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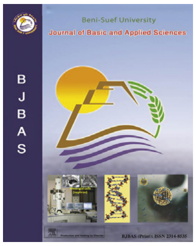

Full Length Article

Numerical solution of two-dimensional fuzzy Fredholm integral equations of the second kind using triangular functions

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ABSTRACT

The main purpose of this paper is to approximate the solution of linear two-dimensional fuzzy Fredholm integral equations of the second kind (2D-FFIE-2). We use fuzzy two-dimensional triangular functions (2D-TFs) to reduce the 2D–FFIE–2 to a system of linear Fredholm integral equations of the second kind with three variables in crisp. More over, we prove the convergence of the method. Finally we illustrate this method with some numerical examples to demonstrate the validity and applicability of the technique.

1. Introduction

Fuzzy integral equations are important in studying and solving a large proportion of the problems in many topics in applied mathematics, in particular in relation to physics, geographic, medical, biology. Usually in many applications some of the parameters in our problems are represented by fuzzy number rather than crisp, and hence it is important to develop mathematical models and numerical procedures that would appropriately treat general fuzzy integral equations and solve them.

Recently, Mirzaee and Bimesl (2014) adapted the matrix method for the Fredholm integral equations. The study of fuzzy integral equations (FIEs) which attracted growing interest for some time, begins with the investigations of Kaleva (1987) and Seikkala (1987) for the fuzzy Volterra integral equation that is equivalent to the initial value problem for first order fuzzy differential equations. These studies continued by Wang (1984), Nanda (1989), Ralescu and Adams (1980), Bede and Gal (2005), Goetschel and Voxman (1986) and others. In Wu (2000) investigated the fuzzy Riemann integral and its numerical integration. Molabahrami et al. (2011) have used the parametric form of a fuzzy number and they have converted a linear fuzzy Fredholm integral equation to two linear systems of integral equations of the second kind in the crisp case.

Recently, some numerical methods have been investigated to solve linear fuzzy Fredholm integral equations of the

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second kind in two-dimensional space. For example, Rivaz and Yousefi (2012) and Ezzati and Ziari (2013) used homotopy perturbation method and fuzzy Bivariate Bernestein polynomials method for solving 2D-FFIE-2, respectively. Deb et al. (2006) introduced a new set of orthogonal functions, a numerical scheme based on such functions was applied for solving variational problem and integral equation by Babolian et al. (2007, 2009; 2010).

In this paper, we apply the triangular functions for approximate the solutions of the linear two-dimensional Fredholm fuzzy integral equations of the second kind for the first time.

This paper is organized as follows. In Section 2, we present some definitions and properties of one and two-dimensional triangular functions which will be used later. In Section 3, we give an overview of elementary concepts of the fuzzy calculus. two-dimensional fuzzy Fredholm integral equation is described in Section 4. In Section 5, we apply 2D-TFs for solving linear two-dimensional fuzzy Fredholm integral equation. Section 6 is concerned with discussing the convergency of the proposed method then this method is implemented for solving two illustrative examples in Section 7 and finally, conclusion is drawn in Section 8.

2. Preliminaries

2.1. A review of one-dimensional triangular functions

Definition 2.1. Two m-sets of triangular functions (*TFs*) are defined over the interval $[0,T]$ as:

$$T1_i(t)=\begin{cases}1-\dfrac{t-ih}{h} & ih\le t\le (i+1)h,\\ 0 & \text{otherwise},\end{cases} \tag{1}$$

$$T2_i(t)=\begin{cases}\dfrac{t-ih}{h} & ih\le t\le (i+1)h,\\ 0 & \text{otherwise},\end{cases} \tag{2}$$

where $i=0,1,\ldots,m-1$, $h=T/m$, with a positive integer value for m. We have

$$\int_0^1 T1_i(t)T1_j(t)dt=\int_0^1 T2_i(t)T2_j(t)dt=\begin{cases}\dfrac{h}{3} & i=j,\\ 0 & i\neq j,\end{cases} \tag{3}$$

and

$$\int_0^1 T1_i(t)T2_j(t)dt=\int_0^1 T2_i(t)T1_j(t)dt=\begin{cases}\dfrac{h}{6} & i=j,\\ 0 & i\neq j.\end{cases} \tag{4}$$

Also, consider $T1_i$ as the i th left-handed triangular function and $T2_i$ as the i th right-handed triangular function. In this paper, it is assumed that $T=1$.

Consider the first m terms of the left-handed triangular functions and the first m terms of the right-handed triangular functions and write them concisely as m-vectors:

$$T1(t)=\left[T1_0(t),T1_1(t),\ldots,T1_{m-1}(t)\right]^T, \tag{5}$$

$$T2(t)=\left[T2_0(t),T2_1(t),\ldots,T2_{m-1}(t)\right]^T, \tag{6}$$

where $T1(t)$ and $T2(t)$ are called left-handed triangular functions (*LHTF*) vector and right-handed triangular functions (*RHTF*) vector, respectively. We have:

$$\int_0^1 T1(t)T1^T(t)dt=\int_0^1 T2(t)T2^T(t)dt=\frac{h}{3}I, \tag{7}$$

$$\int_0^1 T1(t)T2^T(t)dt=\int_0^1 T2(t)T1^T(t)dt=\frac{h}{6}I, \tag{8}$$

which I is an $m\times m$ identity matrix. We denote the 1D-TF vector $T(t)$ as follows

$$T(t)=\begin{bmatrix}T1(t)\\ T2(t)\end{bmatrix}. \tag{9}$$

The expansion of the function $f(t)$ over $[0,1]$ with respect to 1D-TFs may be written as

$$f(t)\approx\sum_{i=0}^{m-1}c_iT1_i(t)+\sum_{i=0}^{m-1}d_iT2_i(t)=C^T\cdot T1(t)+D^T\cdot T2(t)$$
$$=\begin{bmatrix}C\\ D\end{bmatrix}^T\cdot\begin{bmatrix}T1(t)\\ T2(t)\end{bmatrix}=F^T\cdot T(t),$$

where C_i and D_i are samples of f, for example $C_i=f(ih)$ and $D_i=f((i+1)h)$ for $i=0,1,\ldots,m-1$, so there is no need for integration. The vector F is called the 1D-TF coefficient vector.

2.2. Two-dimensional triangular functions and their properties

An $(m_1\times m_2)$-set of the region $(\Omega=[0,1]\times[0,1])$ is defined by

$$T^{1,1}_{i,j}(s,t)=\begin{cases}(1-\dfrac{s-ih_1}{h_1})(1-\dfrac{t-jh_2}{h_2}) & ih_1\le s\le (i+1)h_1,\ jh_2\le t\le (j+1)h_2,\\ 0 & \text{otherwise},\end{cases} \tag{10}$$

$$T^{1,2}_{i,j}(s,t)=\begin{cases}(1-\dfrac{s-ih_1}{h_1})(\dfrac{t-jh_2}{h_2}) & ih_1\le s\le (i+1)h_1,\ jh_2\le t\le (j+1)h_2,\\ 0 & \text{otherwise},\end{cases} \tag{11}$$

$$T^{2,1}_{i,j}(s,t)=\begin{cases}(\dfrac{s-ih_1}{h_1})(1-\dfrac{t-jh_2}{h_2}) & ih_1\le s\le (i+1)h_1,\ jh_2\le t\le (j+1)h_2,\\ 0 & \text{otherwise},\end{cases} \tag{12}$$

$$T^{2,2}_{i,j}(s,t)=\begin{cases}(\dfrac{s-ih_1}{h_1})(\dfrac{t-jh_2}{h_2}) & ih_1\le s\le (i+1)h_1,\ jh_2\le t\le (j+1)h_2,\\ 0 & \text{otherwise},\end{cases} \tag{13}$$

where $i=0,1,\ldots,m_1-1$, $j=0,1,\ldots,m_2-1$ and $h_1=1/m_1$, $h_2=1/m_2$. m_1 and m_2 are arbitrary positive integers. It is clear that

$$\begin{aligned}T^{1,1}_{i,j}(s,t)&=T1_i(s).T1_j(t),\\ T^{1,2}_{i,j}(s,t)&=T1_i(s).T2_j(t),\\ T^{2,1}_{i,j}(s,t)&=T2_i(s).T1_j(t),\\ T^{2,2}_{i,j}(s,t)&=T2_i(s).T2_j(t).\end{aligned} \tag{14}$$

Furthermore,

$$T^{1,1}_{i,j}(s,t)+T^{1,2}_{i,j}(s,t)+T^{2,1}_{i,j}(s,t)+T^{2,2}_{i,j}(s,t)=\Phi_{i,j}(s,t),$$

where $\Phi_{i,j}(s,t)$ is the $\{im_2+j+1\}$ th block-pulse function defined on $ih_1 \le s \le (i+1)h_1$ and $jh_2 \le t \le (j+1)h_2$ as

$$\Phi_{i,j}(s,t) = \begin{cases} 1 & \begin{array}{l} ih_1 \le s \le (i+1)h_1, \\ jh_2 \le t \le (j+1)h_2, \end{array} \\ 0 & \quad \text{otherwise.} \end{cases}$$

From Eq.s (3), (4) and (14) we have:

$$\int_0^1 \int_0^1 T_{i_1,j_1}^{p_1,q_1}(s,t) \cdot T_{i_2,j_2}^{p_2,q_2}(s,t) ds dt = \Delta_{p_1,p_2} \delta_{i_1,i_2} \cdot \Delta_{q_1,q_2} \delta_{j_1,j_2},$$

where δ denotes the Kronecker delta function and

$$\Delta_{\alpha,\beta} = \begin{cases} \dfrac{h}{3} & \alpha = \beta \in \{1,2\}, \\ \dfrac{h}{6} & \alpha \ne \beta. \end{cases}$$

On the other hand, if

$$T11(s,t) = \left[T_{0,0}^{1,1}(s,t), \ldots, T_{0,m_2-1}^{1,1}(s,t), T_{1,0}^{1,1}(s,t), \ldots, T_{m_1-1,m_2-1}^{1,1}(s,t)\right]^T,$$

$$T12(s,t) = \left[T_{0,0}^{1,2}(s,t), \ldots, T_{0,m_2-1}^{1,2}(s,t), T_{1,0}^{1,2}(s,t), \ldots, T_{m_1-1,m_2-1}^{1,2}(s,t)\right]^T,$$

$$T21(s,t) = \left[T_{0,0}^{2,1}(s,t), \ldots, T_{0,m_2-1}^{2,1}(s,t), T_{1,0}^{2,1}(s,t), \ldots, T_{m_1-1,m_2-1}^{2,1}(s,t)\right]^T,$$

$$T22(s,t) = \left[T_{0,0}^{2,2}(s,t), \ldots, T_{0,m_2-1}^{2,2}(s,t), T_{1,0}^{2,2}(s,t), \ldots, T_{m_1-1,m_2-1}^{2,2}(s,t)\right]^T,$$

then $T(s,t)$, the 2D-TF vector, can be defined as

$$T(s,t) = \begin{bmatrix} T11(s,t) \\ T12(s,t) \\ T21(s,t) \\ T22(s,t) \end{bmatrix}_{4m_1m_2 \times 1}. \tag{15}$$

We have:

$$\int_0^1 \int_0^1 T11^T(s,t) T11(s,t) ds dt = \frac{h_1}{3} I_{m_1 \times m_1} \otimes \frac{h_2}{3} I_{m_2 \times m_2},$$

$$\int_0^1 \int_0^1 T11^T(s,t) T12(s,t) ds dt = \frac{h_1}{3} I_{m_1 \times m_1} \otimes \frac{h_2}{6} I_{m_2 \times m_2},$$

$$\int_0^1 \int_0^1 T11^T(s,t) T21(s,t) ds dt = \frac{h_1}{6} I_{m_1 \times m_1} \otimes \frac{h_2}{3} I_{m_2 \times m_2},$$

$$\int_0^1 \int_0^1 T11^T(s,t) T22(s,t) ds dt = \frac{h_1}{6} I_{m_1 \times m_1} \otimes \frac{h_2}{6} I_{m_2 \times m_2},$$

where $\otimes$ denotes the Kronecker product defined for two arbitrary matrices P and Q as

$$P \otimes Q = p_{i,j} Q.$$

The same equations are implied for $T12(s,t)$, $T21(s,t)$ and $T22(s,t)$, by similar computations. Hence, we can carry out double integration of $T(s,t)$:

$$\int_0^1 \int_0^1 T^T(s,t) T(s,t) dt ds = D, \tag{16}$$

which D is $(4m_1m_2 \times 4m_1m_2)$-matrix as follows:

$$D = \begin{bmatrix} \frac{h_1}{3} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{6} I_2 \\ \frac{h_1}{3} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{3} I_2 \\ \frac{h_1}{6} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{6} I_2 \\ \frac{h_1}{6} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{3} I_2 \end{bmatrix}, \tag{17}$$

which $I_1 = I_{m_1 \times m_1}$ and $I_2 = I_{m_2 \times m_2}$.

2.3. *Function expansion with 2D-TFs*

We can approximate the function $f(s,t)$ defined over Ω by 2D-TFs as follows

$$\begin{aligned} f(s,t) \approx & \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} c_{i,j} T_{i,j}^{1,1}(s,t) + \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} d_{i,j} T_{i,j}^{1,2}(s,t) \\ & + \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} e_{i,j} T_{i,j}^{2,1}(s,t) + \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} l_{i,j} T_{i,j}^{2,2}(s,t) \\ & = C^T \cdot T11(s,t) + D^T \cdot T12(s,t) + E^T \cdot T21(s,t) \\ & + L^T \cdot T22(s,t) = F^T \cdot T(s,t), \end{aligned}$$

where F is a $4m_1m_2$-vector given by

$$F = \begin{bmatrix} C^T & D^T & E^T & L^T \end{bmatrix}^T,$$

and $T(s,t)$ is defined in Eq. (15). The 2D-TF coefficients in C, D, E and L can be computed by sampling the function $f(s,t)$ at grid points s_i and t_j such that $s_i = ih_1$ and $t_j = jh_2$, for various i and j. So we have

$$C_k = c_{i,j} = f(s_i, t_j),$$

$$D_k = d_{i,j} = f(s_i, t_{j+1}),$$

$$E_k = e_{i,j} = f(s_{i+1}, t_j),$$

$$L_k = l_{i,j} = f(s_{i+1}, t_{j+1}),$$

where $k = im_2 + j$ and $i = 0,1,\ldots,m_1-1$, $j = 0,1,\ldots,m_2-1$. The $4m_1m_2$-vector F is called the 2D-TF coefficient vector.

Let $u(s,t,r)$ be a function of three variables on $(\Omega \times [0,1])$. It can be approximated with respect to 2D-TFs and 1D-TFs as follows:

$$u(s,t,r) \approx T^T(s,t) \cdot U \cdot T(r), \tag{18}$$

where $T(s,t)$ and $T(r)$ are 2D-TF vector and 1D-TF vector of dimension $4m_1m_2$ and $2m_3$, respectively and U is a $(4m_1m_2 \times 2m_3)$ 2D-TF coefficient matrix. This matrix can be represented as

$$U = \begin{bmatrix} U11 & U12 \\ U21 & U22 \\ U31 & U32 \\ U41 & U42 \end{bmatrix},$$

where each block of U is an $(m_1m_2 \times m_3)$-matrix that can be computed by sampling the function $u(s,t,r)$ at grid points (s_i,t_j,r_k) such that

$$s_i = ih_1, \quad i = 0,1,\ldots,m_1-1, \quad h_1 = \frac{1}{m_1},$$

$$t_j = jh_2, \quad j = 0,1,\ldots,m_2-1, \quad h_2 = \frac{1}{m_2},$$

$$r_k = kh_3, \quad k = 0,1,\ldots,m_3-1, \quad h_3 = \frac{1}{m_3}.$$

Let $l = im_2 + j$, then

$$U11_{l,k} = u(s_i,t_j,r_k),$$
$$U12_{l,k} = u(s_i,t_j,r_{k+1}),$$
$$U21_{l,k} = u(s_i,t_{j+1},r_k),$$
$$U22_{l,k} = u(s_i,t_{j+1},r_{k+1}),$$

$$U31_{l,k} = u(s_{i+1},t_j,r_k),$$
$$U32_{l,k} = u(s_{i+1},t_j,r_{k+1}),$$
$$U41_{l,k} = u(s_{i+1},t_{j+1},r_k),$$
$$U42_{l,k} = u(s_{i+1},t_{j+1},r_{k+1}).$$

Let $k(s,t,x,y)$ be a function of four variables on $(\Omega \times \Omega)$. It can be approximated with respect to 2D-TFs as follows:

$$k(s,t,x,y) \approx T^T(s,t)\cdot K\cdot T(x,y), \tag{19}$$

where $T(s,t)$ and $T(x,y)$ are 2D-TF vectors of dimension $4m_1m_2$ and $4m_3m_4$, respectively and K is a $(4m_1m_2 \times 4m_3m_4)$ 2D-TF coefficient matrix. This matrix can be represented as

$$K = \begin{bmatrix} K11 & K12 & K13 & K14 \\ K21 & K22 & K23 & K24 \\ K31 & K32 & K33 & K34 \\ K41 & K42 & K43 & K44 \end{bmatrix}, \tag{20}$$

where each block of K is an $(m_1m_2 \times m_3m_4)$-matrix that can be computed by sampling the function $k(s,t,x,y)$ at grid points $(s_{i_1},t_{j_1},x_{i_2},y_{j_2})$ such that

$$s_{i_1} = i_1h_1, \quad i_1 = 0,1,\ldots,m_1-1, \quad h_1 = \frac{1}{m_1},$$

$$t_{j_1} = j_1h_2, \quad j_1 = 0,1,\ldots,m_2-1, \quad h_2 = \frac{1}{m_2},$$

$$x_{i_2} = i_2h_3, \quad i_2 = 0,1,\ldots,m_3-1, \quad h_3 = \frac{1}{m_3},$$

$$y_{j_2} = j_2h_4, \quad j_2 = 0,1,\ldots,m_4-1, \quad h_4 = \frac{1}{m_4}.$$

Let $p = i_1m_2 + j_1$ and $q = i_2m_4 + j_2$, then

$$K11_{p,q} = k(s_{i_1},t_{j_1},x_{i_2},y_{j_2}),$$
$$K12_{p,q} = k(s_{i_1},t_{j_1},x_{i_2},y_{j_2+1}),$$
$$K13_{p,q} = k(s_{i_1},t_{j_1},x_{i_2+1},y_{j_2}),$$
$$K14_{p,q} = k(s_{i_1},t_{j_1},x_{i_2+1},y_{j_2+1}),$$

$$K21_{p,q} = k(s_{i_1},t_{j_1+1},x_{i_2},y_{j_2}),$$
$$K22_{p,q} = k(s_{i_1},t_{j_1+1},x_{i_2},y_{j_2+1}),$$
$$K23_{p,q} = k(s_{i_1},t_{j_1+1},x_{i_2+1},y_{j_2}),$$
$$K24_{p,q} = k(s_{i_1},t_{j_1+1},x_{i_2+1},y_{j_2+1}),$$

$$K31_{p,q} = k(s_{i_1+1},t_{j_1},x_{i_2},y_{j_2}),$$
$$K32_{p,q} = k(s_{i_1+1},t_{j_1},x_{i_2},y_{j_2+1}),$$
$$K33_{p,q} = k(s_{i_1+1},t_{j_1},x_{i_2+1},y_{j_2}),$$
$$K34_{p,q} = k(s_{i_1+1},t_{j_1},x_{i_2+1},y_{j_2+1}),$$

$$K41_{p,q} = k(s_{i_1+1},t_{j_1+1},x_{i_2},y_{j_2}),$$
$$K42_{p,q} = k(s_{i_1+1},t_{j_1+1},x_{i_2},y_{j_2+1}),$$
$$K43_{p,q} = k(s_{i_1+1},t_{j_1+1},x_{i_2+1},y_{j_2}),$$
$$K44_{p,q} = k(s_{i_1+1},t_{j_1+1},x_{i_2+1},y_{j_2+1}).$$

In this paper we supposed that $m_1 = m_2 = m_3 = m_4 = M$, for convergence.

3. The basic concepts of fuzzy equations

We now recall some definitions needed through the paper.

Definition 1. A fuzzy number is a set $v{:}R_F \to I = [0,1]$ which satisfies in the following statements:

- v is upper semi continuous,
- $v(x) = 0$ outside of some interval such as $[c,d]$,
- There are real numbers a,b that $c \le a \le b \le d$ and
  - $v(x)$ is monotonic increasing on $[c,a]$,
  - $v(x)$ is monotonic decreasing on $[b,d]$,
  - $v(x) = 1$, $a \le x \le b$.

The set of all such fuzzy number is denoted by R_F (Kaleva, 1987).

Definition 2. Let V be a fuzzy set on R_F. V is called a fuzzy interval if:

- V is normal, there exists $x_0 \in R_F$ such that $V(x_0) = 1$.
- V is convex, it holds that $V(\lambda x + (1-\lambda)t) \ge min\{V(x),V(t)\}$, for all $x,t \in R_F$ and $0 \le \lambda \le 1$.
- V is upper semi-continuous, $V(x_0) \ge \lim\limits_{x \to x_0^{\mp}} V(x)$, for any $x_0 \in R_F$,
- $[V]^0 = Cl\{x \in R_F | V(x) > 0\}$ is a compact subset of R_F.

The α-cut of a fuzzy interval V, with $0 < \alpha \le 1$ is the crisp set, $[V]^\alpha = \{x \in R_F | V(x) > \alpha\}$.

For a fuzzy interval V, its α-cuts are closed intervals in R_F (Kaleva, 1987). Let denote them by $[V]^\alpha = [\underline{V}(\alpha), \overline{V}(\alpha)]$.

An alternative definition or parametric form of a fuzzy number which yields the same R_F is given by Kaleva (1987) as follows:

Definition 3. An arbitrary fuzzy number $\tilde{u}$ in the parametric form is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$ which satisfy the following requirements (Ralescu and Adams (1980)):

- $\underline{u}(r)$ is a bounded left-continuous non-decreasing function over [0,1],
- $\overline{u}(r)$ is a bounded right-continuous non-increasing function over [0,1],

- $\underline{u}(r) \le \overline{u}(r)$, for all $0 \le r \le 1$.

For arbitrary fuzzy numbers $\tilde{v} = (\underline{v}(r), \overline{v}(r))$, $\tilde{w} = (\underline{w}(r), \overline{w}(r))$ and real number λ, we can define the addition and the scalar multiplication of the fuzzy numbers by using the extension principle as follows:

- $\tilde{v} = \tilde{w}$ if and only if $\underline{v}(r) = \underline{w}(r)$ and $\overline{v}(r) = \overline{w}(r)$,
- $\tilde{v} \oplus \tilde{w} = (\underline{v}(r) + \underline{w}(r), \overline{v}(r) + \overline{w}(r))$,
- $(\lambda \otimes \tilde{v}) = \begin{cases} (\lambda \underline{v}(r), \lambda \overline{v}(r)) & \lambda \ge 0, \\ (\lambda \overline{v}(r), \lambda \underline{v}(r)) & \lambda < 0. \end{cases}$

Definition 4. For arbitrary numbers $\tilde{v} = (\underline{v}(r), \overline{v}(r))$ and $\tilde{w} = (\underline{w}(r), \overline{w}(r))$ (Nanda, 1989)

$$D(\tilde{v}, \tilde{w}) = max\left\{ \sup_{0 \le r \le 1} |\overline{v}(r) - \overline{w}(r)|, \sup_{0 \le r \le 1} \left|\underline{v}(r) - \underline{w}(r)\right| \right\},$$

in the distance between $\tilde{v}$ and $\tilde{w}$. It is proved that (R_F,D) is a complete metric space with the properties (Nanda, 1989).

- $\forall \tilde{u}, \tilde{v}, \tilde{w} \in R_F \quad ; \quad D(\tilde{u} \oplus \tilde{w}, \tilde{v} \oplus \tilde{w}) = D(\tilde{u}, \tilde{v})$,
- $\forall \tilde{u}, \tilde{v} \in R_F, \quad \forall k \in R \quad ; \quad D(k \otimes \tilde{u}, k \otimes \tilde{v}) = |k| D(\tilde{u}, \tilde{v})$,
- $\forall \tilde{u}, \tilde{v}, \tilde{w}, \tilde{e} \in R_F \quad ; \quad D(\tilde{u} \oplus \tilde{v}, \tilde{w} \oplus \tilde{e}) \le D(\tilde{u}, \tilde{w}) + D(\tilde{v}, \tilde{e})$.

Definition 5. Let $\tilde{f}, \tilde{g} : [a, b] \to R_F$, be fuzzy real number valued functions. The uniform distance between $\tilde{f}, \tilde{g}$ is defined by Wang (1984):

$$D_U(\tilde{f}, \tilde{g}) = sup\left\{ D(\tilde{f}(x), \tilde{g}(x)) | x \in \left[a, b\right] \right\}. \tag{21}$$

In Bede and Gal (2005), the authors proved that if the fuzzy function, $\tilde{f}(t)$, is continuous in the metric D, its definite integral exists and also,

$$\underline{\int_a^b f(t,r)dt} = \int_a^b \underline{f}(t,r)dt,$$

$$\overline{\int_a^b f(t,r)dt} = \int_a^b \overline{f}(t,r)dt.$$

Definition 6. A fuzzy real number valued function $\tilde{f} : [a, b] \to R_F$ is said to be continuous in $x_0 \in [a,b]$, if for each $\varepsilon > 0$ there is $\delta > 0$ such that $D(\tilde{f}(x), \tilde{f}(x_0)) < \varepsilon$, whenever $x \in [a,b]$ and $|x - x_0| < \delta$ (Wu, 2000). We say that $\tilde{f}$ is fuzzy continuous on $[a,b]$ if $\tilde{f}$ is continuous at each $x_0 \in [a,b]$ and denote the space of all such functions by $C_F([a,b])$.

Definition 7. Let $\tilde{f} : [a, b] \to R_F$ be a bounded function, then function $\omega_{[a,b]}(\tilde{f}, .) : R^+ \cup \{0\} \to R^+$,

$$\omega_{[a,b]}(\tilde{f}, \delta) = sup\left\{ D(\tilde{f}(x), \tilde{f}(y)) \Big| x, y \in \left[a, b\right], |x - y| \le \delta \right\}, \tag{22}$$

where R^+ is the set of positive real numbers, is called the modulus of continuity of $\tilde{f}$ on $[a,b]$ (Wu, 2000).

Definition 8. Let $\tilde{f} : [a, b] \to R_F$, $\tilde{f}$ is fuzzy-Riemann integrable to $I \in R_F$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u,v] ; \xi\}$ of $[a,b]$ with the norms $\Delta(p) < \delta$, we have (Wu, 2000):

$$D(\sum_p {}^*(v - u) \otimes \tilde{f}(\xi), I) < \varepsilon, \tag{23}$$

where $\sum^*$ denotes the fuzzy summation. In this case it is denoted by $I = (FR) \int_a^b \tilde{f}(x)dx$.

Lemma 1. If $\tilde{f}, \tilde{g} : [a, b] \subseteq R \to R_F$ are fuzzy continuous functions, then the function F:$[a,b] \to R_F$ by $F(x) = D(\tilde{f}(x), \tilde{g}(x))$ is continuous on $[a,b]$ and by (Wu, 2000).

$$D\left(\int_a^b \tilde{f}(x)dx, \int_a^b \tilde{g}(x)dx \right) \le \int_a^b D(\tilde{f}(x), \tilde{g}(x))dx.$$

Definition 9. A function $\tilde{f} : R^2 \to R_F$ is called a fuzzy function in two-dimensional space. $\tilde{f}$ is said to be continuous (Rivaz and Yousefi (2012)), if for arbitrary fixed $t_0 \in R^2$ and $\varepsilon > 0$ a $\delta > 0$ exists such that

$$\|t - t_0\| < \delta \Rightarrow D(\tilde{f}(t), \tilde{f}(t_0)) < \varepsilon \quad ; \quad t = (x, y), t_0 = (x_0, y_0).$$

Definition 10. Let $\tilde{f} : [a, b] \times [c, d] \to R_F$. For each partition $p = \{x_1, x_2, \ldots, x_m\}$ of $[a,b]$ and $q = \{y_1, y_2, \ldots, y_n\}$ of $[c,d]$ and for arbitrary ξ:$x_{i-1} \le \xi_i \le x_i$, $2 \le i \le m$ and for arbitrary η_j:$y_{j-1} \le \eta_j \le y_j$, $2 \le j \le n$, let

$$R_p = \sum_{i=2}^{m} \sum_{j=2}^{n} \tilde{f}(\xi_i, \eta_j)(x_i - x_{i-1})(y_j - y_{j-1}).$$

The definite integral of $\tilde{f}(x, y)$ over $[a,b] \times [c,d]$ is

$$\int_c^d \int_a^b \tilde{f}(x, y)dxdy = limR_p,$$

$$\left(\max_{2 \le i \le m} |x_i - x_{i-1}|, \max_{2 \le j \le n} \left|y_j - y_{j-1}\right| \right) \to (0, 0),$$

provided that this limit exists in metric D (Rivaz and Yousefi (2012)).

If the function $\tilde{f}(x, y)$ is continuous in the metric D, it's definite that integral exists (Goetschel and Voxman (1986)). Furthermore

$$\left(\underline{\int_c^d \int_a^b f(x, y, r)dxdy} \right) = \int_c^d \int_a^b \underline{f}(x, y, r)dxdy,$$

$$\left(\overline{\int_c^d \int_a^b f(x, y, r)dxdy} \right) = \int_c^d \int_a^b \overline{f}(x, y, r)dxdy.$$

4. Two-dimensional fuzzy Fredholm integral equation

Two-dimensional Fredholm integral equation of the second kind is defined as the following from (Xie and Lin (2009))

$$u(x,y)=f(x,y)+\lambda\int_a^b\int_c^d K(x,y,s,t)u(s,t)dsdt. \tag{24}$$

The linear two-dimensional fuzzy Fredholm integral equation of the second kind (2D-FFIE-2) is

$$\tilde{u}(x,y)=\tilde{f}(x,y)\oplus\lambda\otimes\int_a^b\int_c^d K(x,y,s,t)\otimes\tilde{u}(s,t)dsdt, \tag{25}$$

where $k(x,y,s,t)$ is an arbitrary kernel function over $S=[a,b]\times[c,d]\times[a,b]\times[c,d]$ and $\tilde{u}(x,y)$ and $\tilde{f}(x,y)$ are fuzzy real valued functions and $\tilde{u}(x,y)$ is unknown.

Now, we introduce parametric form of a 2D-FFIE-2 with respect to definition (3).

Let $(\underline{f}(x,y,r),\overline{f}(x,y,r))$ and $(\underline{u}(x,y,r),\overline{u}(x,y,r))$, $0\le r\le 1$, be parametric form of. $\tilde{f}(x,y)$and $\tilde{u}(x,y)$, respectively. Then parametric from of 2D-FFIE-2 is as follows:

$$\underline{u}(x,y,r)=\underline{f}(x,y,r)+\lambda\int_a^b\int_c^d v_1(x,y,s,t,\underline{u}(s,t,r),\overline{u}(s,t,r))dsdt, \tag{26}$$

$$\overline{u}(x,y,r)=\overline{f}(x,y,r)+\lambda\int_a^b\int_c^d v_2(x,y,s,t,\underline{u}(s,t,r),\overline{u}(s,t,r))dsdt. \tag{27}$$

Where

$$v_1(x,y,s,t,\underline{u}(s,t,r),\overline{u}(s,t,r))=\begin{cases}k(x,y,s,t)\underline{u}(s,t,r) & k(x,y,s,t)\ge 0,\\ k(x,y,s,t)\overline{u}(s,t,r) & k(x,y,s,t)<0,\end{cases}$$

and

$$v_2(x,y,s,t,\underline{u}(s,t,r),\overline{u}(s,t,r))=\begin{cases}k(x,y,s,t)\overline{u}(s,t,r) & k(x,y,s,t)\ge 0,\\ k(x,y,s,t)\underline{u}(s,t,r) & k(x,y,s,t)<0,\end{cases}$$

for $0\le r\le 1$. We can see that Eq.s (26) and (27) are systems of Fredholm integral equations of the second kind with three variables in crisp case.

5. Solving linear two-dimensional fuzzy Fredholm integral equation by 2D-TFs

In this section, we present an effective method for solving a 2D-FFIE-2 by using 2D-TFs. The parametric form of (25), with $a=c=0$, $b=d=1$ and $\lambda=1$, is:

$$\underline{u}(x,y,r)=\underline{f}(x,y,r)+\int_0^1\int_0^1 k(x,y,s,t)\underline{u}(s,t,r)dsdt, \tag{28}$$

$$\overline{u}(x,y,r)=\overline{f}(x,y,r)+\int_0^1\int_0^1 k(x,y,s,t)\overline{u}(s,t,r)dsdt. \tag{29}$$

2D-TFs are applied for Eq.s (28) and (29) respectively. To explain this method for Eq. (28), first let us expand $\underline{u}(x,y,r)$, $\underline{f}(x,y,r)$ and $k(x,y,s,t)$ by 2D-TFs as follows:

$$\underline{u}(x,y,r)\approx T^T(x,y)UT(r), \tag{30}$$

$$\underline{f}(x,y,r)\approx T^T(x,y)FT(r), \tag{31}$$

$$k(x,y,s,t)\approx T^T(x,y)KT(s,t). \tag{32}$$

Where $T(x,y)$ and $T(r)$ are defined in Eq.s (15) and (9), respectively and U and F are $(4M^2\times 2M)$-matrices of 2D-TFs coefficients of $\underline{u}(x,y,r)$ and $\underline{f}(x,y,r)$, respectively, and K is $(4M^2\times 4M^2)$-matrix 2D-TFs coefficients of $k(x,y,s,t)$.

To approximate the solution of Eq. (28), from Eq.s (30)–(32), we have

$$T^T(x,y)UT(r)\approx T^T(x,y)FT(r)+\int_0^1\int_0^1 T^T(x,y)KT(s,t)T^T(s,t)UT(r)dsdt,$$

$$T^T(x,y)UT(r)\approx T^T(x,y)FT(r)+T^T(x,y)K\left(\int_0^1\int_0^1 T(s,t)T^T(s,t)dsdt\right)UT(r),$$

using the Eq. (16) we have

Table 1 – Numerical results of Example 1 with 2D-TFs method.

r	Exact solution $(\underline{u}(x,y,r),\overline{u}(x,y,r))$	2D-TFs method for $x=0.1$, $y=0.1$ and $M=5$	2D-TFs method for $x=0.1$,$y=0.1$ and $M=10$	Absolute error for $x=0.1$,$y=0.1$ and $M=10$
0.0	(0.00000000, 0.02036474)	(0.00000000, 0.02071364)	(0.00000000, 0.02036484)	(0.00000000e-00,9.9976971e-08)
0.1	(0.00056003, 0.01985053)	(0.00062140, 0.02017509)	(0.00056003, 0.01985063)	(2.7493667e-09, 9.7452553e-08)
0.2	(0.00122188, 0.01930578)	(0.00124281, 0.01963653)	(0.00122189, 0.01930587)	(5.9986183e-09, 9.4778169e-08)
0.3	(0.00198556, 0.01869993)	(0.00207136, 0.01897370)	(0.00198557, 0.01870002)	(9.7477547e-09, 9.1803854e-08)
0.4	(0.00285106, 0.01800243)	(0.00289991, 0.01831086)	(0.00285107, 0.01800252)	(1.3996776e-08, 8.8379643e-08)
0.5	(0.00381839, 0.01718275)	(0.00393559, 0.01739946)	(0.00381840, 0.01718284)	(1.8745682e-08, 8.4355570e-08)
0.6	(0.00488753, 0.01621034)	(0.00497127, 0.01648806)	(0.00488756, 0.01621042)	(2.3994473e-08, 7.9581669e-08)
0.7	(0.00605851, 0.01505464)	(0.00621409, 0.01520381)	(0.00605854, 0.01505471)	(2.9743149e-08, 7.3907976e-08)
0.8	(0.00733130, 0.01368511)	(0.00745691, 0.01391957)	(0.00733134, 0.01368517)	(3.5991709e-08, 6.7184525e-08)
0.9	(0.00870593, 0.01207120)	(0.00890686, 0.01213819)	(0.00870597, 0.01207126)	(4.2740155e-08, 5.9261350e-08)

Table 2 – Numerical results of Example 1 with 2D-TFs method.

r	Exact solution $(\underline{u}(x,y,r),\overline{u}(x,y,r))$	2D-TFs method for $x=0.3$, $y=0.6$ and $M=10$	Absolute error for $x=0.3$, $y=0.6$ and $M=10$	Absolute error the method of Rivaz and Yousefi (2012) for $n=52$
0.0	(0.00000000, 0.37477066)	(0.00000000, 0.37477605)	(0.0000000e-00, 5.3987564e-06)	(0.0009, 0.0000)
0.1	(0.01030619, 0.36530770)	(0.01030634, 0.36531296)	(1.4846580e-07, 5.2624378e-06)	(0.0007, 0.0004)
0.2	(0.02248623, 0.35528258)	(0.02248656, 0.35528770)	(3.2392538e-07, 5.1180211e-06)	(0.0003, 0.0007)
0.3	(0.03654013, 0.34413315)	(0.03654066, 0.34413811)	(5.2637875e-07, 4.9574081e-06)	(0.0002, 0.0011)
0.4	(0.05246789, 0.33129726)	(0.05246864, 0.33130203)	(7.5582590e-07, 4.7725007e-06)	(0.0009, 0.0016)
0.5	(0.07026949, 0.31621274)	(0.07027051, 0.31621274)	(1.0122668e-06, 4.5552007e-06)	(0.0020, 0.0008)
0.6	(0.08994495, 0.29831744)	(0.08994625, 0.29832174)	(1.2957015e-06, 4.2974101e-06)	(0.0015, 0.0004)
0.7	(0.11149427, 0.27704921)	(0.11149587, 0.27705320)	(1.6061300e-06, 3.9910307e-06)	(0.0007, 0.0003)
0.8	(0.13491743, 0.25184588)	(0.13491938, 0.25184951)	(1.9435523e-06, 3.6279643e-06)	(0.0001, 0.0012)
0.9	(0.16021445, 0.22214530)	(0.16021676, 0.22214850)	(2.3079683e-06, 3.2001129e-06)	(0.0011, 0.0024)

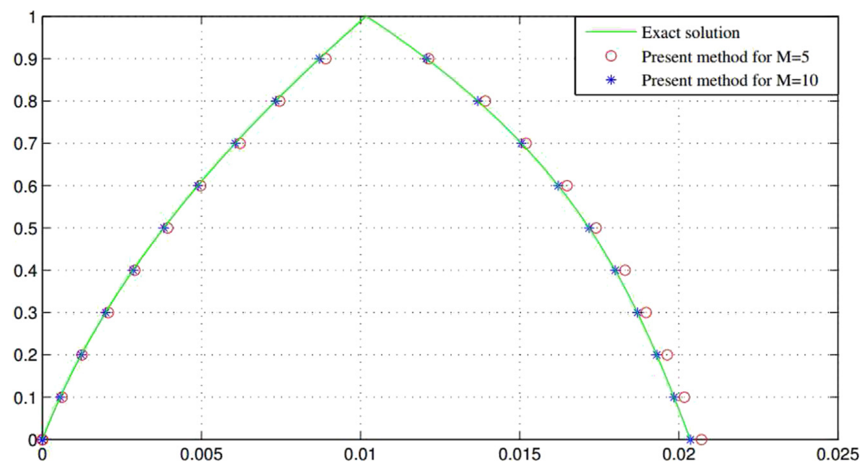

Fig. 1 – Comparison between the exact solution and the approximate solution by present method for $M = 5, 10$.

$$T^T(x,y)UT(r) \approx T^T(x,y)FT(r) + T^T(x,y)KDUT(r),$$

then

$$U = F + KDU,$$

where U and F are $(4M^2 \times 2M)$-matrix and KD is $(4M^2 \times 4M^2)$-matrix, so KDU is $(4M^2 \times 2M)$-matrix, where U is unknown. Then we have

$$(I - KD)U = F,$$

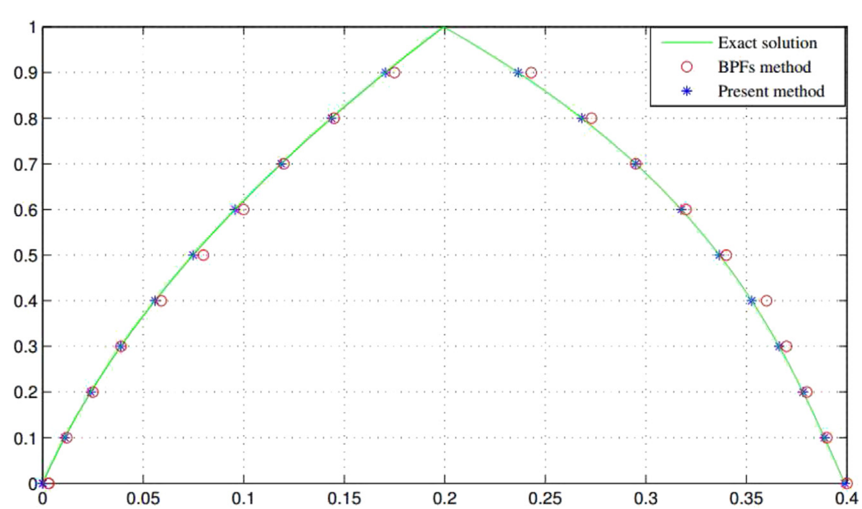

Fig. 2 – Comparison between the exact solution, the approximate solution by BPFs method for $n = 52$ and present method for $M = 10$.

Table 3 – Numerical results of Example 2 with 2D-TFs method method.

r	Exact solution $(\underline{u}(x,y,r),\overline{u}(x,y,r))$	2D-TFs method for $x=0.1$, $y=0.4$ and $M=4$	2D-TFs method for $x=0.1$, $y=0.4$ and $M=12$	Absolute error for $x=0.1$, $y=0.4$ and $M=12$
0.0	(0.00000000, 0.08000000)	(0.00000000, 0.07973560)	(0.00000000, 0.08000985)	(0.0000000e-00, 9.8553102e-06)
0.1	(0.00400000, 0.07600000)	(0.00398678, 0.07574882)	(0.00400049, 0.07600936)	(4.9276551e-07, 9.3625447e-06)
0.2	(0.00800000, 0.07200000)	(0.00797356, 0.07176204)	(0.00800098, 0.07200886)	(9.8553102e-07, 8.8697792e-06)
0.3	(0.01200000, 0.06800000)	(0.01196034, 0.06777526)	(0.01200147, 0.06800837)	(1.4782965e-06, 8.3770137e-06)
0.4	(0.01600000, 0.06400000)	(0.01594712, 0.06378849)	(0.01600197, 0.06400788)	(1.9710620e-06, 7.8842481e-06)
0.5	(0.02000000, 0.06000000)	(0.01993390, 0.05980170)	(0.02000246, 0.06000739)	(2.4638275e-06, 7.3914826e-06)
0.6	(0.02400000, 0.05600000)	(0.02392068, 0.05581492)	(0.02400295, 0.05600689)	(2.9565930e-06, 6.8987171e-06)
0.7	(0.02800000, 0.05200000)	(0.02790746, 0.05182814)	(0.02800344, 0.05200640)	(3.4493585e-06, 6.4059516e-06)
0.8	(0.03200000, 0.04800000)	(0.03189424, 0.04784136)	(0.0320039, 0.04800591)	(3.9421240e-06, 5.9131861e-06)
0.9	(0.03600000, 0.04400000)	(0.03588102, 0.04385458)	(0.03600443, 0.04400542)	(4.4348896e-06, 5.4204206e-06)

where $I_{4M^2\times 4M^2}$ is identity matrix. Then we can write above equation as follows

$$U=(I-KD)^{-1}F.$$

By solving this matrix system we can find matrix $U_{4M^2\times 2M}$ and we can approximate $\underline{u}(x,y,r)$ from

$$\underline{u}(x,y,r)=T^T(x,y)UT(r).$$

The same trend holds for Eq. (29).

6. The convergence of the method

In this section, we obtain error estimate for the numerical method proposed in previous section.

Theorem 1. The solution of two-dimensional Fredholm fuzzy integral equations such as Eq. (25), by using 2D-TFs are convergence to exact solution if

$$P=\max_{0\le x,y,s,t\le 1}|k(x,y,s,t)|<1.$$

Proof. Assume $\tilde{u}(x,y)$ and $\tilde{u}_M(x,y)$ show approximate and exact solution of Eq. (25) respectively. Then

$$\begin{aligned}
D(\tilde{u}(x,y),\tilde{u}_M(x,y)) &= D\Big(\int_0^1\int_0^1 k(x,y,s,t)\tilde{u}(s,t)dsdt, \int_0^1\int_0^1 k(x,y,s,t)\Big(\sum_{i=0}^{M-1}{}^{*}\sum_{j=0}^{M-1}{}^{*}c_{i,j}T^{1,1}_{i,j}(s,t)+\sum_{i=0}^{M-1}{}^{*}\sum_{j=0}^{M-1}{}^{*}d_{i,j}T^{1,2}_{i,j}(s,t)+\sum_{i=0}^{M-1}{}^{*}\sum_{j=0}^{M-1}{}^{*}e_{i,j}T^{2,1}_{i,j}(s,t)\\
&\quad+\sum_{i=0}^{M-1}{}^{*}\sum_{j=0}^{M-1}{}^{*}l_{i,j}T^{2,2}_{i,j}(s,t)\Big)dsdt\Big)\le\int_0^1\int_0^1 D\Big(k(x,y,s,t)\tilde{u}(s,t),k(x,y,s,t)\Big(\sum_{i=0}^{M-1}{}^{*}\sum_{j=0}^{M-1}{}^{*}c_{i,j}T^{1,1}_{i,j}(s,t)+\sum_{i=0}^{M-1}{}^{*}\sum_{j=0}^{M-1}{}^{*}d_{i,j}T^{1,2}_{i,j}(s,t)\\
&\quad+\sum_{i=0}^{M-1}{}^{*}\sum_{j=0}^{M-1}{}^{*}e_{i,j}T^{2,1}_{i,j}(s,t)+\sum_{i=0}^{M-1}{}^{*}\sum_{j=0}^{M-1}{}^{*}l_{i,j}T^{2,2}_{i,j}(s,t)\Big)\Big)dsdt\le P\int_0^1\int_0^1 D\Big(\tilde{u}(s,t),\sum_{i=0}^{M-1}{}^{*}\sum_{j=0}^{M-1}{}^{*}c_{i,j}T^{1,1}_{i,j}(s,t)+\sum_{i=0}^{M-1}{}^{*}\sum_{j=0}^{M-1}{}^{*}d_{i,j}T^{1,2}_{i,j}(s,t)\\
&\quad+\sum_{i=0}^{M-1}{}^{*}\sum_{j=0}^{M-1}{}^{*}e_{i,j}T^{2,1}_{i,j}(s,t)+\sum_{i=0}^{M-1}{}^{*}\sum_{j=0}^{M-1}{}^{*}l_{i,j}T^{2,2}_{i,j}(s,t)\Big)dsdt=P\int_0^1\int_0^1 D(\tilde{u}(s,t),\tilde{u}_M(s,t))dsdt,
\end{aligned}$$

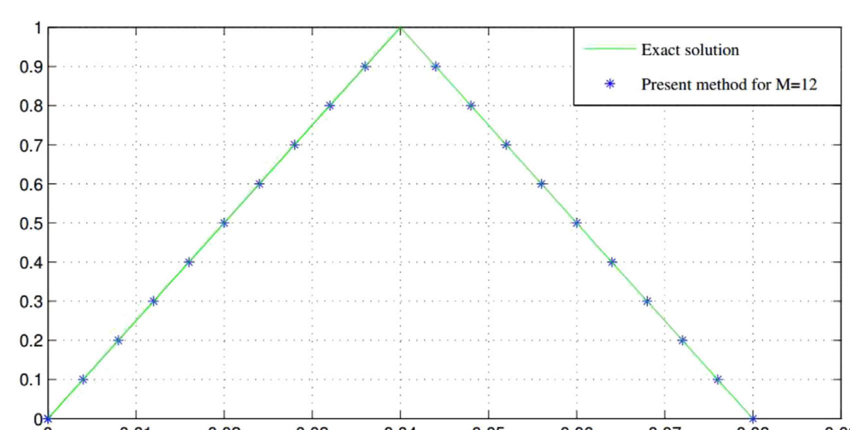

Fig. 3 – Comparison between the exact solution and the approximate solution by present method for M = 12.

therefore we have

$$D(\tilde{u}(x,y),\tilde{u}_M(x,y)) \leq PD(\tilde{u}(x,y),\tilde{u}_M(x,y)),$$

$$\sup_{(x,y)\in\Omega}[0,1)\times[0,1))D(\tilde{u}(x,y),\tilde{u}_M(x,y))$$
$$\leq P\sup_{(x,y)\in\Omega}[0,1)\times[0,1))D(\tilde{u}(x,y),\tilde{u}_M(x,y)).$$

Therefore, if $P<1$, we will have:

$$\lim_{M\to\infty}\sup_{(x,y)\in\Omega} D(\tilde{u}(x,y),\tilde{u}_M(x,y)) = 0.$$

7. Numerical examples

Here, two examples are given to certify the convergence and error bound of the 2D-TFs method for Two-dimensional fuzzy Fredholm integral equations. All results are computed by using a program written in the Matlab. In this regard, we have presented with tables and figures.

Example 1. Consider the following 2D-FFIE-2 (Rivaz and Yousefi (2012)).

$$\underline{f}(x,y,r) = x\sin(\frac{y}{2})(r^2+r),$$

$$\overline{f}(x,y,r) = x\sin(\frac{y}{2})(4-r^3-r),$$

and

$$k(x,y,s,t) = x^2ys, \quad 0\leq x,y,s,t\leq 1 \quad and\ \lambda = 1.$$

That the exact solution is

$$\underline{u}(x,y,r) = (x\sin(\frac{y}{2}) - \frac{16}{21}(\cos(\frac{1}{2})-1)x^2y)(r^2+r),$$

$$\overline{u}(x,y,r) = (x\sin(\frac{y}{2}) - \frac{16}{21}(\cos(\frac{1}{2})-1)x^2y)(4-r^3-r).$$

Tables 1 and 2 show the exact solution and the numerical solutions by the presented method and absolute error of this method. Figs. 1 and 2 show comparison between the exact solution and the numerical solutions by the presented method.

Example 2. Consider the following 2D-FFIE-2

$$\underline{f}(x,y,r) = r(xy + \frac{1}{676}(x^2+y^2-2)),$$

$$\overline{f}(x,y,r) = (2-r)(xy + \frac{1}{676}(x^2+y^2-2)),$$

and

$$k(x,y,s,t) = \frac{1}{169}(x^2+y^2-2)(s^2+t^2-2), \quad 0\leq x,y,s,t \leq 1 \quad and\ \lambda=1.$$

The exact solution is

$$\underline{u}(x,y,r) = rxy,$$

$$\overline{u}(x,y,r) = (2-r)xy.$$

Absolute values of the error function are provided in Table 3 by taking $M = 4, 12$. The numerical histories for $N = 12$ is depicted in Fig. 3.

8. Conclusion

Fuzzy integral equations are important for studying and solving a large proportion of the problems in many topics in applied mathematics, particularly in fuzzy control. In this study, we introduce 2D-TFs method for approximating the solution of linear (2D-FFIE-2). By this method, the original equation is converted into two crisp (2D-FFIE-2). The efficiency and simplicity of this method are illustrated by introducing some numerical examples with known exact solutions. The main advantage of this method is low cost of setting up the equations without using any projection method and any integration.

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