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Fixed Points in Digital Topology (via Helly Posets)

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Abstract

We extend some of our previous results on fixed points of graph multifunctions to posets. The posets of most interest here are the (finite) Khalimsky spaces, in their specialization order. Retracts of Khalimsky spaces coincide with Helly posets. Notions of convexity can be defined in these spaces, providing the basis for certain "geometric" fixed point theorems.

Key words: Fixed point property; Helly posets; Khalimsky spaces; Multifunctions

1 Introduction

In studying fixed point properties in digital topology, we have the choice between two main types of model, namely the graph-theoretic model [10], and the topological (specifically, Khalimsky space) model [5,6]. In previous work [12,13,15] we adopted the first model, focusing on fixed point results for many-valued functions on graphs. (Multifunctions on discrete structures are useful in approximating, or representing, ordinary continuous functions.)

Here we investigate the second model. Khalimsky spaces may be identified with posets of a certain kind, since the topology of a Khalimsky space may be recovered from its specialization order. This is particularly clear in the present context, where we are concerned almost entirely with finite spaces: a finite T_0 space may of course be considered as a finite poset.

An advantage of working with posets rather than graphs is that we can take as our multifunctions the usual upper and lower semi-continuous multifunctions of topology (or domain theory). Thus, there is no need to introduce any new "power structure", as we had to do in the graph context [12,13]. Also,

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the fixed points are exact, rather than the "almost fixed points" of [12,13,15]. For our main fixed point result(s), in Section 5, we need to assume that the multifunctions in question map each point to an image-set which is "convex" in a suitable sense. A suitable notion is most readily available if the posets are required to satisfy a certain Helly condition (see Section 2). As mentioned below, these Helly posets have previously put in an appearance (in computer science) in studies of subtyping, but this is not our motivation for considering them here. Rather, the justification for studying them here is the extremely strong, though hitherto unnoticed, connection between these posets and Khalimsky spaces. This key idea of the paper receives its precise formulation in Section 3: Helly posets are exactly the retracts of Khalimsky spaces.

In Section 4 we adapt to posets a notion which is standard in the theory of "Helly" metric spaces (that is: *hyperconvex* metric spaces [1,14]), namely *admissible* subsets. An admissible set is simply an intersectioin of (closed) balls. It provides our notion of convex set, for Helly posets. As mentioned, Section 5 contains the main result of the paper: every Helly poset has the fixed point property (FPP) for lower and upper semi-continuous poset multifunctions which map points to admissible subsets. Section 6 is concerned with the approximation of continuous real functions by (suitable) functions between Khalimsky spaces (or Helly posets). In that context, the FPP result for Helly posets provides a discrete version of Brouwer's fixed point theorem.

2 Preliminaries

A set S with a distinguished collection $\mathcal{C}$ of subsets may be said to have the (2-) Helly property if, whenever a subset $\mathcal{A}$ of $\mathcal{C}$ is such that each pair of members of $\mathcal{A}$ has non-empty intersection, then $\bigcap \mathcal{A} \neq \emptyset$. In the typical cases of interest in computer science, S is structured as a graph, a poset, or a metric space, with a distance function d, and $\mathcal{C}$ consists of the (closed) metric balls $B_S(x, r)$.

In some recent applications in semantics [2,11], Helly posets have been used to model type structure. Like domains, Helly posets are well-suited to this role because of the wealth of constructions they enjoy: closure under products, disjoint sums, retracts, various subtype constructions, function spaces, fixed points and more. (Not all of these constructs have been employed so far in the semantics literature.)

Of most significance is the fact that, typically, the Helly structures are the injective objects of the appropriate category. (For a recent discussion of the significance of injectivity for semantics, see [4].) It may be helpful here to give some informal explanation as to why we (expect to) have the equation:

$$Helly \;=\; injective.$$

We consider (undirected) graphs equipped with the usual graph distance. (This is the simplest case: the pattern of the argument is similar in the cases of posets and of metric spaces, but involves certain complications.)

Notice that a graph mapping is a homomorphism (i.e. relation-preserving) if and only if it is non-expansive with respect to graph distance. Now suppose that $h : G \to H$ is a graph homomorphism into the Helly graph H, and that G is isometrically embedded (see Section 3 for the exact definition) in the graph G'. We seek to extend h over G'. Let v be an arbitrary vertex of $G' - G$. For any vertex x of G, abbreviate $d_{G'}(x,v)$ by x_v. Then we have, for any vertices x, y of G, $d_H(h(x), h(y)) \leq d_G(x,y) \leq x_v + y_v$. Hence the balls $B_H(h(x), x_v), B_H(h(y), y_v)$ meet. By the Helly property, the entire family of balls $B_H(h(x), x_v)$ as x ranges over G has non-empty meet, and we may select (arbitrarily) a vertex w from this meet to be the image of v. It is easily seen that in this way we have a graph homomorphism from $G \cup \{v\}$ into H. Repeating the construction, we extend h over G'.

The pattern of the argument is similar in the case of metric spaces, but to make it go through the Helly property as stated above has to be supplemented with a certain convexity condition, leading to the notion of a *hyperconvex* metric space [1,14]. The argument for posets is again similar; the complication this time concerns the distance function itself, which cannot be expressed as a mapping into the reals. We return to this point in a moment.

The distance from x to y in a connected poset is measured by the minimal (zig-zag) paths from x to y. The complication lies in the fact that, as between paths of a given length, we have to distinguish the zig-zag which begins with an upward link from that which begins with a downward link. (For what goes wrong when we ignore this distinction and take distance simply to be minimal path-length, see Example 2.2 below.) To be precise, define a code to be a finite alternating sequence of the symbols $+, -$. Then we say that x, y are related by the code $\sigma = (s_0, s_1, \ldots, s_{k-1})$ if there exists a path $x = x_0 x_1 \cdots x_k = y$ of P such that, for all i with $0 \leq i \leq k-1$, $x_i \leq_P x_{i+1}$ if $s_i = +$ and $x_i \geq_P x_{i+1}$ if $s_i = -$.

By repeating points where necessary in a given path we see that if $x \sigma_P y$, and the cardinality $\#\sigma < \#\tau$, then $x \tau_P y$. The only obstacle to taking distance to be minimal path-length thus lies in the fact that, for each $k > 0$, we have two incomparable codes of length k. This leads to the idea of expressing $d_P(x,y)$ as a pair of integers. The pairs in question are ordered in a certain lattice (of width 2). For a detailed presentation of this idea, see [11].

In favour of codes (as against integers, or integer-pairs), it may be pointed out that they can be used to handle "distance" in digraphs just as easily as in posets. In any case we shall define the σ-ball with centre x ($x \in P$) to be the set

$$B_P(x, \sigma) = \{y \in P \mid x \sigma_P y\};$$

that P is Helly then means that it is 2-Helly with respect to these balls.

Example 2.1 Every finite lattice is a Helly poset. Indeed, to check that a lattice P is Helly, we need only consider the codes $+$ and $-$, since a ball whose

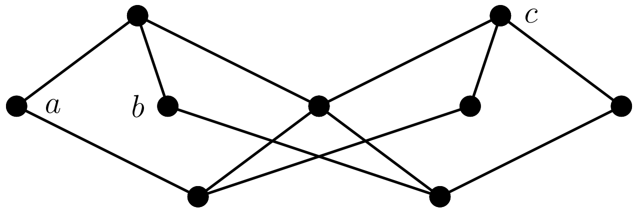

Fig. 1. W

"radius" is a code of length greater than 1 coincides with P. Thus we only have to consider balls which are, in fact, up-sets or down-sets of individual points of P. Let, then, $\mathcal{A}$ be a collection $\{\uparrow_P a_1, \ldots, \uparrow_P a_k, \downarrow_P b_1, \ldots, \downarrow_P b_l\}$ of subsets of P which meet in pairs, where $\uparrow_P x = \{y \in P \mid x \leq_P y\}$, $\downarrow_P x = \{y \in P \mid y \leq_P x\}$. This implies in particular that the set of a_i has a (least) upper bound, and also that each b_j is an upper bound of all the a_i. So we can take the least upper bound $\bigvee_{1\leq i\leq k} a_i$ as the witness that P is Helly.

Example 2.2 The poset W in Fig. 1 is a Helly poset (in fact, a Khalimsky space $\diagdown\!\diagup \times \diagup\!\diagdown$, see Section 3). However, if we were to use minimal path-length as our poset "distance", we would have to say that W is not Helly: consider the collection of balls $\mathcal{A} = \{B_W(x, 1) \mid x = a, b, c\}$. It is clear that each pair of members of $\mathcal{A}$ has non-empty intersection. However, $\bigcap \mathcal{A} = \emptyset$.

Finally, we review briefly the notion of a Khalimsky space (which represents one of the two main approaches to digital topology). This may be described either as a (partially) ordered structure, or as a topological space. The description as a partial order is most concisely achieved by saying that it is the face order of cells resulting from a rectilinear cubical subdivision of a cube in $\mathbb{R}^n$. Specifically, an one-dimensional Khalimsky "space" K is obtained by dividing an interval $[k, l]$ $(k, l \in \mathbb{N}, k < l)$ into the cells $(i, i+1)$ for $k \leq i < l$ and $\{j\}$ for $k \leq j \leq l$, the ordering of K being given by: $c \leq c'$ if c is an end-point of the interval (i.e. cell) c'. (A more formal definition appears in the next section.) Topologically, a Khalimsky space is given as the quotient of a subdivided cube C of $\mathbb{R}^n$, by the equivalence relation $\equiv$, where:

$$x \equiv y \Leftrightarrow x, y \text{ belong to the same cell of } C.$$

3 Khalimsky spaces and Helly posets

A (*binary*) *code* is a finite sequence $(s_0, \ldots, s_k)$ of elements of the set $\{+, -\}$. For codes σ and τ, σ is said to be a *subcode* of τ, denoted by $\sigma \leq_\Sigma \tau$ (called *Higman* ordering), if σ is a subsequence of τ, where Σ is the set of all codes together with the null code ϵ.

It is clear that every path $\Pi = x_0 \cdots x_k$ of the finite poset P can be associated with a minimal code $\sigma = (s_0, \ldots, s_{k-1})$, namely

$$s_i = \begin{cases} +, & \text{if } x_i \leq_P x_{i+1}, \\ -, & \text{if } x_i \geq_P x_{i+1}, \end{cases}$$

for all $i, 0 \leq i \leq k-1$. A code τ is said to be associated with Π if and only if $\sigma \leq_\Sigma \tau$ (thus the null code ϵ is associated with every singleton). For any code $\sigma \in \Sigma$ and any pair of points x, y in P, we write $x\sigma_P y$ if there exists a path $\Pi = x \cdots y$ of P such that σ is associated with Π. The σ-*ball* of the point x in P, denoted by $B_P(x, \sigma)$, is the subset of P which contains all points $z \in P$ such that $x\sigma_P z$. For any finite poset P, denote by $\mathcal{B}(P)$ the set of all σ-balls of P, i.e., $\mathcal{B}(P) = \{B_P(x, \sigma) \mid \forall x \in P, \forall \sigma \in \Sigma\}$:

Definition 3.1 ([9]) The poset P is said to be *Helly* if P is finite connected and $\mathcal{B}(P)$ satisfies the Helly property.

A map $f : Q \to P$ of posets is said to be *non-expansive* (with respect to codes) if, for any code $\sigma \in \Sigma$ and $x, y \in Q$:

$$x\sigma_Q y \Rightarrow f(x)\sigma_P f(y).$$

Clearly, any non-expansive map is monotone, but not conversely. A poset P will be called *injective* if the following holds: for any poset Q and subset $A \subseteq Q$, any non-expansive map from A to P can be extended to a non-expansive map from Q to P. Then we have the following:

Theorem 3.2 ([9, Quilliot]) *A finite poset is injective if and only if it is Helly.*

Some care is needed in interpreting the injectivity condition involved here. The mapping $f : A \to P$, whose extension over Q is sought, is required to be non-expansive with respect to codes inherited from Q. For example, let Q be the 4-element Boolean algebra $\{\bot, \mathrm{t}, \mathrm{f}, \top\}$, with $A = \{\bot, \mathrm{t}, \mathrm{f}\}$. The identity map $\mathrm{id} : A \to A$ obviously cannot be extended to Q. The non-expansiveness condition fails because $\mathrm{t}(+-)\mathrm{f}$ holds in Q but not in A considered as a poset in its own right.

A subset A of the poset P is said to be a *retract* of P if there exists an order-preserving map $r : P \to A$, called *retraction*, such that $r(a) = a$ for all $a \in A$. We say that $A \subseteq P$ is *isometrically embedded* in P if the code-distances inherited from P coincide with those derived from the poset structure of A; that is, if for all codes $\sigma \in \Sigma$, and $x, y \in A$:

$$x\sigma_A y \Leftrightarrow x\sigma_P y.$$

It is easy to see, in particular, that if A is a retract of P, then it is isometrically embedded in P. At the same time a retraction mapping is necessarily non-expansive. From these facts, and related ones about products (and projection maps), we deduce by a standard argument:

Theorem 3.3 *The class of finite injective posets is closed under retracts and products.*

We give an original (topological) definition of Khalimsky spaces: the so-called *Khalimsky* (*integer*) *line* is defined to be the set of all integers $\mathbb{Z}$ with its natural order together with its interval alternating topology defined by sets of types $(-\infty, 2i], [2j, \infty)$ for $i, j \in \mathbb{Z}$ (or alternately $(-\infty, 2i-1], [2j-1, \infty)$ for $i, j \in \mathbb{Z}$) as a subbasis. The *Khalimsky* (*digital*) *arc* then can be defined as a finite connected subspace of the Khalimsky line. The n-dimensional *Khalimsky space* is a topological product of n Khalimsky arcs [5,6].

Using the specialization order of a Khalimsky space, we can view Khalimsky space as a poset. For example, Fig. 3 shows a 2-dimensional Khalimsky space. In the following we denote K_m^n as the n-dimensional Khalimsky space such that K_m, the m-Khalimsky arc, has m open points and $m+1$ closed points for $m \geq 1$.

For any poset P, we know that the *comparability* graph of P, denoted by $\mathrm{cmp}(P)$, is the graph $\mathrm{cmp}(P) = (V, E)$ with point set $V = P$, such that $(x, y) \in E$ in $\mathrm{cmp}(P)$ if and only if $x \leq_P y$ or $y \leq_P x$ in P. A *fence* (or *zig-zag*) is a poset whose comparability graph is a path. Any fence is evidently Helly, and therefore (by Theorem 3.2) injective. What is interesting for us in this is that the fences of poset theory can be identified with the one-dimensional Khalimsky spaces (arcs) of digital topology. Thus Theorem 3.3 has as a corollary:

Corollary 3.4 *Every retract of a Khalimsky space is, in its specialization order, a Helly poset.*

The converse of this (as is well-known) is also true. The key point in the proof of this converse is the following Theorem 3.5. We have not been able to find any detailed exposition of this result (which is of independent interest for digital topology) in the literature, so we provide here some of the details.

We begin with an application of Theorem 3.2 which will be useful in the proof: The ordering of codes is the Higman ordering (so that $\sigma \leq_\Sigma \tau$ if τ can be obtained from σ by prefixing, inserting or suffixing instances of $+$ and $-$). Let P be a finite connected poset, and $x, y \in P$. Consider the geodesics between x and y, that is, the zig-zags whose code is minimal. Two cases are possible: either all the geodesics have the same code, or two distinct codes σ, σ' are associated with these geodesics, where σ' arises from σ by interchanging $+, -$. Let us take the second case, and assume the cardinality $\#\sigma = \#\sigma' = k$. Then it is easy to find in the Khalimsky space $K = K^2_{\lceil \frac{k}{2} \rceil}$ two points u, v and geodesics $\Pi = uw_1w_2 \cdots w_{k-2}w_{k-1}v$, $\Pi' = uz_1w_2 \cdots w_{k-2}z_{k-1}v$, having the codes σ, σ' associated with them. By assigning $x \mapsto u, y \mapsto v$ we determine an order isomorphism of the geodesics Π, Π' onto these two geodesics in K; thus a map $h : \Pi \cup \Pi' \to K$. It is clear that this map h satisfies the condition of Quilliot's theorem. Hence we may extend to a morphism from P to K.

Theorem 3.5 *Every finite connected poset may be isometrically embedded into a Khalimsky space.*

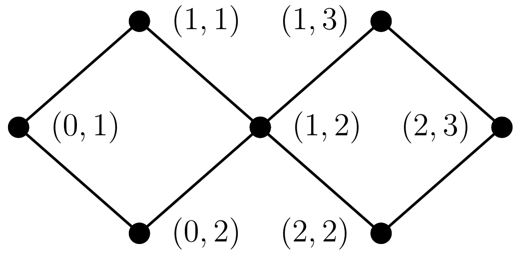

Fig. 2. X

Proof. Let the diameter (maximal length of geodesic) of the poset P be d, and let K be the Khalimsky space K_m^n, where $n = 2{\cdot}\binom{\#P}{2}$ and $m = \lceil \frac{d}{2} \rceil$. With each pair x, y of distinct points of P we associate a two-dimensional Khalimsky space K_{xy}, a product of two components of K. (In case all geodesics between x and y have the same code, it suffices to take an one-dimensional K_{xy}.) By the preceding remarks we have a projection morphism $\pi_{xy} : P \to K_{xy}$, isometric on the set $\{x, y\}$. Combining all these projections, we get an isometry from P into K. $\square$

Corollary 3.6 *Every Helly poset is a retract of a Khalimsky space.*

Proof. In fact, this corollary is just a special case of Theorem 3.2: Let P be any Helly poset. From Theorem 3.5, P can be isometrically embedded into a Khalimsky space, say K. Thus then, from Theorem 3.2, the identity map $\mathrm{id} : P \to P$ has an extension $r : K \to P$. The extension r is clearly a retraction of K onto P. $\square$

Summarizing, we have:

Theorem 3.7 *The Helly posets are exactly the retracts of the Khalimsky spaces.*

Example 3.8 As for an illustration of Theorem 3.7, it is easy to check that the poset X in Fig. 2 is a retract of the Khalimsky space K_2^2 in Fig. 3. Clearly, X is a Helly poset.

4 Admissible subsets

In this section, we define a special collection of subsets called “admissible subsets” for any finite poset P, which contains $\mathcal{B}(P)$ and all finite intersections of elements of $\mathcal{B}(P)$. We show that every admissible subset of a finite poset P is order convex, and especially, the set of admissible subsets of P forms a graph convexity if P is Helly.

Definition 4.1 The *admissible* subsets of the finite poset P are the sets of the form

$$\bigcap_i B_P(x_i, \sigma_i), x_i \in P, \sigma_i \in \Sigma.$$

Example 4.2 In the case of a lattice (Example 2.1), the admissible subsets coincide with the intervals of the lattice P.

The j^{th} *projection* $\text{pr}_j : \prod_{1 \leq i \leq n} P_i \to P_j$, from the product $\prod_{1 \leq i \leq n} P_i = P_1 \times \cdots \times P_j \times \cdots \times P_n$ to its j^{th} factor P_j, is the surjective order-preserving map satisfying $\text{pr}_j((x_1, \ldots, x_j, \ldots, x_n)) = x_j$ for all $j, 1 \leq j \leq n$.

Lemma 4.3 *Every admissible subset of a finite product of finite posets is equal to the product of its projections.*

Proof. Note that, given $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ of a finite product of posets $P = \prod_{1 \leq i \leq n} P_i$, clearly $x \leq_P y$ if and only if $x_i \leq_{P_i} y_i$ for each $i, 1 \leq i \leq n$. Since the poset relations are reflexive, therefore it is easy to check that, for any ball $B_P(x, \sigma)$ we have

$$\begin{aligned} B_P(x,\sigma) &= B_{P_1}(x_1,\sigma) \times \cdots \times B_{P_n}(x_n,\sigma) \\ &= \text{pr}_1(B_P(x,\sigma)) \times \cdots \times \text{pr}_n(B_P(x,\sigma)) \\ &= \textstyle\prod_{1 \leq i \leq n} \text{pr}_i(B_P(x,\sigma)). \end{aligned} \tag{1}$$

Let A be an admissible subset of P. It is clear that $A \subseteq \prod_{1 \leq i \leq n} \text{pr}_i(A)$. Thus, we show the converse: Since A is admissible, A can be represented as a finite intersection of finite balls. Assume that $A = \bigcap_{j \in J} B_P(z_j, \sigma_j)$ for some finite index set J. Hence

$$\begin{aligned} \textstyle\prod_{1 \leq i \leq n} \text{pr}_i(A) &= \textstyle\prod_{1 \leq i \leq n} \text{pr}_i(\bigcap_{j \in J} B_P(z_j, \sigma_j)) \\ &\subseteq \textstyle\prod_{1 \leq i \leq n} \bigcap_{j \in J} \text{pr}_i(B_P(z_j, \sigma_j)). \end{aligned} \tag{2}$$

Let $w = (w_1, \ldots, w_n)$ be any point of $\prod_{1 \leq i \leq n} \bigcap_{j \in J} \text{pr}_i(B_P(z_j, \sigma_j))$. It is clear that $w_i \in \bigcap_{j \in J} \text{pr}_i(B_P(z_j, \sigma_j))$ for each $i, 1 \leq i \leq n$; hence $w_i \in \text{pr}_i(B_P(z_j, \sigma_j))$ for each $j, j \in J$. From Eq. 1, we have $w \in B_P(z_j, \sigma_j)$ for each $j, j \in J$. Thus $w \in \bigcap_{j \in J} B_P(z_j, \sigma_j) = A$. Then from Eq. 2, we have $\prod_{1 \leq i \leq n} \text{pr}_i(A) \subseteq A$. □

Lemma 4.4 *Every admissible subset of a finite poset is order convex.*

Proof. For this, we need only to point out that every ball is an up-set, a down-set, or a singleton. In more detail: for any non-null code $\sigma \in \Sigma^* = \Sigma - \{\epsilon\}$, define the function $\text{tl} : \Sigma^* \to \{+, -\}$ by

$$\text{tl}(\sigma) = \begin{cases} +, & \text{if } \sigma = \rho+ \text{ for some } \rho \in \Sigma, \\ -, & \text{if } \sigma = \rho- \text{ for some } \rho \in \Sigma. \end{cases}$$

It is clear that, for any $x \in P$ and any non-null code $\sigma \in \Sigma^*$, $B_P(x, \sigma)$ is, respectively, an up-set if $\text{tl}(\sigma) = +$, and a down-set if $\text{tl}(\sigma) = -$. Since $B_P(x, \epsilon) = \{x\}$, therefore every admissible subset of P is the intersection of singletons, up-sets and down-sets of P, hence must be order convex. □

Evidently, order convex sets of connected posets need not themselves be connected. But we do at least have the following:

Lemma 4.5 *Every admissible subset of the Helly poset is connected.*

Proof. If $\sigma = (s_1, \ldots, s_k)$ is a non-null code, we denote by σ^{-1} the code $(-s_k, \ldots, -s_1)$, e.g., $(+-+)^{-1} = (-+-)$ and $(-+-+)^{-1} = (-+-+)$. If the cardinality of σ is $k \geq 1$, for any $j, 1 \leq j \leq k$, let $j|\sigma$ denotes the subcode $(s_1, \ldots, s_j)$, and $\sigma|j$ denotes the subcode $(s_{k-j+1}, \ldots, s_k)$, e.g., $4|(+-+-+-+-) = (+-+-), (+-+-+-+)|3 = (+-+)$, and $((+-+-+-+)|3)^{-1} = (-+-)$.

Suppose A is an admissible subset of the Helly poset P which is not connected. Since A is admissible, A can be represented as the intersection of finite balls of P , we assume that $A = \bigcap_{i \in I} B_P(x_i, \sigma_i)$ for some finite index set I. From the assumption that A is disconnected, and P is finite connected, there exist x, y belonging to different components of A such that the codes of geodesics between x and y are minimal. Let τ be one of the minimal codes. We define two balls: $B_1 = B_P(x, \lceil \frac{\#\tau}{2} \rceil | \tau)$ and $B_2 = B_P(y, (\tau | \lceil \frac{\#\tau}{2} \rceil)^{-1})$. Clearly, any two elements of the family of balls $\mathcal{B} = \{B_P(x_i, \sigma_i) \mid i \in I\} \cup \{B_1, B_2\}$ are intersection non-empty. However, it is easy to check that $\bigcap \mathcal{B} = \emptyset$. Hence P is not Helly: contradiction $\Box$

Recall that a *convexity* on a non-empty set X is a family $\mathcal{C}$ of subsets of X satisfying

- $\emptyset, X \in \mathcal{C}$,
- $\bigcap S \in \mathcal{C}$ for any $S \subseteq \mathcal{C}$,
- $\bigcup T \in \mathcal{C}$ for any chain $T \subseteq \mathcal{C}$.

The ordered pair $(X, \mathcal{C})$ is called a *convexity space*. A *graph convexity space* [3] is an ordered pair $(G, \mathcal{C})$ formed by a connected graph $G = (V, E)$, and a convexity $\mathcal{C}$ on V such that $(V, \mathcal{C})$ is a convexity space satisfying the additional condition:

- C induces a connected subgraph of G for any $C \in \mathcal{C}$.

Let $\mathcal{H}(P)$ denote the collection of admissible subsets of the finite poset P. From Lemma 4.5 and some simple observations, it is easy to check that we have

Proposition 4.6 $(\mathrm{cmp}(P), \mathcal{H}(P))$ *is a graph convexity space if P is Helly.*

We remark here that in [14,15], we called $\mathcal{H}(G)$ the *neighbourhood convexity* of the Helly graph G.

From the original definition of one-dimensional Khalimsky space (Khalimsky arc, see [5,6]), we know that K_m also inherits the natural order from $\mathbb{Z}$. Thus, we may consider K_m^n as a point lattice under the usual pointwise order $\leq_{\mathbb{N}^n}$, in which every point x of K_m^n is represented with n coordinates $(x_1, \ldots, x_n)$, all integers between 0 and $2m$ (see Fig. 3 for an example). Then,

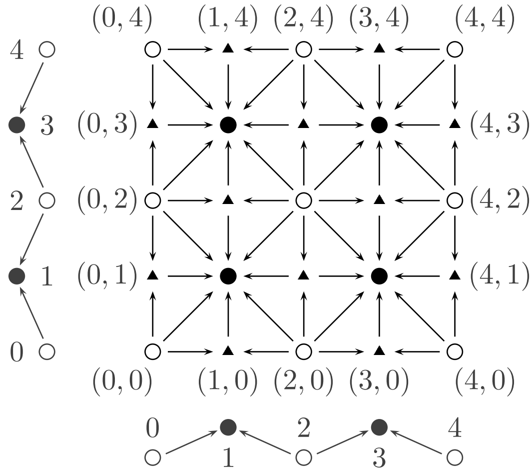

Fig. 3. A two-dimensional Khalimsky space K_2^2 with coordinates

if $x, y \in K_m^n$ such that $x \leq_{\mathbb{N}^n} y$, we write $[x, y]$ for the *segment*

$$[x, y] = \{z \in K_m^n \mid x \leq_{\mathbb{N}^n} z \leq_{\mathbb{N}^n} y\}.$$

Lemma 4.7 *Every admissible subset of the Khalimsky space is a segment, but the converse is not true.*

Proof. The first part of the assertion is just a direct consequence of Lemmas 4.5 and 4.3. On the other hand, it is easy to check that the segment $[(0, 0), (4, 0)]$ in Fig. 3 is not an admissible subset of K_2^2. □

5 Fixed point properties for Helly posets

In this section, we introduce (poset) multifunctions which can be regarded as, respectively, lower-semi continuous, upper-semi continuous, and continuous multifunctions in the sense of topological spaces (or domain theory). We show that every Helly poset has the fixed point property for both lower and upper semi-continuous multifunctions which map each point to an admissible subset of the poset.

Let P, Q be posets. By a *multifunction* $f : P \to Q$ we understand a mapping that assigns to a point $x \in P$ a non-empty subset $f(x) \subseteq Q$:

Definition 5.1 The multifunction $f : P \to Q$ is said to be *lower semi-continuous* if

$$x \leq_P y \Rightarrow \forall x' \in f(x), \exists y' \in f(y), x' \leq_Q y'.$$

f is *upper semi-continuous* if

$$x \leq_P y \Rightarrow \forall y' \in f(y), \exists x' \in f(x), x' \leq_Q y'.$$

Finally, f is said to be *continuous* if f is lower semi-continuous and upper semi-continuous.

It is clear that each class of multifunctions is closed under compositions. Also, in case f is single-valued, each of the three conditions reduces to monotonicity.

Recall that a (closed) *n-simplex* S^n is the convex hull of $n+1$ affinely independent points $\{v_0, \ldots, v_n\}$ in $\mathbb{R}^m$, for $m \geq n$; the points are called the vertices of the simplex. A k-face of S^n, $k \leq n$, is a k-simplex all of whose vertices are vertices of S^n. For $i \in \mathcal{N}(= \{0, 1, \ldots, n\})$, let S_i^n denotes the face of S^n opposite the vertex v_i. A *triangulation* T of S^n is a finite collection of (distinct) n-simplices satisfying: (1) the n-simplices cover S^n; and (2) if two n-simplices meet, their intersection is a common face. Then *Sperner's lemma* applies in the form: Let T be a triangulation of S^n with each vertex of T labeled with an integer in $\mathcal{N}$ such that no vertex in S_i^n is labeled i. (Such a labeling is called *Sperner* or *admissible*.) Then there is a simplex in T whose vertices carry all the labels in $\mathcal{N}$ (called a *complete-labeled* simplex).

Definition 5.2 A graph $G = (V, E)$ is said to be an n-dimensional *triangulation* graph if, there exists a triangulation T of S^n with V_T the 0-face set and E_T the 1-face set such that $V = V_T$ and $E = E_T$.

The Sperner lemma for simplicial complexes can be reformulated for triangulation graphs: Any Sperner labeling of an n-dimensional triangulation graph contains a complete-labeled clique (simplex).

Lemma 5.3 $\mathrm{cmp}(K_m^n)$ *is an n-dimensional triangulation graph.*

It is clear that every point of $\mathrm{cmp}(K_m^n)$ carries the same coordinate representation as it is in K_m^n. We say a function $L : \mathrm{cmp}(K_m^n) \to \mathbf{i}^n$ from the point set of $\mathrm{cmp}(K_m^n)$ to the n-product of $\mathbf{i} = \{0, 1\}$, $\mathbf{i}^n$, is a *labeling* of $\mathrm{cmp}(K_m^n)$ if

$$L_i(x) = \begin{cases} 0, & x_i = 0, \\ 1, & x_i = 2m, \end{cases}$$

for all $i, 1 \leq i \leq n$, where $x = (x_1, \ldots, x_n)$ and $L(x) = (L_1(x), \ldots, L_n(x))$.

Lemma 5.4 *Let L be any labeling of* $\mathrm{cmp}(K_m^n)$*. Then there exist maximal cliques* (*simplexes*) Δ *and* Λ*, where clearly* $\#\Delta = \#\Lambda = n+1$*, such that*

1. $(0, \ldots, 0) \in L(\Delta)$*, and for every coordinate* $i, 1 \leq i \leq n$*, there exists an individual point* $x \in \Delta$ *such that* $L_i(x) = 1$*,*
2. $(1, \ldots, 1) \in L(\Lambda)$*, and for every coordinate* $j, 1 \leq j \leq n$*, there exists an individual point* $y \in \Lambda$ *such that* $L_j(y) = 0$*.*

Proof. Similar to the proof of Theorems 4.4 and 4.5 in [15] (where we used Sperner's lemma to prove our theorems). □

Let $f : P \to P$ be a self-mapping multifunction of the poset P, and x any point of P. Then x is said to be a *fixed point* of f if $x \in f(x)$. Given posets P and Q. Let ϕ be a property of subsets of Q, then the multifunction $f : P \to Q$ is said to be a ϕ-multifunction if f sends points of P into subsets of Q satisfying ϕ. Finally, the poset P is said to have the *fixed point property* (*FPP*) for ϕ-multifunctions if every ϕ-multifunction of P has a fixed point.

Lemma 5.5 *K_m^n has the FPP for lower semi-continuous multifunctions which map each point to a segment of K_m^n.*

Proof. Let $f : K_m^n \to K_m^n$ be any self-mapping lower semi-continuous multifunction which maps each point of K_m^n to a segment of K_m^n. We define the multifunction $g : K_{m+1}^n \to K_{m+1}^n$ by $x \mapsto f(x)$ if $x \in K_m^n$, and $x \mapsto f(y)$ if $x \in K_{m+1}^n - K_m^n$, where $y \in K_m^n$ such that $\mathrm{pr}_i(y) = \mathrm{pr}_i(x)$ if $\mathrm{pr}_i(x) \leq 2m$, $\mathrm{pr}_i(y) = \mathrm{pr}_i(x) - 2$ if $\mathrm{pr}_i(x) > 2m$. The extension of g in this way to K_{m+1}^n enables some tedious case analysis to be avoided, later on.

It is easy to check that g is a lower semi-continuous multifunction which maps each point of K_{m+1}^n to a segment of K_{m+1}^n (in fact, K_m^n). Notice that if $z \in K_m^n$ is a fixed point of g, that is, $z \in g(z) \subseteq K_m^n$, then it is clear that z is also a fixed point of f, and hence we complete the proof. Let us, then, show that g has a fixed point in the subset K_m^n of K_{m+1}^n: We define the function $L : \mathrm{cmp}(K_{m+1}^n) \to \dot{\mathbf{1}}^n$ by

$$L_i(x) = \begin{cases} 0, & \mathrm{pr}_i(x) \leq y \text{ for some } y \in (\mathrm{pr}_i \circ g)(x) \text{ and } \mathrm{pr}_i(x) \neq 2m+2, \\ 1, & \mathrm{pr}_i(x) > y \text{ for every } y \in (\mathrm{pr}_i \circ g)(x) \text{ or } \mathrm{pr}_i(x) = 2m+2. \end{cases}$$

It is easy to check that $L(x) = (L_1(x), \ldots, L_n(x))$ is a labeling. Hence from Lemma 5.4, there exists a maximal clique Δ of $\mathrm{cmp}(K_{m+1}^n)$, $\#\Delta = n+1$, such that $(0, \ldots, 0) \in L(\Delta)$, and for every coordinate $i, 1 \leq i \leq n$, there exists an individual point $x \in \Delta$ such that $L_i(x) = 1$. Note that Δ is a chain of K_{m+1}^n, and for any point $x = (x_1, \ldots, x_j, \ldots, x_n)$ such that $x_j \geq 2m+1$, we have $L_j(x) = 1$.

Let z be the point of Δ which is labeled $(0, \ldots, 0)$. Clearly, $z \in K_m^n$. We show that z is a fixed point of g: suppose not, then there exists $j, 1 \leq j \leq n$, such that $\mathrm{pr}_j(z) < \mathrm{pr}_j(y)$ for all $y \in g(z)$. Let $w \in \Delta$ be the (unique) point such that $L_j(w) = 1$. By the definition of L, clearly we have $\mathrm{pr}_j(w) > \mathrm{pr}_j(y)$ for all $y \in g(w)$. Since Δ is a chain in K_{m+1}^n, we have either $z \leq_{K_{m+1}^n} w$ or $w \leq_{K_{m+1}^n} z$. Thus we have

(1) $z \leq_{K_{m+1}^n} w$ and $\mathrm{pr}_j(w) \leq \mathrm{pr}_j(z)$; or $w \leq_{K_{m+1}^n} z$ and $\mathrm{pr}_j(w) \leq \mathrm{pr}_j(z)$: Since $g(z)$ and $g(w)$ are segments of K_{m+1}^n, therefore it is easy to check that $g(z)$ and $g(w)$ are incomparable (with respect to the lower semi-continuous power ordering). Contradiction.

(2) $z \leq_{K_{m+1}^n} w$ and $\mathrm{pr}_j(z) \leq \mathrm{pr}_j(w)$: It is easy to check that we must have $\max(\mathrm{pr}_j(g(w))) \leq \mathrm{pr}_j(z)$ and $\min(\mathrm{pr}_j(g(z))) \geq \mathrm{pr}_j(w)$. However, since g is a lower semi-continuous multifunction, therefore for any $x \in g(z)$, there exists $y \in g(w)$ such that $x \leq_{K_{m+1}^n} y$. Then, picking up any point $x \in g(z)$ such that $\mathrm{pr}_j(x) = \min(\mathrm{pr}_j(g(z)))$. It is clear that no point of $g(w)$ would be greater than or equal to x: contradiction.

(3) $w \leq_{K_{m+1}^n} z$ and $\mathrm{pr}_j(z) \leq \mathrm{pr}_j(w)$: This can be proved in a similar way as (2) above.

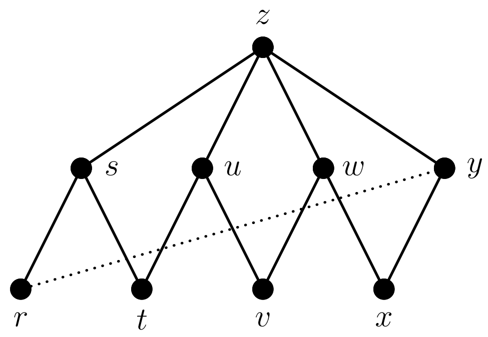

Fig. 4. K_1^2

Therefore we complete the proof. □

Example 5.6 Note that K_m^n does not have the FPP for lower semi-continuous multifunctions which map points to connected order convex sets of K_m^n. As for a simple example, let us consider K_1^2 (see Fig. 4) and the multifunction $f : K_1^2 \to K_1^2$ defined by

$$\begin{aligned} f(r) &= \{t, u, v, w, x\}, & f(s) &= \{u, v, w, x, y\}, \\ f(t) &= \{r, v, w, x, y\}, & f(u) &= \{r, s, w, x, y\}, \\ f(v) &= \{r, s, t, x, y\}, & f(w) &= \{r, s, t, u, y\}, \\ f(x) &= \{r, s, t, u, v\}, & f(y) &= \{s, t, u, v, w\}, \\ f(z) &= \{r, s, t, u, v, w, x, y\}. \end{aligned}$$

It is easy to check that f is a lower semi-continuous multifunction with images the connected order convex sets of K_m^n. However, it is clear that f has no fixed point.

Corollary 5.7 *K_m^n has the FPP for upper semi-continuous multifunctions which map each point to a segment of K_m^n.*

Proof. Let $f : K_m^n \to K_m^n$ be any self-mapping upper semi-continuous multifunction which maps points of K_m^n to segments of K_m^n. Define the multifunction $g : \hat{K}_m^n \to \hat{K}_m^n$ by

$$g(x) = f(x),$$

for any $x \in \hat{K}_m^n$, where $\hat{K}_m^n$ is the dual poset of K_m^n. It is easy to check that $\hat{K}_m^n$ is a Khalimsky space, and any segment of K_m^n is also a segment of $\hat{K}_m^n$. By duality, g is a lower semi-continuous multifunction. Hence, from Lemma 5.5, g has a fixed point, say $z \in \hat{K}_m^n$. Clearly, z is also a fixed point of f. □

Note that every admissible subset of the Khalimsky space is a segment:

Theorem 5.8 *Every Helly poset has the FPP for lower semi-continuous multifunctions which map points to admissible subsets of itself.*

Proof. Let P be a Helly poset and $f : P \to P$ any lower semi-continuous multifunction which maps points of P to admissible subsets of P. From Theorem 3.7, the poset P is a retract of K^n_m, for some $m, n \in \mathbb{N}$. Let $r : K^n_m \to P$ be the retraction. For any admissible subset A of P, denote H_A to be the least admissible subset of K^n_m such that $A \subseteq H_A$. We define the multifunction $g : K^n_m \to K^n_m$ by

$$g(x) = \bigcap\{A \mid (f \circ r)(x) \subseteq A, A \in \mathcal{H}(K^n_m)\} = H_{(f \circ r)(x)}$$

for any $x \in K^n_m$. Clearly, g is a multifunction which maps points of K^n_m to admissible subsets of K^n_m. We claim that g is lower semi-continuous: let $x, y \in K^n_m$ such that $x \leq_{K^n_m} y$. For any $z \in g(x)$, we have, for every i in the range $1, \ldots, n$, $\mathrm{pr}_i(z) = \mathrm{pr}_i(a(i))$ for some $a(i) \in (f \circ r)(x)$. Since for any $a \in (f \circ r)(x)$, there exists $b \in (f \circ r)(y)$ such that $a \leq_P b$ (hence $a \leq_{K^n_m} b$), therefore every $a(i)$ is less than or equal to some element of $(f \circ r)(y)$, say $b(i)$ in P. Hence z is less than or equal to $w = (\mathrm{pr}_1(b(1)), \ldots, \mathrm{pr}_n(b(n)))$, and clearly $w \in g(y)$.

Thus from Lemmas 4.7 and 5.5, g has a fixed point, say $z \in K^n_m$. We show as follows that $r(z)$ is a fixed point of f: indeed, since $z \in g(z) = H_{(f \circ r)(z)}$, therefore it is sufficient to show that $r(H_{(f \circ r)(z)}) \subseteq (f \circ r)(z)$. Note that for any $x \in P$ and $\sigma \in \Sigma$, we have $r(B_{K^n_m}(x, \sigma)) \subseteq B_P(x, \sigma)$, since $r(x) = x$ and r is a non-expansive map. Now, we know that $(f \circ r)(z) = \bigcap_{j \in J} B_P(x_j, \sigma_j)$, for some suitable finite index set J, where $x_j \in P, \sigma_j \in \Sigma$. Hence the set $C = \bigcap_{j \in J} B_{K^n_m}(x_j, \sigma_j)$ is mapped into $(f \circ r)(z)$ by r. But C is an admissible subset of K^n_m, and hence contains $H_{(f \circ r)(z)}$. Therefore we have $r(H_{(f \circ r)(z)}) \subseteq (f \circ r)(z)$. □

By duality, we have

Corollary 5.9 *Every Helly poset has the FPP for upper semi-continuous multifunctions which map points to admissible subsets of itself.*

Corollary 5.10 *Every Helly poset has the FPP for continuous multifunctions which map points to admissible subsets of itself.*

6 Approximation of continuous real functions

In order to relate our fixed point theorem with classical fixed point theory, we consider the approximation of continuous real functions by (suitable) functions between Khalimsky spaces (or Helly posets). Given the n-dimensional unit cube I^n, we shall understand by the *Khalimsky k-partition*, denoted by P^n_k, the subdivision of I^n into cubical cells of length 2^{-k} and their faces of dimension $n, \ldots, 0$. The k-partition P^n_k is partially ordered by the incidence order of its cells.

We begin with a simple lemma about Khalimsky partitions. This lemma could be formulated for some much more general notion of "cellular partition", but the restricted version is sufficient for our purposes here.

Lemma 6.1 *Let P $(= P^n_k)$ be a Khalimsky partition of I^n, $\alpha, \beta \in P$, $\alpha \leq_P \beta$ and $x \in \alpha$. Then:*

(1) $\bigcup(\uparrow_P \alpha)$ *is a topological (open) neighbourhood of x;*

(2) *For every topological neighbourhood N of x, we have $N \cap \beta \neq \emptyset$.*

Proof. The cell α lies within the boundary of every cell of which it is a face (in particular, of β). □

Given a real function $f : I^m \to I^n$, where I^m, I^n have the Khalimsky partitions Q, P respectively, we want to "approximate" f by a function from Q to P. The idea is that the approximating function (actually a multifunction) maps the cell $\alpha \in Q$ to the smallest segment (admissible subset) of P which contains all cells $\beta \in P$ such that every β meets $f[\alpha] \equiv \{f(x) \mid x \in \alpha\}$. Formally:

$$\phi_f : Q \to P, \alpha \mapsto H_{D(\alpha)},$$

where $D(\alpha) = \{\beta \in P \mid \beta \cap f[\alpha] \neq \emptyset\}$.

Proposition 6.2 *If f is continuous, then ϕ_f is a lower semi-continuous multifunction.*

Proof. The proof is divided into two steps: Let $\alpha, \beta \in Q$ with $\alpha \leq_Q \beta$.

(1) We show that, for any element (cell) μ of $D(\alpha)$, there exists $\nu \in D(\beta)$ such that $\mu \leq_P \nu$: If $x \in \alpha$ and $f(x) \in \mu$, then (by (1) of Lemma 6.1), $f^{-1}(\bigcup(\uparrow_P \mu))$ is an open neighbourhood of x which (by (2) of Lemma 6.1) must meet β. It follows that $f[\beta]$ meets $\bigcup(\uparrow_P \mu)$, and hence $D(\beta)$ meets $\uparrow_P \mu$.

(2) We show that, for any element μ of $H_{D(\alpha)}$, there exists $\nu \in H_{D(\beta)}$ such that $\mu \leq_P \nu$: Since $H_{D(\alpha)}$ is the minimal segment of P containing $D(\alpha)$, therefore for every index $i, i = 1, \ldots, n$, there exists $\omega(i) \in D(\alpha)$ such that $\mathrm{pr}_i(\omega(i)) = \mathrm{pr}_i(\mu)$. From (1) above, there exists $\upsilon(i) \in D(\beta)$ such that $\omega(i) \leq_P \upsilon(i)$ for every $\omega(i)$. Hence, it is clear that we have $\mu = (\mathrm{pr}_1(\omega(1)), \ldots, \mathrm{pr}_n(\omega(n)) \leq_P (\mathrm{pr}_1(\upsilon(1)), \ldots, \mathrm{pr}_n(\upsilon(n))$. Let $\nu = (\mathrm{pr}_1(\upsilon(1)), \ldots, \mathrm{pr}_n(\upsilon(n))$. Clearly, we have $\nu \in H_{D(\beta)}$ since $H_{D(\beta)}$ is the minimal segment of P containing $D(\beta)$.

Thus ϕ_f is lower semi-continuous. □

Let $Q_0, \ldots$ be the successive Khalimsky partitions of I^m; likewise $P_0, \ldots$ for I^n. For any $x \in I^m$, we have the (unique) approximating sequence of cells $\alpha_0 \in Q_0, \ldots$ such that $x \in \alpha_i$ $(i = 0, \ldots)$; moreover, $\{x\} = \bigcap_i \alpha_i$. Let $\langle \phi^i_f : Q_i \to P_i \rangle_{i=0,\ldots}$ be a sequence of multifunctions, and suppose that, for each $x \in I^m$ with approximating sequence $\alpha_0, \ldots$, $\langle \bigcup \phi^i_f(\alpha_i) \rangle_i$ is a decreasing sequence of subsets of I^n with singleton intersection, say $f(x)$. Then we say that $\langle \phi^i_f \rangle$ is an approximating sequence for $f : I^m \to I^n$.

Suppose now we start with a continuous map $f : I^m \to I^n$, and consider its approximating multifunctions $\phi^i_f : Q_i \to P_i$. By uniform continuity it follows that $\langle \phi_f^i \rangle$ is an approximating sequence, and indeed it is easy to see that we have

Proposition 6.3 (1) *$\langle \phi_f^i \rangle$ is an approximating sequence for f,*

(2) *Suppose that $i > 0$, and α_{i+1}, α_i are cells of the partitions Q_{i+1}^m, Q_i^m with $\alpha_{i+1} \subseteq \alpha_i$. Then $\bigcup \phi_f^{i+1}(\alpha_{i+1}) \subseteq \bigcup \phi_f^i(\alpha_i)$.*

Taking now the case that $m = n$, we have (by Theorem 5.8) that each ϕ_f^i has a fixed point, and by Proposition 6.3(2) that each fixed point of ϕ_f^{i+1} is contained in a fixed point of ϕ_f^i. By König's lemma we can select a decreasing sequence $\alpha_0 \supseteq \alpha_1 \supseteq \cdots$ of fixed points of $\phi_f^0, \phi_f^1, \ldots$, and by Proposition 6.3(1) we find that $\alpha_0, \alpha_1, \ldots$ converges to a fixed point of f. Thus we get a version of Brouwer's fixed point theorem: the unit cube I^n has the fixed point property for continuous functions.

7 Concluding remarks

From the preceding Section, we see that Theorem 5.8 (and Corollary 5.9) may be regarded as providing a discrete version of Brouwer's fixed point theorem. Since we work in this paper with posets rather than graphs, we have exact fixed point results, rather than the "almost fixed point property" of [12,13,15].

In future work we plan to examine the possible implications of the ideas discussed here for domain theory and semantics. In Example 2.1 we pointed out that every finite lattice is a Helly poset. Here we observe that the argument readily extends to (finite!) coherent CPO's in the sense of Markowsky and Rosen [7], Plotkin [8]. (In the finite case, a coherent CPO is a poset P with least element, such that the collection of up-sets $\{\uparrow_P x \mid x \in P\}$ has the Helly property.) In this paper we have required Helly posets to be finite, but this is in fact an unnecessary restriction. If "finite" is deleted from Definition 3.1, then we have an evident generalization of Example 2.1: every complete lattice is Helly. We may ask: can these ideas be extended in a useful way to coherent domains?

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