Computers Math. Applic. Vol. 25, No. 1, pp. 89-104, 1993
Printed in Great Britain

0097-4943/93 $5.00 + 0.00
Pergamon Press Ltd

ERROR ANALYSIS OF THE TAU METHOD: DEPENDENCE OF THE APPROXIMATION ERROR ON THE CHOICE OF PERTURBATION TERM

S. NAMASIVAYAM AND E. L. ORTIZ
Mathematics Department, Imperial College
London, England

(Received March 1992)

Abstract—We consider a system of ordinary differential equations with constant coefficients and deduce asymptotic estimates for the Tau Method approximation error vector per step for different choices of the perturbation term $H_n(x)$. The cases considered are Legendre polynomials, Chebyshev polynomials, powers of x and polynomials of the form $(x^2 - r^2)^n$, $-r \leq x \leq r$. The first two are standard choices for the Tau Method, for Chebyshev and Legendre series expansion techniques and also for collocation; the third one realizes the classical power series expansion techniques in the framework of the Tau Method and the last is related to the trial functions used in weighted residuals methods; we shall refer to it as the *weighted residuals* choice. We show that the resulting Tau Method implementations can be arranged into the following scale of increasing error estimates at the end point $x = r$:

$$\text{Legendre} < \text{Chebyshev} << \text{Power series} < \text{Weighted residuals}.$$

For the interesting case of Legendre Tau approximations, we offer upper and lower error bounds for the end point of the interval of approximation. In particular, this last estimates solve a conjecture on increased accuracy at the end point of the interval of approximation formulated by Lanczos in 1956. Such conjecture has equivalent forms for other polynomial methods for the numerical solution of differential equations.

Although formulated in the convenient framework of the recursive Tau Method (see Ortiz [1]), the results given here apply, without essential modifications, to Chebyshev or Legendre series expansion techniques for differential equations, collocation and spectral methods.

We give numerical examples which confirm the sharpness of the lemmas and theorems given in this paper. Finally, we discuss in an example the application of our results to the analysis of singularly perturbed differential equations.

1. INTRODUCTION

This investigation is motivated by the fact that the members of a large family of numerical methods can be regarded as special realizations of the Tau Method corresponding to suitable choices of perturbation terms $\underline{H}_n(x)$ (see [2] for a detailed discussion of this problem). They include methods based on the representation of the approximate solution vector in a preassigned form, usually a truncated series expansion with free coefficients which may be given in terms of powers of x; a finite segment of a Fourier expansion expressed in terms Chebyshev, Legendre, Hermite (as in [3]) or other special polynomials, as used in spectral methods, or alternatively, it can be given in terms of a special basis which takes care of the supplementary conditions.

In these techniques, the free parameters are determined

(i) by a direct substitution of the approximating expression in the given equation and supplementary conditions, equating the result with the right hand side of the differential equation and supplementary conditions to fix the free parameters;
(ii) by point evaluation techniques based on an interpolation process in the image space of the differential operator to determine the coefficients of the approximate solution. These

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TeX

techniques are referred to as Lanczos' method of selected points (see [4]), collocation or pseudo-spectral techniques;

(iii) or through an averaging or weighted residuals procedure, which uses the geometric idea of orthogonality between the hyperplane of approximation and a function of the residual term, such as in projection and Galerkin's methods.

The arguments given in this paper apply, without essential modifications, to the error analysis of these numerical techniques. Further details are given in a separate paper.

2. THE RECURSIVE FORMULATION OF THE TAU METHOD

Let us consider the system of ordinary differential equations

$$\mathbf{D}\,\underline{y}(x) := \frac{d}{dx}\underline{y}(x) + \mathbf{A}\,\underline{y}(x) = \underline{f}(x), \qquad \text{with} \quad -r \le x \le r \quad \text{and} \quad \underline{y}(-r) = \underline{a}, \tag{1}$$

where $\mathbf{D} := ((D_{ij}))$, $i, j = 1\,(1)\,m$ is a matrix differential operator of order $m \times m$; its elements D_{ij} are linear differential operators with constant coefficients:

$$D_{ij} := \begin{cases} a_{ij}, & \text{for } i \neq j \\ a_{ij} + \left[\frac{d}{dx}\right], & \text{for } i = j. \end{cases}$$

$\mathbf{A} := ((a_{ij}))$, $i, j = 1\,(1)\,m$, is a nonsingular coefficient matrix and $\underline{y}$, $\underline{f}$ and $\underline{a}$ are vectors of $\mathbf{R^m}$.

We associate with (1) the **Tau Problem**:

$$\mathbf{D}\,\underline{y}_n(x) := \frac{d}{dx}\underline{y}_n(x) + \mathbf{A}\,\underline{y}_n(x) = \underline{f}(x) + \underline{H}_n(x); \qquad -r \le x \le r \quad \underline{y}_n(-r) = \underline{a}, \tag{2}$$

where $\underline{H}_n(x)$ is a *vector polynomial* perturbation term which forces the exact solution of (2) to become a vector polynomial (see [5]). We assume that a norm of $\underline{H}_n(x)$ satisfies given minimal conditions. Let $\underline{H}_n(x)$ be defined by $\tau_{n,V} v_n(x)$, where

$$\mathbf{V} := V \cdot (1, x, x^2, \dots)^{\mathsf{T}} = \{v_n(x)\}, \qquad n \in \mathbf{N}$$

is a polynomial basis, defined by a nonsingular matrix V, with elements $v_n(x)$, which are polynomials of degree n and

$$\tau_{n,V} := [\tau_{n,V}^{(1)}, \dots, \tau_{n,V}^{(s)}]^{\mathsf{T}}.$$

With the matrix operator $\mathbf{D}$, we associate a sequence of *vector canonical polynomials* (see [1,5,6])

$$\mathbf{Q} := \{\underline{Q}_n^{(i)}(x)\}, \qquad i = 1\,(1)\,m, \quad n \in \mathbf{N} \setminus \mathbf{S},$$

such that

$$\mathbf{D}\,\underline{Q}_n^{(j)}(x) := x^n\,\underline{E}_{(j)}, \qquad j = 1\,(1)\,m,$$

possibly, plus residual terms; $\underline{E}_{(j)} := (e_{1j}, e_{2j}, e_{3j}, \dots, e_{mj})^{\mathsf{T}}$ and $e_{ij} := 1$ for $i = j$, and zero otherwise.

3. THE MAIN RESULTS: PERTURBATION TERMS IN THE TAU METHOD

As far back as 1956, Lanczos [7] produced examples showing that the error of Tau Method approximations could be reduced considerably at the end point of the interval in which the solution is sought by expressing the perturbation term as a linear combination of *Legendre polynomials* instead of the traditional choice of Chebyshev polynomials. Later, the same author gave further examples and an heuristic explanation of this remarkable phenomenon [8].

Ortiz [9] used Legendre polynomials as perturbation terms to formulate a *step-by-step Tau Method* with a minimized error vector at matching points and applied it to initial value problems; Onumanyi and Ortiz [10] used this technique to treat singularly perturbed and nonlinear boundary value problems in the presence of stiffness in cases where the solution is not unique.

El Misiery and Ortiz have considered types of perturbation terms of this form in the context of their hybrid *Tau-Lines Method* [11], which they used for the numerical approximation of singular boundary value problems for partial differential equations related to problems in fracture mechanics; Hosseini Ali Abadi and Ortiz have used them in the numerical solution of systems of nonlinear partial differential equations related to the numerical simulation of soliton interactions (see [12] and the earlier references given therein).

In this paper, we discuss, from an analytic point of view, the question of the dependence of the error on the basis chosen for the representation of the vector perturbation term in the Tau Method. We assume that our problem is defined on $-r \le x \le r$, and consider four possible choices for the representation of $\underline{H}_n(x)$:

(i) Chebyshev polynomials $T_n(x) := \cos[n \arccos(x/r)]$;
(ii) Legendre polynomials $P_n(x)$;
(iii) Powers of x; and
(iv) the polynomials defined by $(x^2 - r^2)^n$, $-r \le x \le r$;

other choices can be treated with techniques similar to that developed in this paper.

The first two are the most commonly used perturbation terms for the Tau Method. If $\underline{H}_n(x)$ is given as a linear combination of powers of x, the Tau Method can be made to generate the approximate solution corresponding to classical power series expansion techniques. Finally, the choice $(x^2 - r^2)^n$, $-r \le x \le r$, is relevant to simulation through the Tau Method of approximate solutions obtained by using weighted residuals principles in which the last expression appears in the formulation of the trial functions.

In the proof of Theorems 1, 3 and 6, our main analytic tool is the recursive formulation of the Tau Method as given in [1], which we use to give sharp estimates of the tau-parameter vector for each of the four choices for the representation of $\underline{H}_n(x)$ mentioned before. On the basis of these estimates in Theorems 2, 4 and 7, we give individual estimates of the truncation error vector corresponding to each of our choices.

Our proofs are constructed for the special case of (1), that is for, a system of linear ordinary differential equations with constant coefficients. In Theorems 2–4, we consider the case of $m = 2$, giving *upper* and *lower* error bounds for some cases of particular interest. Using the same technique as for $m = 2$, we extend these result to the case of an $m \times m$ system with $m > 2$ in Theorems 6 and 7. In Theorems 5 and 8, we show that for $m \times m$ systems, $m \ge 2$ and for sufficiently large n, the Tau Method vector approximation errors corresponding to perturbation terms $\underline{H}_n(x)$ defined in terms of a linear combination of either x^n, $T_n(x)$ or $P_n(x)$ can be arranged into a hierarchical scale, as follows, for the end point $x = r$ of the interval $[-r, r]$:

$$\underline{e}_{n,\text{Powers}}(r) = \mathbf{O}\left(\frac{r^n\, \mathbf{A}^n\, \underline{k}}{(n+1)!}\right), \qquad \text{where } \underline{k} \text{ is a constant vector;}$$

$$\underline{e}_{n,\text{Chebyshev}}(r) = \mathbf{O}\left(\underline{e}_{n,\text{Powers}}(r)\, \frac{1}{n\, 2^n}\right);$$

$$\underline{e}_{n,\text{Legendre}}(r) = \mathbf{O}\left(\underline{e}_{n,\text{Chebyshev}}(r)\, \frac{n\,((2n)!!)^3\, r^n\, \mathbf{A}^n)}{((2n)!)^2}\right).$$

Our results show that the weighted residuals choice leads to the worst error vector per step. At the other end of the scale is a Tau Method based on the choice of Legendre polynomials for the representation of $\underline{H}_n(x)$. The Chebyshev choice follows in accuracy and the power series one comes after it, but with an error increased by a large factor: $n\, 2^n$.

In Corollary 1, we show that linear combinations of Legendre polynomials lead to a considerable improvement in accuracy at the *end point* of the interval $-r \le x \le r$, as conjectured by Lanczos in [7]. Such conjecture has equivalent forms for other polynomial methods for the numerical solution of differential equations (see [13] and the references to earlier related work by the same authors given therein). In Corollary 2, we make a remark on the behaviour of a Tau Method approximation for a singularly perturbed problem. In Corollary 3, we show that the weighted residuals choice is not even acceptable in terms of accuracy for the center point $x = 0$.

We give, in this paper, a number of numerical examples which show that the estimates used in our lemmas and theorems are very sharp. They are used as one-step error estimates for the step-by-step formulation of the Tau Method of Ortiz [9], showing the accuracy of both our error estimates and the step-by-step Tau Method.

Estimates of the uniform norm $\|H_n\|$ of the perturbation term, which is the error introduced in the equation by the Tau Method, follow from Theorems 1, 3 and 6. Such estimates are useful in the design of adaptive software for the Tau Method (see [10]).

Our estimates have also a theoretical interest: they suggest that the extraordinary accuracy at the end point of the interval which is obtained by using a Tau Method based on a Legendre perturbation term is not due to a significantly *smaller* uniform norm of the perturbation term, but to a *geometric* feature related to the orthogonality properties of these polynomials. The relevant results are contained in Lemmas 3 and 4.

The estimates given in this paper, valid for systems of $m \times m$, $m \geq 2$ ordinary differential equations with constant coefficients, combined with the results on differential equations with approximate coefficients given by El Daou, Namasivayam and Ortiz in [14] make it possible to reformulate the estimates given here in the context of non-constant coefficients. Such analysis is carried out in a separate paper.

4. AN ERROR ANALYSIS OF THE TAU METHOD FOR A 2 × 2 SYSTEM OF DIFFERENTIAL EQUATIONS

Let us consider initially the second order differential equation defined by

$$\begin{aligned} &D\underline{y}(x) := y''(x) + B\,y'(x) + C\,y(x) = f(x), \qquad -r \leq x \leq r, \\ &y(-r) = a_1, \quad y'(-r) = a_2, \end{aligned} \tag{3}$$

where B and C are constant coefficients given by

$$B := \frac{g_1 + g_2}{g_1 \cdot g_2}, \quad \text{and} \quad C := \frac{1}{g_1 \cdot g_2},$$

with $g_1,\, g_2 \neq 0$. Let us introduce the matrix differential operator $\mathbf{D}$ to express (3) in the form of a 2×2 system of first order differential equations:

$$\mathbf{D}\underline{y}(x) := \begin{pmatrix} \frac{d}{dx} & -1 \\ C & B + \frac{d}{dx} \end{pmatrix} \cdot \begin{pmatrix} y(x) \\ z(x) \end{pmatrix} = \underline{f}(x) = \begin{pmatrix} 0 \\ f(x) \end{pmatrix};\ \underline{y}(-r) = \begin{pmatrix} y(-r) \\ z(-r) \end{pmatrix} = \underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \tag{4}$$

and associate with it the **Tau Problem**:

$$\mathbf{D}\underline{y}_n(x) = \underline{f}(x) + \underline{H}_n(x), \qquad \text{for } -r \leq x \leq r, \quad \underline{y}_n(-r) = \underline{a}. \tag{5}$$

In the case of (3) the vector height $\underline{h}$ is zero, the set $\mathbf{S}$ is empty, since the coefficient C cannot be equal to zero, and so is the subspace of residuals $\mathbf{R_S}$ (for details see [1]). Therefore, vector canonical polynomials are defined for all indices $n \in \mathbb{N}$. The vector perturbation term is defined by

$$\underline{H}_n(x) := \tau_{n,V}\, v_n(x) := \left[\tau^{(1)}_{n,V},\, \tau^{(2)}_{n,V}\right]^{\mathsf{T}} v_n(x),$$

where $v_n(x)$ is a polynomial of degree $\leq n$ in x and the subindex V identifies the polynomial basis which has be chosen. The two components of the free parameter vector $\tau_{n,V}$ are fixed to satisfy exactly the initial conditions of problem (3).

We shall choose for $\mathbf{V}$ the following four polynomial bases:

$$\mathbf{T}_n := \{T_n(x)\}; \quad \mathbf{P}_n := \{P_n(x)\}; \quad \mathbf{X}_n := \{x^n\}, \quad \text{and} \quad \mathbf{W}_{2n} := \{(x^2 - r^2)^n\}.$$

Since canonical polynomials associated with (3) are defined for all $n \in \mathbb{N}$, we can assume without loss of generality that $\underline{f}(x)$ is a constant. Otherwise, we only have to consider additive terms in our arguments, induced by the representation of a non-constant $\underline{f}$ in terms of canonical polynomials.

To construct the vector canonical polynomial sequence $\mathbf{Q} := \{\underline{Q}_n^{(k)}(x)\}$, $k := 1,2$; $n \in \mathbf{N}$, associated with $\mathbf{D}$ of (5), we begin by introducing the auxiliary polynomial:

$$S_n(x) := (-1)^n g_2^n\, n! \sum_{j=0}^{n} \left(\frac{g_1}{g_2}\right)^j \sum_{t=0}^{n-j} \frac{(-1)^t x^t}{g_2^t\, t!},$$

which is such that

$$\left[\frac{d^k}{dx^k}\right] S_n(x) = \frac{n!}{(n-k)!} S_{n-k}(x). \tag{6}$$

The last identity shows that, but for a numerical factor, the polynomials $S_n(x)$ are closed under differentiation. In this sense, they resemble the behaviour of x^n in the one dimensional case (see [1]), when an ordinary linear differential operator D with polynomial coefficients is applied to x^n to form a generating polynomial $D x^n$. Since differentiation is the only functional operation in the expression of (4), this convenient property of S_r is used to generate the canonical polynomials associated with the matrix operator $\mathbf{D}$. It can be verified that

$$\underline{Q}_n^{(1)}(x) = \begin{pmatrix} [B + \frac{d}{dx}]\,(S_n(x))/C \\ -S_n(x) \end{pmatrix} \quad \text{and} \quad \underline{Q}_n^{(2)}(x) = \frac{1}{C}\begin{pmatrix} x^n - [B + \frac{d}{dx}]\,(S_n'(x))/C \\ S_n'(x) \end{pmatrix}.$$

Let

$$v_n(x) := \sum_{j=0}^{n} c_j^{(n)} x^j,$$

we shall use the notation

$$v_n(u(x)) := \sum_{j=0}^{n} c_j^{(n)} u_j(x)$$

to indicate a linear combination of the functions u_j, $j = 0\,(1)\,n$, with the coefficients $c_j^{(n)}$ of $v_n(x)$.

Taking into account the remark made above on the behaviour of the functions S_r under differentiation (5) and the expression of $\underline{H}_n(x)$, we shall choose $u_j(x) := S_j(x)$. Therefore,

$$\underline{y}_n(x) - \begin{pmatrix} f/C \\ 0 \end{pmatrix} = \tau_{n,V}^{(1)} \begin{pmatrix} \frac{B\, v_n(S(x)) + v_n'(S(x))}{C} \\ -v_n(S(x)) \end{pmatrix} + \tau_{n,V}^{(2)} \begin{pmatrix} \frac{C\, v_n(x) - B\, v_n'(S(x)) - v_n''(S(x))}{C} \\ v_n'(S(x)) \end{pmatrix}.$$

Let $G_{n,V}(x)$ be defined by

$$G_{n,V}(x) := \frac{1}{C}\begin{pmatrix} B\, v_n(S(x)) + v_n'(S(x)) & \frac{C\, v_n(x) - B\, v_n'(S(x)) - v_n''(S(x))}{C} \\ -C\, v_n(S(x)) & v_n'(S(x)) \end{pmatrix}.$$

Then, on putting $x = -r$ in $\underline{y}_n - (f/C, 0)^{\mathrm{T}}$ and making use of the initial condition $\underline{y}_n(-r) = \underline{a}$, we obtain

$$G_{n,V}\,(-r)\, \tau_{n,V} = \underline{a} - \underline{f}/C.$$

Let $\Delta_{n,V}$ stand for the determinant of $G_{n,V}(-r)$, then

$$\Delta_{n,V} = \frac{v_n'(S(-r))^2 - v_n(S(-r))\, v_n''(S(-r)) + (-1)^n C\, v_n(S(-r))}{C^2}. \tag{7}$$

5. ESTIMATES OF THE TAU PARAMETERS

We shall now take for $\mathbf{V}$ the four different choices indicated before, starting with perturbation terms given in terms of Chebyshev or Legendre polynomials. Let

$$W(j,n) := \begin{cases} \prod_{s=0}^{j-1} (n-2s)(n+2s), & \text{if } n \text{ is even,} \\ n \prod_{s=1}^{j} (n-2s+1)(n+2s-1), & \text{if } n \text{ is odd,} \end{cases}$$

and

$$Z(j,n) := \begin{cases} \prod_{s=0}^{j}(n-2s+2)(n+2s-1), & \text{if } n \text{ is even,} \\ (n+1)\prod_{s=0}^{j}(n-2s+1)(n+2s), & \text{if } n \text{ is odd.} \end{cases}$$

Chebyshev polynomials defined on $[-r, r]$ are such that (see [15]):

$$T_n^{(j)}(0) = j!\, c_j^{(n)}, \qquad \text{for } j \le n,$$

while for $j = 0\,(1)[(n-1)/2]$, $c_{2j}^{(n)} = 0$ for odd n and $c_{2j+1}^{(n)} = 0$ for even n.

In terms of the functions $W(j,n)$ and $Z(j,n)$, we define:

$$T_n^{(j)}(0) := \begin{cases} (-1)^{[n/2]+[j/2]} \frac{W([j/2],n)}{r^j}, & \text{if } n+j \text{ is even,} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$P_n^{(j)}(0) := \begin{cases} (-1)^{[n/2]+[j/2]} \frac{n!\, Z([j/2],n)}{r^j\, 2^n\, [n/2]!\, [(n+1)/2]!}, & \text{for } n+j \text{ even,} \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

where $[k]$ stands for the integer part of k. We also recall the notation $n!! := n(n-2)\ldots 1$, if n is odd and $n!! := n(n-2)\ldots 2$ if n is even.

It will simplify the statement of our proofs to introduce the following notation

$$p_{k,j}(z) := \frac{(r+z)^k}{(2g_j)^k\, k!}, \qquad j = 1, 2; \quad k = 0, 1, \ldots;$$

$$\Theta_v^{(j)}(z) := \sum_{t=0}^{j-v} \left(\frac{g_1}{g_2}\right)^t \sum_{k=0}^{j-v-t} p_{k,2}(z); \qquad -r \le z \le r$$

and

$$\Phi^{(n)}(z) := g_1^2\, \{[\Theta_1^{(n)}(z)]^2 - \Theta_2^{(n)}(z)\, \Theta_0^{(n)}(z)\}; \qquad -r \le z \le r. \tag{9}$$

The following lemma gives asymptotic estimates of the derivatives of $T_n(S(x))$ and $P_n(S(x))$ at $x = -r$, which will be required later to fix the initial conditions for $\underline{y}_n(x)$.

LEMMA 1. *For sufficiently large n and $0 \le v \le n$, $v \in \mathbf{N}$:*

$$T_n^{(v)}(S(-r)) = (-1)^{n-v} \frac{(2n)!!\, g_2^{n-v}}{2\, r^n}\, \Theta_v^{(n)}(r) \left[1 + O\left(\frac{1}{n}\right)\right],$$

$$P_n^{(v)}(S(-r)) = (-1)^{n-v} \frac{(2n-1)!!\, g_2^{n-v}}{r^n}\, \Theta_v^{(n)}(r) \left[1 + O\left(\frac{1}{n}\right)\right].$$

PROOF. From (6) and (9), it follows that

$$T_n^{(k)}(S(-r)) = \sum_{j=k}^{n} c_j^{(n)} \frac{j!}{(j-k)!} S_{j-k}(-r) = (-1)^k \sum_{j=k}^{n} (-1)^j g_2^{j-k}\, j!\, c_j^{(n)}\, \Theta_k^{(j)}(r)$$

taking into account that $c_j^{(n)} j! = T_n^{(j)}(0)$, $j \le n$, and that the latter is the first of the identities (8), we find that

$$T_n^{(v)}(S(-r)) = (-1)^{v+[(n+1)/2]} \frac{g_2^{n-v}}{r^n} \sum_{i=0}^{[n/2]} (-1)^i\, W(i,n)\, \Theta_v^{(2i)}(r)$$

if n is even, and

$$T_n^{(v)}(S(-r)) = (-1)^{v+[(n+1)/2]}\frac{g_2^{n-v}}{r^n}\sum_{i=0}^{[n/2]}(-1)^i W(i,n)\,\Theta_v^{(2i+1)}(r)$$

if n is odd. Hence,

$$T_n^{(v)}(S(-r)) = (-1)^{n-v}\frac{{g_2}^{n-v}\,2^{n-1}\,n!}{r^n}\,\Theta_v^{(n)}(r)\left[1+O\left(\frac{1}{n}\right)\right].$$

The proof of the statement for Legendre polynomials follows similarly using the second identity in (8). ∎

Let n be the degree of $v_n(x)$ in the perturbation term $\underline{H}_n(x)$ of (5). Without loss of generality, we shall confine the proofs explicitly developed in this paper to the cases

$$\begin{aligned}&g_2 < g_1 < 0; \quad r \le |\, g_1 \,|, \qquad \text{for } n \text{ odd, or}\\ &g_1 < 0; \quad g_2 \ge 0; \qquad |\, g_1 \,| \le |\, g_2 \,|; \quad r \le |\, g_1 \,|, \text{ for } n \text{ even.}\end{aligned} \tag{10}$$

The remaining cases follow using the same arguments.

Let us consider (7) again, taking for **V** either the Chebyshev or the Legendre polynomials, and making use of the functions Θ and Φ introduced in (9).

LEMMA 2. *For sufficiently large n,*

$$\Delta_{n,V} = J_V\left(\frac{g_2}{r}\right)^{2n}\Phi^{(n)}(r)\left[1+O\left(\frac{1}{n}\right)\right],$$

where

$$J_{\text{Chebyshev}} := \frac{((2n)!!)^2}{4} \quad \textit{and} \quad J_{\text{Legendre}} := ((2n-1)!!)^2$$

if any of the conditions (10) are satisfied.

PROOF. We wish to show first that $\Phi^{(n)}(z) > 0$ if the first of conditions (10) is satisfied and $\Phi^{(n)}(z) < 0$ if the second one holds. From (9),

$$\Theta_v^{(n)}(z) = \left(\frac{g_1}{g_2}\right)^{n-v}\sum_{k=0}^{n-v}p_{k,1}(z) + \Theta_{v+1}^{(n)}(z).$$

Therefore,

$$\begin{aligned}\Theta_0^{(n)}(z)\cdot\Theta_2^{(n)}(z) = \Theta_1^{(n)}(z)&\left[\Theta_1^{(n)}(z) + \left(\frac{g_1}{g_2}\right)^n\sum_{k=0}^{n}p_{k,1}(z) - \left(\frac{g_1}{g_2}\right)^{n-1}\sum_{k=0}^{n-1}p_{k,1}(z)\right]\\ &- \left(\frac{g_1}{g_2}\right)^{2n-1}\sum_{k=0}^{n-1}p_{k,1}(z)\sum_{k=0}^{n}p_{k,1}(z),\end{aligned}$$

and

$$\begin{aligned}\frac{\Phi^{(n)}(z)}{g_1^2} &= \left(\frac{g_1}{g_2}\right)^{2n-1}\sum_{k=0}^{n}p_{k,1}(z)\sum_{k=0}^{n-1}p_{k,1}(z)\\ &\quad + \Theta_1^{(n)}(z)\left\{\left[1-\frac{g_1}{g_2}\right]\left(\frac{g_1}{g_2}\right)^{n-1}\sum_{k=0}^{n-1}p_{k,1}(z) - p_{n,2}(z)\right\},\end{aligned}$$

which gives the required signs for $\Phi^{(n)}(z)$ and shows that it is different from zero if any of the conditions (10) is satisfied. The last identity enables us to estimate $\Phi^{(n)}(z)$:

$$\Phi^{(n)}(z) = O\left(\left(\frac{g_1}{g_2}\right)^n\right), \qquad \text{for } | g_1 | \leq | g_2 | .$$

We return to the proof of Lemma 2; from (7), assuming (10) and taking for $\mathbf{V}$ Chebyshev polynomials, we find that:

$$\Delta_{n,\text{Chebyshev}} = \frac{1}{C^2}\left\{[T_n'(S(-r))]^2 - T_n(S(-r))\,T_n''(S(-r)) + (-1)^n C\,T_n(S(-r))\right\}.$$

For the the Chebyshev polynomials choice, Lemma 2 follows replacing $T_n^{(v)}(S(-r))$ by the estimate given in Lemma 1. The same argument proves the statement for the Legendre polynomials choice. ∎

The Tau vectors $\tau_{n,V}$ are implicitly defined by the system of linear algebraic equations $\underline{a} - (1/C)\underline{f} = G_{n,V}(-r)\,\tau_{n,V}$, which is constructed setting the initial conditions of (4) into the expression of the Tau approximation $\underline{y}_n(x)$.

Let $R := [a_1 C + a_2 B - f]/C$, then

$$\tau_{n,V} := \frac{1}{C\,\Delta_{n,V}}\begin{pmatrix} R\,v_n'(S(-r)) + a_2\,v_n''(S(-r))/C - v_n(-r)] \\ R\,C\,v_n(S(-r)) + a_2\,v_n'(S(-r)) \end{pmatrix}. \tag{11}$$

The results of Lemmas 1 and 2 shall now be used to estimate the tau-parameters in (11) with the choices of the Chebyshev and the Legendre polynomials for $\mathbf{V}$. Let

$$\underline{L}_n(z) := [L_n^{(1)}(z),\, -L_n^{(2)}(z)]^{\mathsf{T}},$$

where

$$L_n^{(j)}(z) := [-R\,\Theta_{2-j}^{(n)}(z) + a_2\,g_1\,\Theta_{3-j}^{(n)}(z)]\,\frac{g_1^{2-j}}{\Phi^{(n)}(z)}.$$

THEOREM 1. *For sufficiently large n, and assuming conditions (10) are satisfied,*

$$\tau_{n,V} := (-1)^n J_{n,V}\,\underline{L}_n(r)\left(\frac{r}{g_2}\right)^n\left[1 + O\left(\frac{1}{n}\right)\right],$$

where

$$J_{n,\text{Chebyshev}} := \frac{2}{(2n)!!}$$

and

$$J_{n,\text{Legendre}} := \frac{1}{(2n-1)!!}.$$

PROOF. Let us consider first the Chebyshev choice. Using the estimates given in Lemma 1 and the first component of $\tau_{n,V}$ in (11), we find that

$$R\,T_n'(S(-r)) + a_2\left[\frac{T_n''(S(-r))}{C - T_n(-r)}\right] = (-1)^{n-1}\,\frac{(2n)!!\,{g_2}^{n-1}}{2\,r^n}\,[R\,\Theta_1^{(n)}(r) - a_2\,g_1\,\Theta_2^{(n)}(r)]\left[1 + O\left(\frac{1}{n}\right)\right].$$

Using the estimate for $\Delta_{n,V}$ given by Lemma 2, with either the Chebyshev or the Legendre choices, we conclude the proof of Theorem 1. ∎

6. ERROR ESTIMATES FOR CHEBYSHEV AND LEGENDRE PERTURBATION TERMS

Having estimated the tau-parameters, we are in a position to give estimates of the error in the solution. We need some auxiliary results before proceeding to discuss these error estimates.

LEMMA 3. *Let $F(x) \in \mathbf{C}^{(6)}[-r,r]$, and let*

$$b_0 := r\left[(-1)^{n+1} F(-r) - F(r)\right]; \quad b_1 := 3\,r^2\left[(-1)^n F'(-r) - F'(r)\right].$$

Then for sufficiently large n, $n > 6$:

$$\int_{-r}^{r} F(x)\,T_n(x)\,dx = \frac{b_0}{n^2-1} + \frac{b_1}{(n^2-1)(n^2-4)} + O\left(\frac{1}{n^6}\right).$$

PROOF. Recalling that $\int T_n(x)\,dx = (r/2)[T_{n+1}(x)/(n+1) - T_{n-1}/(n-1)]$ plus a constant and integrating by parts it follows that

$$\int_{-r}^{r} F(x)\,T_n(x)\,dx := I$$
$$= \frac{b_0}{n^2-1} - \frac{r}{2(n+1)}\int_{-r}^{r} F'(x)\,T_{n+1}(x)\,dx + \frac{r}{2(n-1)}\int_{-r}^{r} F'(x)\,T_{n-1}(x)\,dx.$$

Repeating the same argument with the last two integrals,

$$I = \frac{b_0}{n^2-1} + \frac{b_1}{(n^2-1)(n^2-4)} + 15r^3\,\frac{(-1)^{n+1}F''(-r) - F''(r)}{(n^2-1)(n^2-4)(n^2-9)} + \text{ remainder}.$$

The order of the remainder is at most $1/n^6$, which proves the Lemma. ∎

LEMMA 4. *Let $F(x) \in \mathbf{C}^{(n)}[-r,r]$ and let $[d^n/dx^n]\,F(x)$ be a monotonic function with one sign in $[-r,r]$.*

Then,

$$\left[\frac{d^n}{dx^n}\right] F(-tr) \le \frac{(2n+1)!!}{2\,r^{n+1}} \int_{-r}^{r} F(x)\,P_n(x)\,dx \le \left[\frac{d^n}{dx^n}\right] F(tr), \tag{12}$$

where $t = +1$ or -1 according to $[d^n/dx^n]F(x)$ being an increasing or a decreasing function in $[-r,r]$.

PROOF. From Rodrigues' formula

$$\int_{-r}^{r} F(x)\,P_n(x)\,dx := J = \frac{1}{(2n)!!\,r^n}\int_{-r}^{r} F(x)\left\{\left[\frac{d^n}{dx^n}\right]\left[(x^2-r^2)^n\right]\right\}dx,$$

integrating by parts n times,

$$J = \frac{1}{(2n)!!\,r^n}\int_{-r}^{r}\left[(r^2-x^2)^n\right]\left[\frac{d^n}{dx^n}\right]F(x)\,dx,$$

taking into account that

$$\int_{-r}^{r}(r^2-x^2)^n\,dx = 2\,r^{2n+1}\,\frac{(2n)!!}{(2n+1)!!},$$

we complete the proof of (12). ∎

Let us return to the estimation of the error vector per step $\underline{e}_{n,V}(x) := \underline{y}_n(x) - \underline{y}(x)$ in a given basis $\mathbf{V}$ of a Tau Method approximation of the solution of (4). Since $\mathbf{D}$ is a linear operator, $\underline{e}_{n,V}(x)$ is implicitly defined by a system of differential equations similar to (5), namely

$$\mathbf{D}\underline{e}_{n,V}(x) := \underline{H}_n(x), \qquad -r \le x \le r,$$

but with homogeneous supplementary conditions: $\underline{e}_{n,V}(-r) := \underline{0}$. Assuming the first of the conditions (10) we have, that

$$\underline{e}_{n,V}(r) = \left\{ \frac{1}{|g_1 - g_2|} \cdot \binom{|g_1|}{1} [\tau^{(2)}_{n,V}\, |g_2| - \tau^{(1)}_{n,V}] \cdot \int_{-r}^{r} \left[\exp\left(\frac{t-r}{g_1}\right) - \exp\left(\frac{t-r}{g_2}\right)\right] v_n(t)\, dt \right\} + \tau_{n,V} \int_{-r}^{r} \exp\left(\frac{t-r}{g_2}\right) v_n(t)\, dt. \tag{13}$$

Let us assume that in addition to the first of conditions (10), either

(i)

$$R = a_1 + \frac{(a_2 B - f)}{C} \geq 0, \quad a_2 \geq 0,$$

which implies that

$$L_n^{(1)}(z) \geq 0,\ L_n^{(2)}(z) \leq 0 \quad \text{and} \quad -|g_2|\, L_n^{(2)}(z) - L_n^{(1)}(z) \geq 0;$$

or (ii)

$$R = a_1 + \frac{(a_2 B - f)}{C} \leq 0, \quad a_2 \leq 0,$$

which in turn implies that

$$L_n^{(1)}(z) \leq 0, \quad L_n^{(2)}(z) \geq 0 \quad \text{and} \quad -|g_2|\, L_n^{(2)}(z) - L_n^{(1)}(z) \leq 0. \tag{14}$$

We shall introduce specific notation for functions which will appear frequently in our discussion; for $i = 1,\ 2$:

$$G(j,i,n) := \frac{1}{|g_i|^n} \exp\left(\frac{j\, r}{|g_i|}\right)$$

$$U_i(n) := 2\, L_n^{(i)}(r)\, [G(2,2,0) - 1] - \frac{2\, |g_1|^i}{|g_1 - g_2|} [|g_2|\, L_n^{(2)}(r) + L_n^{(1)}(r)]\, [G(2,1,0) - G(2,2,0)];$$

for $i = 1,\ 2$ and $j \geq 0$;

$$W_{ji}(n) := 2\, |L_n^{(i)}(r)| G(j,2,n) - 2\, |g_1|^i\ [|g_2| L_n^{(2)}(r) + L_n^{(1)}(r)]\, \frac{G(j,1,n) - G(j,2,n)}{|g_1 - g_2|};$$

for $i = 1,\ 2$ and $j \geq 0$;

$$\Lambda(n) := \frac{r^{2n+1}}{g_2^n\, (2n-1)!!\, (2n+1)!!}; \quad \text{and} \quad \Xi(n) := \frac{r^{n+1}}{n^2\, g_2^n\, (2n)!!}$$

and with them the vector functions

$$\underline{U}(n) := (U_1(n),\ U_2(n))^{\mathsf{T}} \quad \text{and} \quad \underline{W}_j(n) := (W_{j1}(n),\ W_{j2}(n))^{\mathsf{T}}.$$

We can now formulate:

THEOREM 2. *Assuming (10) and (14), for sufficiently large n*

$$\underline{e}_{n,\text{Chebyshev}}(r) = -\underline{U}(n)\, \Xi(n) \left[1 + O\left(\frac{1}{n}\right)\right]$$

and

$$\underline{W}_2(n)\,\Lambda(n)\left[1+O\left(\frac{1}{n}\right)\right] \le \underline{e}_{n,\mathrm{Legendre}}(r) \le \underline{W}_0(n)\,\Lambda(n)\left[1+O\left(\frac{1}{n}\right)\right].$$

PROOF.

(i) Assume that the first two of conditions (10) and (14) are satisfied, then for $i = 1, 2$ and $j \ge 0$:

$$U_i(n),\ W_{ji}(n) \ge 0; \quad U_i(n) = O\left(\left(\frac{g_2}{g_1}\right)^n\right); \quad W_{ji} = O\left(\left(\frac{g_2}{g_1^2}\right)^n\right). \tag{15}$$

From Lemma 3, it follows that for any sufficiently differentiable function $F(x)$:

$$\int_{-r}^{r} F(x)\,T_n(x)\,dx = \frac{b_0}{n^2-1}\left[1+O\left(\frac{1}{n^2}\right)\right].$$

Let us replace this estimate in (13) and let us take $v_n(x) := T_n(x)$. If we now use the first of the estimates of Theorem 1, we obtain the first of the estimates of Theorem 2. The second follows with a similar argument, taking for **V** the Legendre polynomials choice.

(ii) Let us now assume that the second two of conditions (10) and (14) are satisfied. For $i = 1$, 2; $j \ge 0$ let us introduce the starred functions U^* and W^*:

$$U_i^*(n) := \frac{2}{g_2-g_1}\left\{\left[\frac{(i-2)\,L_n^{(2)}(r)}{|C|} + (-g_1-g_2)^{2-i}\,L_n^{(1)}(r)\right][G(2,1,0)-G(-2,2,0)] - \frac{L_n^{(2-i)}(r)}{|C|}[G(2,1,1)+G(-2,2,1)] + \frac{1}{|g_1|}+\frac{1}{|g_2|}\right\}$$

and

$$W_{ji}^*(n) := \frac{8}{g_2-g_1}\left\{\frac{(-1)^{i+1}\,L_n^{(2-i)}(r)}{|C|}[G(j,1,n+1)+G(j,2,n+1)] + \left[(-1)^{2-i}(g_1+g_2)^{2-i}\,L_n^{(1)}(r) - \frac{(2-i)L_n^{(2)}(r)}{|C|}\right][G(j,1,n)-G(j,2,n)]\right\}.$$

In analogy with (15), for $i = 1$, 2 and $j \ge 0$,

$$U_i^*(n),\ W_{ji}^*(n) \ge 0, \quad U_i^*(n) = O\left(\left(\frac{g_2}{g_1}\right)^n\right), \quad \text{and } W_{ji}^*(n) = O\left(\left(\frac{g_2}{g_1^2}\right)^n\right).$$

The proof of the second case of (10) and (14) follows immediately by using the same arguments, but considering the corresponding starred functions instead of U and W. ∎

From Theorem 2, we can immediately deduce an interesting comparison of the effect of using Chebyshev or Legendre perturbation terms on the error per step at the end point of the interval in a Tau Method approximation. Such comparison explains Lanczos' conjecture concerning the remarkable accuracy of Tau approximations based on Legendre polynomial perturbations terms at the end point of the interval of approximation. The following proof, which holds true for *vector* Tau approximations and hence also for the case of simultaneous approximation of function and derivatives is immediately applicable to collocation, since for the constant coefficients case both methods can be made equivalent (see [2]).

COROLLARY 1. *For sufficiently large n,*

$$\underline{e}_{n,\mathrm{Legendre}}(r) = O\left(\underline{e}_{n,\mathrm{Chebyshev}}\,\frac{n\,((2n!!)^3\,r^n}{((2n)!)^2\,g_1^n}\right).$$ ∎

Table 1. Estimates of the components of the tau vector $\tau_{n,V} := (\tau_{n,V}^{(1)}, \tau_{n,V}^{(2)})^T$ obtained using the results of Theorem 1.

Exact $\tau_{n,V}^{(i)}$ / Estimated $\tau_{n,V}^{(i)}$				
	V := Chebyshev basis		V := Legendre basis	
Example	$i=1$	$i=2$	$i=1$	$i=2$
1	4.976 D-33 4.970 D-33	4.976 D-31 4.970 D-31	1.935 D-32 1.932 D-32	1.935 D-30 1.932 D-30
2	1.477 D-08 1.476 D-06	1.477 D-06 2.999 D-08	3.001 D-08 2.999 D-08	3.001 D-08
3	6.255 D-12 6.250 D-12	3.132 D-12 3.130 D-12	1.493 D-11 1.492 D-11	7.476 D-12 7.470 D-12

- Example 1: $a_1 = 2.0$; $a_2 = 1.0$; $f = 0$; $n = 5$; $g_1 = -0.01$; $g_2 = -1.0$; $r = 0.003$.
- Example 2: $a_1 = 2.0$; $a_2 = 1.0$; $f = 0$; $n = 5$; $g_1 = -0.01$; $g_2 = -1.0$; $r = 0.0012$.
- Example 3: $a_1 = 10.0$; $a_2 = 1.0$; $f = 0$; $n = 7$; $g_1 = -2.0$; $g_2 = -5.0$; $r = 0.28$.

Table 2a. Estimates of the components of the error vector $\underline{e}_{n,\text{Chebyshev}}$ obtained using the results of Theorem 2.

Example	Exact	Estimate
1	1.3578 D-27 1.3578 D-25	1.3250 D-27 1.3250 D-25
2	5.7099 D-12 2.8555 D-12	5.3542 D-12 2.6771 D-12
3	2.5180 D-13 2.5876 D-14	2.3937 D-13 2.4602 D-14

- Example 1: $a_1 = 2.0$; $a_2 = 1.0$; $f = 0$; $n = 19$; $g_1 = -0.01$; $g_2 = -1.0$; $r = 0.009$;
- Example 2: $a_1 = 10.0$; $a_2 = 1.0$; $f = 0$; $n = 11$; $g_1 = -2.0$; $g_2 = -5.0$; $r = 1.9$;
- Example 3: $a_1 = 176.0$; $a_2 = 1.0$; $f = 1.4$; $n = 13$; $g_1 = -10.0$; $g_2 = -11.0$; $r = 9.8$.

Table 2b. Upper and lower estimates of the components of the error vector linebreak $\underline{e}_{n,\text{Legendre}}$ obtained using the results of Theorem 2.

Example	Lower estimate	Exact	Upper estimate
1	-7.7677 D-68 -7.7677 D-66	-5.701 D-68 -5.701 D-66	-4.2134 D-68 -4.2134 D-66
2	-5.7624 D-24 -2.8812 D-24	-5.0521 D-24 -2.4874 D-24	-4.3551 D-24 -2.1776 D-24
3	-7.3335 D-22 -7.4758 D-23	-4.9170 D-22 -5.0197 D-23	-3.2421 D-22 -3.3309 D-23

- Example 1: $a_1 = 2.0$; $a_2 = 1.0$; $f = 0$; $n = 19$; $g_1 = -10.0$; $g_2 = -1.0$; $r = 0.003$;
- Example 2: $a_1 = 10.0$; $a_2 = 1.0$; $f = 0$; $n = 7$; $g_1 = -2.0$; $g_2 = -5.0$; $r = 0.28$;
- Example 3: $a_1 = 176.0$; $a_2 = 1.0$; $f = 1.4$; $n = 9$; $g_1 = -10.0$; $g_2 = -11.0$; $r = 4.0$.

7. SOME NUMERICAL ESTIMATES FOR CHEBYSHEV AND LEGENDRE PERTURBATION TERMS

In Tables 1 and 2, respectively, we report numerical results which illustrate the sharpness of Lemmas 1–4, on which Theorems 1 and 2 depend. In these and other tables, $D-k$ stands for 10^{-k}.

8. ERROR ESTIMATES FOR POWER SERIES AND WEIGHTED RESIDUALS PERTURBATION TERMS

Using the same arguments as in the proof of Theorem 2, we deduce:

THEOREM 3. *For sufficiently large n,*

$$\tau_{n,\mathrm{Power}} = \frac{(-1)^n \underline{L}_n(r)}{g_2^n\, n!}\left[1+\left(\frac{1}{n}\right)\right] \quad \text{and} \quad \tau_{2n,\mathrm{Weighted}} = \frac{\underline{L}_{2n}(r)}{g_2^{2n}(2n)!}\left[1+\left(\frac{1}{n}\right)\right], \tag{16}$$

if the first or second of conditions (10), respectively, are satisfied. ∎

We introduce some auxiliary functions to simplify the formulation of our results. Let:

$$\underline{M}_{ij}(2n) := \left(M_{ij}^{(1)}(2n),\, M_{ij}^{(2)}(2n)\right)^{\mathrm{T}},$$

where

$$M_{ij}^{(1)}(2n) := \frac{1}{(g_2-g_1)}\left\{ j\,[G(j,1,0)-G(-j,2,0)]\left[\frac{-B\,L_{2n}^{(1)}(r)}{C} - \frac{L_{2n}^{(2)}(r))}{|C|}\right] + i\,L_{2n}^{(1)}(r)\,\frac{G(j,1,1)+G(-j,2,1)}{|C|}\right\},$$

and

$$M_{ij}^{(2)}(2n) := \frac{1}{g_2-g_1}\left\{ j\,L_{2n}^{(1)}(r)\,[G(j,1,0)-G(-j,2,0)] - \frac{i\,L_{2n}^{(2)}(r)}{|C|}\,[G(j,1,1)+G(-j,2,1)]\right\}.$$

We shall assume that $R = a_1 + (a_2B - f)/C \le 0$, $a_2 \le 0$ and $n \ge 1$ and remark that

$$M_{ij}^{(k)}(2n) \ge 0 \quad \text{and} \quad M_{ij}^{(k)}(2n) = O\left(\left(\frac{g_2}{g_1}\right)^{2n}\right), \qquad \text{for } i,j,k = 1,2.$$

The same arguments as those used in the proof of Theorem 2 lead to the following upper and lower error estimates for $\underline{e}_{n,\mathrm{Power}}(r)$:

THEOREM 4. *For sufficiently large n,*

$$\underline{N}_0(n) \le \underline{e}_{n,\mathrm{Power}}(r) \le \underline{N}_2(n),$$

where

$$\underline{N}_j(n) := \frac{\underline{M}_{2j}(n)\, r^{n+1}}{g_2^n\,(n+1)!}\left[1+O\left(\frac{1}{n}\right)\right], \qquad j = 0,2,$$

and

$$\underline{N}_{20}^*(2n) \le \underline{e}_{2n,\mathrm{Weighted}}(r) \le \underline{N}_{22}^*(2n),$$

where

$$\underline{N}_{ij}^*(2n) := \frac{(-1)^n\, \underline{M}_{ij}(2n)\,\Lambda(n)}{2}\left[1+O\left(\frac{1}{n}\right)\right], \qquad i = 1,2; \quad j = 0,2,$$

if the second conditions of (10) and (14) are satisfied.

The same analysis applies when the alternative conditions in (10) and (14) are chosen. ∎

9. AN ANALYSIS OF THE RELATIVE ACCURACY TO BE OBTAINED USING DIFFERENT PERTURBATION TERMS IN THE TAU METHOD

If we take take $2n$ instead of n in the first of the estimates (16) given by Theorem 3 for the tau-parameters and compare it with the second, we find that they are of the same order in n. A comparison of the corresponding error bounds for $x = r$ shows, instead, that

$$\underline{e}_{2n,\text{Power}}(r) = \underline{e}_{2n,\text{Weighted}}(r)\, O\left(\frac{(2n-1)!!}{(2n)!!}\right);$$

this indicates that the choice $(x^2 - r^2)^n$ asymptotically is at disadvantage at the end point of the interval if compared with the vector error corresponding to power series. In the following theorem, we compare the latter with those resulting from the choice of Chebyshev or Legendre polynomials for the perturbation term.

THEOREM 5. *For sufficiently large n,*

- $\underline{e}_{n,\text{Power}}(r) = O(r^n \underline{k}/g_1^n\,(n+1)!)$, *where $\underline{k}$ is a constant vector;*
- $\underline{e}_{n,\text{Chebyshev}}(r) = O(\underline{e}_{n,\text{Power}}(r)\, 1/2^n\, n)$ *and*
- $\underline{e}_{n,\text{Legendre}}(r) = O(\underline{e}_{n,\text{Chebyshev}}(r)\; n\,((2n)!!)^3\, r^n/((2n)!)^2\, g_1^n)$. ∎

COROLLARY 2. The asymptotic behaviour of the functions $U_k(n), R_{jk}(n), M_{ij}^{(k)}(n)$ shows that the size of g_1 plays an important role in the rate of convergence of Tau Method approximations when Legendre polynomials are chosen for the basis $\mathbf{V}$. Clearly the case when $|g_1| \to 0$ corresponds to the singularly perturbed problem

$$\epsilon\, y''(x) + \left(\frac{B}{C}\right) y'(x) + y(x) = \epsilon\, f(x),$$

with $\epsilon = g_1 g_2 = 1/C$ and $g_1 + g_2 = B/C$. ∎

The following result shows that the choice $(x^2 - r^2)^n$ is not even attractive at $x = 0$. An estimation of $\underline{e}_{2n,\text{Weighted}}(0)$ gives the following upper and lower error estimates:

COROLLARY 3. *For sufficiently large n,*

$$\frac{\underline{N}_{10}^*(2n)}{g_2^n} \le \underline{e}_{2n,\text{Weighted}}(0) \le \frac{\underline{N}_{11}^*(2n)}{g_2^n},$$

while

$$\underline{e}_{2n,\text{Chebyshev}}(0) = \underline{E}_{2n}\, \frac{r^{2n+1}}{g_2^{2n}\, n^2\, (4n)!!}\left[1 + O\left(\frac{1}{n}\right)\right],$$

where

$$\underline{E}_n := \frac{1}{2(g_2 - g_1)} \begin{pmatrix} -\alpha_2/|C| - \beta_1 \\ \alpha_1 + (\beta_2 + \gamma)/|C| \end{pmatrix} \tag{17}$$

and

$$\alpha_i := \underline{L}_n^i(r)\left[G(-1,2,0) - G(1,1,0) + (-1)^{1+n/2}\, r\left(\frac{1}{|g_1|} - \frac{1}{|g_2|}\right)\right];$$

$$\beta_i := \underline{L}_n^i(r)\,[G(1,1,2i-3) + G(-1,2,2i-3)];$$

$$\gamma := -(-1)^{1+n/2}\, r\left(\frac{1}{|g_1|^2} - \frac{1}{|g_2|^2}\right).$$ ∎

10. THE CASE OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

Let us now return to the system of ordinary differential equations defined by (1) and to the associated Tau Problem defined by (2). The results discussed in the previous sections can now be formulated in the context of systems of linear ordinary differential equations with constant coefficients of order $m \times m$, with $m \geq 2$.

We begin, as we did before in the case $m = 2$, with the estimation of the tau-parameters. Let

$$\mathbf{G} := [\exp(-r\,\mathbf{A})]\ (\underline{a} - \mathbf{A}^{-1}\underline{f});$$

let $\mathbf{I}$ be a unit matrix and let $\mathbf{O}(n^k)$ stand for a matrix of order $m \times m$ with no entry larger than $O(n^k)$. Then,

THEOREM 6. *For sufficiently large n,*

$$\tau_{n,\text{Powers}} = \frac{(-1)^n}{n!}\left\{\mathbf{I} + \mathbf{O}\left(\frac{r^{n+1}\,\mathbf{A}^{n+1}}{(n+1)!}\right)\right\}\mathbf{A}^{n+1}\ \mathbf{G};$$
$$\tau_{n,\text{Chebyshev}} = \frac{2\,r^n}{(2n)!!}\,\mathbf{A}^{n+1}\left\{\mathbf{I} + \frac{r^2\,\mathbf{A}^2}{4(n-1)} + \mathbf{O}\left(\frac{1}{n^2}\right)\right\}\ \mathbf{G};$$
$$\tau_{n,\text{Legendre}} = \frac{r^n}{(2n-1)!!}\,\mathbf{A}^{n+1}\left\{\mathbf{I} + \frac{r^2\,\mathbf{A}^2}{2(2n-1)} + \mathbf{O}\left(\frac{1}{n^2}\right)\right\}\ \mathbf{G}. \qquad \blacksquare$$

The previous results enable us to estimate the error per step for each of these three choices of perturbation term in a step-by-step formulation of the Tau Method.

THEOREM 7. *For sufficiently large n:*

$$\underline{e}_{n,\text{Powers}}(x) = \frac{r^{n+1}\,\mathbf{A}^{n+1}}{(n+1)!}\left[\exp(-(x+r)\,\mathbf{A}) + (-1)^n\left(\frac{x}{r}\right)^{n+1}\mathbf{I}\right]\left[\mathbf{I} + \mathbf{O}\left(\frac{1}{n}\right)\right]\ \mathbf{G};$$
$$\underline{e}_{n,\text{Chebyshev}}(r) = \frac{-2\,r^{n+1}\,\mathbf{A}^{n+1}}{(n^2-1)\,(2n)!!}[\mathbf{I} + (-1)^n\,\exp(-2r\,\mathbf{A})]\left[\mathbf{I} + \frac{r^2\,\mathbf{A}^2}{4(n-1)} + \mathbf{O}\left(\frac{1}{n^2}\right)\right]\ \mathbf{G};$$
$$\underline{e}_{n,\text{Legendre}}(r) = \frac{2\,r^{2n+1}\,\mathbf{A}^{2n+1}\,\mathbf{B}_n}{(2n-1)!!\,(2n+1)!!}\left\{\mathbf{I} + \frac{r^2\,\mathbf{A}^2}{2(2n-1)} + \mathbf{O}\left(\frac{1}{n^2}\right)\right\}\ \mathbf{G},$$

where

$$\mathbf{B}_n := \sum_{j=0}^{\infty}\frac{(-2r\,\mathbf{A})^j}{j!}\cdot\prod_{i=1}^{j}\frac{n+i}{2n+i+1}. \qquad \blacksquare$$

With this last Theorem it is possible to compare the relative effect of the choice of power series, Chebyshev or Legendre polynomials for the representation of the vector perturbation term $\underline{H}_n(x)$ on the one step error of a Tau Method approximate solution of an $m \times m, m \geq 2$ system of ordinary differential equations with constant coefficients defined by problem (1):

THEOREM 8. *For sufficiently large n, the one step error at the end point $x = r$ of a step-by-step formulation of the Tau Method with a perturbation term defined by a linear combination of terms of one of the three choices: $x^n, T_n(x)$ or $P_n(x)$, is given by the following scale of estimates:*

- $\underline{e}_{n,\text{Powers}}(r) = \mathbf{O}(\,r^n\,\mathbf{A}^n\,\underline{k}/(n+1)!\,)$, *where $\underline{k}$ is a constant vector;*
- $\underline{e}_{n,\text{Chebyshev}}(r) = \mathbf{O}(\,\underline{e}_{n,\text{Powers}}(r)\,1/n\,2^n\,)$;
- $\underline{e}_{n,\text{Legendre}}(r) = \mathbf{O}(\,\underline{e}_{n,\text{Chebyshev}}(r)\,n\,((2n)!!)^3\,r^n\,\mathbf{A}^n/((2n)!)^2)$,

respectively. $\blacksquare$

Further details of the proofs of theorems 6–8 are given in [16].

REFERENCES

1. E.L. Ortiz, The Tau Method, *SIAM J. Numer. Analysis* **6**, 480–491 (1969).
2. M. El Daou, E.L. Ortiz and H. Samara, A unified approach to the Tau Methods and to Chebyshev and Legendre series expansion techniques, *Imperial College Research Report* **NAS 7-89**, 1–9 (1989).
3. E.L. Ortiz, A recursive method for the approximate expansion of functions in a series of polynomials, *Comp. Phys. Comm.* **4**, 151–156 (1972).
4. C. Lanczos, Trigonometric interpolation of empirical and analytical functions, *J. Math. Phys.* **17**, 123–199 (1938).
5. J.H. Freilich and E.L. Ortiz, Numerical solution of systems of differential equations with the Tau Method: An error analysis, *Maths. Comp.* **39**, 189–203 (1984).
6. M.R. Crisci and E. Russo, An extension of Ortiz' recursive formulation of the Tau Method to certain linear systems of ordinary differential equations, *Maths. Comput.* **41**, 27–42 (1983).
7. C. Lanczos, *Applied Analysis*, Prentice Hall, New Jersey, (1956).
8. C. Lanczos, Legendre vs. Chebyshev polynomials, In *Topics in Numerical Analysis*, (Edited by J.J.H. Miller), pp.191–201, Academic Press, New York, (1973).
9. E.L. Ortiz, Step by step Tau Method: Piecewise polynomial approximations, *Comp. and Maths. with Appli.* **1**, 381–392 (1975).
10. P. Onumanyi and E.L. Ortiz, Numerical solution of stiff and singularly perturbed boundary value problems with a segmented-adaptive formulation of the Tau Method, *Math. Comput.* **43**, 189–203 (1984).
11. A.E.M. El Misiery and E.L. Ortiz, Tau-Lines: A new hybrid approach to the numerical treatment of crack problems based on the Tau Method, *Comp. Meth. in Appl. Mech. and Engng.* **56**, 265–282 (1986).
12. M. Hosseini Ali Abadi and E.L. Ortiz, A Tau Method based on non-uniform space-time elements for the numerical simulation of solitons, *Comp. and Maths. with Appli.* **22**, 7–19 (1991).
13. C. de Boor and B. Swartz, Local piecewise polynomial projection methods for ODE which give higher-order convergence at knots, *Math. Comput.* **36**, 21–33 (1981).
14. M. El Daou, S. Namasivayam and E.L. Ortiz, Differential equations with piecewise approximate coefficients: Discrete and continuous estimation for initial and boundary value problems, *Comp. and Maths. with Appli.* (1992, in press).
15. T.J. Rivlin, *The Chebyshev Polynomials*, 2nd ed., John Wiley & Sons, New York, (1990).
16. S. Namasivayam and E.L. Ortiz, A hierarchy of truncation error estimates for the numerical solution of a system of ordinary differential equations with techniques based on the Tau Method, In *Numerical treatment of differential equations*, (Edited by K. Strehmel), pp. 113–121, Teubner, Leipzig, (1988).
17. E.L. Ortiz, On the numerical solution of nonlinear and functional differential equations with the Tau Method, In *Numerical Treatment of Differential Equations in Applications*, (Edited by R. Ansorge and W. Töring), pp. 127–139, Springer-Verlag, Berlin, (1978).