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Note

Oberwolfach rectangular table negotiation problem

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Abstract

We completely solve certain case of a "two delegation negotiation" version of the Oberwolfach problem, which can be stated as follows. Let $H(k, 3)$ be a bipartite graph with bipartition $X = \{x_1, x_2, \ldots, x_k\}, Y = \{y_1, y_2, \ldots, y_k\}$ and edges $x_1y_1, x_1y_2, x_ky_{k-1}, x_ky_k$, and $x_iy_{i-1}, x_iy_i, x_iy_{i+1}$ for $i = 2, 3, \ldots, k-1$. We completely characterize all complete bipartite graphs $K_{n,n}$ that can be factorized into factors isomorphic to $G = mH(k, 3)$, where k is odd and $mH(k, 3)$ is the graph consisting of m disjoint copies of $H(k, 3)$.

Keywords: Oberwolfach problem; Graph decomposition

There are many modifications of the well-known Oberwolfach problem (for an overview, see [1]). One of them is a bipartite version which asks: For what values of k, n it is possible to decompose the complete bipartite graph $K_{n,n}$ into graphs isomorphic to mC_{2k}, where $m = n/k$ and mC_{2k} is a vertex-disjoint union of m cycles of length $2k$? This problem was solved by Piotrowski [3], and can be also described as follows. Suppose we have two delegations with n people each, and we want to find a seating arrangement over n nights such that every night the members of the delegations sit alternately around m round tables, each table accommodating $2k$ people, and every person sits next to each member of the other delegation exactly once. One could agree that while such an arrangement is good for social occasions, it is not particularly suitable if we assume that the delegations are involved in negotiations. Then it would be more natural to have rectangular tables with members of each delegation sitting along one of the long sides of the table while the tables would not be too big. Even if we have tables with three people on each of the two long sides of the table, it is reasonable to assume that the people sitting at the opposite corners cannot easily communicate with each other. Therefore, we may translate this modification into terms of graph decompositions as follows.

We say that a graph B has a *G-decomposition* if there are subgraphs $G_0, G_1, G_2, \ldots, G_s$ of B, all isomorphic to G, such that each edge of B belongs to exactly one G_i. If the graph G (more precisely, each $G_i, i = 0, 1, \ldots, s$) contains all vertices of B, then we say that B has a *G-factorization*.

Let $H(k, 3)$ be a bipartite graph with bipartition $X = \{x_1, x_2, \ldots, x_k\}, Y = \{y_1, y_2, \ldots, y_k\}$ and edges $x_1y_1, x_1y_2, x_ky_{k-1}, x_ky_k$, and $x_iy_{i-1}, x_iy_i, x_iy_{i+1}$ for $i = 2, 3, \ldots, k-1$. We always assume that k is odd and $k \geq 3$. We want to characterize all complete bipartite graphs $K_{n,n}$ that can be factorized into factors isomorphic to $G = mH(k, 3)$, where $mH(k, 3)$ is the graph consisting of m vertex-disjoint copies of $H(k, 3)$. Since the number of edges of G equals $m(3k-2)$ and the number of its vertices in each partite set equals mk, the necessary conditions

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are $n = mk$ and $n^2 \equiv 0 \pmod{m(3k-2)}$. We will show that then m must be a multiple of $3k-2$ which yields $n \equiv 0 \pmod{k(3k-2)}$. We will show that these necessary conditions are also sufficient for the existence of a G-factorization of $K_{n,n}$.

Therefore, the smallest meaningful case is a decomposition of $K_{21,21}$ into $7H(3,3)$, where the graph $7H(3,3)$ consists of seven vertex-disjoint copies of the bipartite graph $H(3,3)$ with vertex bipartition $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and edges $x_1y_1, x_1y_2, x_2y_1, x_2y_2, x_2y_3$, and x_3y_2, x_3y_3.

In fact, this problem was originally stated in terms of a decomposition of the complete graph into the graph arising from $H(k,3)$ by adding a path with k vertices into each partite set with the endvertices being the vertices x_1, x_k and y_1, y_k, respectively—see [2]. The general version seems to be more difficult, and only very partial results are known.

Lemma 1. *Let G be isomorphic to $mH(k,3)$, with k odd, $k \geq 3$. If there exists a G-factorization of $K_{n,n}$, then $n = mk$ and $m \equiv 0 \pmod{3k-2}$.*

Proof. The equality $n = mk$ is evident. To prove that $m \equiv 0 \pmod{3k-2}$, we first observe that $H(k,3)$ has $3k-2$ edges and therefore $m(3k-2) | n^2$. This yields

$$n^2 = m^2k^2 = m(3k-2)q$$

for some integer q, which is the number of isomorphic factors in the G-factorization. Dividing both sides of the equality by m, we get

$$mk^2 = (3k-2)q.$$

Obviously, $mk^2 \equiv 0 \pmod{k^2}$. On the other hand, $3k-2$ and k are coprime for $k \geq 3$. Therefore, it follows that $q \equiv 0 \pmod{k^2}$, say $q = sk^2$. But then

$$mk^2 = (3k-2)sk^2$$

and

$$m = (3k-2)s,$$

which we wanted to prove. □

Now we show that $K_{3k-2,3k-2}$ can be decomposed into $3k-2$ copies of $H(k,3)$. We use a *bigraceful labeling* introduced by Ringel, Llado, and Serra [4] to do that.

Definition 2. Let G be a bipartite graph with k edges, $V(G) = V_0 \cup V_1$, $V_0 \cap V_1 = \emptyset$, and $|V_0| \leq |V_1| \leq k$. Let λ be an injection, $\lambda : V_i \to \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, $i = 0, 1$. For any pair of vertices $x_0 \in V_0$ and $y_1 \in V_1$ with $\lambda(x_0) = a_0$ and $\lambda(y_1) = b_1$ we define the *length* of an edge x_0y_1 as $\ell(x_0y_1) = b - a \pmod{k}$. We say that G has a *bigraceful labeling* if $\{\ell(x_0y_1) | x_0y_1 \in E(G)\} = \{0, 1, 2, \ldots, k-1\}$.

We will often identify a vertex with its label and say "a vertex j_i" rather than "a vertex x_i with label $\lambda(x_i) = j_i$".

It is an easy observation that if G with k edges has a bigraceful labeling, then G allows a decomposition of the complete bipartite graph $K_{k,k}$ into k isomorphic copies when the vertices in each of the partite sets V_0, V_1 rotate concurrently.

Lemma 3. *A complete bipartite graph $K_{3k-2,3k-2}$ with k odd can be decomposed into $3k-2$ copies of the graph $H(k,3)$.*

Proof. Let $H = H(k,3)$ have vertices $0_0, 2_0, \ldots, (2k-2)_0, (2k-2)_1, (2k-1)_1, \ldots, (3k-3)_1$ and edges $(2k-2)_0(2k-2)_1$, $(2k-2)_0(2k-1)_1$ of lengths 0 and 1, $(2k-4)_0(2k-2)_1$, $(2k-4)_0(2k-1)_1$, $(2k-4)_0(2k)_1$, of lengths 2,3,4, $(2k-6)_0(2k-1)_1$, $(2k-6)_0(2k)_1$, $(2k-6)_0(2k+1)_1$, of lengths 5,6,7, etc.,

...

$(2k-2i)_0(2k+i-4)_1$, $(2k-2i)_0(2k+i-3)_1$, $(2k-2i)_0(2k+i-2)_1$, of lengths $3i-4, 3i-3, 3i-2$ for each $1 < i \leq k-1$,

...

and $0_0(3k-4)_1$, $0_0(3k-3)_1$, of lengths $3k-4$ and $3k-3$.

Then H has a bigraceful labeling and therefore decomposes $K_{3k-2,3k-2}$. We describe this decomposition formally as $\mathcal{H} = \{H_0, H_1, \ldots, H_{3k-3}\}$, where $H_j = \phi_j(H)$ and $\phi_j(a_0) = (a+j)_0$, $\phi_j(b_1) = (b+j)_1$ for $j = 0, 1, \ldots, 3k-3$. □

Next we show that $K_{k(3k-2),k(3k-2)}$ can be factorized into k^2 copies of the graph G that consists of $3k-2$ disjoint copies of H.

Lemma 4. *Let k be odd, $k \geq 3$, $m = 3k-2$, and $n = k(3k-2)$. Then there exists a factorization of $K_{n,n}$ into k^2 copies of the graph $G = mH(k,3)$.*

Proof. First we observe that in the bi-cyclic $H(k,3)$-decomposition of $K_{3k-2,3k-2}$ every vertex appears in k copies of $H(k,3)$ and each time is an image of a different vertex of $H(k,3)$.

We denote the vertices of $K_{k(3k-2),k(3k-2)}$ by $(a,b)_i$, where $0 \leq a \leq 3k-3, 0 \leq b \leq k-1, 0 \leq i \leq 1$. We are in a sense "blowing up" each vertex a_i of $K_{3k-2,3k-2}$ into the k-tuple $(a,0)_i, (a,1)_i, \ldots, (a,k-1)_i$ in $K_{k(3k-2),k(3k-2)}$ and placing copies of H_j into $K_{k(3k-2),k(3k-2)}$ in such a way that out of the k copies that were incident with a_i in the decomposition $\mathcal{H}$ of $K_{3k-2,3k-2}$, each one will be incident with exactly one of the vertices $(a,0)_i, (a,1)_i, \ldots, (a,k-1)_i$ in the factorization $\mathcal{G}$ of $K_{k(3k-2),k(3k-2)}$.

More formally we describe the factorization $\mathcal{G}$ of $K_{k(3k-2),k(3k-2)}$ as follows. We set $\mathcal{G} = \{G_{r,s} | 0 \leq r \leq k-1, 0 \leq s \leq k-1\}$, where $G_{r,s} = \psi_{r,s}(G)$ for $0 \leq r \leq k-1, 0 \leq s \leq k-1$. The graph G is composed of $3k-2$ disjoint copies of $H(k,3)$, denoted by $H'_0, H'_1, \ldots, H'_{3k-3}$. We first construct H' with vertices $(0,0)_0, (2,1)_0, \ldots, (2k-2,k-1)_0, (2k-2,0)_1, (2k-1,1)_1, \ldots, (3k-3,k-1)_1$. Two vertices $(2c,c)_0$ and $(2k-2+d,d)_1$ for $c,d = 0,1,\ldots,k-1$ are joined by an edge in H' if the vertices $(2c)_0$ and $(2k-2+d)_1$ were joined by an edge in H. In other words, H' is defined by a mapping $\theta : H \to H'$, where $\theta((2c)_0) = (2c,c)_0$ and $\theta((2k-2+d)_1) = (2k-2+d,d)_1$.

Then we define a mapping $\sigma_t : H' \to H'_t$ for $t = 0,1,\ldots,3k-3$ as $\sigma_t(a,b)_i = (a+t,b)_i$ for $i = 0,1$, where the addition is taken modulo $3k-2$. The graphs $H'_0, H'_1, \ldots, H'_{3k-3}$ essentially mimic the $H(k,3)$-decomposition of $K_{3k-2,3k-2}$ in the sense that there is an edge $(a,c)_0(b,d)_1$ in H'_t for some values c,d (which we specify below) if and only if there is the edge a_0b_1 in H_t. This correspondence can be described as follows. We define mappings $\theta_t : H_t \to H'_t$ as $\theta_t(a_0) = (a,c)_0$, where $c = (a-t)/2 \pmod{3k-2}$ and $\theta_t(b_1) = (b,d)_1$, where $d = b-t-(2k-2) \pmod{3k-2}$. Recall that H_t is a member of the decomposition $\mathcal{H}$ of $K_{3k-2,3k-2}$.

We notice that for every pair of k-tuples $(a,0)_0, (a,1)_0, \ldots, (a,k-1)_0$ and $(b,0)_1, (b,1)_1, \ldots, (b,k-1)_1$ there is exactly one edge $(a,c)_0(b,d)_1$ in the graph G. To prove this claim, it is enough to show that for every such a pair of k-tuples there is at least one edge between them, because G has $(3k-2)^2$ edges and there are $(3k-2)^2$ such pairs. However, every edge a_0b_1 of $K_{3k-2,3k-2}$ appears in some graph H_t, as $\mathcal{H}$ is an $H(k,3)$-decomposition of $K_{3k-2,3k-2}$. Therefore, some edge $(a,c)_0(b,d)_1$ appears in H'_t as $(a,c)_0(b,d)_1 = \theta_t(a_0b_1)$ for some choice of c,d as described above.

By now we have constructed the basic copy of G. Now we need to define the isomorphisms $\psi_{r,s}(G)$ for $0 \leq r \leq k-1, 0 \leq s \leq k-1$ that yield the G-factorization. We set $\psi_{r,s}((a,c)_0) = (a,c+r)_0$, $\psi_{r,s}((b,d)_1) = (b,d+s)_1$ for $r,s = 0,1,\ldots,k-1$. The addition here is always taken modulo $k-1$. Then we observe that since there was exactly one edge $(a,c)_0(b,d)_1$ between the k-tuples $(a,0)_0, (a,1)_0, \ldots, (a,k-1)_0$ and $(b,0)_1, (b,1)_1, \ldots, (b,k-1)_1$, the k^2 images $\psi_{r,s}((a,c)_0(b,d)_1)$ for $r,s = 0,1,\ldots,k-1$ will induce the complete bipartite graph $K_{k,k}$ with the bipartition $X_a = \{(a,0)_0, (a,1)_0, \ldots, (a,k-1)_0\}$, $Y_b = \{(b,0)_1, (b,1)_1, \ldots, (b,k-1)_1\}$. Therefore, every edge of $K_{k(3k-2),k(3k-2)}$ appears in one of the factors $G_{r,s}$, which completes the proof. □

Now we observe that if there exists a decomposition of the complete bipartite graph $K_{n,n}$ into q copies of a graph G, then there exists a decomposition of the graph $K_{pn,pn}$ into pq copies of the graph pG. Of course, one can factorize the graph $K_{pn,pn}$ into p copies of the graph $pK_{n,n}$. But because we can decompose $K_{n,n}$ into q copies of G, we get this way q copies of pG in each $pK_{n,n}$. The decomposition of $K_{pn,pn}$ is then obvious. We formally state these observations as

Proposition 5. *Let there exist a decomposition of the complete bipartite graph $K_{n,n}$ into q copies of a graph G. Then there exists a decomposition of the graph $K_{pn,pn}$ into pq copies of the graph pG for any $p \geq 1$.*

Combining Lemma 4 and Proposition 5, we immediately get our result.

Theorem 6. *Let $H = H(k, 3)$ be the graph described above and k be an odd integer, $k \geq 3$. Then there exists an mH-factorization of the complete graph $K_{n,n}$ for some positive integer m if and only if $m \equiv 0 \pmod{3k-2}$ and $n \equiv 0 \pmod{k(3k-2)}$.*

Proof. The necessity is obvious. The sufficiency follows directly from Lemma 4 and Proposition 5. □

Naturally, one can ask the same question for a more general version of this problem. First of all, the same problem for k even remains open. We can also require that the tables accommodate k people on each side, and that each person can communicate upto $2\ell + 1$ people on the other side of the table—the person right across the table, an ℓ persons immediately on the left and right from that person across the table. A graph describing this arrangement can be denoted by $H(k, 2\ell + 1)$ and we can ask for which values n, k, ℓ, m there exists a factorization of $K_{n,n}$ into $mH(k, 2\ell + 1)$.

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