JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **131**, 412–420 (1988)

A Class of Distortion Theorems Involving Certain Operators of Fractional Calculus*

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Submitted by A. Schumitzky

Received October 23, 1985

The object of the present paper is to investigate a general class of fractional integral operators involving the Gauss hypergeometric function. Several interesting distortion theorems for various subclasses of analytic and univalent functions are proved in terms of these operators of fractional calculus. Some special cases of the results presented here are also indicated. © 1988 Academic Press, Inc.

1. INTRODUCTION AND DEFINITIONS

Among several interesting definitions of fractional integrals given in the literature (cf., e.g., [2, Chap. 13; 5; 8, p. 28 *et seq.*; 12]), we find it to be convenient to recall here the following definitions:

DEFINITION 1 (Owa [3]; see also Srivastava and Owa [10]). The fractional integral of order λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} \, d\zeta, \tag{1.1}$$

* Supported, in part, by NSERC (Canada) under Grant A-7353. The present investigation was initiated at the University of Victoria while the third author was on study leave from Kinki University, Osaka, Japan.

0022-247X/88 $3.00

where $\lambda > 0, f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

DEFINITION 2 (Saigo [6]; see also Srivastava and Saigo [13]). For real numbers $\alpha > 0$, β, and η, the fractional integral operator $I_{0,x}^{\alpha,\beta,\eta}$ is defined by

$$I_{0,x}^{\alpha,\beta,\eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} F\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)\, dt \tag{1.2}$$

for a real-valued function $f(x)$ which is continuous on the open interval $(0, \infty)$ with the order

$$f(x) = O(x^{\varepsilon}), \qquad x \to 0,$$

where

$$\varepsilon > \max\{0, \beta-\eta\} - 1.$$

It follows from Definition 1 and Definition 2 that

$$D_x^{-\alpha} f(x) = I_{0,x}^{\alpha,-\alpha,\eta} f(x). \tag{1.3}$$

Furthermore, for a complex-valued function $f(z)$, Definition 2 may be written in the modified form:

DEFINITION 3. For real numbers $\alpha > 0$, β, and η, the fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}$ is defined by

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} \times F\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\zeta}{z}\right) f(\zeta)\, d\zeta, \tag{1.4}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^{\varepsilon}), \qquad z \to 0,$$

where

$$\varepsilon > \max\{0, \beta-\eta\} - 1,$$

and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed as in Definition 1 above.

It is easy to observe that [cf. Eq. (1.3)]

$$D_z^{-\alpha} f(z) = I_{0,z}^{\alpha,-\alpha,\eta} f(z). \tag{1.5}$$

Let $\mathscr{A}(n)$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in \mathscr{N} = \{1, 2, 3, ...\}), \tag{1.6}$$

which are analytic in the unit disk $\mathscr{U} = \{z: |z| < 1\}$. Further, let $\mathscr{S}(n)$ denote the class of all functions in $\mathscr{A}(n)$ which are univalent in the unit disk $\mathscr{U}$. Then a function $f(z)$ belonging to the class $\mathscr{S}(n)$ is said to be in the subclass $\mathscr{S}_\delta(n)$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \delta \qquad (z \in \mathscr{U}) \tag{1.7}$$

for some δ $(0 \leqq \delta < 1)$. Also, a function $f(z)$ belonging to the class $\mathscr{S}(n)$ is said to be in the subclass $\mathscr{K}_\delta(n)$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta \qquad (z \in \mathscr{U}) \tag{1.8}$$

for some δ $(0 \leqq \delta < 1)$.

We note that $f(z) \in \mathscr{K}_\delta(n)$ if and only if $zf'(z) \in \mathscr{S}_\delta(n)$, and that

$$\mathscr{S}_\delta(n) \subseteq \mathscr{S}_0(n), \qquad \mathscr{K}_\delta(n) \subseteq \mathscr{K}_0(n), \qquad \text{and} \qquad \mathscr{K}_\delta(n) \subset \mathscr{S}_\delta(n) \tag{1.9}$$

for $0 \leqq \delta < 1$.

The classes $\mathscr{S}_\delta(n)$ and $\mathscr{K}_\delta(n)$ were studied recently by Srivastava, Owa, and Chatterjea [11]. For $n = 1$, $\mathscr{S}_\delta(1)$ and $\mathscr{K}_\delta(1)$ become the classes $\mathscr{S}^*(\delta)$ and $\mathscr{K}(\delta)$, respectively, which were introduced earlier by Robertson [4].

Let $\mathscr{T}(n)$ be the subclass of $\mathscr{S}(n)$ consisting of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \qquad (a_k \geqq 0). \tag{1.10}$$

Denote by $\mathscr{T}_\delta(n)$ and $\mathscr{C}_\delta(n)$ the classes obtained by taking intersections, respectively, of the classes $\mathscr{S}_\delta(n)$ and $\mathscr{K}_\delta(n)$ with $\mathscr{T}(n)$; that is,

$$\mathscr{T}_\delta(n) = \mathscr{S}_\delta(n) \cap \mathscr{T}(n) \qquad (0 \leqq \delta < 1; n \in \mathscr{N}) \tag{1.11}$$

and

$$\mathscr{C}_\delta(n) = \mathscr{K}_\delta(n) \cap \mathscr{T}(n) \qquad (0 \leqq \delta < 1; n \in \mathscr{N}). \tag{1.12}$$

The classes $\mathscr{T}_\delta(n)$ and $\mathscr{C}_\delta(n)$ were considered by Chatterjea [1]. In particular, $\mathscr{T}_\delta(1)$ and $\mathscr{C}_\delta(1)$ are the classes $\mathscr{T}^*(\delta)$ and $\mathscr{C}(\delta)$, respectively, which were introduced by Silverman [7].

In this paper we aim at presenting several interesting distortion theorems for the fractional integrals of functions belonging to the general classes $\mathscr{T}_\delta(n)$ and $\mathscr{C}_\delta(n)$.

2. Preliminaries

In order to prove our results for functions belonging to the general classes $\mathscr{T}_\delta(n)$ and $\mathscr{C}_\delta(n)$, we shall need the following lemmas given by Chatterjea [1]:

Lemma 1. *Let the function $f(z)$ be defined by* (1.10). *Then $f(z)$ is in the class $\mathscr{T}_\delta(n)$ if and only if*

$$\sum_{k=n+1}^{\infty} \left(\frac{k-\delta}{1-\delta}\right) a_k \leqq 1 \qquad (n \geqq 1). \tag{2.1}$$

Lemma 2. *Let the function $f(z)$ be defined by* (1.10). *Then $f(z)$ is in the class $\mathscr{C}_\delta(n)$ if and only if*

$$\sum_{k=n+1}^{\infty} \left(\frac{k(k-\delta)}{1-\delta}\right) a_k \leqq 1 \qquad (n \geqq 1). \tag{2.2}$$

Remark 1. Lemma 1 follows immediately from a result due to Silverman [7, p. 110, Theorem 2] upon setting $a_k = 0$ $(k = 2, 3, ..., n)$. Lemma 2, on the other hand, is a similar consequence of another result due to Silverman [7, p. 111, Corollary 2].

We shall also need the following result in our investigation.

Lemma 3. *If $\alpha > 0$ and $\kappa > \beta - \eta - 1$, then*

$$I_{0,z}^{\alpha,\beta,\eta} z^\kappa = \frac{\Gamma(\kappa+1)\,\Gamma(\kappa-\beta+\eta+1)}{\Gamma(\kappa-\beta+1)\,\Gamma(\kappa+\alpha+\eta+1)} z^{\kappa-\beta}. \tag{2.3}$$

Proof. By Definition 2, we have

$$\begin{aligned} I_{0,z}^{\alpha,\beta,\eta} z^\kappa &= \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} F\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\zeta}{z}\right) \zeta^\kappa \, d\zeta \\ &= \frac{z^{\kappa-\beta}}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-t)^\kappa F(\alpha+\beta, -\eta; \alpha; t)\, dt \\ &= \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\alpha+1)} z^{\kappa-\beta} F(\alpha+\beta, -\eta; \kappa+\alpha+1; 1) \\ &= \frac{\Gamma(\kappa+1)\,\Gamma(\kappa-\beta+\eta+1)}{\Gamma(\kappa-\beta+1)\,\Gamma(\kappa+\alpha+\eta+1)} z^{\kappa-\beta}, \end{aligned} \tag{2.4}$$

where we have employed the formulas [9, p. 287, Eq. (44)]

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\lambda)\,\Gamma(\gamma-\lambda)} \int_0^1 t^{\lambda-1}(1-t)^{\gamma-\lambda-1}\, F(\alpha, \beta; \lambda; zt)\, dt,$$

$$\operatorname{Re}(\gamma) > \operatorname{Re}(\lambda) > 0, \tag{2.5}$$

and [9, p. 19, Eq. (20)]

$$F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}, \qquad \operatorname{Re}(\gamma-\alpha-\beta) > 0. \tag{2.6}$$

3. Distortion Theorems for the Classes $\mathcal{T}_\delta(n)$ and $\mathcal{C}_\delta(n)$

Applying Lemma 1 and Lemma 3, we shall prove

THEOREM 1. *Let α, β, and η satisfy the inequalities*:

$$\alpha > 0, \qquad \beta < 2, \qquad \alpha+\eta > -2, \qquad \text{and} \qquad \beta-\eta < 2. \tag{3.1}$$

Choose a positive integer n such that

$$n \geqq \frac{\beta(\alpha+\eta)}{\alpha} - 2. \tag{3.2}$$

Also let the function $f(z)$ defined by (1.10) *be in the class $\mathcal{T}_\delta(n)$. Then*

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \geqq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta)\,\Gamma(2+\alpha+\eta)}\, |z|^{1-\beta} \times \left\{1 - \frac{(1-\delta)(-\beta+\eta+2)_n (n+1)!}{(n+1-\delta)(-\beta+2)_n(\alpha+\eta+2)_n}\, |z|^n\right\} \tag{3.3}$$

and

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \leqq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta)\,\Gamma(2+\alpha+\eta)}\, |z|^{1-\beta} \times \left\{1 + \frac{(1-\delta)(-\beta+\eta+2)_n (n+1)!}{(n+1-\delta)(-\beta+2)_n(\alpha+\eta+2)_n}\, |z|^n\right\} \tag{3.4}$$

for

$$z \in \mathcal{U} \text{ if } \beta \leqq 1 \qquad \text{and} \qquad z \in \mathcal{U} - \{0\} \text{ if } \beta > 1, \tag{3.5}$$

where $(\lambda)_k$ *is the Pochhammer symbol defined by*

$$(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } k=0, \\ \lambda(\lambda+1)\cdots(\lambda+k-1), & \forall k \in \mathcal{N}. \end{cases} \tag{3.6}$$

Equalities in (3.3) *and* (3.4) *are attained by the function*

$$f(z) = z - \frac{1-\delta}{n+1-\delta} z^{n+1} \tag{3.7}$$

at certain values of z, *where* β *is assumed to be a rational number for the case* (3.4).

Proof. By virtue of Formula (2.3), we have

$$\begin{aligned} I_{0,z}^{\alpha,\beta,\eta} f(z) &= \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta)\,\Gamma(2+\alpha+\eta)} z^{1-\beta} \\ &\quad - \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\,\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+1)\,\Gamma(k+\alpha+\eta+1)} a_k z^{k-\beta}. \end{aligned} \tag{3.8}$$

Now define the function $\Phi(z)$ by

$$\begin{aligned} \Phi(z) &= \frac{\Gamma(2-\beta)\,\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)} z^{\beta} I_{0,z}^{\alpha,\beta,\eta} f(z) \\ &= z - \sum_{k=n+1}^{\infty} \Psi(k)\, a_k z^k, \end{aligned} \tag{3.9}$$

where, for convenience,

$$\Psi(k) = \frac{(-\beta+\eta+2)_{k-1}\, k!}{(-\beta+2)_{k-1}(\alpha+\eta+2)_{k-1}} \qquad (k = n+1, n+2, n+3, \ldots). \tag{3.10}$$

It is easily seen from the assumptions in (3.1) and (3.2) that $\Psi(k)$ is non-increasing for integers $k \geqq n+1$, and we have

$$0 < \Psi(k) \leqq \Psi(n+1) = \frac{(-\beta+\eta+2)_n (n+1)!}{(-\beta+2)_n (\alpha+\eta+2)_n}. \tag{3.11}$$

In view of Lemma 1, we also have

$$\sum_{k=n+1}^{\infty} a_k \leqq \frac{1-\delta}{n+1-\delta}. \tag{3.12}$$

Making use of (3.11) and (3.12) in (3.9), we see that

$$
\begin{aligned}
|\Phi(z)| &\geqq |z| - |z|^{n+1} \sum_{k=n+1}^{\infty} \Psi(k)\, a_k \\
&\geqq |z| - \Psi(n+1)|z|^{n+1} \sum_{k=n+1}^{\infty} a_k \\
&\geqq |z| - \frac{1-\delta}{n+1-\delta}\, \Psi(n+1)|z|^{n+1},
\end{aligned}
$$

which implies the assertion (3.3) of Theorem 1.

The assertion (3.4) of Theorem 1 can be proved similarly.

Finally, in view of the formula (2.3), it is not difficult to verify that the function given by (3.7) does indeed attain the equality in (3.3) for $z = |z|$. If β is a rational number, we can seek integers $n > 0$, m_1, and m_2 such that

$$n(1 + 2m_1)/(m_1 - m_2) = 2\beta - 2.$$

Put

$$\theta = (2m_1 + 1)\,\pi/(1 - \beta) = (2m_2 + 1)\,\pi/(n + 1 - \beta).$$

Then we can see that (3.7) attains the equality in (3.4) at the value $z = |z|\, e^{i\theta}$.

Corollary 1. *Let the function* $f(z)$ *defined by* (1.10) *be in the class* $\mathcal{T}_\delta(n)$. *Then*

$$|D_z^{-\lambda} f(z)| \geqq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{1 - \frac{(1-\delta)(n+1)!}{(n+1-\delta)(2+\lambda)_n} |z|^n\right\} \tag{3.13}$$

and

$$|D_z^{-\lambda} f(z)| \leqq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{1 + \frac{(1-\delta)(n+1)!}{(n+1-\delta)(2+\lambda)_n} |z|^n\right\} \tag{3.14}$$

for $\lambda > 0$ *and* $z \in \mathcal{U}$. *Equalities in* (3.13) *and* (3.14) *are attained by the function given by* (3.7) *at certain values of* z, *where* λ *is assumed to be a rational number for the case* (3.14).

Proof. In view of the relationship (1.5), Corollary 1 follows readily from Theorem 1 in the special case when

$$\alpha = -\beta = \lambda.$$

Remark 2. Letting $\lambda \to 0$ in Corollary 1, we obtain the corresponding result due to Srivastava, Owa, and Chatterjea [11, p. 117, Theorem 1].

Similarly, by applying Lemma 2 (instead of Lemma 1) to the function $f(z)$ belonging to the class $\mathscr{C}_\delta(n)$, we can derive

THEOREM 2. *Under the assumptions* (3.1) *and* (3.2) *of Theorem* 1, *let the function defined by* (1.10) *be in the class* $\mathscr{C}_\delta(n)$. *Then*

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \geqq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta)\,\Gamma(2+\alpha+\eta)} |z|^{1-\beta} \times \left\{1 - \frac{(1-\delta)(-\beta+\eta+2)_n\, n!}{(n+1-\delta)(-\beta+2)_n(\alpha+\eta+2)_n} |z|^n\right\} \tag{3.15}$$

and

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \leqq \frac{\Gamma(2-\beta+\eta)}{\Gamma(2-\beta)\,\Gamma(2+\alpha+\eta)} |z|^{1-\beta} \times \left\{1 + \frac{(1-\delta)(-\beta+\eta+2)_n\, n!}{(n+1-\delta)(-\beta+2)_n(\alpha+\eta+2)_n} |z|^n\right\} \tag{3.16}$$

for z *given precisely by* (3.5). *Equalities in* (3.15) *and* (3.16) *are attained by the function*

$$f(z) = z - \frac{1-\delta}{(n+1)(n+1-\delta)} z^{n+1} \tag{3.17}$$

at certain values of z, *where* β *is assumed to be a rational number for the case* (3.16).

Finally, by virtue of the relationship (1.5), a special case of Theorem 2 when

$$\alpha = -\beta = \lambda$$

may be stated as

COROLLARY 2. *Let the function* $f(z)$ *defined by* (1.10) *be in the class* $\mathscr{C}_\delta(n)$. *Then*

$$|D_z^{-\lambda} f(z)| \geqq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{1 - \frac{(1-\delta)\, n!}{(n+1-\delta)(2+\lambda)_n} |z|^n\right\} \tag{3.18}$$

and

$$|D_z^{-\lambda} f(z)| \leqq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{1 + \frac{(1-\delta)\, n!}{(n+1-\delta)(2+\lambda)_n} |z|^n\right\} \tag{3.19}$$

for $\lambda > 0$ *and* $z \in \mathcal{U}$. *Equalities in* (3.18) *and* (3.19) *are attained by the function given by* (3.17) *at certain values of* z, *where* λ *is assumed to be a rational number for the case* (3.19).

Remark 3. Letting $\lambda \to 0$ in Corollary 2, we obtain the corresponding result due to Srivastava, Owa, and Chatterjea [11, p. 119, Theorem 2].

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