* $\mathrm{St}(R)$ *for all prime ideals Q and T that do not intersect $\mathscr{C}(P)$.*

Proof. Since P is localizable, $\mathscr{C}(P)$ is an Ore set and $R_P = R_{\mathscr{C}(P)}$ is a quasi-local ring (see [J]). So, by Proposition 4.3, $\mathrm{St}(R_P)$ is a singleton. On the other hand, if Q and T are prime ideals in R not intersecting $\mathscr{C}(P)$, then Q_P and T_P are prime ideals in R_P, hence $s_{Q_P} = s_{T_P}$, and thus by Proposition 4.1, $s_Q = (s_{Q_P})\,\Delta = (s_{T_P})\,\Delta = s_T$.

Recall that a ring is called semi-pefect if it semi-local and idempotents can be lifted modulo the Jacobson radical, see [Sn]. For the case of localizable semiprime ideals, the following propositions will be useful.

PROPOSITION 4.5. *If R is semi-perfect, then* $\mathrm{St}(R) \approx \mathrm{St}(R/J)$. *So* $\mathrm{St}(R)$ *is affinely homeomorphic to a finite dimensional simplex.*

Proof. By [B, III.2.12], the map $R \to R/J$ induces an injective map ϕ between the semigroups of isomorphism classes of f.g. projective left modules $\phi : \mathbf{P}(R) \to \mathbf{P}(R/J)$ which is surjective if we can lift a finite set of orthogonal idempotents. So we get a group isomorphism $\Phi : K_0(R) \to K_0(R/J)$ with $([R])\,\Phi = [R/J]$ and $(K_0^+(R))\,\Phi = K_0^+(R/J)$. Hence $\mathrm{St}(R)$ is affinely homeomorphic to $\mathrm{St}(R/J)$. Furthermore, since R/J is semisimple artinian, $\mathrm{St}(R/J)$ is affinely homeomorphic to a finite dimensional simplex (see [G1]).

But it is not so common that we can lift all idempotents in a semilocal ring. Under a weaker condition we get the following theorem:

THEOREM 4.6. *Let R be a semilocal ring. Assume that central idempotents can be lifted modulo the Jacobson radical J. Then the natural map* $\mathrm{St}(R/J) \to \mathrm{St}(R)$ *is an affine embedding.*

Proof. The natural projection map $R \to R/J$ induces an order-preserving group homomorphism $\phi : K_0(R) \to K_0(R/J)$ and an affine continuous map $\Phi : \mathrm{St}(R/J) \to \mathrm{St}(R)$. Moreover, ϕ is a monomorphism by [B, III.2.12]. We claim that $\mathrm{Coker}(\phi) = K_0(R/J)/K_0(R)$ is a torsion group. It is enough to show that for every f.g. projective left R/J-module M, there exists an integer $n > 0$ and a f.g. projective left R-module N, such that $n[M] = [R/J \otimes_R N]$. Since R/J is semisimple artinian, then $[M] = n_1[R/Jf_1] + n_2[R/Jf_2] + \cdots + n_k[R/Jf_k]$ where the f_i's are a set of primitive orthogonal idempotents of R/J, thus every f_i is a summand of a central idempotents e_i with $R/J\, f_i \subset R/J\, e_i$. Thus $m_i[R/J f_i] = [R/J\, e_i]$ for some positive integer m_i, for each $i = 1, ..., k$. Let $n = m_1 m_2 \cdots m_k > 0$. Then

$$\begin{aligned} n[M] &= nn_1[R/Jf_1] + nn_2[R/Jf_2] + \cdots + nn_k[R/Jf_k] \\ &= n_1 m_2 \cdots m_k[R/J\, e_1] + m_1 n_2 \cdots m_k[R/J\, e_2] \\ &\quad + \cdots + m_1 m_2 \cdots n_k[R/J\, e_k] \\ &= \alpha_1[R/J\, e_1] + \alpha_2[R/J\, e_2] + \cdots + \alpha_k[R/J\, e_k]. \end{aligned}$$

By hypothesis all central idempotents can be lifted, thus there exist idempotent elements $a_1, a_2, ..., a_k$ in R that are mapped to $e_1, e_2, ..., e_k$, respectively, under the map $R \to R/J$, so $R/J \otimes_R Ra_i \cong R/J\, e_i$, and

$$\begin{aligned} n[M] &= \alpha_1[R/J \otimes_R Ra_1] + \alpha_2[R/J \otimes_R Ra_2] + \cdots + \alpha_k[R/J \otimes_R Ra_k] \\ &= [R/J \otimes_R (\bigoplus \alpha_i(Ra_i))], \end{aligned}$$

where $N = (\bigoplus \alpha_i(Ra_i))$ is a f.g. projective left R-module. So we proved the claim. Now we prove that Φ is injective. Let s, t be states of R/J with $(s)\,\Phi = (t)\,\Phi$, let $X \in K_0(R/J)$. Since $\mathrm{Coker}(\phi)$ is a torsion group, then there exists an integer $n > 0$ such that $nX \in (K_0(R))\,\phi$, so there exists a $Y \in K_0(R/J)$ with $(Y)\,\phi = nX$. Hence, $(X)\,s = (1/n)(Y)\,\phi s = (1/n)(Y)\,s\Phi = (1/n)(Y)\,t\Phi = (1/n)(Y)\,\phi t = (X)\,t$. Thus $s = t$ and $\mathrm{St}(R/J) \to \mathrm{St}(R)$ is an injection.

COROLLARY 4.7. *If R is a semilocal noetherian ring of finite Krull dimension, and central idempotents can be lifted modulo the Jacobson radical, then* $\mathrm{St}(R) \approx \mathrm{St}(R/J)$. *Hence,* $\mathrm{St}(R)$ *is affinely homeomorphic to a finite dimensional simplex.*

Proof. By the theorem we know that the map $\Phi : \mathrm{St}(R/J) \to \mathrm{St}(R)$ is injective affine and continuous. Thus we just need to prove that Φ is also surjective. Since R is noetherian of finite Krull dimension and has only a finite number of maximal prime ideals, then every extreme point of $\mathrm{St}(R)$ is a state s_Q for some maximal prime ideal Q of R [St, Theorem 6.4]. But then Q/J is a minimal prime ideal of the semisimple artinian ring R/J and $(s_{Q/J})\,\Phi = s_Q$. So the map Φ is surjective.

5. Finite Algebras over Commutative Noetherian Rings

A *finite algebra* R over a commutative noetherian ring A is a ring R and a ring homomorphism $f: A \to R$ such that R is f.g. as a left and right A-module and $(A)f$ is contained in the center of R. We now discuss some properties of the state space of finite algebras. The main tool we use is Mueller's theorem which is stated originally for rings finite over their center in [Mu]. We give here Jategaonkar's result and Mueller's result for finite algebras over commutative rings.

Proposition 5.1 [J, Proposition A.2.1]. *Let R be a finite algebra over a commutative noetherian ring A. Then the natural map $\chi : \mathrm{Spec}(R) \to \mathrm{Spec}(A)$, defined by $(P)\chi = P \cap A$ is surjective with finite fibres. Moreover, R is a fully bounded noetherian (FBN) ring.*

A χ-fibre is the preimage of a prime ideal of the ring A under the map χ of Proposition 5.1.

Proposition 5.2 [Mu, Theorem 7]. *Let R be a finite algebra over a commutative noetherian ring A. Assume A is the center of R. Let $P \in \mathrm{Spec}(R)$. Then the clique of P is a clan and coincides with the set $\{Q \in \mathrm{Spec}(R)/Q \cap A = P \cap A\}$. Furthermore, the localization of R at $\mathscr{C}_A(P \cap A)$ is also the localization of R at the clique of P.*

For details about cliques and clans, see [J] or [MR]. Roughly speaking, we will say that the clique of a prime ideal P of R is the minimal set of prime ideals, including P, at which an Ore localization can be performed. If we assume that the ring R is left regular, we can put together these results with Corollary 4.2 and obtain:

Theorem 5.3. *Let R be a finite algebra over a commutative noetherian ring A, assume R is left regular and A is the center of R. Let q be a prime ideal of A, and let S be the intersection of the χ-fibre of q. Then $\mathrm{St}(R_S) \to \mathrm{St}(R)$ is an embedding.*

Proof. The χ-fibre of q is the set $\mathscr{S}=\{Q\in \mathrm{Spec}(R)/Q\cap A=q\}$, and it is finite by Proposition 5.1. Then by Mueller's theorem, $\mathscr{S}$ is a finite clan (whose intersection is the semiprime ideal S), and $\mathscr{C}_R(S)$ is an Ore set. Thus, by Corollary 4.2, $\mathrm{St}(R_S)\to \mathrm{St}(R)$ is an embedding.

Moreover, if R is a prime ring, every localization R_s is also a prime ring and so by Theorem 4.1, the state $s_0\in \mathrm{St}(R_S)$ is mapped to the state $s_0\in \mathrm{St}(R)$ for every embedding $\mathrm{St}(R_S)\to \mathrm{St}(R)$. Thus the state s_0 of $\mathrm{St}(R)$ lies in every state space $\mathrm{St}(R_S)$ for S an intersection of a χ-fibre of R. Hence we obtain:

THEOREM 5.4. *Let R be a finite algebra over a commutative noetherian ring A. Assume that R is a prime left regular ring, has finite Krull dimension, and A is the center of R. Then, there exists a surjective affine continuous map*

$$\overline{CH}\Big(\bigvee_{S\in I} \mathrm{St}(R_S)\Big)\to \mathrm{St}(R),$$

where $\bigvee$ is the topological wedge product over s_0 of the compact convex sets $\mathrm{St}(R_S)$, *I is the set of semiprime ideals S, and S is the intersection of a χ-fibre of R. ($\overline{CH}$ denotes the closure of the convex hull.) By Mueller's theorem, we can write equivalently*

$$\overline{CH}\Big(\bigvee_{p\in \mathrm{Spec}(A)} \mathrm{St}(R_p)\Big)\to \mathrm{St}(R).$$

Proof. (For definition of the topological wedge product, see [*CV*].) Every $S\in I$ is of the form $S=\bigcap\{P\in \mathrm{Spec}(R)/P\cap A=p\}$ for some prime ideal $p\in \mathrm{Spec}(A)$ (Mueller's theorem). Also S is a localizable semiprime ideal, so R_S exists and is a semi-local ring with finitely many maximal prime ideals since the χ-fibres are finite (Proposition 5.1). Then, by [St, Theorem 6.4], the extreme points of $\mathrm{St}(R_S)$ are of the form s_{P_S} for P_S maximal prime ideals of R_S. Furthermore, by Theorem 5.3, the maps $\Gamma_S:\mathrm{St}(R_S)\to \mathrm{St}(R)$ are injective for every $S\in I$, and by the comments preceding the theorem, they all agree on s_0. Thus we can define an affine map Γ from the topological wedge product over s_0 of the compact convex sets $\mathrm{St}(R_S)$, to the compact convex set $\mathrm{St}(R)$ such that $\Gamma|_{\mathrm{St}(R_S)}=\Gamma_S$, and then extend by linearity to the convex hull. We just need to show that Γ is surjective. Let $s\in \mathrm{St}(R)$. Since R is noetherian with finite Krull dimension, then s is an affine linear combination of the states s_P for P some J-prime ideals of R (see [St]). Let $p=P\cap A$, so P belongs to a χ-fibre of R by Proposition 5.1. Let T_P denote the intersection of the χ-fibre. Thus the state s_P is the image of the state $s_N\in \mathrm{St}(R_{T_P})$ under the affine continuous map Γ_{T_P} where $N=PR_{T_P}$ (Theorem 4.1). Thus an affine linear combination of the states s_P is the image of an affine linear combination of the states s_N under the affine continuous map Γ, and hence Γ is surjective.

EXAMPLE 5.5. Let

$$R=\begin{pmatrix} k & k & k \\ & k & 0 \\ & & k \end{pmatrix}$$

a k-subalgebra of $T_3(k)$. There is only one clique, given by the prime ideals

$$P=\begin{pmatrix} 0 & k & k \\ & k & 0 \\ & & k \end{pmatrix}, \quad Q=\begin{pmatrix} k & k & k \\ & 0 & 0 \\ & & k \end{pmatrix}, \quad \text{and} \quad T=\begin{pmatrix} k & k & k \\ & k & 0 \\ & & 0 \end{pmatrix}$$

with $P\cap k=Q\cap k=T\cap k=(0)$. Thus the map in Theorem 5.4 is a homeomorphism. Also if $S=P\cap Q\cap T$, then $R_S=R_{(0)}=R$. Thus R is semilocal and we can lift the central idempotents of $R/J\cong k\oplus k\oplus k$, so $\mathrm{St}(R)$ is affinely homeomorphic to a 2-dimensional simplex with extreme points s_P, s_Q, and s_T.

EXAMPLE 5.6. Let A be a commutative noetherian domain, and let I be a nonzero prime ideal of A. Let $R=\left(\begin{smallmatrix} A & I \\ A & A \end{smallmatrix}\right)$ a prime A-subalgebra of the ring $M_2(A)$. Let P be a prime ideal of A. If $I\subseteq P$, set $P'=\left(\begin{smallmatrix} P & I \\ A & A \end{smallmatrix}\right)$ and $P''=\left(\begin{smallmatrix} A & I \\ A & P \end{smallmatrix}\right)$. If $I\not\subseteq P$, set $P^*=\left(\begin{smallmatrix} P & I\cap P \\ P & P \end{smallmatrix}\right)$. It can be seen that all the prime ideals of R have one of these forms. The cliques are of the form $\{0^*\}$, $\{P^*\}$, $\{P', P''\}$. The localizations at $\{0^*\}$ and $\{P^*\}$ give us quasi-local rings, so the state spaces reduce to a point. On the other hand, the localization at P gives us a 1-dimensional simplex for state space with extreme points $s_{P'}$ and $s_{P''}$. So if I is a maximal ideal, then $I=P$ and $\mathrm{St}(R)$ is a 1-dimensional simplex. Otherwise we have to consider an amalgamation of 1-dimensional simplices arising from maximal prime ideals containing I.

6. SKEW GROUP RINGS OF COMMUTATIVE NOETHERIAN DOMAINS

In this section we apply the previous results to the case of a skew group ring when the coefficient ring is a commutative noetherian domain. We relate the ideas of ramification with the structure of the state space of the skew group ring via localization at the semiprime ideals arising from G-prime ideals of the ring. Hence we give an improvement of a result of Stafford [St, Theorem 6.4] concerning the extreme points of the state space in the case of skew group rings.

Let S be a commutative noetherian domain and let G be a finite group acting faithfully as automorphisms of S. Assume $|G|^{-1} \in S$. We can see that center $(S * G) = S^G$, which is also a commutative noetherian domain (see [Mo1]). Also, since S is noetherian and $|G|^{-1} \in S$, S is a f.g. S^G-module [Mo 1, Corollary 5.9] and hence the skew group ring $S * G$ is a prime finite algebra over its center S^G which is a commutative noetherian domain. So, applying the above results we obtain:

PROPOSITION 6.1. *Let S and G be as above. If P is a prime ideal of S then $P^0 * G$ is a localizable semiprime ideal of $S * G$ and if $p = P \cap S^G$ then $S * G_{P^0 * G} = S_{P^0} * G$.*

Proof. If $\mathscr{P}$ is a minimal prime over $P^0 * G$, then $\mathscr{P} \cap S^G = \mathscr{P} \cap S \cap S^G = P^0 \cap S^G = p$ and thus the set of prime ideals of $S * G$ that intersect to the prime ideal p in S^G is exactly the minimal prime ideals over $P^0 * G$. Then, except the last equality, this is a direct application of Theorem 5.2. To prove the last equality we show that the elements of $\mathscr{C}(P^0 * G)$ are invertible in $S_{P^0} * G$ and that $S_{P^0} * G \subseteq S * G_{P^0 * G}$. Let $x \in \mathscr{C}(P^0 * G)$, then $x = [x + P^0 * G]$ is regular in $S * G/P^0 * G = S/P^0 * G$ and hence invertibel in $\mathbf{Q}(S/P^0) * G \cong S_{P^0}/P^0 S_{P^0} * G \cong S_{P^0} * G/P^0 S_{P^0} * G = S_{P^0} * G/J(S_{P^0}) * G$. But by [Pa2, Theorem 4.2], $J(S_{P^0} * G) = J(S_{P^0}) * G$. Thus x is invertible in $S_{P^0} * G/J(S_{P^0} * G)$ and hence x is invertible in $S_{P^0} * G$. On the other hand, let $y \in S_{P^0} * G$, say $y = \sum r_g g$ with $r_g \in S_{P^0}$. Then $r_g = h_g^{-1} s_g$ for some $h_g \in \mathscr{C}_S(P^0)$ and $s_g \in S$ for every g in the support of y (supp(y)). Since $\mathscr{C}_S(P^0)$ is an Ore set we can apply the common denominator property, see [J], and find an element $h \in \mathscr{C}_S(P^0)$ and elements $t_g \in S$ such that $r_g = h^{-1} t_g$ for all g in supp(y); thus $y = h^{-1} \sum t_g g$ with $h \in \mathscr{C}_S(P^0)$ and $t_g \in S$. But obviously $\mathscr{C}_S(P^0) \subseteq \mathscr{C}(P^0 * G)$ and then $h \in \mathscr{C}(P^0 * G)$, thus $y \in S * G_{P^0 * G}$.

PROPOSITION 6.2. *Let S and G be as above, $P \in \mathrm{Spec}(S)$. If P is unramified then $s_{P^0 * G} = s_0$ in $\mathrm{St}(S * G)$.*

Proof. Since P is unramified, $P^0 * G$ is a prime ideal of $S * G$ by Theorem 2.3. Then, by Proposition 6.1, $P^0 * G$ is a localizable prime ideal in the prime ring $S * G$, and hence by Theorem 4.4, $s_{P^0 * G} = s_0$.

For the next theorem we need a general lemma:

LEMMA 6.3. *Let R be a prime noetherian ring. Let e be a non-trivial idempotent in R. If $ReR \neq R$ then $\mathrm{St}(R)$ is not a singleton.*

Proof. There exists a maximal prime ideal P such that $ReR \subseteq P$, hence $([Re])\, s_P = 0$. But $([Re])\, s_0 = \mathrm{length}(\mathbf{Q}(R) \otimes_R Re)/\mathrm{length}(\mathbf{Q}(R)) = \mathrm{length}(\mathbf{Q}(R)\, e)/\mathrm{length}(\mathbf{Q}(R)) \neq 0$. Thus $s_P \neq s_0$.

With all this information, we can state one of the main theorems of this paper which tells us some of the information that is stored in the state space.

THEOREM 6.4. *Let S be a commutative noetherian domain with finite Krull dimension, and let G be a finite group acting faithfully as automorphisms of S with $|G|^{-1} \in S$. The action is G-Galois if and only if* $\mathrm{St}(S * G)$ *is a singleton.*

Proof. ($\Rightarrow$) Since S is unramified (Corollary 2.5), every prime ideal of $S * G$ is of the form $P^0 * G$ for some prime ideal P of S. Thus by Proposition 6.2, $s_{P^0 * G} = s_0$ for all primes in $S * G$. Hence by [St, Theorem 6.4], the state space reduces to a point.

($\Leftarrow$) Recall that e, the sum of elements of G in $S * G$ multiplied by $|G|^{-1}$, is an idempotent. Then, by Lemma 6.3, $S * GeS * G = S * G$; so the action is G-Galois on S.

Now we study the case when S is ramified. Let P be a ramified prime and let $\mathscr{P}$ be a prime ideal minimal over $P^0 * G$, then $\mathscr{P} \cap S^G = \mathscr{P} \cap S \cap S^G = P^0 \cap S^G = p$, where p is a prime ideal of S^G, and $P^0 * G = \bigcap \{\mathscr{P}$ prime ideals of $S * G / \mathscr{P} \cap S^G = p\}$ of $S * G$. Thus a direct interpretation of Theorems 5.3 and 5.4 gives us:

THEOREM 6.5. *Let S and G be as in Theorem* 6.4. *Assume S is left regular and P is a prime ideal of S, then*:

(1) $\mathrm{St}(S * G_{P^0 * G}) \to \mathrm{St}(S * G)$ *is an embedding.*

(2) $\overline{CH}(\bigvee \mathrm{St}(S * G_{Q^0 * G})) \to \mathrm{St}(S * G)$ *is an affine continuous surjective map, where the topological wedge product is taken over the state s_0 and all semiprime ideals $Q^0 * G$ for Q ramified maximal prime ideals of S.*

Proof. To apply Theorems 5.3 and 5.4 we just need to check that $S * G$ is left regular with finite Krull dimension. But this follows from [MR, Proposition 5.5] since G is finite. If Q is unramified, then the state space $\mathrm{St}(S * G_{Q^0 * G})$ is a point by Proposition 6.2. In the wedge product all these points are identified with the state s_0 and hence we may consider only the ramified prime ideals of S. If there are no ramified primes then the wedge product is a point, namely the state s_0, and also $\mathrm{St}(S * G)$ is a point by Theorem 6.4. The fact that we only consider maximal prime ideals will be justified after Theorem 6.9.

This means that the state space $\mathrm{St}(S * G)$ comes from an amalgamation of convex compact sets corresponding to localization of the ring at semiprime ideals of $S * G$ that arise from ramified prime ideals of S. In particular, if the ramified prime is G-invariant, this compact convex set is a simplex as we now show.

THEOREM 6.6. *Let S be a commutative noetherian domain with finite Krull dimension, and let G be a finite group acting faithfully as automorphisms of S with $|G|^{-1} \in S$. Assume that S contains the $|G|$th roots of unity. If P is a G-invariant ramified prime of S, then* $\mathrm{St}(S * G_{P^0 * G})$ *is affinely homeomorphic to an $(m-1)$-dimensional simplex, where* $m = \mathrm{ram}(P)$.

Proof. By Proposition 6.1, $P^0 * G$ is a localizable semiprime ideal and thus $S * G_{P^0 * G}$ is a semi-local ring with exactly $m = \mathrm{ram}(P)$ maximal prime ideals. If, in $S * G_{P^0 * G}$, we can lift central idempotents modulo $J(S * G_{P^0 * G})$, then by Corollary 4.7 we get $\mathrm{St}(S * G_{P^0 * G}) \approx \mathrm{St}(S * G_{P^0 * G}/J(S * G_{P^0 * G}))$, which is affinely homeomorphic to an $(m-1)$-dimensional simplex. Thus we only need to prove that we can lift central idempotents. Proposition 6.1 and [Pa2, Theorem 4.2] yield $S * G_{P^0 * G}/J(S * G_{P^0 * G}) \cong S_{P^0} * G/J(S_{P^0} * G) \cong S_{P^0} * G/J(S_{P^0}) * G \cong S_{P^0}/J(S_{P^0}) * G$. Since P is G-invariant, $P = P^0$ and $S_{P^0}/J(S_{P^0}) = S_P/J(S_P) = S_P/PS_P = \mathbf{Q}(S/P)$ which is a field, say k. Thus $S * G_{P^0 * G}/J(S * G_{P^0 * G}) = k * G$ a semisimple artinian ring with m maximal prime ideals, $|G|^{-1} \in k$, and k contains the $|G|$ th roots of unity. Let the decomposition of 1 as a sum of central orthogonal idempotents of $k * G$ be $1 = e_1 + \cdots + e_m$. Then each e_i is in $k^G H$, where H is the inertia group of k. Thus e_i is the sum of primitive orthogonal idempotents of $k^G H$, whose coefficients, by [Pa1], are sums of some $|H|$th roots of unity, thus sums of some $|G|$th roots of unity. Hence the coefficients of the e_i's are sums of some $|G|$th roots of unity which in fact, are in S_{P^0}, and thus $e_i \in S_{P^0} * G$, giving a trivial lifting of the central idempotents.

In the case of ramified primes that are not G-invariant, we can not give such a specific structure of $\mathrm{St}(S * G_{P^0 * G})$. But, in general, we can compare these "pieces" for G-comparable prime ideals in S. We call P and Q two *G-comparable* prime ideals if $P^g \subseteq Q$ or $Q^h \subseteq P$ for some g, h of G.

THEOREM 6.7. *Let S be a commutative noetherian domain and let G be a finite group acting faithfully as automorphisms of S with $|G|^{-1} \in S$. Assume S is left regular. If $P^g \subseteq Q$ are two G-comparable prime ideals in S, then there is an embedding* $\mathrm{St}(S * G_{P^0 * G}) \to \mathrm{St}(S * G_{Q^0 * G})$.

Proof. Since P and Q are G-comparable primes, then $P^0 * G \subset Q^0 * G$. By "going up" [LP2], for each prime ideal $\mathscr{P}_i$ minimal over $P^\circ * G$, there exists a prime ideal $\mathscr{Q}_i$ minimal over $Q^0 * G$ with $\mathscr{P}_i \subset \mathscr{Q}_i$. Thus $\mathscr{P}_i \cap \mathscr{C}(Q^0 * G) \subseteq \mathscr{P}_i \cap \mathscr{C}(\mathscr{Q}_i) = \varnothing$ and hence $\mathscr{P}_i S * G_{Q^0 * G}$ is a prime ideal of $S * G_{Q^0 * G}$, name it $\mathscr{P}_i$, with $\bigcap \mathscr{P}_i = (\bigcap \mathscr{P}_i)\, S * G_{Q^0 * G} = (P^0 * G)\, S * G_{Q^0 * G} = P^0 S_{Q^0} * G$. Moreover, $S_{Q^0} * G$ is a finite algebra over its center $\mathscr{Z}$, the fixed subring; thus by Mueller's theorem $P^0 S_{Q^0} * G$ is a localizable semiprime ideal of $S_{Q^0} * G$ and $(S * G_{Q^0 * G})_{P^0 S_{Q^0} \cap \mathscr{Z}} = (S * G_{Q^0 * G})_{P^0 S_{Q^0} * G} =$

$(S_{Q^0})_{P^0 S_{Q^0}} * G$. But, since S is commutative $(S_{Q^0})_{P^0 S_{Q^0}} \cong S_{P^0}$, and hence $(S * G_{Q^0 * G})_{P^0 S_{Q^0}} * G \cong S_{P^0} * G = S * G_{P^0 * G}$. Furthermore, since S is left regular, then $S * G$ is left regular, and by [MR, Proposition 7.7.3] $S * G_{Q^0 * G}$ is left regular. Then, by Theorem 5.3, the map $\mathrm{St}(S * G_{P^0 * G}) = \mathrm{St}((S * G_{Q^0 * G})_{P^0 S_{Q^0} * G}) \to \mathrm{St}(S * G_{Q^0 * G})$ is an embedding.

COROLLARY 6.8. *The following diagram is commutative*

$$\begin{array}{ccc} & \mathrm{St}(S * G_{Q^0 * G}) & \\ \nearrow & & \searrow \\ \mathrm{St}(S * G_{P^0 * G}) & \longrightarrow & \mathrm{St}(S * G). \end{array}$$

Proof. Let s be a state in $\mathrm{St}(S * G_{P^0 * G})$, and let α be the embedding map of the theorem. Let γ and β be the correspondent maps of Theorem 6.5 for $P^0 * G$ and $Q^0 * G$, respectively. Let M be a f.g. projective left $S * G$-module. Then

$$\begin{aligned} ([M])\, s\gamma &= ([S * G_{P^0 * G} \otimes_{S * G} M])\, s \\ &= ([S * G_{P^0 * G} \otimes_{S * G_{Q^0 * G}} S * G_{Q^0 * G} \otimes_{S * G} M])\, s \\ &= ([S * G_{Q^0 * G} \otimes_{S * G} M])\, s\alpha = ([M])\, s\, \alpha\beta. \end{aligned}$$

Because of the commutativity of the diagram in the previous corollary, the extreme points of the state space of the skew group ring arise from generalized rank functions at maximal prime ideals, improving in this case a result of Stafford [St, Theorem 6.4], so we have:

THEOREM 6.9. *Let S be a commutative noetherian domain with finite Krull dimension, and let G be a finite group acting faithfully as automorphisms of S with $|G|^{-1} \in S$. Assume S is left regular, then $\partial_e\, St(S * G) \subseteq \{s_{\mathscr{P}} / \mathscr{P}$* maximal prime ideal of $S * G\}$.

Furthermore, this theorem allows us to consider only ramified maximal prime ideals in Theorem 6.5.

EXAMPLE 6.10. Let $S = k[x, y]$ and $G = \langle \alpha, \beta \rangle$ as in Example 1.6. In that example we show that $\mathrm{St}(S * G)$ is a 3-dimensional simplex. Now we can give more detailed information about the extreme points. We can see that $S^G = k[x^2, y^2]$ and $S\pi S \cap S = (xy)$. Thus the ramified primes are $P = (x, y)$, $Q(\pm\alpha) = (x \pm \alpha, y)$, $T(\pm\beta) = (x, y \pm \beta)$, (x), and (y); with ramification numbers $\mathrm{ram}(P) = 4$, $\mathrm{ram}(Q(\pm\alpha)) = 2$, $\mathrm{ram}(T(\pm\beta)) = 2$, $\mathrm{ram}((x)) = 2$, and $\mathrm{ram}((y)) = 2$. Let $Q = Q(\pm\alpha)^0 = (x^2 - \alpha^2, y)$ and $T = T(\pm\beta)^0 = (x, y^2 - \beta^2)$. By Theorems 6.6 and 6.7, the 1-dimensional simplex $\mathrm{St}(k[x,y]_{(x)} * G)$ embeds into $\mathrm{St}(k[x,y]_T * G)$ which is also a 1-dimensional simplex; furthermore, since the intersection of those prime ideals of $S * G$ minimal over $(x) * G$ and $T * G$ coincide in kG, the map $\mathrm{St}(kG) \to \mathrm{St}(S * G)$ of Example 1.6 gives us an identification of the extreme points of $\mathrm{St}(k[x,y]_{(x)} * G)$ and $\mathrm{St}(k[x,y]_T * G)$ inside $\mathrm{St}(k[x,y] * G)$. Hence, by Theorem 6.9, we just need to consider the ramified prime ideal $P=(x,y)$; but P is G-invariant, so theorems 6.5 and 6.6 imply that $\mathrm{St}(k[x,y] * G)$ is affinely homeomorphic to a 3-dim simplex with extreme points corresponding to the 4 prime ideals minimal over $P * G$.

7. Dedekind Domains

Let S be a Dedekind domain and let G be a finite group acting faithfully as automorphisms of S. Assume $|G|$ is invertible in S. Then $S * G$ is an hereditary noetherian prime ring (HNP ring) and we can apply the structure theorem of [GW] and improve Theorem 6.5.

Theorem 7.1. *Let S be a Dedekind domain and let G be a finite group acting faithfully as automorphisms of S with $|G|^{-1} \in S$. Then*

$$\mathrm{St}(S * G) \approx \overline{CH}\left(\bigvee_{i \in I} \mathrm{St}(S/P_i^0 * G)\right),$$

where I is an index set for the ramified primes of S and the topological wedge product is taken over the common state s_0. Furthermore, each $\mathrm{St}(S/P_i^0 * G)$ is affinely homeomorphic to an $(n_i - 1)$-dimensional simplex where $n_i = \mathrm{ram}(P_i)$. If S unramified, then $\mathrm{St}(S * G)$ is a point.

Proof. Applying Mueller's theorem to the HNP ring $S * G$, we get that the clans are the set of prime ideals minimal over a semiprime ideal $P^0 * G$ for some prime ideal P of the ring S. But in a HNP ring the clans coincide with the cycles of idempotent maximal ideals, see [J]. Then, using the structure theorem of [GW] for HNP rings, we have a simplex K_P for every one of these clans, with the extreme points in a one-to-one correspondence to the $n = \mathrm{ram}(P)$ prime ideals minimal over $P^0 * G$, for P a ramified prime ideal of S; and $\mathrm{St}(S * G) \approx \overline{CH}(\bigvee K_P)$ where the wedge product is taken over the common origin and all ramified primes of S. On the other hand $S/P_i^0 * G = S * G/P_i^0 * G$ is a semisimple artinian ring with exactly $n_i = \mathrm{ram}(P_i)$ minimal prime ideals and its state space is affinely homomorphic to an $(n_i - 1)$-dimensional simplex, hence affinely homeomorphic to K_{P_i} with the state s_0 corresponding to the origin.

Thus $\mathrm{St}(S/P_i^0 * G) \approx K_{P_i}$ and $\mathrm{St}(S * G) \approx \overline{CH}\,(\bigvee_{i \in I} \mathrm{St}(S/P_i^0 * G))$ where i runs over all ramified primes of S. If there are no ramified primes, then Theorem 6.4 says that the state space $\mathrm{St}(S * G)$ is a point.

EXAMPLE 7.2. Let $S = k[x]$ with k a field such that $\operatorname{char}(k) \nmid n$ and k contains a primitive nth root of unity ω. Define the automorphism g of S by $(x)g = \omega x$ and let $G = \langle g \rangle$, so $|G| = n$. Some few computations show that $S\pi S \cap S = (x^{n-1})$, so the only ramified prime ideal is $P = (x)$. In fact, the skew group ring is isomorphic to

$$S * G \cong \begin{pmatrix} \mathbb{C}[x^n](x^n) \cdots & (x^n) \\ & \\ \cdots & \cdots (x^n) \\ \mathbb{C}[x^n] & \cdots \mathbb{C}[x^n] \end{pmatrix}$$

and the semiprime ideal $(x) * G$ is the intersection of the prime ideals $\mathscr{P}_1, ..., \mathscr{P}_n$ with

$$\mathscr{P}_i = \begin{pmatrix} \mathbb{C}[x^n] & \cdots & (x^n) \\ \cdots & (x^n) & \cdots \\ \mathbb{C}[x^n] & \cdots & \mathbb{C}[x^n] \end{pmatrix}$$

with (x^n) in the (i, i)th entry. Thus $\operatorname{ram}((x)) = n$, and because we have only one ramified prime, Theorem 7.1 implies that $\operatorname{St}(k[x] * G)$ is affinely homeomorphic to an $(n-1)$-dimensional simplex with extreme points $s_{\mathscr{P}_1}, ..., S_{\mathscr{P}_n}$

ACKNOWLEDGMENTS

This paper constitutes part of the author's dissertation at the University of Cincinnati. I thank my advisor Professor Timothy Hodges for his invaluable assistance.

REFERENCES

[AR] M. AUSLANDER AND D. S. RIM, Ramification index and multiplicity, *Illinois J. Math.* **7** (1963), 566–581.

[B] H. BASS, "Algebraic K-Theory," Benjamin, New York, 1968.

[CV] C. O. CHRISTENSON AND W. VOXMAN, "Aspects of Topology," Monographs in Pure and Applied Mathematics, Dekker, New York, 1977.

[DM] F. DEMEYER, Some notes on the general galois theory of rings, *Osaka J. Math.* **2** (1965), 117-127.

[ER] D. Eisenbud and J. C. Robson, Hereditary Noetherian prime rings, *J. Algebra* **16** (1970), 86–104.

[G1] K. R. Goodearl, The state space of K_0 of a ring, *in* "Ring Theory," Lecture Notes in Math., Vol. 734, Springer-Verlag, New York/Berlin, 1978.

[G2] K. R. Goodearl, Linked injectives and Ore localizations, *J. London Math. Soc.* (2) **37** (1988), 404–420.

[GW] K. Goodearl and R. B. Warfield, Jr., State spaces of Ko of Noetherian rings, *J. Algebra* **71** (1981), 322–378.

[H1] T. Hodges, Equivariant K-theory for Noetherian rings, *J. London Math. Soc.* (129) **39** (1989).

[H2] T. Hodges, K-theory and right ideal class groups for HNP rings, *Trans. Amer. Math. Soc.* **302**, No. 2 (1987).

[J] A. V. Jategaonkar, Localization in Noetherian rings, *in* "London Math. Soc. Lecture Note Series," Vol. 89, Cambridge Univ. Press, Cambridge, 1985.

[LP1] M. Lorenz and D. Passman, Prime ideals in group algebras of polycyclic-by-finite groups, *Proc. London Math. Soc.* (3) **43** (1981), 520–543.

[LP2] M. Lorenz and D. Passman, Prime ideals in crossed products, *Israel J. Math.* **33** (1979), 89–132.

[Mo1] S. Montgomery, Fixed rings of finite automorphism groups of associative rings, *in* "Lecture Notes in Math.," Vol. 818, Springer, Berlin, 1980.

[Mo2] S. Montgomery, Prime ideals in fixed rings, *Comm. Algebra* **9**, No. 4 (1981), 423–449.

[MR] J. C. McConell and J. C. Robson, "Noncommutative Noetherian Rings," Wiley, New York, 1988.

[Mu] B. Mueller, Localizations in non-commutative noetherian rings, *Canad. J. Math.* **28**, No. 3 (1976), 600–610.

[Pa1] D. Passman, "The Algebraic Structure of Group Rings," Wiley, New York, 1977.

[Pa2] D. Passman, "Infinite Crossed Products," Academic Press, Orlando, FL, 1989.

[Q] D. Quillen, Higher algebraic K-theory, *in* "Lectures Notes in Math.," Vol. 341, Springer-Verlag, Berlin/New York, 1973.

[Re] I. Reiner, "Maximal Orders," Academic Press, New York, 1975.

[Sm] P. F. Smith, Localization and the AR property, *Proc. London Math. Soc.* (3) **22** (1971), 39–68.

[Sn] B. Stenstrom, "Rings of Quotients," Springer-Verlag, New York, 1975.

[St] J. T. Stafford, Generating modules efficiently: Algebraic K-theory for noncommutative rings, *J. Algebra* **69** (1981), 312–346.

[W] S. M. J. Wilson, Twisted group rings and ramification, *Proc. London Math. Soc.* (3) **31** (1975), 311–330.