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A multi-variable theta product

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ABSTRACT

A multi-variable theta product is examined. It is shown that, under very general choices of the parameters, the quotient of two such general theta products is a root of unity. Special cases are explicitly determined. The second main theorem yields an explicit evaluation of a sum of series of cosines, which greatly generalizes one of Ramanujan's theorems on certain sums of hyperbolic cosines.

1. INTRODUCTION

Motivated by a fascinating identity involving a quasi-theta product in Ramanujan's lost notebook [8], [4], we examine in this paper a certain, very general multi-variable theta product. For $k+1$ complex variables $u_1, \ldots, u_k, w$, $k \geq 1$, define

$$F_k(u_1, \ldots, u_k; w) := \prod_{j_1, \ldots, j_k = 0}^{\infty} \frac{1 + (-1)^{j_1 + \cdots + j_k} w u_1^{2j_1+1} \cdots u_k^{2j_k+1}}{1 - (-1)^{j_1 + \cdots + j_k} w u_1^{2j_1+1} \cdots u_k^{2j_k+1}}. \tag{1.1}$$

Throughout the paper, we assume that $0 < |u_1|, \ldots, |u_k| < 1$. The product on the right side of (1.1) converges for any w not of the form $(-1)^{j_1+\cdots+j_k} u_1^{-2j_1-1} \cdots u_k^{-2j_k-1}$ with $j_1, \ldots, j_n \in \mathbf{N}$, and so, as a function of w, $F_k(u_1, \ldots, u_k; w)$ is meromorphic.

When $k = 1$, $u_1 = q$, and $w = q$, then

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$$F_1(q;q)=\prod_{j=0}^{\infty}\frac{1+(-1)^j q^{2j+2}}{1-(-1)^j q^{2j+2}}=\prod_{j=0}^{\infty}\frac{1+q^{4j+2}}{1-q^{4j+2}}\prod_{j=1}^{\infty}\frac{1-q^{4j}}{1+q^{4j}}$$
$$=\prod_{j=1}^{\infty}\frac{1-q^{8j-4}}{1-q^{8j}}\frac{(1-q^{4j})^2}{(1-q^{4j-2})^2}=\frac{\psi^2(q^2)}{\psi(q^4)},$$

where we have used the well-known product representation [3, p. 36, Chap. 16, Entry 22(ii)] for the theta function $\psi(q)$ (in Ramanujan's notation),

$$\psi(q):=\sum_{n=0}^{\infty}q^{n(n+1)/2},\qquad |q|<1.$$

Thus, in this instance, (1.1) reduces to a quotient of classical theta functions.

Section 2 is devoted to proving our two main results. Our first theorem shows that for very general choices of the parameters $0<|u_1|,\ldots,|u_k|<1$, the quotient of two functions (1.1) is a root of unity when we take the two choices of w to be reciprocals of each other. Our second theorem gives an evaluation of a sum of $k+1$ infinite series involving cosines.

In Section 3, we provide applications. First, we show that Theorem 2.2 yields a vast generalization of the following identity of Ramanujan found in Entry 15 of Chapter 14 of his second notebook [7], [2, p. 262]. Let $\alpha,\beta>0$ with $\alpha\beta=\pi^2/4$. Then

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\cosh\{(2n+1)\alpha\}}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\cosh\{(2n+1)\beta\}}=\frac{\pi}{4}. \tag{1.2}$$

We next examine special instances of Theorem 2.1. In particular, we give new explicit evaluations of certain infinite products.

2. MULTI-VARIABLE PRODUCTS

Our first objective is to prove the following transformation formula.

Theorem 2.1. *Let $A_1,\ldots,A_{k+1}$ and A be complex numbers such that $A_1,\ldots,A_{k+1}$ are nonzero and the quotient of any two of them is nonreal. For each distinct pair (l,j), $1\le l,j\le k+1$, denote $q_{l,j}=\exp(\pm\pi iA_l/(2A_j))$, where the sign $\pm$ is chosen so that $|q_{l,j}|<1$. We also set $q_j=i\exp(\pi A/(2A_j))$ for $1\le j\le k+1$. Then*

$$\prod_{j=1}^{k+1}\frac{F_k(q_{1,j},\ldots,q_{j-1,j},q_{j+1,j},\ldots,q_{k+1,j};q_j)}{F_k(q_{1,j},\ldots,q_{j-1,j},q_{j+1,j},\ldots,q_{k+1,j};\frac{1}{q_j})}=\exp\left(\frac{\pi i}{2^k}\right). \tag{2.1}$$

The form of (2.1) is not surprising, for in the 'inversion' formulas for classical theta functions, roots of unity arise. We will derive Theorem 2.1 from the following result.

Theorem 2.2. *Let $A_1,\ldots,A_{k+1}$ be nonzero complex numbers with the quotient of* *any two of them nonreal. Then for any complex number A, with $|A|$ small enough in terms of $A_1, \ldots, A_{k+1}$,*

$$(2.2)\qquad \sum_{j=1}^{k+1}\sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{\frac{(2n+1)\pi A}{2A_j}}}{(2n+1)\prod_{1\le l\le k+1, l\ne j}\cos\frac{(2n+1)\pi A_l}{2A_j}} = \frac{\pi}{2}.$$

Proof of Theorem 2.1. We first derive a convenient expression for $\log F_k(u_1, \ldots, u_k; w)$. Taking logarithms on both sides of (1.1), we find that

(2.3)

$$\log F_k(u_1,\ldots,u_k;w) = \sum_{j_1,\ldots,j_k=0}^{\infty} \log(1+(-1)^{j_1+\cdots+j_k} w u_1^{2j_1+1}\cdots u_k^{2j_k+1})$$
$$- \sum_{j_1,\ldots,j_k=0}^{\infty} \log(1-(-1)^{j_1+\cdots+j_k} w u_1^{2j_1+1}\cdots u_k^{2j_k+1}) =: S_1 - S_2.$$

Here and in the next step, we have ignored branches of the logarithm. The justification lies in our eventual proof of Theorem 2.2. Using the Taylor series of $\log(1+z)$ about $z=0$, we find that

$$(2.4)\qquad \begin{aligned} S_1 &= -\sum_{j_1,\ldots,j_k=0}^{\infty}\sum_{m=1}^{\infty} \frac{((-1)^{1+j_1+\cdots+j_k} w u_1^{2j_1+1}\cdots u_k^{2j_k+1})^m}{m} \\ &= -\sum_{m=1}^{\infty} \frac{(-wu_1\cdots u_k)^m}{m} \sum_{j_1,\ldots,j_k=0}^{\infty} (-1)^{m(j_1+\cdots+j_k)} u_1^{2mj_1}\cdots u_k^{2mj_k} \\ &= -\sum_{m=1}^{\infty} \frac{(-wu_1\cdots u_k)^m}{m} \left(\sum_{j_1=0}^{\infty} (-u_1^2)^{mj_1}\right)\cdots\left(\sum_{j_k=0}^{\infty}(-u_k^2)^{mj_k}\right) \\ &= -\sum_{m=1}^{\infty} \frac{(-wu_1\cdots u_k)^m}{m(1-(-u_1^2)^m)\cdots(1-(-u_k^2)^m)}. \end{aligned}$$

By replacing w by $-w$ in (2.4), we find that

$$(2.5)\qquad S_2 = -\sum_{m=1}^{\infty} \frac{(wu_1\cdots u_k)^m}{m(1-(-u_1^2)^m)\cdots(1-(-u_k^2)^m)}.$$

Thus, substituting (2.4) and (2.5) into (2.3), we deduce that

$$(2.6)\qquad \begin{aligned} \log F_k(u_1,\ldots,u_k;w) &= \sum_{m=1}^{\infty} \frac{(wu_1\cdots u_k)^m(1-(-1)^m)}{m(1-(-u_1^2)^m)\cdots(1-(-u_k^2)^m)} \\ &= 2\sum_{\substack{m=1\\ m \text{ odd}}}^{\infty} \frac{(wu_1\cdots u_k)^m}{m(1+u_1^{2m})\cdots(1+u_k^{2m})}. \end{aligned}$$

Since $u_1, \ldots, u_k$ are nonzero, we may write (2.6) in the form

$$(2.7)\qquad \begin{aligned} &\log F_k(u_1,\ldots,u_k;w) \\ &= 2\sum_{n=0}^{\infty} \frac{w^{2n+1}}{(2n+1)(u_1^{2n+1}+u_1^{-2n-1})\cdots(u_k^{2n+1}+u_k^{-2n-1})}. \end{aligned}$$

If we apply (2.7) with w replaced by $1/w$ and use the symmetry of the denominator on the right side of (2.7) with respect to the transformation $2n+1 \to -(2n+1)$, we find that

$$\begin{aligned}
&\log F_k\left(u_1, \ldots, u_k; \frac{1}{w}\right) \\
(2.8)\qquad &= -2 \sum_{n=-1}^{-\infty} \frac{w^{2n+1}}{(2n+1)(u_1^{2n+1} + u_1^{-2n-1}) \cdots (u_k^{2n+1} + u_k^{-2n-1})}.
\end{aligned}$$

Note that the series on the right side of (2.7) converges absolutely for any w with $|w| < 1/|u_1 \cdots u_k|$, while the series on the right side of (2.8) converges for any w with $|w| > |u_1 \cdots u_k|$. It follows that

$$\begin{aligned}
&\log F_k(u_1, \ldots, u_k; w) - \log F_k\left(u_1, \ldots, u_k; \frac{1}{w}\right) \\
(2.9)\qquad &= 2 \sum_{n=-\infty}^{\infty} \frac{w^{2n+1}}{(2n+1)(u_1^{2n+1} + u_1^{-2n-1}) \cdots (u_k^{2n+1} + u_k^{-2n-1})},
\end{aligned}$$

for any w in the annulus

$$(2.10)\qquad |u_1 \cdots u_k| < |w| < \frac{1}{|u_1 \cdots u_k|}.$$

Now recall that $A_1, \ldots, A_{k+1}$ are nonzero complex numbers with the quotient of any two of them nonreal. Also, recall that A is a complex number with $|A|$ small enough in terms of $A_1, \ldots, A_{k+1}$, such that (2.2) is valid and such that for any $1 \le j \le k+1$, (2.10) holds with $u_1, \ldots, u_k$ and w replaced by $q_{1,j}, \ldots, q_{j-1,j}$, $q_{j+1,j}, \ldots, q_{k+1,j}$, and q_j, respectively. Then, taking into account that

$$(-1)^n \exp\left(\frac{(2n+1)\pi A}{2A_j}\right) = -iq_j^{2n+1}$$

and

$$2\cos\frac{(2n+1)\pi A_l}{2A_j} = q_{l,j}^{2n+1} + q_{l,j}^{-2n-1},$$

from (2.2) and (2.9), we deduce that

$$\begin{aligned}
&\sum_{j=1}^{k+1} \Big(\log F_k(q_{1,j}, \ldots, q_{j-1,j}, q_{j+1,j}, \ldots, q_{k+1,j}; q_j) \\
(2.11)\qquad &- \log F_k\left(q_{1,j}, \ldots, q_{j-1,j}, q_{j+1,j}, \ldots, q_{k+1,j}; \frac{1}{q_j}\right)\Big) = \frac{\pi i}{2^k}.
\end{aligned}$$

By exponentiating both sides of (2.11), we obtain (2.1) for any A with $|A|$ small enough in terms of $A_1, \ldots, A_{k+1}$. Then the general form of (2.1) follows by analytic continuation, and this completes the proof of Theorem 2.1. $\square$

Proof of Theorem 2.2. Let $A_1, \ldots, A_{k+1}$ and A be as in the statement of Theorem 2.2 and define the function

$$f(z) = \frac{e^{Az}}{z \cos A_1 z \cdots \cos A_{k+1} z}.$$

The function $f(z)$ is meromorphic in the entire complex plane with a simple pole at $z = 0$ and simple poles at $z = (2n+1)\pi/(2A_j)$ for each integer n and each integer j, $1 \leq j \leq k+1$. Let γ_{R_m} be a sequence of positively oriented circles centered at the origin and with radii R_m tending to ∞ as $m \to \infty$, where the radii R_m are chosen so that the circles remain at a bounded distance from all the poles of $f(z)$. From the definition of f, it is easy to see that $f(z)$ decays exponentially as $|z| \to \infty$ provided z remains at a bounded distance from the poles of f and $|A|$ is small enough in terms of $A_1, \ldots, A_{k+1}$. It follows that

$$\left| \int_{\gamma_{R_m}} f(z) dz \right| \to 0, \tag{2.12}$$

as $R_m \to \infty$.

Let $R(a)$ denote the residue of $f(z)$ at a pole a. Then, brief calculations show that

$$R(0) = 1, \tag{2.13}$$

$$R\left(\frac{(2n+1)\pi}{2A_j}\right) = \frac{2(-1)^{n+1} e^{\frac{(2n+1)\pi A}{2A_j}}}{\pi(2n+1) \prod_{1 \leq l \leq k+1, l \neq j} \cos \frac{(2n+1)\pi A_l}{2A_j}}, \tag{2.14}$$

for each integer n and each $j \in \{1, \ldots, k+1\}$. Hence, using (2.13), (2.14), and the residue theorem, we deduce that

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\gamma_{R_m}} f(z) dz \\ &= 1 + \sum_{j=1}^{k+1} \sum_{|2n+1| < 2R_m A_j/\pi} \frac{2(-1)^{n+1} e^{\frac{(2n+1)\pi A}{2A_j}}}{\pi(2n+1) \prod_{1 \leq l \leq k+1, l \neq j} \cos \frac{(2n+1)\pi A_l}{2A_j}}. \end{aligned} \tag{2.15}$$

Letting R_m tend to ∞ in (2.15) and employing (2.12), we conclude that

$$0 = 1 + \sum_{j=1}^{k+1} \sum_{n=-\infty}^{\infty} \frac{2(-1)^{n+1} e^{\frac{(2n+1)\pi A}{2A_j}}}{\pi(2n+1) \prod_{1 \leq l \leq k+1, l \neq j} \cos \frac{(2n+1)\pi A_l}{2A_j}}. \tag{2.16}$$

This gives (2.2), and the theorem is proved. □

3. APPLICATIONS

In this section we derive some applications of Theorems 2.1 and 2.2. We begin with Theorem 2.2.

If we set $k = 1$ in Theorem 2.2, then, subject to the prescribed conditions on A, A_1, and A_2, (2.2) takes the shape

$$(3.1)\qquad \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{\frac{(2n+1)\pi A}{2A_1}}}{(2n+1)\cos\frac{(2n+1)\pi A_2}{2A_1}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{\frac{(2n+1)\pi A}{2A_2}}}{(2n+1)\cos\frac{(2n+1)\pi A_1}{2A_2}} = \frac{\pi}{2}.$$

Set

$$A = 0, \qquad \frac{\pi A_2}{2A_1} = i\alpha \quad (\alpha > 0), \qquad \text{and} \qquad \frac{\pi A_1}{2A_2} = -i\beta \quad (\beta > 0).$$

Thus, $\alpha\beta = \pi^2/4$, and (3.1) reduces to the identity

$$(3.2)\qquad \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(2n+1)\cosh\{(2n+1)\alpha\}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(2n+1)\cosh\{(2n+1)\beta\}} = \frac{\pi}{2},$$

which is obviously equivalent to (1.2). As mentioned in the Introduction, this identity can be found in Ramanujan's second notebook [7, Chap. 14, Entry 15], [2, p. 262]. The first proof was given in 1925 by S. L. Malurkar [5], but because the notebooks were not made available to the general public until 1957, he did not realize that he had proven a result from Ramanujan's notebooks. In 1951, T. S. Nanjundiah [6] gave another proof. Finally, Berndt [1, pp. 176–177, Proposition 4.5] offered a further proof in 1977. We emphasize that all the authors cited above, in fact, found generalizations of (3.2). The generalizations proved in this paper are different and seem to be new.

For a second example, let $k = 2$ and $A = 0$. For $\omega := \exp(2\pi i/3)$, set

$$\frac{A_2}{A_1} = \alpha\omega, \qquad \frac{A_1}{A_3} = \beta\omega, \qquad \text{and} \qquad \frac{A_3}{A_2} = \gamma\omega,$$

where α, β and γ are positive real numbers. Thus, by Theorem 2.2, if α, β, and γ are positive real numbers such that $\alpha\beta\gamma = 1$, then

$$(3.3)\qquad \begin{aligned} &\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(2n+1)\cos\left(\frac{(2n+1)\pi\alpha\omega}{2}\right)\cos\left(\frac{(2n+1)\pi\omega^2}{2\beta}\right)} \\ &+ \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(2n+1)\cos\left(\frac{(2n+1)\pi\beta\omega}{2}\right)\cos\left(\frac{(2n+1)\pi\omega^2}{2\gamma}\right)} \\ &+ \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(2n+1)\cos\left(\frac{(2n+1)\pi\gamma\omega}{2}\right)\cos\left(\frac{(2n+1)\pi\omega^2}{2\alpha}\right)} = \frac{\pi}{2}. \end{aligned}$$

In particular, if $\alpha = \beta = \gamma = 1$, then (3.3) reduces to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\cos\left(\frac{(2n+1)\pi\omega}{2}\right)\cos\left(\frac{(2n+1)\pi\omega^2}{2}\right)} = \frac{\pi}{12}.$$

The latter two equalities and all such multi-sum identities arising from Theorem 2.2 are apparently new.

We now consider applications of Theorem 2.1.

We see from the definition of F_k that

$$(3.4)\qquad F_k(u_1,\ldots,u_k;-w)=\frac{1}{F_k(u_1,\ldots,u_k;w)}.$$

If we set $A=0$ in (2.1), then $q_j=i$ for $1\le j\le k+1$. Using also (3.4), we find that

$$(3.5)\qquad \left(\prod_{j=1}^{k+1} F_k(q_{1,j},\ldots,q_{j-1,j},q_{j+1,j},\ldots,q_{k+1,j};i)\right)^2=\exp\left(\frac{\pi i}{2^k}\right).$$

Taking square roots on both sides of (3.5), we obtain

$$(3.6)\qquad \prod_{j=1}^{k+1} F_k(q_{1,j},\ldots,q_{j-1,j},q_{j+1,j},\ldots,q_{k+1,j};i)=\pm\exp\left(\frac{\pi i}{2^{k+1}}\right).$$

In order to determine the sign $\pm$ on the right side of (3.6), we use (3.4) with $w=i$ to deduce that

$$(3.7)\qquad \log F_k(u_1,\ldots,u_k;1/i)=\log F_k(u_1,\ldots,u_k;-i)=-\log F_k(u_1,\ldots,u_k;i).$$

Then, combining (2.9) and (3.7), we obtain

$$(3.8)\qquad \begin{aligned}&\log F_k(u_1,\ldots,u_k;i)\\ &\quad=i\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(2n+1)(u_1^{2n+1}+u_1^{-2n-1})\cdots(u_k^{2n+1}+u_k^{-2n-1})}.\end{aligned}$$

In particular, for any $j\in\{1,\ldots,k+1\}$, the equality (3.8) gives

$$(3.9)\qquad \begin{aligned}&\log F_k(q_{1,j},\ldots,q_{j-1,j},q_{j+1,j},\ldots,q_{k+1,j};i)\\ &\quad=\frac{i}{2^k}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(2n+1)\prod_{1\le l\le k+1,l\ne j}\cos\frac{(2n+1)\pi A_l}{2A_j}}.\end{aligned}$$

If we add the equalities (3.9) for $j=1,\ldots,k+1$, then exponentiate both sides and use (2.2), we obtain

$$(3.10)\qquad \prod_{j=1}^{k+1} F_k(q_{1,j},\ldots,q_{j-1,j},q_{j+1,j},\ldots,q_{k+1,j};i)=\exp\left(\frac{\pi i}{2^{k+1}}\right).$$

An interesting particular case arises when $A_j=\exp\bigl(2\pi ij/(k+1)\bigr)$, $1\le j\le k+1$. We need the quotient of any two of them to be nonreal, and for this reason we choose k to be even, say $k=2r$. Note that for any $j\in\{1,\ldots,2r+1\}$ the numbers $A_1/A_j,\ A_2/A_j,\ldots,A_{j-1}/A_j,\ A_{j+1}/A_j,\ldots,A_{2r+1}/A_j$ coincide in a certain order with the numbers $A_1,\ldots,A_{2r}$. Therefore, the set $\{q_{1,j},\ldots,q_{j-1,j},q_{j+1,j},\ldots,q_{2r+1,j}\}$ is the same for any j and coincides with the set $\{\exp(\pm\pi iA_1/2),\ldots,\exp(\pm\pi iA_{2r}/2)\}$, where the signs are chosen such that $|\exp(\pm\pi iA_j/2)|<1$, $1\le j\le 2r$. Since $\mathrm{Im}\,A_j=\sin\bigl(2\pi j/(2r+1)\bigr)$ is positive for $1\le j\le r$ and is negative for $r+1\le j\le 2r$, the choice of the signs above will be 'plus' for $1\le j\le r$ and 'minus' for $r+1\le j\le 2r$. From the definition (1.1) we see that $F_k(u_1,\ldots,u_k;w)$ is symmetric in $u_1,\ldots,u_k$. In our particular case, it follows that all the factors on the left side of (3.10) are equal, and coincide with

$$F_{2r}\left(e^{\pi iA_1/2},\dots,e^{\pi iA_r/2},e^{-\pi iA_{r+1}/2},\dots,e^{-\pi iA_{2r}/2};i\right).$$

Thus (3.10) gives

$$(3.11)\qquad \left(F_{2r}\left(e^{\pi iA_1/2},\dots,e^{\pi iA_r/2},e^{-\pi iA_{r+1}/2},\dots,e^{-\pi iA_{2r}/2};i\right)\right)^{2r+1}=\exp\left(\frac{\pi i}{2^{2r+1}}\right).$$

Then from (3.11) it follows that there is an integer $h, 0\le h\le 2r$, such that

$$(3.12)\qquad \begin{aligned}&F_{2r}\left(e^{\pi iA_1/2},\dots,e^{\pi iA_r/2},e^{-\pi iA_{r+1}/2},\dots,e^{-\pi iA_{2r}/2};i\right)\\&=\exp\left(\frac{\pi i}{(2r+1)2^{2r+1}}+\frac{2h\pi i}{2r+1}\right).\end{aligned}$$

In order to determine the value of h in (3.12), we again rely on Theorem 2.2. Since in our case the left side of (3.9) is the same for any $j\in\{1,\dots,2r+1\}$ and coincides with

$$\log F_{2r}\left(e^{\pi iA_1/2},\dots,e^{\pi iA_r/2},e^{-\pi iA_{r+1}/2},\dots,e^{-\pi iA_{2r}/2};i\right),$$

by combining (3.9) and (2.2) we find that

$$(3.13)\qquad \log F_{2r}\left(e^{\pi iA_1/2},\dots,e^{\pi iA_r/2},e^{-\pi iA_{r+1}/2},\dots,e^{-\pi iA_{2r}/2};i\right)=\frac{\pi i}{(2r+1)2^{2r+1}}.$$

Exponentiating both sides of (3.13), we find that

$$(3.14)\quad F_{2r}\left(e^{\pi iA_1/2},\dots,e^{\pi iA_r/2},e^{-\pi iA_{r+1}/2},\dots,e^{-\pi iA_{2r}/2};i\right)=\exp\left(\frac{\pi i}{(2r+1)2^{2r+1}}\right).$$

Note that the left side of (3.14) has the form

$$F_{2r}\left(\rho_1e^{i\theta_1},\rho_2e^{i\theta_2},\dots,\rho_re^{i\theta_r},\rho_re^{-i\theta_r},\dots,\rho_2e^{-i\theta_2},\rho_1e^{-i\theta_1};i\right),$$

where $\rho_j=\exp(-\frac{\pi}{2}\sin\frac{2\pi j}{2r+1})$ and $\theta_j=\frac{\pi}{2}\cos\frac{2\pi j}{2r+1}$, for $1\le j\le r$. From the definition (1.1) we see that for any real numbers $\rho_1,\dots,\rho_r,\theta_1,\dots,\theta_r$ with $0<\rho_j<1$ for $1\le j\le r$,

$$(3.15)\qquad \begin{aligned}&F_{2r}\left(\rho_1e^{i\theta_1},\rho_2e^{i\theta_2},\dots,\rho_re^{i\theta_r},\rho_re^{-i\theta_r},\dots,\rho_2e^{-i\theta_2},\rho_1e^{-i\theta_1};i\right)\\&=\prod_{j_1,l_1,\dots,j_r,l_r=0}^{\infty}\\&\quad\times\frac{1+i(-1)^{j_1+\dots+j_r+l_1+\dots+l_r}\prod_{m=1}^r\left(\rho_me^{i\theta_m}\right)^{2j_m+1}\left(\rho_me^{-i\theta_m}\right)^{2l_m+1}}{1-i(-1)^{j_1+\dots+j_r+l_1+\dots+l_r}\prod_{m=1}^r\left(\rho_me^{i\theta_m}\right)^{2j_m+1}\left(\rho_me^{-i\theta_m}\right)^{2l_m+1}}\\&=\prod_{j_1,l_1,\dots,j_r,l_r=0}^{\infty}\\&\times\frac{1+i(-1)^{j_1+\dots+j_r+l_1+\dots+l_r}\rho_1^{2(j_1+l_1+1)}\dots\rho_r^{2(j_r+l_r+1)}e^{2i\theta_1(j_1-l_1)}\dots e^{2i\theta_r(j_r-l_r)}}{1-i(-1)^{j_1+\dots+j_r+l_1+\dots+l_r}\rho_1^{2(j_1+l_1+1)}\dots\rho_r^{2(j_r+l_r+1)}e^{2i\theta_1(j_1-l_1)}\dots e^{2i\theta_r(j_r-l_r)}}.\end{aligned}$$

In our case we may write

$$(3.16)\qquad \rho_1^{2(j_1+l_1+1)}\cdots\rho_r^{2(j_r+l_r+1)}=\exp\left(-\pi\sum_{m=1}^{r}(j_m+l_m+1)\sin\frac{2\pi m}{2r+1}\right)$$

and

$$(3.17)\qquad e^{2i\theta_1(j_1-l_1)}\cdots e^{2i\theta_r(j_r-l_r)}=\exp\left(\pi i\sum_{m=1}^{r}(j_m-l_m)\cos\frac{2\pi m}{2r+1}\right).$$

Combining (3.14)–(3.17), we obtain the following result.

Theorem 3.1. *For each integer $r\geq 1$,*

$$(3.18)\qquad \begin{aligned}&\prod_{j_1,l_1,\ldots,j_r,l_r=0}^{\infty}\\ &\times\frac{1+i(-1)^{j_1+\cdots+j_r+l_1+\cdots+l_r}e^{-\pi\sum_{m=1}^{r}(j_m+l_m+1)\sin\frac{2\pi m}{2r+1}+\pi i\sum_{m=1}^{r}(j_m-l_m)\cos\frac{2\pi m}{2r+1}}}{1-i(-1)^{j_1+\cdots+j_r+l_1+\cdots+l_r}e^{-\pi\sum_{m=1}^{r}(j_m+l_m+1)\sin\frac{2\pi m}{2r+1}+\pi i\sum_{m=1}^{r}(j_m-l_m)\cos\frac{2\pi m}{2r+1}}}\\ &=\exp\left(\frac{\pi i}{(2r+1)2^{2r+1}}\right).\end{aligned}$$

Let us write down explicitly the left side of (3.18) for the first two values of r. If $r=1$, then (3.18) reduces to

$$(3.19)\qquad \begin{aligned}&\prod_{j,l=0}^{\infty}\frac{1+i(-1)^{j+l}\exp(-\pi(j+l+1)\sin(2\pi/3)+\pi i(j-l)\cos(2\pi/3))}{1-i(-1)^{j+l}\exp(-\pi(j+l+1)\sin(2\pi/3)+\pi i(j-l)\cos(2\pi/3))}\\ &=e^{\pi i/24}.\end{aligned}$$

On the left side of (3.19), make the substitution $n=j+l+1$ and note that

$$\exp\left(\pi i(j-l)\cos\frac{2\pi}{3}\right)=\exp\left(\frac{\pi i(l-j)}{2}\right)=i^{l-j}=i^{n-1-2j}=i^{n-1}(-1)^j.$$

Then (3.19) becomes

$$(3.20)\qquad \prod_{n=1}^{\infty}\prod_{j=0}^{n-1}\frac{1+i^n(-1)^{n-1+j}e^{-\pi n\sqrt{3}/2}}{1-i^n(-1)^{n-1+j}e^{-\pi n\sqrt{3}/2}}=e^{\pi i/24}.$$

In the inner product on the left side of (3.20), the contributions of any two consecutive values of j cancel each other. Since j takes exactly n values, we deduce that, if n is even, then the inner product above equals 1, while if n is odd, say $n=2k+1$, then the inner product equals

$$\frac{1+i^n(-1)^{n-1}e^{-\pi n\sqrt{3}/2}}{1-i^n(-1)^{n-1}e^{-\pi n\sqrt{3}/2}}=\frac{1+i(-1)^k e^{-\pi(2k+1)\sqrt{3}/2}}{1-i(-1)^k e^{-\pi(2k+1)\sqrt{3}/2}}.$$

Thus (3.20) reduces to

$$(3.21)\qquad \prod_{k=0}^{\infty}\frac{1+i(-1)^k e^{-\pi(2k+1)\sqrt{3}/2}}{1-i(-1)^k e^{-\pi(2k+1)\sqrt{3}/2}}=e^{\pi i/24}.$$

The evaluation (3.21) can be formulated in terms of Dedekind eta-functions.

Recall that, for $q = \exp(2\pi i\tau), \operatorname{Im} \tau > 0$, the Dedekind eta-function $\eta(\tau)$ is defined by

$$\eta(\tau) := e^{2\pi i\tau/24} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau}).$$

A brief calculation shows that

$$\prod_{n=0}^{\infty} (1 - q^{n+1/2}) = q^{1/48} \frac{\eta(\frac{1}{2}\tau)}{\eta(\tau)}. \tag{3.22}$$

Observe that if we set $\tau = \pm\frac{1}{2} + \frac{1}{2}i\sqrt{3}$ in (3.22), we obtain, respectively, the denominator and numerator in the product (3.21).

This last part of the reasoning works in more generality. To be precise, for any complex number u with $|u| < 1$ and any w for which $F_2(iu, u; w)$ is defined, we can use the same reasoning as above to show that

$$F_2(iu, u; w) = \prod_{k=0}^{\infty} \frac{1 + iwu^{4k+2}}{1 - iwu^{4k+2}}.$$

Returning to (3.18), in the case $r = 2$, we obtain

$$\begin{aligned} &\prod_{j_1, j_2, l_1, l_2 = 0}^{\infty} \\ &\times \frac{1 + i(-1)^{j_1+j_2+l_1+l_2} e^{-\pi \sum_{m=1}^{2} (j_m + l_m + 1)\sin(2\pi m/5) + \pi i \sum_{m=1}^{2} (j_m - l_m)\cos(2\pi m/5)}}{1 - i(-1)^{j_1+j_2+l_1+l_2} e^{-\pi \sum_{m=1}^{2} (j_m + l_m + 1)\sin(2\pi m/5) + \pi i \sum_{m=1}^{2} (j_m - l_m)\cos(2\pi m/5)}} \\ &= e^{\pi i/160}. \end{aligned} \tag{3.23}$$

We now use the equalities

$$\sin\frac{2\pi}{5} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}, \quad \sin\frac{4\pi}{5} = \frac{\sqrt{10 - 2\sqrt{5}}}{4},$$

and

$$\cos\frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}, \quad \cos\frac{4\pi}{5} = \frac{-1 - \sqrt{5}}{4},$$

and make the substitutions $n_1 = j_1 + l_1 + 1$, $n_2 = j_2 + l_2 + 1$, $n = j_1 - j_2$, and $j = j_1$. Then the numerator on the left side of (3.23) equals

$$\begin{aligned} &1 + i(-1)^{n_1+n_2} e^{-\frac{\pi}{4}(n_1\sqrt{10+2\sqrt{5}} + n_2\sqrt{10-2\sqrt{5}}) + \frac{\pi i}{4}((n_2 - n_1 + 2j_1 - 2j_2)\sqrt{5} + n_1 + n_2 - 2j_1 - 2j_2 - 2)} \\ &= 1 + i^{-j_1-j_2}(-1)^{n_1+n_2} e^{-\frac{\pi}{4}(n_1\sqrt{10+2\sqrt{5}} + n_2\sqrt{10-2\sqrt{5}}) + \frac{\pi i}{4}((n_2 - n_1 + 2j_1 - 2j_2)\sqrt{5} + n_1 + n_2)} \\ &= 1 + i^{n}(-1)^{n_1+n_2+j} e^{-\frac{\pi}{4}(n_1\sqrt{10+2\sqrt{5}} + n_2\sqrt{10-2\sqrt{5}}) + \frac{\pi i}{4}((n_2 - n_1 + 2n)\sqrt{5} + n_1 + n_2)}. \end{aligned} \tag{3.24}$$

If we denote

$$\mathcal{A} = \{(n_1, n_2, n, j) : j_1, j_2, l_1, l_2 \in \mathbf{N}\}, \tag{3.25}$$

then, by (3.24), (3.23) becomes

$$\prod_{(n_1,n_2,n,j)\in\mathcal{A}} \times \frac{1+i^n(-1)^{n_1+n_2+j}e^{-\frac{\pi}{4}(n_1\sqrt{10+2\sqrt5}+n_2\sqrt{10-2\sqrt5})+\frac{\pi i}{4}((n_2-n_1+2n)\sqrt5+n_1+n_2)}}{1-i^n(-1)^{n_1+n_2+j}e^{-\frac{\pi}{4}(n_1\sqrt{10+2\sqrt5}+n_2\sqrt{10-2\sqrt5})+\frac{\pi i}{4}((n_2-n_1+2n)\sqrt5+n_1+n_2)}} = e^{\pi i/160}. \tag{3.26}$$

A brief calculation shows that

$$\begin{aligned}\mathcal{A} =\{&(n_1,n_2,n,j)\in\mathbf{Z}^4 : n_1,n_2\geq 1, 1-n_2\leq n\leq n_1-1,\\ &\max\{n,0\}\leq j\leq \min\{n+n_2-1,n_1-1\}\}.\end{aligned} \tag{3.27}$$

Let us remark that for fixed n_1, n_2, n the contributions of any two consecutive values of j on the left side of (3.26) cancel each other. If we denote

$$n^* = \min\{n+n_2-1, n_1-1\} - \max\{n,0\}, \tag{3.28}$$

then j takes exactly n^*+1 values. We deduce that for any fixed n_1, n_2, and n as above, the product over j on the left side of (3.26) equals 1 if n^* is odd. If n^* is even, then the value of this product can be obtained by letting j equal any one of its two extreme values, that is, $j=\max\{n,0\}$ or $j=\min\{n+n_2-1,n_1-1\}$. Thus, choosing the first of these values for j and denoting for convenience

$$\mathcal{B} = \{(n_1,n_2,n)\in\mathbf{Z}^3 : n_1,n_2\geq 1, 1-n_2\leq n\leq n_1-1, n^* \text{ even}\}, \tag{3.29}$$

we finally obtain

$$\prod_{(n_1,n_2,n)\in\mathcal{B}} \times \frac{1+i^n(-1)^{n_1+n_2+\max\{n,0\}}e^{-\frac{\pi}{4}(n_1\sqrt{10+2\sqrt5}+n_2\sqrt{10-2\sqrt5})+\frac{\pi i}{4}((n_2-n_1+2n)\sqrt5+n_1+n_2)}}{1-i^n(-1)^{n_1+n_2+\max\{n,0\}}e^{-\frac{\pi}{4}(n_1\sqrt{10+2\sqrt5}+n_2\sqrt{10-2\sqrt5})+\frac{\pi i}{4}((n_2-n_1+2n)\sqrt5+n_1+n_2)}} = e^{\pi i/160}. \tag{3.30}$$

4. CONCLUDING REMARKS

The product on the left side of (3.21) involves one variable k, while the product on the left side of (3.30) involves three variables n_1, n_2 and n. For a general r, if one uses the definition (1.1) on the left side of (3.14), this will involve products of the form

$$\begin{aligned}&\left(e^{\pi i A_1/2}\right)^{2j_1+1}\cdots\left(e^{\pi i A_r/2}\right)^{2j_r+1}\left(e^{-\pi i A_{r+1}/2}\right)^{2j_{r+1}+1}\cdots\\ &\left(e^{-\pi i A_{2r}/2}\right)^{2j_{2r}+1} := e^{-\pi i\gamma/2}\end{aligned} \tag{4.1}$$

say, where γ belongs to the ring of integers of the cyclotomic field $\mathbf{Q}(2\pi i/(2r+1))$. The degree of $\mathbf{Q}(2\pi i/(2r+1))$ over $\mathbf{Q}$ equals $\varphi(2r+1)$, and each γ as above can be written uniquely in the form

$$\gamma = a_0 + a_1 e^{\frac{2\pi i}{2r+1}} + \cdots + a_{\varphi(2r+1)-1} e^{\frac{2\pi i(\varphi(2r+1)-1)}{2r+1}}$$

with $a_0, a_1, \ldots, a_{\varphi(2r+1)-1} \in \mathbf{Z}$. Thus one may write the left side of (3.14) as a product over the *independent* integral variables $a_0, a_1, \ldots, a_{\varphi(2r+1)-1}$, and a *finite* inner product, over those $(j_1, \ldots, j_{2r}) \in \mathbf{N}^{2r}$ which correspond to a given $(a_0, a_1, \ldots, a_{\varphi(2r+1)-1}) \in \mathbf{Z}^{\varphi(2r+1)}$. Moreover a_0 can be eliminated in the sense that its contribution to the right side of (4.1) can be easily described reasoning mod 4. So we are left with $a_1, \ldots, a_{\varphi(2r+1)-1}$. This explains why in our most simplified concrete equalities (3.21) and (3.30) we had products over one and respectively three variables.

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