The local well-posedness and existence of weak solutions for a generalized Camassa–Holm equation

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\textbf{Abstract}

A generalization of the Camassa–Holm equation, a model for shallow water waves, is investigated. Using the pseudoparabolic regularization technique, its local well-posedness in Sobolev space $H^s(\mathbb{R})$ with $s > \frac{3}{2}$ is established via a limiting procedure. In addition, a sufficient condition for the existence of weak solutions of the equation in lower order Sobolev space $H^s$ with $1 < s \leq \frac{3}{2}$ is developed.

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\section{1. Introduction}

In the study of shallow water waves, Camassa and Holm [4] used the Hamiltonian method to derive a completely integrable wave equation

$$u_t - u_{xxt} + 2ku_x + 3uux = 2uxuxx + uu_{xxx}$$

by retaining two terms that are usually neglected in the small amplitude, shallow water limit. Alternative derivations of the equation as a model for water waves were established in Constantin and Lannes [10] and Johnson [26].

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Since the birth of the Camassa–Holm equation (1), a huge amount of work has been carried out to study dynamic properties of Eq. (1). For \( k = 0 \), Eq. (1) has traveling wave solutions of the form \( ce^{-|x-ct|} \), called peakons, which capture an essential feature of the traveling waves of largest amplitude (see Constantin [8], Constantin and Escher [13,14] and Toland [40]). For \( k > 0 \), its solitary waves are stable solitons (see Constantin and Strauss [17,18] and Johnson [27]). It was shown in Constantin, Gerdjikov and Ivanov [11], and Constantin and McKean [12] that the inverse spectral or scattering approach was a powerful tool to handle Camassa–Holm equation and analyzed its various fluid dynamics. It should be addressed that Eq. (1) gives rise to geodesic flow of a certain invariant metric on the Bott–Virasoro group (see Constantin [9], Kouranbaeva [29] and Misiolek [33], Mclachlan and Zhang [35]), and this geometric illustration leads to a proof that the Least Action Principle holds (see Constantin and Kolev [15], Constantin et al. [16]). Solitary wave solutions of (1) that include peakons, kinks, compactons, solitary pattern solutions and plane periodic solutions were obtained analytically in Wazwaz’s paper [42]. Determinant formulas of N-soliton solutions of the continuous and semi-discrete Camassa–Holm equations are presented in Ohta et al. [36] to generate multi-soliton, multi-cuspon and multi-soliton–cuspon solutions (also see McKean [34]). It is worthwhile to mention that Xin and Zhang [44] proved that the global existence of the weak solution in the energy space \( H^1(\mathbb{R}) \) without any sign conditions on the initial value, and the uniqueness of this weak solution is obtained under some assumptions on the solution [45]. The sharpest results for the global existence and blow-up solutions are found in Bressan and Constantin [2,3], Li and Olver [32] established the local well-posedness in the Sobolev space \( H^s(\mathbb{R}) \) with \( s > \frac{3}{2} \) for Eq. (1) and gave conditions on the initial data that lead to finite time blow-up of certain solutions. It was shown in Constantin and Escher [19] that the blow-up occurs in the form of breaking waves, namely, the solution remains bounded but its slope becomes unbounded in finite time. Lai and Xu [30] analyzed the compact and noncompact structures for two types of generalized Camassa–Holm–KP equations. For other methods to handle the problems relating to various dynamic properties of the Camassa–Holm equation, the reader is referred to [5–7,20–23,25,30,31,37,38,43,46–48] and the references therein.

In fact, many different types of solutions for various generalized forms of the Camassa–Holm equation have been investigated. Tian and Song [39] studied the modified Camassa–Holm equation

\[
\begin{align*}
  ut - u_{txx} + au^nu_x + 2ku_x &= 2u_xu_{xx} + uu_{xxx},
\end{align*}
\]  

(2)

where \( a > 0, k \in \mathbb{R} \) and \( n \) is a natural number. When \( n = 3 \), two types of exact traveling wave solutions to Eq. (2) were acquired in [39]. For any exponent \( n \) of the nonlinearity, the integral expressions of traveling wave solutions for Eq. (2) were analyzed. In this situation, however, it is difficult to obtain the exact traveling wave solutions since one cannot work out the integral easily.

Recently, Hakkaev and Kirchev [24] investigated the generalized form of the Camassa–Holm equation (1)

\[
\begin{align*}
  ut - u_{txx} + 2ku_x + (m + 2)(m + 1)2^{-m}u_x &= \left(mu^{m-1}\frac{u_x^2}{2} + u^m u_{xx}\right)_x, 
\end{align*}
\]  

(3)

where \( m \geq 1 \). In [24], the local well-posedness of a Cauchy problem for Eq. (3) was established in Sobolev space \( H^s \) with \( s > \frac{3}{2} \), and the stability and instability of the solitary waves for the equation were discussed under suitable assumptions.

In this article, motivated by the work in [24,32], we study the following generalized Camassa–Holm equation

\[
\begin{align*}
  ut - u_{txx} + 2ku_x + au^mu_x &= \left(nu^{n-1}\frac{u_x^2}{2} + u^n u_{xx}\right)_x + \beta \partial_x[(u_x)^{2N-1}],
\end{align*}
\]  

(4)

where \( m \geq 1, n \geq 1 \) and \( N \geq 1 \) are natural numbers, and \( a, k \) and \( \beta \geq 0 \) are constants. Obviously, Eq. (4) reduces to Eq. (3) if we set \( a = \frac{(m+2)(m+1)}{2} \), \( n = m \) and \( \beta = 0 \). Eq. (4) becomes Eq. (1) by letting
\[ a = 3, \ m = 1, \ n = 1 \] and \( \beta = 0 \). Actually, Wu and Yin [43] consider a nonlinearly dissipative Camassa–Holm equation which includes a nonlinearly dissipative term \( L(u) \), where \( L \) is a differential operator or a quasi-differential operator. Therefore, we can regard the term \( \beta \partial_x[(u_x)^{2N-1}] \) as a nonlinearly dissipative term for the dissipative Camassa–Holm equation (4).

Due to the term \( \beta \partial_x[(u_x)^{2N-1}] \) in Eq. (4), the conservation laws in previous works [24,32] for Eq. (1) lose their powers to obtain some bounded estimates of the solution for Eq. (4). A new conservation law different from those presented in [24,32] is established to prove the local existence and uniqueness of the solution to Eq. (4) subject to initial value \( u_0(x) \in H^s(\mathbb{R}) \) with \( s > \frac{3}{2} \). We should address that all the generalized versions of the Camassa–Holm equation in previous works (see [2,3, 8–11,22,24,28,32,37,43]) do not involve the nonlinear term \( \beta \partial_x[(u_x)^{2N-1}] \). In addition, for an arbitrary positive Sobolev exponent, a new lemma (see Lemma 5 in Section 2), which is similar to that presented in [1] where the Sobolev exponent is required to be greater than \( \frac{3}{2} \), is established to prove the existence of weak solutions of the problem in lower order Sobolev space \( H^s \) with \( 1 < s \leq \frac{3}{2} \).

The main tasks of this work are two-fold. One is to prove the existence of weak solutions for the generalized Camassa–Holm equation (4) in lower order Sobolev space \( H^s \) with \( 1 < s \leq \frac{3}{2} \) (see Theorem 1 in Section 2). The other is to establish the local existence and uniqueness of solutions for Eq. (4) in Sobolev space \( H^s \) with \( s > \frac{3}{2} \) (see Theorem 2 in Section 3). The ideas of proving these two theorems come from those presented in Hakkaev and Kirchev [24], Li and Olver [32].

2. Lower order regularity

Firstly, we give some notations.

The space of all infinitely differentiable functions \( \phi(x,t) \) with compact support in \( \mathbb{R} \times [0,+\infty) \) is denoted by \( C_0^\infty \). We let \( L^p = L^p(\mathbb{R}) \) (\( 1 \leq p < +\infty \)) be the space of all measurable functions \( h \) such that \( \|h\|_{L^p}^p = \int_{\mathbb{R}} |h(x,t)|^p \, dx < \infty \). We define \( L^\infty = L^\infty(\mathbb{R}) \) with the standard norm \( \|h\|_{L^\infty} = \inf_{m(\varepsilon) = 0} \sup_{x \in \mathbb{R} \setminus [\varepsilon]} |h(x,t)|. \) For any real number \( s \), we let \( H^s = H^s(\mathbb{R}) \) denote the Sobolev space with the norm defined by

\[
\|h\|_{H^s} = \left( \int_{\mathbb{R}} (1 + |\xi|^2)^s \left| \hat{h}(\xi,t) \right|^2 \, d\xi \right)^{\frac{1}{2}} < \infty.
\]

where \( \hat{h}(\xi,t) = \int_{\mathbb{R}} e^{-i\xi x} h(x,t) \, dx \). Let \( C([0,T]; H^s(\mathbb{R})) \) denote the class of continuous functions from \( [0,T] \) to \( H^s(\mathbb{R}) \) and \( \Lambda = (1-\partial_x^2)^{\frac{1}{2}} \). Here we note that the norms \( \|h\|_{L^p}, \|h\|_{L^\infty} \) and \( \|h\|_{H^s} \) depend on time \( t \in [0,\infty) \). For simplicity, throughout this article, we let \( c \) denote any positive constant which is independent of parameter \( \varepsilon \).

For the equivalent form of Eq. (4), we consider its Cauchy problem

\[
\begin{cases}
u_t - u_{xxt} = \partial_x \left( -2ku - \frac{a}{m+1}u^{m+1} \right) + \frac{1}{n+1} \partial_x^3(u^{n+1}) - \frac{n}{2} \partial_x(u^{n-1}u_x^2) + \beta \partial_x[(u_x)^{2N-1}], \\ u(x,0) = u_0(x).\end{cases}
\]

Now, we have the theorem.

**Theorem 1.** Suppose that \( u_0(x) \in H^s \) with \( 1 < s \leq \frac{3}{2} \) and \( \|u_{0x}\|_{L^\infty} < \infty \). Then there exists a \( T > 0 \) such that Eq. (4) subject to initial value \( u_0(x) \) has a weak solution \( u(x,t) \in L^2([0,T], H^s) \) in the sense of distribution and \( u_x \in L^\infty([0,T]\times\mathbb{R}). \)
In order to prove Theorem 1, we consider the regularized problem in the form

\[
\begin{aligned}
&\left\{ \begin{array}{l}
    u_t - u_{xxt} + \varepsilon u_{xxxx} = \partial_x \left( -2ku - \frac{a}{m+1} u^{m+1} \right) + \frac{1}{n+1} \partial_x^3 (u^{n+1}) \\
    \quad - \frac{n}{2} \partial_x (u^{n-1} u_x^2) + \beta \partial_x [(u_x)^{2N-1}], \\
    u(x, 0) = u_0(x),
\end{array} \right.
\end{aligned}
\]

(6)

where \(0 < \varepsilon < \frac{1}{4}, m \geq 1, n \geq 1\) and \(N \geq 1\) are natural numbers, \(a, k, \beta \geq 0\) are constants.

Before giving the proof of Theorem 1, we give several lemmas.

**Lemma 1.** Let \(u_0(x) \in H^s(R)\) with \(s > \frac{3}{2}\). Then the Cauchy problem (6) has a unique solution \(u(x, t) \in C([0, T] ; H^s(R))\) where \(T > 0\) depends on \(\|u_0\|_{H^s(R)}\). If \(s \geq 2\), the solution \(u \in C([0, +\infty); H^3)\) exists for all time. In particular, when \(s \geq 4\), the corresponding solution is a classical globally defined solution of problem (6).

**Proof.** Assuming \(D = (1 - \partial^2_x + \varepsilon \partial^2_x)^{-1}\), we know that \(D : H^s \to H^{s+4}\) is a bounded linear operator. Applying the operator \(D\) to both sides of the first equation of system (6) and then integrating the resultant equation with respect to \(t\) over the interval \((0, t)\) lead to

\[
u(x, t) = u_0(x) + \int_0^t D \left[ \partial_x \left( -2ku - \frac{a}{m+1} u^{m+1} \right) + \frac{1}{n+1} \partial_x^3 (u^{n+1}) \\
\quad - \frac{n}{2} \partial_x (u^{n-1} u_x^2) + \beta \partial_x [(u_x)^{2N-1}] \right] dt.
\]

(7)

Suppose that both \(u\) and \(v\) are in the closed ball \(B_{M_0}(0)\) of radius \(M_0\) about the zero function in \(C([0, T]; H^s(R))\) and \(A\) is the operator in the right-hand side of (7). For fixed \(t \in [0, T]\), we get

\[
\left\| \int_0^t \left[ \partial_x \left( -2ku - \frac{a}{m+1} u^{m+1} \right) + \frac{1}{n+1} \partial_x^3 (u^{n+1}) - \frac{n}{2} \partial_x (u^{n-1} u_x^2) + \beta \partial_x [(u_x)^{2N-1}] \right] dt \right\|_{H^s}
\]

\[
\leq TC_1 (2k \sup_{0 \leq t \leq T} \|u - v\|_{H^s}) + \sup_{0 \leq t \leq T} \left\| u^{m+1} - v^{m+1} \right\|_{H^s}
\]

\[
+ \sup_{0 \leq t \leq T} \left\| u^{n+1} - v^{n+1} \right\|_{H^s} + \sup_{0 \leq t \leq T} \left\| D \partial_x [\partial_x (u^n) \partial_x u - \partial_x (v^n) \partial_x v] \right\|_{H^s}
\]

\[
+ \sup_{0 \leq t \leq T} \| D[\partial_x (u_x^{2N-1}) - \partial_x (v_x^{2N-1})] \|_{H^s},
\]

(8)

where \(C_1\) may depend on \(\varepsilon\). The algebraic property of \(H^m(R)\) with \(s_0 > \frac{1}{2}\) derives the following
The global existence result follows from the integral from (7) and (15).
Lemma 2. (See Kato and Ponce [28].) If \( r > 0 \), then \( H^r \cap L^\infty \) is an algebra. Moreover

\[
\|uv\|_r \leq c(\|u\|_{L^\infty} \|v\|_r + \|u\|_r \|v\|_{L^\infty}),
\]

where \( c \) is a constant depending only on \( r \).

Lemma 3. (See Kato and Ponce [28].) Let \( r > 0 \). If \( u \in H^r \cap W^{1,\infty} \) and \( v \in H^{r-1} \cap L^\infty \), then

\[
\left\| [A^r, u]v \right\|_{L^2} \leq c(\|\partial_x u\|_{L^\infty} \|A^{r-1} v\|_{L^2} + \|A^r u\|_{L^2} \|v\|_{L^\infty}).
\]

Lemma 4. Let \( s \geq 4 \) and the function \( u(x, t) \) be a solution of problem (6) and the initial data \( u_0(x) \in H^s \). Then the following inequality holds

\[
\|u\|^2_{H^1} \leq c \int_R \left( u^2 + u_x^2 + \varepsilon u_{xx}^2 + 2\beta \int_0^t u_x^{2N} \, d\tau \right) \, dx
\]

\[
= c \int_R (u_0^2 + u_0^{2x} + \varepsilon u_0^{2xx}) \, dx.
\]

(16)

For \( q \in (0, s - 1] \), there is a constant \( c \) independent of \( \varepsilon \) such that

\[
\int_R (A^{q+1} u)^2 \, dx \leq \int_R \left[ \left( (A^{q+1} u_0)^2 + \varepsilon (A^q u_{0xx})^2 \right) \right] \, dx
\]

\[
+ c \int_0^t \|u_x\|_{L^\infty} \left( \|u\|^2_{H^q} \left( \|u\|^2_{L^\infty} + \|u\|_{L^\infty}^{n-1} \right) + \|u\|^{2}_{H^{q+1}} \|u\|_{L^\infty}^{n-1}) \, d\tau
\]

\[
+ c \int_0^t \|u\|^2_{H^{q+1}} \|u_x\|_{L^\infty}^{2N-2} \, d\tau.
\]

(17)

For \( q \in [0, s - 1] \), there is a constant \( c \) independent of \( \varepsilon \) such that

\[
(1 - 2\varepsilon)\|u_t\|_{H^0} \leq c \|u\|_{H^{q+1}} \left( 1 + \left( \|u\|^2_{L^\infty} + \|u\|_{L^\infty}^{n-1} \right) \right) \|u\|_{H^1} + \|u_x\|_{L^\infty}^{2N-2}.
\]

(18)

Proof. Using \( \|u\|^2_{H^1} \leq c \int_R (u^2 + u_x^2) \, dx \) and (15) derives (16).

Using \( \partial_x^2 = -\Lambda^2 + 1 \) and the Parseval equality gives rise to

\[
\int_R A^q u A^3(2) \, dx = -2 \int_R (A^{q+1} u) A^{q+1}(u_u_x) \, dx + 2 \int_R (A^q u) A^q(u_{uu_x}) \, dx.
\]

For \( q \in (0, s - 1] \), applying \( (A^q u) A^q \) to both sides of the first equation of system (6) and integrating with respect to \( x \) by parts, we have the identity

\[
\frac{1}{2} \frac{d}{dt} \int_R \left( (A^q u)^2 + (A^q u_x)^2 + \varepsilon (A^q u_{xx})^2 \right) \, dx
\]
\[
\frac{1}{2} \frac{d}{dt} \int_R \left( (A^q u)^2 + (A^q u_x)^2 + \varepsilon (A^q u_x^2)^2 \right) dx \\
\leq c \|u\|_{H^q}^2 (\|u\|_{L^\infty}^{m-1} + \|u\|_{L^\infty}^n) + \|u\|_{H^{q+1}}^2 \|u_x\|_{L^\infty}^{2n-2}.
\]
To estimate the norm of \( u_t \), we apply the operator \((1 - \partial^2_x)^{-1}\) to both sides of the first equation of system \((6)\) to obtain the equation

\[
(1 - \varepsilon) u_t - \varepsilon u_{xx}t = (1 - \partial^2_x)^{-1} \left[ -\varepsilon u_t - 2ku_x \\
+ \partial_x \left( -\frac{a}{m+1} u^{m+1} + \frac{1}{n+1} \partial^2_x (u^{n+1}) - \frac{n}{2} u^{n-1} u_x^2 \right) + \beta \partial_x [(u_x)^{2N-1}] \right].
\]

(25)

Applying \((\Lambda^q u_t) \Lambda^q\) to both sides of Eq. \((25)\) for \(q \in (0, s - 1)\) gives rise to

\[
(1 - \varepsilon) \int_R (\Lambda^q u_t)^2 \, dx + \varepsilon \int_R (\Lambda^q u_{xx})^2 \, dx \\
= \int_R (\Lambda^q u_t) \Lambda^{q-2} \left[ -\varepsilon u_t + \partial_x \left( -2ku_x - \frac{a}{m+1} u^{m+1} + \frac{1}{n+1} \partial^2_x (u^{n+1}) - \frac{n}{2} u^{n-1} u_x^2 \right) \\
+ \beta \partial_x [(u_x)^{2N-1}] \right] \, dt.
\]

(26)

For the right-hand of Eq. \((26)\), we have

\[
\int_R (\Lambda^q u_t) \Lambda^{q-2} (-\varepsilon u_t - 2ku_x) \, dx \leq \varepsilon \| u_t \|^2_{H^q} + 2k \| u_t \|_{H^q} \| u \|_{H^q}
\]

(27)

and

\[
\int_R (\Lambda^q u_t) (1 - \partial^2_x)^{-1} \Lambda^q \partial_x \left( -\frac{a}{m+1} u^{m+1} - \frac{n}{2} u^{n-1} u_x^2 \right) \, dx \\
\leq c \| u_t \|_{H^q} \left( \int_R (1 + \xi^2)^{-q} \left[ \int_R \left[ -\frac{a}{m+1} \tilde{u}^m (\xi - \eta) \tilde{u}(\eta) - \frac{n}{2} u^{n-1} u_x (\xi - \eta) \tilde{u}_x (\eta) \right] d\eta \right]^2 \right)^{1/2} \\
\leq c \| u_t \|_{H^q} \| u \|_{H^{1+q}} \left( \| u \|_{L^1}^{m-1} + \| u \|_{L^1}^{n-1} \right).
\]

(28)

Since

\[
\int_R (\Lambda^q u_t) (1 - \partial^2_x)^{-1} \Lambda^q \partial^2_x (u^n u_x) \, dx \\
= -\int_R (\Lambda^q u_t) \Lambda^q (u^n u_x) \, dx + \int_R (\Lambda^q u_t) (1 - \partial^2_x)^{-1} \Lambda^q (u^n u_x) \, dx,
\]

(29)

using Lemma 2, \( \| u^{n-1} u_x \|_{H^q} \leq c \| u^n \|_{H^q} \leq c n \| u \|_{L^1}^{n-1} \| u \|_{H^{q+1}} \) and \( \| u \|_{L^\infty} \leq \| u \|_{H^1} \), we have

\[
\int_R (\Lambda^q u_t) \Lambda^q (u^n u_x) \, dx \leq c \| u_t \|_{H^q} \| u^n u_x \|_{H^q} \\
\leq c \| u_t \|_{H^q} \| u \|_{L^1}^{n-1} \| u \|_{H^1} \| u \|_{H^{q+1}}
\]

(30)
and
\[
\int (A^q u_t) \left(1 - \partial_x^2\right)^{-1} A^q (u^m u_x) \, dx \leq c \|u_t\|_{H^q} \|u\|_{L^\infty}^{n-1} \|u\|_{H^1} \|u\|_{H^{q+1}}.
\] (31)

Using the Cauchy–Schwartz inequality and Lemma 2 yields
\[
\left| \int (A^q u_t) \left(1 - \partial_x^2\right)^{-1} A^q \partial_x(u^{2N-1}) \, dx \right| \leq c \|u_t\|_{H^q} \|u_x\|_{L^\infty}^{2N-2} \|u\|_{H^{q+1}}.
\] (32)

Applying (27)–(32) into (26) yields the inequality
\[
(1 - 2\varepsilon)\|u_t\|_{H^q} \leq c \|u\|_{H^{q+1}} (1 + (\|u\|_{L^\infty}^{m-1} + \|u\|_{L^\infty}^{n-1}) \|u\|_{H^1} + \|u_x\|_{L^\infty}^{2N-2})
\] (33)
for a constant $c > 0$. This completes the proof of Lemma 4. □

For a real number $s$ with $s > 0$, suppose that the function $u_0(x)$ is in $H^s(R)$, and let $u_{\varepsilon 0}$ be the convolution $u_{\varepsilon 0} = \phi_{\varepsilon} \ast u_0$ of the function $\phi_{\varepsilon}(x) = e^{-\frac{x^2}{2}}\phi(\varepsilon^{-\frac{1}{2}} x)$ and $u_0$ such that the Fourier transform $\hat{\phi}$ of $\phi$ satisfies $\hat{\phi} \in C_0^\infty$, $\hat{\phi}(\xi) \geq 0$, and $\hat{\phi}(\xi) = 1$ for any $\xi \in (-1, 1)$. Then we have $u_{\varepsilon 0}(x) \in C^\infty$. It follows from Lemma 1 that for each $\varepsilon$ satisfying $0 < \varepsilon < \frac{1}{4}$, the Cauchy problem
\[
\begin{aligned}
&u_t - u_{xxt} + \varepsilon u_{xxxxx} = \partial_x \left( -2ku - \frac{a}{m+1} u^{m+1} \right) + \frac{1}{n+1} \partial_x^2 (u^{n+1}) \\
&\quad - \frac{n}{2} \partial_x (u^{n-1} u_x^2) + \beta \partial_x \left( u_x^{2N-1} \right),
\end{aligned}
\] (34)

has a unique solution $u_{\varepsilon}(x, t) \in C^\infty([0, \infty); H^\infty)$.

For an arbitrary positive Sobolev exponent, we give the following lemma and its proof that is similar to that of Lemma 5 in [1] where the Sobolev exponent $s > \frac{1}{2}$ is required.

**Lemma 5.** Under the assumptions of problem (34), the following estimates hold for any $\varepsilon$ with $0 < \varepsilon < \frac{1}{4}$ and $s > 0$

\[
\|u_{\varepsilon 0}\|_{H^q} \leq c \quad \text{if} \quad q \leq s,
\] (35)
\[
\|u_{\varepsilon 0}\|_{H^q} \leq ce^{\frac{s-q}{4}}, \quad \text{if} \quad q > s,
\] (36)
\[
\|u_{\varepsilon 0} - u_0\|_{H^s} \leq ce^{\frac{s-q}{4}}, \quad \text{if} \quad q \leq s,
\] (37)
\[
\|u_{\varepsilon 0} - u_0\|_{H^1} = O(1),
\] (38)

where $c$ is a constant independent of $\varepsilon$.

**Proof.** Using the Fourier transform leads to
\[
\hat{\phi}_{\varepsilon}(\xi) = \int e^{-i\xi x} e^{-\frac{1}{4}\phi(\varepsilon^{-\frac{1}{2}} x)} \, dx = \int e^{i(\varepsilon^{-\frac{1}{2}} x)(\xi + \frac{1}{4}\varepsilon)} \phi(\varepsilon^{-\frac{1}{2}} x) \, d(\varepsilon^{-\frac{1}{2}} x)
\]
\[
= \hat{\phi}(\varepsilon^{\frac{1}{4}} \xi).
\]
Thus, we have

$$\hat{u}_{\varepsilon 0}(\xi) = \hat{\phi}_1 u_0 = \hat{\phi}_1(\xi)\hat{u}_0(\xi) = \hat{\phi}(\varepsilon^{\frac{1}{2}} \xi)\hat{u}_0(\xi)$$

and

$$\|u_{\varepsilon 0}\|_{H^q}^2 = \int_R \left(1 + |\xi|^2\right)^q |\hat{\phi}(\varepsilon^{\frac{1}{2}} \xi)\hat{u}_0(\xi)|^2 \, d\xi$$

$$\leq \int_R \left(1 + |\xi|^2\right)^q \left(1 + |\xi|^2\right) |\hat{\phi}(\varepsilon^{\frac{1}{2}} \xi)|^2 \left(1 + |\xi|^2\right) |\hat{u}_0(\xi)|^2 \, d\xi$$

$$\leq \|u_0\|_{H^q}^2 \sup_{\xi \in R} \left[\left(1 + |\xi|^2\right)^q |\hat{\phi}(\varepsilon^{\frac{1}{2}} \xi)|^2 \right].$$

If $q \leq s$, we get

$$\sup_{\xi \in R} \left[\left(1 + |\xi|^2\right)^q |\hat{\phi}(\varepsilon^{\frac{1}{2}} \xi)|^2 \right] \leq \sup_{\xi \in R} |\hat{\phi}(\varepsilon^{\frac{1}{2}} \xi)|^2 \leq c,$$

which derives that inequality (35) holds.

If $q > s$, it results

$$\sup_{\xi \in R} \left[\left(1 + |\xi|^2\right)^q |\hat{\phi}(\varepsilon^{\frac{1}{2}} \xi)|^2 \right] \leq \sup_{\xi \in R} \left[\left(1 + |\xi|^2\right)^q |\hat{\phi}(K)|^2 \right]$$

$$\leq \varepsilon^{-\frac{q}{2s}} \sup_{K \in R} \left[\left(1 + |\xi|^2\right)^{q-s} |\hat{\phi}(K)|^2 \right]$$

$$\leq ce^{-\frac{q}{2s}},$$

from which we know that (36) holds.

For $q \leq s$, we have

$$\|u_{\varepsilon 0} - u_0\|_{H^q}^2 = \int_R \left(1 + |\xi|^2\right)^q |\hat{\phi}(\varepsilon^{\frac{1}{2}} \xi)\hat{u}_0(\xi) - \hat{u}_0(\xi)|^2 \, d\xi$$

$$\leq \int_R \left(1 + |\xi|^2\right)^q \left(1 + |\xi|^2\right)^s |\hat{u}_0(\xi)|^2 \left|\hat{\phi}(\varepsilon^{\frac{1}{2}} \xi) - 1\right|^2 \, d\xi$$

$$\leq \|u_0\|_{H^q}^2 \sup_{R \in R} \left[\left(1 + |\xi|^2\right)^q \left|\hat{\phi}(\varepsilon^{\frac{1}{2}} \xi) - 1\right|^2 \right]$$

$$\leq \|u_0\|_{H^q}^2 \varepsilon^{-\frac{s}{2s}} \sup_{\xi \in R} \left[\left(1 + |\xi|^2\right)^{q-s} |\hat{\phi}(K) - 1|^2 \right]$$

$$\leq ce^{-\frac{q}{2}},$$

which results in inequality (37). The expression (38) is a common result since $u_{\varepsilon 0}$ uniformly converges to $u_0$ in the space $H^s(R)$ with $s > 0$. \qed
Remark. For $s \geq 1$, using $\|u_\varepsilon\|_{L^\infty} \leq c\|u_\varepsilon\|_{H_+^{\frac{1}{2}}}$, $\|u_\varepsilon\|_{H^1_+}$, $\|u_\varepsilon\|^2_{H^1}$, we know

$$\|u_\varepsilon\|^2_{L^\infty} \leq c\|u_\varepsilon\|_{H^1} \leq c \int_R (u_\varepsilon^2 + u_{\varepsilon 0x}^2 + \varepsilon u_{\varepsilon 0xx}^2) \, dx$$

$$\leq c(\|u_{\varepsilon 0}\|^2_{H^1} + \varepsilon\|u_{\varepsilon 0x}\|^2_{H^2})$$

$$\leq c(\varepsilon + c\varepsilon \times \varepsilon^{\frac{1}{s-2}})$$

$$\leq c_0,$$  \hspace{1cm} (39)

where $c_0$ is independent of $\varepsilon$.

**Lemma 6.** If $u_0(x) \in H^s(R)$ with $s \in [1, \frac{3}{2}]$ such that $\|u_{0x}\|_{L^\infty} < \infty$. Let $u_{\varepsilon 0}$ be defined as in system (34). Then there exist two constants $T > 0$ and $c$ independent of $\varepsilon$ such that the solution $u_\varepsilon$ of problem (34) satisfies $\|u_{\varepsilon x}\|_{L^\infty} \leq c$ for any $t \in [0, T]$.

**Proof.** Using notation $u = u_\varepsilon$ and differentiating both sides of the first equation of problem (34) or Eq. (25) with respect to $x$ give rise to

$$(1 - \varepsilon)u_{xx} - \varepsilon u_{xxx} + \frac{1}{n+1}\partial_x^2 u^{n+1} - \frac{n}{2}u^{n-1}u_X^2$$

$$= 2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{n+1}u^{n+1} - \beta u_X^{2N-1}$$

$$- \Lambda^{-2}\left[\varepsilon u_{xx} + 2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{n+1}u^{n+1} + \frac{n}{2}u^{n-1}u_X^2 - \beta u_X^{2N-1}\right].$$  \hspace{1cm} (40)

Letting $p > 0$ be an integer and multiplying the above equation by $(u_X)^{2p+1}$ and then integrating the resulting equation with respect to $x$ yield the equality

$$\frac{1 - \varepsilon}{2p + 2} \frac{d}{dt} \int_R (u_X)^{2p+2} \, dx - \varepsilon \int_R (u_X)^{2p+1}u_{xxx} \, dx + \frac{pn}{2p + 2} \int_R (u_X)^{2p+3}u^{n-1} \, dx$$

$$= \int_R (u_X)^{2p+1}\left(2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{n+1}u^{n+1} - \beta u_X^{2N-1}\right) \, dx$$

$$- \int_R (u_X)^{2p+1}\Lambda^{-2}\left[\varepsilon u_{xx} + 2ku + \frac{a}{m+1}u^{m+1} - \frac{u^{n+1}}{n+1} + \frac{n}{2}u^{n-1}u_X^2 - \beta u_X^{2N-1}\right] \, dx.$$  \hspace{1cm} (41)

Applying the Hölder’s inequality yields

$$\frac{1 - \varepsilon}{2p + 2} \frac{d}{dt} \int_R (u_X)^{2p+2} \, dx$$

$$\leq \left\{\varepsilon \left(\int_R |u_{xxx}|^{2p+2} \, dx\right)^{\frac{1}{2p+2}} + |2k| \left(\int_R |u|^{2p+2} \, dx\right)^{\frac{1}{2p+2}}\right\}.$$
Using the algebraic property of $H^s(R)$ with $s_0 > \frac{1}{2}$ and the inequality (39) yields

$$\|u^{m+1}\|_{L^\infty} \leq c\|u^{m+1}\|_{H^{s+1}} \leq c\|u^{m+1}\|_{H^s} \leq c\|u\|^{m+1}_{H^1} \leq c$$

and
\[
\|G\|_{L^\infty} \leq c \|G\|_{H^{\frac{1}{2}+}}
\]
\[
= c \left\| \Lambda^{-2} \left[ \varepsilon u_{xx} + 2ku + \frac{a}{m+1}u^{m+1} - \frac{u^{n+1}}{n+1} + \frac{n}{2}u^n u_x^2 - \beta u_x^{2N-1} \right] \right\|_{H^{\frac{1}{2}+}}
\]
\[
\leq c \left( \|\Lambda^{-2}u_{xx}\|_{H^{\frac{1}{2}+}} + \|\Lambda^{-2}(u^{n+1}u_x^2)\|_{H^{\frac{1}{2}+}} + \|\Lambda^{-2}(u_x^{2N-1})\|_{H^{\frac{1}{2}+}} \right) + c
\]
\[
\leq c \left( \|u_t\|_{L^2} + \|u_{xx}\|_{L^2} \right) + c
\]
\[
\leq c \left( \|u_t\|_{L^2} + \|u_x\|_{L^\infty} \right) + c
\]
\[
\leq c \left( \|u_t\|_{L^2} + \|u_x\|_{L^2}^2 + \|u_x\|_{L^\infty}^{2N-2} \right) + c,
\]
(46)

where \(c\) is a constant independent of \(\varepsilon\). Using (18), (45) and (46), we have
\[
\int_0^t \|G\|_{L^\infty} \, d\tau \leq c + c \int_0^t \left( 1 + \|u_x\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^{2N-2} \right) \, d\tau,
\]
(47)

where \(c\) is a constant independent of \(\varepsilon\). Moreover, for any fixed \(r \in (\frac{1}{2}, 1)\), there exists a constant \(c_r\) such that \(\|u_{xxx}\|_{L^\infty} \leq c_r \|u_{xxx}\|_{H^r} \leq c_r \|u_t\|_{H^{r+3}}\). Using (18) and (39) yields
\[
\|u_{xxx}\|_{L^\infty} \leq c \|u\|_{H^{r+4}} (1 + \|u_x\|_{L^\infty}^{2N-2}).
\]
(48)

Making use of the Gronwall’s inequality to (17) with \(q = r + 3\), \(u = u_t\) and (39) gives rise to
\[
\|u\|_{H^{r+4}}^2 \leq \left( \int_R (A^{r+4}u_0)^2 + \varepsilon (A^{r+3}u_{0xx})^2 \right) \exp \left[ c \int_0^t (\|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^{2N-2}) \, d\tau \right].
\]
(49)

From (35), (36), (48) and (49), one has
\[
\|u_{xxx}\|_{L^\infty} \leq c \varepsilon^{\frac{r+4}{4}} (1 + \|u_x\|_{L^\infty}^{2N-2}) \exp \left[ c \int_0^t (\|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^{2N-2}) \, d\tau \right].
\]
(50)

For \(\varepsilon < \frac{1}{4}\), it follows from (44), (47) and (50) that
\[
\|u_x\|_{L^\infty} \leq \|u_{0x}\|_{L^\infty} + c \int_0^t \varepsilon^{\frac{r}{2}} (1 + \|u_x\|_{L^\infty}^{2N-2}) \exp \left( c \int_0^\tau (\|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^{2N-2}) \, d\xi \right)
\]
\[
+ 1 + \|u_x\|_{L^2}^2 + \|u_x\|_{L^\infty}^{2N-2} + \|u_x\|_{L^\infty}^{2N-1} \, d\tau.
\]
(51)

It follows from the contraction mapping principle that there is a \(T > 0\) such that the equation
\[
\|W\|_{L^\infty} = \|u_{0x}\|_{L^\infty} + c \int_0^t \varepsilon^{\frac{r}{2}} (1 + \|W\|_{L^\infty}^{2N-2}) \exp \left( c \int_0^\tau (\|W\|_{L^\infty} + \|W\|_{L^\infty}^{2N-2}) \, d\xi \right)
\]
has a unique solution \( W \in C[0, T] \). Using the theorem presented at page 51 in [32] or Theorem II in Section I.1 presented in [41] yields that there are constants \( T > 0 \) and \( c > 0 \) independent of \( \varepsilon \) such that \( \| u_x \|_{L^\infty} \leq W(t) \) for arbitrary \( t \in [0, T] \), which leads to the conclusion of Lemma 6. \( \square \)

Using Lemma 4, Lemma 6, (17), (18), notation \( u_\varepsilon = u \) and Gronwall’s inequality results in the inequalities

\[
\| u_\varepsilon \|_{H^q} \leq \| u_\varepsilon \|_{H_{q+1}} \leq c \exp \left[ c \int_0^t (1 + \| u_x \|_{L^\infty} + \| u_x \|_{L_{q+2}^{N-2}}^2) \right] dt \leq c
\]

and

\[
\| u_{\varepsilon t} \|_{H^r} \leq c \left( 1 + \| u_x \|_{L^\infty}^{2N-2} \right) \| u_\varepsilon \|_{H_{q+1}} \leq c,
\]

where \( q \in (0, s) \), \( r \in (0, s-1) \) and any \( t \in [0, T] \). It follows from Aubin’s compactness theorem that there is a subsequence of \( \{ u_\varepsilon \} \), denoted by \( \{ u_{\varepsilon_n} \} \), such that \( \{ u_{\varepsilon_n} \} \) and their temporal derivatives \( \{ u_{\varepsilon_n t} \} \) are weakly convergent to a function \( u(x, t) \) and its derivative \( u_t \) in \( L^2([0, T], H^2) \) and \( L^2([0, T], H^{s-1}) \), respectively. Moreover, for any real number \( R_1 > 0 \), \( \{ u_{\varepsilon_n} \} \) is convergent to the function \( u \) strongly in the space \( L^2([0, T], H^q(-R_1, R_1)) \) for \( q \in (0, s) \) and \( \{ u_{\varepsilon_n t} \} \) converges to \( u_t \) strongly in the space \( L^2([0, T], H^r(-R_1, R_1)) \) for \( r \in (0, s-1) \). Thus, we can prove the existence of a weak solution to Eq. (4).

**Proof of Theorem 1.** From Lemma 6, we know that \( \{ u_{\varepsilon_n x} \} \) \( (\varepsilon_n \to 0) \) is bounded in the space \( L^\infty \). Thus, the sequences \( \{ u_{\varepsilon_n} \} \) and \( \{ u_{\varepsilon_n x} \} \) are weakly convergent to \( u \) and \( u_x \) in \( L^2([0, T], H^r(-R, R)) \) for any \( r \in [0, s-1] \), respectively. Therefore, \( u \) satisfies the equation

\[
- \int_0^T \int_R \int_R \left( 2ku + \frac{a}{m+1} u^{m+1} + \frac{n}{2} (u^{n-1} u_x^2) \right) g_x \\
- \frac{1}{n+1} u^{n+1} g_{xxx} - \beta (u_x)^{2N-1} g_x \right) dx dt,
\]

with \( u(x, 0) = u_0(x) \) and \( g \in C_0^\infty \). Since \( X = L^1([0, T] \times R) \) is a separable Banach space and \( \{ u_{\varepsilon_n x} \} \) is a bounded sequence in the dual space \( X^* = L^\infty([0, T] \times R) \) of \( X \), there exists a subsequence of \( \{ u_{\varepsilon_n x} \} \), still denoted by \( \{ u_{\varepsilon_n x} \} \), weakly star convergent to a function \( v \) in \( L^\infty([0, T] \times R) \). It derives from the \( \{ u_{\varepsilon_n x} \} \) weakly convergent to \( u_x \) in \( L^2([0, T] \times R) \) that \( u_x = v \) almost everywhere. Thus, we obtain \( u_x \in L^\infty([0, T] \times R) \). \( \square \)

**3. Local well-posedness**

Now, we give the theorem to describe the existence and uniqueness of solutions for problem (5).

**Theorem 2.** Suppose that the initial function \( u_0(x) \) belongs to the Sobolev space \( H^s(R) \) with \( s > \frac{3}{2} \). Then there is a \( T > 0 \), which depends on \( \| u_0 \|_{H^r} \), such that there exists a unique solution \( u(x, t) \) of the problem (5) and \( u(x, t) \in C([0, T]; H^s(R)) \cap C^1([0, T]; H^{s-1}(R)) \).

We cite the following lemma, which is proved by Li and Olver [32].
Lemma 7. If $u$ and $f$ are functions in $H^{q+1} \cap \|u\|_{L^\infty} < \infty$, then

$$\left| \int_{\mathbb{R}} A^q u A^q (uf)_x \, dx \right| \leq \begin{cases} c_q \|f\|_{H^{q+1}} \|u\|_{H^q}^2, & q \in (\frac{1}{2}, 1], \\ c_q (\|f\|_{H^{q+1}} \|u\|_{H^q} \|u\|_{L^\infty} + \|f\|_{H^q} \|u\|_{H^q} \|u_x\|_{L^\infty}), & q \in (0, \infty). \end{cases}$$

(54)

Lemma 8. For $u, v \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, $w = u - v$, $q > \frac{1}{2}$, and a natural number $n$, it holds that

$$\left| \int_{\mathbb{R}} A^s w A^s (u^{n+1} - v^{n+1}) \, dx \right| \leq c (\|w\|_{H^s} \|w\|_{H^q} \|v\|_{H^{q+1}} + \|w\|_{H^s}^2).$$

Proof. We have

$$u^{n+1} - v^{n+1} = (w + v)^{n+1} - v^{n+1} = w^{n+1} + \sum_{j=1}^{n} C^j_{n+1} w^{n+1-j} v^j.$$

Using Lemmas 2 and 3, the algebraic property of $H^s$ with $s_1 > \frac{1}{2}$ and the inequalities

$$\|u_x\|_{L^\infty} \leq \|u_x\|_{H^{s_1}} \leq c \|u\|_{H^s} \leq c, \quad \|u\|_{L^\infty} \leq c,$$

$$\|v_x\|_{L^\infty} \leq c \|v\|_{H^s} \leq c, \quad \|u\|_{L^\infty} \leq c \|u\|_{H^s},$$

$$\|u_x v_x\|_{L^\infty} \leq c \|u\|_{H^s} \|v\|_{H^s} \leq c,$$

we have

$$\int_{\mathbb{R}} A^s w A^s [(w^{n-j} v^j) w_x] \, dx$$

$$= \int_{\mathbb{R}} A^s w (A^s [(w^{n-j} v^j) w_x] - (w^{n-j} v^j) A^s w_x) \, dx + \int_{\mathbb{R}} (A^s w) w^{n-j} v^j A^s w_x \, dx$$

$$\leq c (\|w\|_{H^s} (\|\partial_x (v^j w^{n-j})\|_{L^\infty} \|A^{s-1} w_x\|_{L^2})$$

$$+ \|A^s (v^j w^{n-j})\|_{L^2 \|w_x\|_{L^\infty}} + \frac{1}{2} \|w\|_{H^s}^2 \|v^j w^{n-j}\|_{L^\infty})$$

$$\leq c \|w\|_{H^s}^2$$

and

$$\int_{\mathbb{R}} A^s w A^s (w^{n+1-j} v^{j-1} v_x) \, dx$$

$$\leq c \|w\|_{H^s} (\|w^{n+1-j} v^{j-1}\|_{L^\infty} \|v_x\|_{H^s} + \|w^{n+1-j} v^{j-1}\|_{H^s} \|v_x\|_{L^\infty})$$

$$\leq c \|w\|_{H^s} (\|w^{n+1-j} v^{j-1}\|_{H^s} \|v\|_{H^{q+1}} + \|w\|_{H^s})$$

$$\leq c (\|w\|_{H^s} \|w\|_{H^q} \|v\|_{H^{q+1}} + \|w\|_{H^s}^2).$$
Applying the above inequalities, we have

\[
\left| \int_{\mathbb{R}} A^s w A^s (u^{n+1} - v^{n+1})_x \, dx \right| \\
= \left| \int_{\mathbb{R}} A^s w A^s \left( w^{n+1} + \sum_{j=1}^{n} C_{n+1}^j w^{n+1-j} v^j \right)_x \, dx \right| \\
= \left| \int_{\mathbb{R}} A^s w A^s \left[ (w^{n+1})_x + \sum_{j=1}^{n} C_{n+1}^j (n+1-j) w^{n-j} w_x v^j + j w^{n+1-j} v^j v_x \right] \, dx \right| \\
\leq c \left( \|w\|_{H^s}^2 \|w\|_{L^\infty}^{n-1} \|w_x\|_{L^\infty} + \|w\|_{H^s} \|w\|_{H^q} \|v\|_{H^{s+1}} + \|w\|_{H^s}^2 \right) \\
\leq c \left( \|w\|_{H^s} \|w\|_{H^q} \|v\|_{H^{s+1}} + \|w\|_{H^s}^2 \right)
\]

which completes the proof. \( \Box \)

**Lemma 9.** For problem (34), \( s > \frac{3}{2} \) and \( u_0 \in H^s(\mathbb{R}) \), there exist two positive constants \( c \) and \( M \), which are independent of \( \varepsilon \), such that the inequalities

\[
\|u_{\varepsilon t}\|_{H^s} \leq Me^{ct}, \\
\|u_{\varepsilon t}\|_{H^{s+k_1}} \leq \varepsilon^{-k_1} Me^{ct}, \quad k_1 > 0, \\
\|u_{\varepsilon t}\|_{H^{s+k_1}} \leq \varepsilon^{-\frac{(k_1+1)}{4}} Me^{ct}, \quad k_1 > -1,
\]

hold for any sufficiently small \( \varepsilon \) and \( t \in [0, T) \).

**Proof.** If \( s > \frac{3}{2} \), \( u_0 \in H^3 \), we easily obtain

\[
u_0 \in H^{s_1} \quad \text{with} \quad 1 \leq s_1 \leq \frac{3}{2}, \\
\|u_{0x}\|_{L^\infty} \leq c \|u_{0x}\|_{H^{s_1}} \leq c \|u_0\|_{H^s} \leq c.
\]

From Lemma 6, we know that there exist two constants \( T \) and \( c \) independent of \( \varepsilon \) such that

\[
\|u_{\varepsilon x}\|_{L^\infty} \leq c \quad \text{for any} \quad t \in [0, T).
\]

Applying the inequality (17) with \( q + 1 = s \) and the bounded property of solution \( u \) (see (39) and (58)), we have

\[
\int_{\mathbb{R}} \left( \Lambda^s u_\varepsilon \right)^2 \, dx \leq \int_{\mathbb{R}} \left[ \left( \Lambda^s u_{\varepsilon 0} \right)^2 + \varepsilon \left( \Lambda^{s-1} u_{\varepsilon 0xx} \right)^2 \right] \, dx + c \int_{0}^{t} \left( \|u_{\varepsilon t}\|_{H^{s-1}}^2 + \|u_{\varepsilon t}\|_{H^s}^2 \right) \, d\tau \\
\leq \int_{\mathbb{R}} \left[ \left( \Lambda^s u_{\varepsilon 0} \right)^2 + \varepsilon \left( \Lambda^{s-1} u_{\varepsilon 0xx} \right)^2 \right] \, dx + c \int_{0}^{t} \|u_{\varepsilon t}\|_{H^s}^2 \, d\tau
\]

\[ A = \int_{\mathbb{R}} \left[ (A^s u_0)^2 + \varepsilon (A^{s-1} u_{0xx})^2 \right] dx \leq \|u_0\|_{H^s}^2 + \|u_{0x}\|_{H^{s+1}}^2 \]
\[ \leq c + c\varepsilon^{\frac{-1}{2}} \leq 2c, \quad (60) \]
in which we have used (35) and (36).

From (59) and (60) and using the Gronwall’s inequality, we get
\[ \|u_\varepsilon\|_{H^s} \leq 2ce^{ct}, \]
from which we know that (55) holds.

In a similar manner, for \( q + 1 = s + k_1 \) and \( k_1 > 0 \), applying (35), (36), (39) and (58) to (17), we have
\[ \|u_\varepsilon\|_{H^{s+k_1}}^2 \leq \left( ce^{-\frac{k_1}{2}} + ce^{-\frac{k_1+1}{2}} \varepsilon \right) + c \int_0^t \|u_\varepsilon\|_{H^{s+k_1}}^2 d\tau, \quad (61) \]
which results in (56) by using Gronwall’s inequality.

From (18), for \( q = s + k_1 \), we have
\[ (1 - 2\varepsilon)\|u_\varepsilon\|_{H^{s+k_1}} \leq c\|u_\varepsilon\|_{H^{s+k_1+1}}, \quad (62) \]
which leads to (57) by using (56) and (58). \( \Box \)

**Lemma 10.** If \( \frac{1}{2} < q < \min\{1, s - 1\} \) and \( s > \frac{3}{2} \), then for any functions \( w, f \) defined on \( \mathbb{R} \), it holds that
\[ \left| \int_{\mathbb{R}} A^q w A^{q-2}(wf)_x dx \right| \leq c \|w\|_{H^q}^2 \|f\|_{H^s}, \quad (63) \]
\[ \left| \int_{\mathbb{R}} A^q w A^{q-2}(wfx)_x dx \right| \leq c \|w\|_{H^q}^2 \|f\|_{H^s}. \quad (64) \]

**Proof.** Using the algebraic property of \( H^{s_0}(\mathbb{R}) \) with \( s_0 > \frac{1}{2} \) and the Schwarz inequality, we derive
\[ \left| \int_{\mathbb{R}} A^q w A^{q-2}(wf)_x dx \right| \leq \|A^q w\|_{L^2} \|A^{q-2}(wf)_x\|_{L^2} = \|w\|_{H^q} \|A^{q-2}(wf)_x\|_{L^2} \leq c \|w\|_{H^q} \|A^q(wf)\|_{L^2} \leq c \|w\|_{H^q}^2 \|f\|_{H^s}. \]
which results in (63). The proof of (64) can be found at page 43 in [32]. Here we use other method to prove it. In fact, for any \( f_1 \in \mathcal{L}^\infty, f_2 \in H^2 \) and \( h \in H^{−2} \) with \( z \leq 0 \), we have
\[
\left| \int_R f_1 f_2 h \, dx \right| \leq \|f_1\|\|f_2\|_H^2 \leq \|f_1\|\|f_2\|_H^2, \]
from which we obtain
\[
\|f_1 f_2\|_H^2 \leq c \|f_1\|_\mathcal{L}^\infty \|f_2\|_H^2 \quad \text{for any } z \leq 0. \tag{65}
\]
Using (65), \( q - 1 \leq 0 \) and \( s > \frac{3}{2} \), we get
\[
\left| \int_R A^q w A^{q-2} (w_x f_x) \, dx \right| \leq \|A^q w\|_L^2 \|A^{q-2} (w_x f_x)\|_L^2
\leq c \|w\|_{H^q} \|A^{q-1} (w_x f_x)\|_L^2
\leq c \|w\|_{H^q} \|f_x\|_\mathcal{L}^\infty \|w_x\|_{H^{q-1}}
\leq c \|w\|_{H^q} \|f\|_{H^s},
\]
from which we obtain (64). \( \square \)

Our next step is to demonstrate that \( u_\varepsilon \) is a Cauchy sequence. Let \( u_\varepsilon \) and \( u_\delta \) be solutions of problem (34), corresponding to the parameters \( \varepsilon \) and \( \delta \), respectively, with \( 0 < \varepsilon < \delta < \frac{1}{4} \), and let \( w = u_\varepsilon - u_\delta \). Then \( w \) satisfies the problem
\[
(1 - \varepsilon) w_t - \varepsilon w_{xxt} + (\delta - \varepsilon)(u_{\delta t} + u_{\delta xxt})
= (1 - \frac{\delta}{\varepsilon})^{-1} \left[ -\varepsilon w_t + (\delta - \varepsilon) u_{\delta t} - 2k w_x - \frac{a}{m+1} \partial_x (u_{\varepsilon}^{m+1} - u_{\delta}^{m+1}) \right]
+ \frac{1}{n+1} \partial_x (u_{\varepsilon}^{n+1} - u_{\delta}^{n+1}) \right] - \frac{1}{2} \partial_x \partial_x (u_{\varepsilon} w + u_{\varepsilon} - u_{\delta} \partial_x u_{\delta})
+ \beta \partial_x \left[ (u_{\varepsilon} x)^{2N-1} - (u_{\varepsilon} x)^{2N-1} \right] - \frac{1}{n+1} \partial_x (u_{\varepsilon}^{n+1} - u_{\delta}^{n+1}), \tag{66}
\]
\[
w(x, 0) = w_0(x) = u_{\varepsilon 0}(x) - u_{\delta 0}(x). \tag{67}
\]

Lemma 11. For \( s > \frac{3}{2} \), \( u_0 \in H^s(R) \), there exists \( T > 0 \) such that solution \( u_\varepsilon \) of (34) is a Cauchy sequence in the space \( C([0, T]; H^s(R)) \cap C^1([0, T]; H^{s-1}(R)) \).

Proof. For \( q \) with \( \frac{1}{2} < q < \min(1, s - 1) \), multiplying both sides of Eq. (66) by \( A^q w A^q \) and then integrating with respect to \( x \) give rise to
\[
\frac{1}{2} \int_R \left[ (1 - \varepsilon)(A^q w)^2 + \varepsilon (A^q w_x)^2 \right] dx
= (\varepsilon - \delta) \int_R (A^q w) \left[ (A^q u_{\delta t} + (A^q u_{\delta xxt}) \right] dx - \varepsilon \int_R A^q w A^{q-2} w_t \, dx
\]
It follows from the Schwarz inequality that

\[ \frac{d}{dt} \int [ (1 - \varepsilon) (A^q w)^2 + \varepsilon (A^q w_x)^2 ] dx \leq c \left\| \Lambda^q w \right\|_{L_2} \left( (\delta - \varepsilon) \left( \left\| \Lambda^q u_{\delta t} \right\|_{L_2} + \left\| \Lambda^q u_{\delta xx} \right\|_{L_2} \right) + \varepsilon \left\| A^q - 2 w_t \right\|_{L_2} \right) + \int R A^q w A^q - 2 (u_{\delta}^{m+1} - u_{\delta}^{m+1}) x dx + \int R A^q w A^q (u_{\delta}^{n+1} - u_{\delta}^{n+1}) x dx + \int R A^q w A^q - 2 \left( \partial_x (u_{\delta}^{n}) \partial_x w \right)_x dx + \int R A^q w A^q - 2 \left( \partial_x (u_{\delta}^{n}) \partial_x u_{\delta} \right)_x dx + \int R A^q w A^q - 2 \left( w_x \sum_{j=0}^{2N-2} (u_{\delta x})^j (u_{\delta x})^{2N-2-j} \right)_x dx \right\} \right) \]  

(69)

Using the first inequality in Lemma 7, we have

\[ \int R A^q w A^q (u_{\delta}^{n+1} - u_{\delta}^{n+1}) x dx = \int R A^q w A^q (w g_n)_x dx \leq c \| w \|^2_{H^2} \| g_n \|_{H^{q+1}}, \]  

(70)

where \( g_n = \sum_{j=0}^{n} u_{\delta}^{n-j} u_{\delta}^{j} \). For the last three terms in (69), using Lemma 10, the algebraic property of \( H^s \) with \( s_0 > \frac{1}{2} \) and (39), we have

\[ \int R A^q w A^q - 2 \left( \partial_x (u_{\delta}^{n}) \partial_x w \right)_x dx \leq c \| w \|^2_{H^2} \| u_{\delta}^{n} \|_{H^2}, \]  

(71)
\[
\left| \int_R A^q w A^{q-2} \left( \partial_x (u_x^m - u_x^m) \partial_x u_\delta \right)_x \, dx \right| \leq c \| w \|_{H^q} \| u_\delta \|_{H^q} \| u_x^m - u_x^m \|_{H^q}
\leq c \| w \|^2_{H^q} \| u_\delta \|_{H^q},
\]  
(72)

\[
\left| \int_R A^q w A^{q-2} \left[ w_x \sum_{j=0}^{2N-2} (u_{xj}) (u_{xj})^{2N-2-j} \right]_x \, dx \right|
\leq c \| w \|_{H^q} \left\| A^{q-2} \left[ w_x \sum_{j=0}^{2N-2} (u_{xj}) (u_{xj})^{2N-2-j} \right] \right\|_{L^2}
\leq c \| w \|_{H^q} \left\| A^{q-1} \left[ w_x \sum_{j=0}^{2N-2} (u_{xj}) (u_{xj})^{2N-2-j} \right] \right\|_{L^2}
\leq c \| w \|^2_{H^q} \sum_{j=0}^{2N-2} \| u_x \|^2_{H^q} \| u_\delta \|^2_{H^q},
\]  
(73)

in which we have used (65) and \( s > \frac{3}{2} \). Using (63), we derive that the inequality

\[
\left| \int_R A^q w A^{q-2} (u^m_x - u^{m+1}_x) \, dx \right| = \left| \int_R A^q w A^{q-2} (w g_m)_x \, dx \right|
\leq c \| g_m \|_{H^q} \| w \|^2_{H^q}
\]  
(74)

holds for some constant \( c \), where \( g_m = \sum_{j=0}^m u_x^{m-j} u_x^j \). Using the algebraic property of \( H^q \) with \( q > \frac{1}{2} \), \( q + 1 < s \) and Lemma 9, we have \( \| g_m \|_{H^{q+1}} \leq c \) for \( t \in (0, \bar{T}) \). Then it follows from (55)–(57) and (69)–(74) that there is a constant \( c \) depending on \( \bar{T} \) such that the estimate

\[
\frac{d}{dt} \int_R \left[ (1 - \varepsilon) (A^q w)^2 + \varepsilon (A^q w_x)^2 \right] \, dx \leq c (\delta \gamma \| w \|_{H^q} + \| w \|^2_{H^q})
\]  
(75)

holds for any \( t \in [0, \bar{T}) \), where \( \gamma = 1 \) if \( s \geq 3 + q \) and \( \gamma = \frac{1+s-q}{4} \) if \( s < 3 + q \). Integrating (75) with respect to \( t \), one obtains the estimate

\[
\frac{1}{2} \| w \|^2_{H^q} = \frac{1}{2} \int_R (A^q w)^2 \, dx
\leq \int_R \left[ (1 - \varepsilon) (A^q w)^2 + \varepsilon (A^q w_x)^2 \right] \, dx
\leq \int_R \left[ (A^q w_0)^2 + \varepsilon (A^q w_{0x})^2 \right] \, dx + \int_0^t (\delta \gamma \| w \|_{H^q} + \| w \|^2_{H^q}) \, d\tau.
\]  
(76)
Applying the Gronwall inequality, (35) and (36) yields
\[ \|u\|_{H^q} \leq c \delta^{\frac{q-s}{4}} e^{ct} + \delta \gamma (e^{ct} - 1) \] (77)
for any \( t \in [0, \tilde{T}) \).

Multiplying both sides of (66) by \( \Lambda^s w \Lambda^s \) and integrating it with respect to \( x \), one obtains
\[ \frac{1}{2} \frac{d}{dt} \int_R \left[ (1 - \varepsilon)(\Lambda^s w)^2 + \varepsilon (\Lambda^s w_x)^2 \right] dx \]
\[ = (\varepsilon - \delta) \int_R (\Lambda^s w) \left[ (\Lambda^s u_{8t}) + (\Lambda^s u_{8xt}) \right] dx - \varepsilon \int_R \Lambda^s w \Lambda^{s-2} w_t dx \]
\[ + (\delta - \varepsilon) \int_R \Lambda^s w \Lambda^{s-2} u_{8t} dx - \frac{1}{n+1} \int_R (\Lambda^s w) \Lambda^s (u_{e}^{n+1} - u_{e}^{n+1})_x dx \]
\[ - \frac{a}{m+1} \int_R \Lambda^s w \Lambda^{s-2} (u_{e}^{m+1} - u_{e}^{m+1})_x dx + \frac{1}{n+1} \int_R (\Lambda^s w) \Lambda^{s-2} (u_{e}^{n+1} - u_{e}^{n+1})_x dx \]
\[ - \frac{1}{2} \int_R \Lambda^s w \Lambda^{s-2} \left[ \partial_x (u_{e}^n) \partial_x w \right]_x dx - \frac{1}{2} \int_R \Lambda^s w \Lambda^{s-2} \left[ \partial_x (u_{e}^n - u_{e}^m) \partial_x u_{8} \right]_x dx \]
\[ + \beta \int_R \Lambda^s w \Lambda^{s-2} \left[ (u_{ex})^{2N-1} - (u_{8x})^{2N-1} \right]_x dx. \] (78)

From Lemma 10, we have
\[ \left| \int_R \Lambda^s w \Lambda^{s-2} (u_{e}^{m+1} - u_{e}^{m+1})_x dx \right| \leq c_2 \|g_m\|_{H^s} \|w\|_{H^2}, \] (79)
where \( g_m = \sum_{j=0}^{m} u_{e}^j u_{8}^{m-j} \).

From Lemma 8, it holds that
\[ \left| \int_R \Lambda^s w \Lambda^{s} (u_{e}^{n+1} - u_{e}^{n+1})_x dx \right| \leq c (\|w\|_{H^s} \|w\|_{H^s} \|u_{8}\|_{H^{s+1}} + \|w\|_{H^2}^2). \] (80)

Using the Cauchy–Schwartz inequality, the algebraic property of \( H^{s_0} \) with \( s_0 > \frac{1}{2} \), for \( s > \frac{1}{2} \), we have
\[ \left| \int_R \Lambda^s w \Lambda^{s-2} \left[ \partial_x (u_{e}^n) \partial_x w \right]_x dx \right| = \left| \int_R \Lambda^q w \Lambda^{s-2} \left[ \partial_x (u_{e}^n) \partial_x w \right]_x dx \right| \]
\[ \leq c \left\| \Lambda^q w \right\|_{L^2} \left\| \Lambda^{s-2} \left( \partial_x (u_{e}^n) \partial_x w \right) \right\|_{L^2} \]
\[ \leq c \|w\|_{H^s} \left\| \partial_x (u_{e}^n) \partial_x w \right\|_{H^{s-1}} \]
\[ \leq c \left\| u_{e}^n \right\|_{H^s} \|w\|_{H^2}, \] (81)
\[
\left| \int_R A^5 w A^{s-2} \left[ \partial_x (u_\delta^m - u_\delta^n) \partial_x u_\delta \right] dx \right| \leq c \| w \|_{H^s} \| A^{s-2} \left[ \partial_x (u_\delta^m - u_\delta^n) \partial_x u_\delta \right] \|_{L^2}
\]

\[
\leq c \| u_\delta \|_{H^s} \| g_{n-1} \|_{H^s} \| w \|_{H^s}^2, \tag{82}
\]

and

\[
\left| \int_R A^5 w A^{s-2} \left[ (u_{\delta x})^{2N-1} - (u_{\delta x})^{2N-1} \right] dx \right|
\]

\[
\leq c \| w \|_{H^s} \left\| A^{s-2} \left[ w_x \sum_{j=0}^{2N-2} (u_{\delta x})^j (u_{\delta x})^{2N-2-j} \right] \right\|_{L^2}
\]

\[
\leq c \| w \|_{H^s} \left\| A^{s-1} \left[ w_x \sum_{j=0}^{2N-2} (u_{\delta x})^j (u_{\delta x})^{2N-2-j} \right] \right\|_{L^2}
\]

\[
\leq c \| w \|_{H^s} \| w_x \|_{H^{s-1}} \left\| \sum_{j=0}^{2N-2} (u_{\delta x})^j (u_{\delta x})^{2N-2-j} \right\|_{H^{s-1}}
\]

\[
\leq c \| w \|_{H^s}^2 \sum_{j=0}^{2N-2} \| u_{\delta x} \|_{H^{s-1}} \| u_{\delta x} \|_{H^{s-1}}^{2N-2-j}.
\tag{83}
\]

Using the bounded property of \( \| u_\delta \|_{H^s} \) and \( \| u_\delta \|_{H^s} \) (see Lemma 9), it follows from (78)–(83) and the inequalities (55)–(57) and (77) that there exists a constant \( c \) depending on \( m \), \( n \) and \( N \) such that

\[
\frac{d}{dt} \int_R \left[ (1 - \varepsilon)(A^5 w)^2 + \varepsilon (A^5 w_x)^2 \right] dx
\]

\[
\leq 2\delta \left( \| u_{\delta t} \|_{H^s} + \| u_{\delta xx} \|_{H^s} + \| A^{s-2} u_t \|_{L^2} + \| A^{s-2} u_{\delta t} \|_{L^2} \right) \| w \|_{H^s}
\]

\[
+ c \left( \| w \|_{H^s}^2 + \| w \|_{H^s} \| w \|_{H^s} \| u_{\delta t} \|_{H^{s+1}} \right)
\]

\[
\leq c \left( \delta^{\gamma_1} \| w \|_{H^s} + \| w \|_{H^s}^2 \right), \tag{84}
\]

where \( \gamma_1 = \min \left( \frac{1}{4}, \frac{s-n-1}{4} \right) > 0 \). Integrating (84) with respect to \( t \) leads to the estimate

\[
\frac{1}{2} \| w \|_{H^s}^2 \leq \int_R \left[ (1 - \varepsilon)(A^5 w)^2 + \varepsilon (A^5 w_x)^2 \right] dx
\]

\[
\leq \int_R \left[ (A^5 w_0)^2 + \varepsilon (A^5 w_{0x})^2 \right] dx + c(\delta^{\gamma_1} \| w \|_{H^s} + \| w \|_{H^s}^2). \tag{85}
\]

It follows from Gronwall inequality and (85) that
$$\|w\|_{H^s} \leq \left( 2 \int_R \left[ (\Lambda^s w_0)^2 + \epsilon (\Lambda^s w_{0x})^2 \right] dx \right)^{\frac{1}{2}} e^{\epsilon t} + \delta^\gamma (e^{\epsilon t} - 1)$$

$$\leq c_1 (\|w_0\|_{H^s} + \delta^\frac{3}{2} e^{\epsilon t} + \delta^\gamma (e^{\epsilon t} - 1)), \quad (86)$$

where $c_1$ is independent of $\epsilon$ and $\delta$.

Then (38) and the above inequality show that

$$\|w\|_{H^s} \rightarrow 0 \quad as \quad \epsilon \rightarrow 0, \; \delta \rightarrow 0.$$ 

Next, we consider the convergence of the sequence $\{u_{\epsilon t}\}$. Multiplying both sides of Eq. (66) by $\Lambda^{s-1} w_t \Lambda^{s-1}$ and integrating the resultant equation with respect to $x$, we obtain

$$(1 - \epsilon) \|w_t\|^2_{x \infty} + \frac{1}{n+1} \int_R (\Lambda^{s-1} w_t) \Lambda^{s-1} (u^{n+1}_t - u^{n+1}_\delta)_x dx$$

$$+ \int_R [-\epsilon (\Lambda^{s-1} w_t) (\Lambda^{s-1} w_{xxt}) + (\delta - \epsilon) (\Lambda^{s-1} w_t) \Lambda^{s-1} (u_{\delta t} + u_{\delta xxt})] dx$$

$$= \int_R (\Lambda^{s-1} w_t) \Lambda^{s-3} \left[ -\epsilon w_t + (\delta - \epsilon) u_{\delta t} - 2kwx - \frac{a}{m+1} \partial_x (u^{m+1}_\delta - u^{m+1}_\delta) \right.$$ 

$$+ \frac{1}{n+1} \partial_x (u^{n+1}_\delta - u^{n+1}_\delta) - \frac{1}{2} \partial_x \left[ \partial_x (u^n) \partial_x w + \partial_x (u^n - u^n) \partial_x u \right]$$

$$+ \beta \left[ (u_{\delta x})^{2N-1} - (u_{\delta x})^{2N-1} \right] \right] dx. \quad (88)$$

It follows from the inequalities (55)–(57) and the Schwartz inequality that there is a constant $c$ depending on $\tilde{T}$ and $m$ such that

$$(1 - \epsilon) \|w_t\|^2_{H^{s-1}} \leq c \left( \delta^\frac{1}{2} + \|w\|_{H^s} + \|w\|_{H^{s-1}} \right) \|w_t\|_{H^{s-1}} + \epsilon \|w_t\|^2_{H^{s-1}}. \quad (89)$$

Hence

$$\frac{1}{2} \|w_t\|^2_{H^{s-1}} \leq (1 - 2\epsilon) \|w_t\|^2_{H^{s-1}}$$

$$\leq c \left( \delta^\frac{1}{2} + \|w\|_{H^s} + \|w\|_{H^{s-1}} \right) \|w_t\|_{H^{s-1}},$$

which results in

$$\frac{1}{2} \|w_t\|^2_{H^{s-1}} \leq c \left( \delta^\frac{1}{2} + \|w\|_{H^s} + \|w\|_{H^{s-1}} \right). \quad (90)$$

It follows from (77) and (87) that $w_t \rightarrow 0$ as $\epsilon, \delta \rightarrow 0$ in the $H^{s-1}$ norm. This implies that $u_\epsilon$ is a Cauchy sequence in the spaces $C([0, T); H^s)$ and $C([0, T); H^{s-1})$, respectively. The proof is completed. \hfill $\Box$
Proof of Theorem 2. We consider the problem
\[
\begin{align*}
(1 - \varepsilon)u_t - \varepsilon u_{xxt} &= (1 - \partial_x^2)^{-1}\left[-\varepsilon u_t + \partial_x\left(-2ku - \frac{a}{m+1}u^{m+1} + \frac{1}{n+1}\partial_x^2(u^{n+1}) - \frac{n}{2}u^{n-1}u_x^2 + \beta(u_x)^{2N-1}\right)\right], \\
u(x, 0) &= u_{\varepsilon 0}(x).
\end{align*}
\] (91)

Letting \(u(x, t)\) be the limit of the sequence \(u_{\varepsilon}\) and taking the limit in problem (91) as \(\varepsilon \to 0\), from Lemma 11, it is easy to see that \(u\) is a solution of the problem
\[
\begin{align*}
u_t &= (1 - \partial_x^2)^{-1}\partial_x\left[-2ku - \frac{a}{m+1}u^{m+1} + \frac{1}{n+1}\partial_x^2(u^{n+1}) - \frac{n}{2}u^{n-1}u_x^2 + \beta(u_x)^{2N-1}\right], \\
u(x, 0) &= u_0(x),
\end{align*}
\] (92)

and hence \(u\) is a solution of problem (92) in the sense of distribution. In particular, if \(s \geq 4\), \(u\) is also a classical solution. Let \(u\) and \(v\) be two solutions of (92) corresponding to the same initial data \(u_0\) such that \(u, v \in C([0, T]; H^s)\). Then \(w = u - v\) satisfies the Cauchy problem
\[
\begin{align*}
w_t &= (1 - \partial_x^2)^{-1}\partial_x\left[-2kw - \frac{a}{m+1}w^{m+1} + \frac{1}{n+1}\partial_x^2(w^{n+1}) - \frac{n}{2}w^{n-1}w_x^2 + \beta(w_x)^{2N-1}\right] \\
w(x, 0) &= 0.
\end{align*}
\] (93)

For any \(\frac{1}{2} < q < \min\{1, s - 1\}\), applying the operator \(\Lambda^q w\Lambda^q\) to both sides of Eq. (93) and integrating with respect to \(x\), we obtain the equality
\[
\frac{1}{2} \frac{d}{dt} \| w \|^2_{H^q} = \int_R (\Lambda^q w) \Lambda^{q-2}\partial_x\left[-2kw - \frac{a}{m+1}w^{m+1} + \frac{1}{n+1}\partial_x^2(w^{n+1}) - \frac{n}{2}\partial_x(w^{n+1})\partial_x w - \frac{1}{2}\partial_x(u^n)\partial_x v + \beta w_x \sum_{j=0}^{2N-2} u_x^j v_x^{2N-2-j}\right] dx.
\] (94)

By the similar estimates presented in Lemma 11, we have
\[
\frac{d}{dt} \| w \|^2_{H^q} \leq \tilde{c}\| w \|^2_{H^q}.
\] (95)

Using the Gronwall inequality leads to the conclusion that
\[
\| w \|_{H^q} \leq 0 \times e^{2t} = 0
\] (96)
for \(t \in [0, \tilde{T})\). This completes the proof. \(\Box\)
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References