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An obstruction to embedding a simplicial n-complex into a $2n$-manifold

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Abstract

Let K be a finite, connected, simplicial n-complex ($n \geqslant 3$) and M a 1-connected, smooth, orientable $2n$-manifold without boundary. If $f:|K| \to M$ is an arbitrary map, we define a first obstruction $\gamma(f) \in H^{2n}(J^*K; Z)$, where J^*K is the reduced deleted product of K and show that the vanishing of this obstruction is necessary and sufficient for f to be homotopic to an embedding.

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1. Introduction

Let K be a finite, connected, simplicial n-complex ($n \geqslant 3$) and M a 1-connected, smooth, orientable $2n$-manifold without boundary. If $f:|K| \to M$ is an arbitrary map, we shall define a first obstruction $\gamma(f) \in H^{2n}(J^*K; Z)$, where J^*K is the reduced deleted product of K and show that the vanishing of this obstruction is necessary and sufficient for f to be homotopic to an embedding. The heart of the paper is the construction of a homotopy of f to another map in its cohomology class via tubular neighborhoods and coordinatizing maps. Therefore, any f with 0 cohomology class can be homotoped to a map f_1 whose co-cycle is 0. The self-intersections of f_1 can then be removed by using the appropriate theorems established in the polyhedral category by Zeeman [4,13], Hudson [3, 4] and Weber [10].

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2. The obstruction

Throughout this paper we let K be a finite, connected, simplicial n-complex ($n \geqslant 3$), $|K|$ the underlying topological space, and M a smooth, orientable $2n$-manifold without boundary. It is well known [8] that given any map $f:|K| \to M$ and any $\varepsilon > 0$, there exists an ε-homotopy between f and a general map from $|K|$ to M. Therefore, we assume in this paper that our given map is general.

Definition 1 [6]. A general map $f:|K| \to M$ is one which satisfies:
(i) for each $\sigma \in K$, $f|\sigma$ is a smooth embedding;
(ii) for each pair of simplices $\sigma^p, \tau^q \in K$ with $p+q < 2n$, $f(\sigma^p) \cap f(\tau^q) = \emptyset$;
(iii) no point of $f(\sigma^n) \cap f(\tau^n)$ lies in the image of any other simplex;
(iv) for each pair $\sigma^n, \tau^n \in K$, $f(\sigma^n) \cap f(\tau^n)$ consists of a finite number of transverse intersections.

Definition 2. Let $f:|K| \to M$ be a general map. Let $\{U_\alpha, \phi_a\}$ be an atlas of coordinate neighborhoods on M. Call f *Euclidean with respect to intersections* if for all n-simplices σ^n, τ^n of K, $f(\sigma^n) \cap f(\tau^n) \neq \emptyset \Longrightarrow f(\sigma^n) \cup f(\tau^n) \subset U_\alpha$ for some α.

Let $f:|K| \to M$ be a general map. Assume K is a fine enough subdivision so that f is Euclidean with respect to intersections. Let $JK = \{\sigma \times \tau \mid \sigma, \tau \in K \text{ and } \sigma \cap \tau = \emptyset\}$. We wish to define a $2n$-cochain on JK with integral coefficients. Since f is general, it is a proper map and the only simplices of K whose images under f intersect in M are n-dimensional. Since M is orientable, it has a covering $\{U_\alpha, \phi_\alpha\}$ of coherently oriented coordinate neighborhoods on M. We may assume that for each α, $\phi_\alpha(U_\alpha)$ is an open ball in R^{2n} containing the origin. Then, if $f(\sigma^n) \cap f(\tau^n) \neq \emptyset$ and $f(\sigma^n) \cup f(\tau^n) \subset U_\alpha$, we define the value of the cochain $c_K(f)$ on $\sigma^n \times \tau^n$ by

$$c_K(f)\big(\sigma^n \times \tau^n\big) = \phi_\alpha f\big(\sigma^n\big) \wedge \phi_\alpha f\big(\tau^n\big)$$

for every such pair and $c_K(f) = 0$ for all other pairs of simplices in JK. (Here $\wedge$ is the intersection number of the simplex images in R^{2n}.) If $f(\sigma^n) \cup f(\tau^n) \subset U_\beta$ for another coordinate neighborhood, then $f(\sigma^n) \cup f(\tau^n) \subset U_\alpha \cap U_\beta$. Since M is orientable, $\phi_\alpha f(\sigma^n) \wedge \phi_\alpha f(\tau^n) = \phi_\beta f(\sigma^n) \wedge \phi_\beta f(\tau^n)$. Therefore, the cochain is well-defined and $c_K(f) \in C^{2n}(JK; Z)$.

We wish to show that $c_K(f) \in C^{2n}(J^*K; Z)$ where $J^*K = JK/(\sigma \times \tau \sim \tau \times \sigma)$. The cell exchange map $T: JK \to JK$ defined by $T(\sigma^p \times \tau^p) = \tau^p \times \sigma^p$ induces a map $T^\#$ on any cochain c having the property: $T^\# c(\sigma^p \times \tau^q) = (-1)^{pq} c(\tau^q \times \sigma^p)$ where the product $\sigma^p \times \tau^q$ has an orientation induced by the orientations of σ^p and τ^q. In particular, for n-simplices σ and τ, and $f:|K| \to M$ a general map,

$$\begin{aligned} T^\# c_K(f)(\sigma \times \tau) &= (-1)^{n^2} c_K(f)(\tau \times \sigma) \\ &= (-1)^{n^2} \phi_\alpha f(\tau) \wedge \phi_\alpha f(\sigma) \\ &\quad \text{for } f(\sigma) \cap f(\tau) \neq \emptyset \text{ and } f(\sigma) \cup f(\tau) \subset U_\alpha \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{n^2}(-1)^{n^2}\phi_\alpha f(\sigma)\wedge\phi_\alpha f(\tau)\\
&= \phi_\alpha f(\sigma)\wedge\phi_\alpha f(\tau)\\
&= c_K(f)(\sigma\times\tau).
\end{aligned}
$$

Therefore, $c_K(f)$ is invariant under $T^{\#}$. Define J^*K to be the decomposition complex $J^*K = JK/(\sigma\times\tau\sim\tau\times\sigma)$. The group $C^{2n}(J^*K;Z)$ may be considered to be the subgroup of $C^{2n}(JK;Z)$ consisting of $T^{\#}$-invariant cochains since the value of such cochains is well-defined on the equivalence class $[\sigma\times\tau]\in J^*K$. Hence, $c_K(f)\in C^{2n}(J^*K;Z)$. Since $c_K(f)$ is a top dimensional cochain, it is a cocycle, henceforth known as the obstruction cocycle.

Definition 3. Let $\gamma_K(f)\in H^{2n}(J^*K;Z)$ be the cohomology class of $c_K(f)$. This is called the *obstruction* to homotoping a map $f:|K|\to M$ to an embedding.

We see next that the obstruction is independent of the subdivision of K. For if K' is any subdivision, then the isomorphism $\Psi: H^{2n}(J^*K';Z)\to H^{2n}(J^*K;Z)$ is defined in the following way. Let $\sigma=\sigma_1\cup\sigma_2\cup\cdots\cup\sigma_p$ and $\tau=\tau_1\cup\tau_2\cup\cdots\cup\tau_q$ be subdivisions of n-simplices $\sigma,\tau\in K$. If $i:K\to K'$ is inclusion, then

$$
\begin{aligned}
&(Ji)^{\#}\big(c_{K'}(f)\big)[\sigma\times\tau]\\
&\quad= c_{K'}(f)\big[(Ji)_{\#}(\sigma\times\tau)\big]\\
&\quad= c_{K'}(f)[\sigma_1\times\tau_1+\sigma_2\times\tau_2+\cdots+\sigma_p\times\tau_q]\\
&\quad= c_{K'}(f)[\sigma_1\times\tau_1]+c_{K'}(f)[\sigma_2\times\tau_2]+\cdots+c_{K'}(f)[\sigma_p\times\tau_q].
\end{aligned}
$$

This last sum must equal $c_K(f)[\sigma\times\tau]$ since all the double point pairs which lie in $\sigma_i\times\tau_j$ for all i,j is precisely the same set which lies in $\sigma\times\tau$. Moreover, $(Ji)^{\#}$ is clearly an isomorphism at the cochain level and passing to cohomology, it induces the required map Ψ. As a result, for any map $f:|K|\to M$, K may be assumed to be of sufficiently fine mesh so that f is Euclidean with respect to intersections. So we henceforth assume all maps to possess this property and thus their obstructions will be defined. When there is no confusion, we shall denote $c_K(f)$ by $c(f)$ and $\gamma_K(f)$ by $\gamma(f)$. As in most obstructions of this type, $\gamma(f)$ is independent of the homotopy class of f. By the general position theorem we may assume that all homotopies are $general$ in the following sense.

Definition 4 [6]. $\{f_t\}:|K|\to M$ is a *general homotopy* if:

(i) F is a differentiable map from $|K|\times I$ to M where $F(x,t)=f_t(x)$.

(ii) The set $\{t\in I\colon\ f_t$ is not general$\}$ is finite.

(iii) If f_r is not general for some r, $0<r<1$, then f_r fails to be general by having a double point pair $x_1\in\sigma^p$, $x_2\in\tau^q$, where $p+q=2n-1$. For any such double point, there exist neighborhoods $U_1\times U$ of (x_1,r) in $\sigma^p\times I$ and $V_2\times V$ of (x_2,r) in $\tau^q\times I$, such that $F(U_1\times U)$ intersects $F(V_2\times V)$ transversely in $M\times I$.

The other double points of f_r are contained in n-dimensional simplices.

Theorem 1. *If two general maps f and $f_1:|K|\to M$ are homotopic, then $\gamma(f)=\gamma(f_1)$.*

Proof. We may assume a general homotopy $\{f_t\}: |K| \to M$ with $f_0 = f$. For any double point pair $(x_0, y_0) \in \sigma^n \times \tau^n$ of f, the self-intersection $z_0 = f(x_0) = f(y_0)$ is the starting point of a path of self-intersections in M, call it $z_t = f_t(x_t) = f(y_t)$ as t varies from 0 to 1. Let $G = \{t \in I\colon\ f_t \text{ is not general}\}$. If G is empty, it means that double point pair (x_t, y_t) corresponding to z_t never passes through the boundary of $\sigma^n \times \tau^n$. In other words, (x_t, y_t) remains in $\sigma^n \times \tau^n$ for all t. Similarly, all the double point pairs of f reside in the same n-simplex pairs as those of f_1 and so $c(f) = c(f_1)$. If G is not empty, it is sufficient to consider the case where there is just one value $r \in G$. This means that f_r fails to be general by having a double point pair (x_r, y_r) in a product of simplices of total dimension $2n-1$, call it, say, $\Sigma^{n-1} \times \tau^n$. Let (x_t, y_t), $0 \leqslant t \leqslant 1$, be the double point path in $K \times K$ containing (x_r, y_r). To construct the $(2n-1)$-coboundary which relates $c(f)$ to $c(f_1)$, we need only consider the n-simplices in $\mathrm{star}(\Sigma, K)$. Just one of these simplices, call it σ, will contain x_0. Condition (iii) of Definition 2.5 guarantees that $x_t \in \sigma$ for $t \leqslant r$ and $x_t \notin \sigma$ for $t > r$. In other words, $f(\sigma) \cap f(\tau)$ contains one more point than $f_1(\sigma) \cap f_1(\tau)$. Therefore, $c(f)[\sigma \times \tau]$ differs from $c(f_1)[\sigma \times \tau]$ by either $+1$ or -1 depending on the orientation of the original intersection. Define $d \in C^{2n-1}(J^*K; Z)$ by $d[\Sigma \times \tau] = 1$ and zero elsewhere. Then

$$
\begin{aligned}
\delta d[\sigma \times \tau] &= d\big[\partial(\sigma \times \tau)\big] \\
&= d[\partial\sigma \times \tau + \sigma \times \partial\tau] \\
&= d[\Sigma \times \tau + \cdots + \sigma \times \partial\tau] \\
&= d[\Sigma \times \tau] \\
&= 1 \\
&= \pm\big(c(f) - c(f_1)\big)[\sigma \times \tau].
\end{aligned}
$$

On the other hand, every other n-simplex $\omega \in \mathrm{star}(\Sigma, K)$ will not contain x_o, but the (general) homotopy must introduce one more self-intersection in the images of ω and τ under f_t for $t > r$. So, in this case, $f(\omega) \cap f(\tau)$ contains one less point than $f_1(\omega) \cap f_1(\tau)$. Therefore, as above, $c(f)[\omega \times \tau]$ differs from $c(f_1)[\omega \times \tau]$ by either 1 or -1 depending on the orientation of the introduced self-intersection. Using the above d, routine computation again gives us $\delta d[\omega \times \tau] = \pm(c(f) - c(f_1))[\omega \times \tau]$. Thus, $c(f)$ and $c(f_1)$ are co-homologous. □

We have the following immediate corollary.

Corollary 1. *If f is homotopic to an embedding, then $\gamma(f) = 0$.*

3. Double point pairs from adjacent simplices

The main objective of this paper is to prove the converse of Corollary 1. Our goal is to eliminate all double point pairs from maps f for which $\gamma(f) = 0$. We divide the double point pairs of a general map f into two types: those residing in adjacent simplices of K

and those that reside in remote (non-adjacent) simplices. Because J^*K is a complex not containing pairs of simplices $[\sigma \times \tau]$ where $\sigma \cap \tau \neq \emptyset$, it is a fact that our $\gamma(f)$ does not detect double points contained in adjacent pairs. However, when M is 1-connected, this is not a problem.

Theorem 2. *Let $f:|K| \to M$ be a general map, where K is a finite n-dimensional simplicial complex, $n \geqslant 3$, and M is a 1-connected, smooth, $2n$-manifold. If σ_1 and σ_2 are adjacent n-simplices of f containing a single double point (x_1, x_2), then f is homotopic to a general map $g:|K| \to M$ where $g(\sigma_1) \cap g(\sigma_2) = \emptyset$ and $g = f$ outside a regular neighborhood of $\sigma_1 \cup \sigma_2$.*

Proof. Let x_0 be in the simplex $\sigma_1 \cap \sigma_2$ and let $\xi : I \to K$ be an arc from x_1 to x_2 such that $\xi[0, \frac{1}{2}] = x_1 * x_0$ and $\xi[\frac{1}{2}, 1] = x_0 * x_2$, where $*$ indicates the join. Since $\pi_1 M = 0$, the loop $f(\xi)$ bounds a 2-cell D in M. Since $2n \geqslant 6$, D may be chosen so that $D \cap f[K - (\sigma_1 \cup \sigma_2)] = \emptyset$. Next, subdivide K by starring at x_0, x_1, and x_2. Then ξ will be a subcomplex of K. There exists a triangulation (T, μ) of M (T a $2n$-simplicial complex and $\mu : |T| \to M$ a homeomorphism), and a further subdivision of K such that $f' = \mu^{-1} f$ is simplicial and $\mu^{-1}(D)$ is a subcomplex of T. Take second deriveds of K and T such that f' is still simplicial. Let $N_0 = |N(\xi, K'')|$ and $N_1 = |N(\mu^{-1}(D), T''|$ be second derived neighborhoods of ξ and $\mu^{-1}(D)$. Then N_0 and N_1 are regular neighborhoods of collapsible polyhedrons and therefore are balls [3,9]. Also $f'(\partial N_0) \subset \partial N_1$ and $f'(N_0) \subset N_1$. If we identify $N_0 \cong I^n$ and $N_1 \cong I^{2n}$, then every x in N_0 can be represented by λx_0 for $0 \leqslant \lambda \leqslant 1$ where $\lambda = 1 \Longleftrightarrow x \in \partial N_0$. Define $g' : |K| \to |T|$ by $g'(\lambda x_0) = \lambda f'(x_0)$ and $g' = f'$ on $\mathrm{cl}(K - N_0)$. Then $g'(\sigma_1) \cap g'(\sigma_2) = \emptyset$, $g' = f'$ on ∂N_0, and f' is homotopic to g' by the standard straight-line homotopy, namely h_t where $h_t(x) = tf'(x) + (1-t)g'(x)$ for $x \in N_0$ and $h_t(x) = f'(x)$ on $\mathrm{cl}(K - N_0)$. The desired $g : |K| \to M$ is then $g = \mu g'$. $\square$

Since K is a finite simplex and the double point set $S(f)$ must be finite for a general map f, we may apply the above theorem a finite number of times to obtain the following result.

Corollary 2. *Let $f : |K| \to M$ be a general map with K and M as above. If the only double point pairs of f occur in adjacent simplices of K, then f is homotopic to an embedding.*

4. Remote double points

We now consider *remote* double point pairs, i.e., those which exist in non-adjacent n-simplices of K. We wish to be able to homotope a map f to an embedding in the event that $c(f) = 0$. Recall that $c(f)$ vanishes precisely when, for $\sigma \cap \tau = \emptyset$, $f(\sigma) \cap f(\tau)$ is either empty or consists of pairs of intersection points of opposite orientation. The following theorem by Weber [10] gives us our desired result.

Theorem 3. *Let M^m be a semi-linear manifold without boundary. Let $m = p + q$. Let $f : \sigma^p \to M$ and $g : \tau^q \to M$ be two semi-linear embeddings such that $f(\sigma^p)$ and $g(\tau^q)$ intersect in general position in exactly two points A and B. We suppose that*:

(i) $p \geqslant 3, q \geqslant 3$;

(ii) *There exists a path α in the interior of $f(\sigma^p)$ connecting A to B and a path α' in the interior of $g(\tau^q)$ connecting A to B such that $\alpha^{-1}\alpha'$ is homotopic to zero in M*;

(iii) *The intersection number $f(\sigma^p) \wedge g(\tau^q) = 0$.*

Then there exists an ambient isotopy h_t in M, whose support is a combinatorial ball of dimension m, leaving $f(\partial\sigma^p)$ and $g(\partial\tau^q)$ fixed, such that $h_1(g(\tau^q)) \cap f(\sigma^p) = \emptyset$.

If $c(f) = 0$ for a general map $f : |K| \to M$, it is straightforward to subdivide K so that whenever $f(\sigma)$ intersects $f(\tau)$, (for $\sigma \cap \tau = \emptyset$), then $f(\sigma) \cap f(\tau)$ consists of exactly two points whose coordinate frames are of opposite orientation. Since the complex K and the manifold M of this paper satisfy the conditions of the above theorem, we can apply the result repeatedly to eliminate all intersections occurring in the images of pairs of non-adjacent n-simplices. In other words, f can be homotoped to a map f' having no remote double point pairs. We then apply Corollary 2 to find a homotopy from f' to an embedding. Therefore, we have the following theorem.

Theorem 4. *Let $f : |K| \to M$ be a general map, where K is a finite n-dimensional simplicial complex, $n \geqslant 3$, and M is a 1-connected, smooth, $2n$-manifold. If $c(f) = 0$, then f is homotopic to an embedding.*

5. Coordinatizing maps

Next, given a general map $f : |K| \to M$, we wish to conclude that if $\gamma(f) = 0$, then f is homotopic to an embedding. If $\gamma(f) = 0$, but $c(f) \neq 0$, then there exists an f_1 whose cochain is 0 and which is in the same cohomology class as $c(f)$. If we can find a homotopy from f to f_1, we will be done because we know from Theorem 4 that f_1 is homotopic to an embedding. In order to construct the required homotopy (from f to f_1), we will need the following tools.

Definition 5 [9]. For S a submanifold of M, a *tubular neighborhood* $W(S, \varepsilon)$ of radius $\varepsilon > 0$ is defined as follows. For a tangent vector v to M, let $\exp(v)$ be the point of M (if it exists) at a distance equal to the length of v along a geodesic in the direction of v. Let $N(M \mid S)$ be those tangent vectors to M at points of S which are orthogonal to S, i.e., the normal bundle along S. If $\mathrm{cl}(S)$ is compact, the implicit function theorem asserts that, for a suitably small ε, the exponential map, when restricted to vectors of $N(M \mid S)$ of length $< \varepsilon$, is a homeomorphism. We shall denote its image by $W(S, \varepsilon)$ and denote by h_ε the inverse of the exponential map so that $h_\varepsilon : W(S, \varepsilon) \to N(M \mid S)$ is a differentiable homeomorphism. If π is the natural projection of $N(M \mid S)$ on S, then $\pi h_\varepsilon : W(S, \varepsilon) \to S$ is a fiber map, where each fiber is a cell.

Definition 6. A *normalizing map* for a vector bundle is a map of the bundle into a vector space L, which induces a linear isomorphism of each fiber onto L. It is well known that there exists a normalizing map for a vector bundle if and only if it is a product bundle.

We wish to consider the normal bundle $N(M \mid \alpha)$ for $\alpha = p((-\delta, 1+\delta))$, a smooth arc in M. Then $N(M \mid \alpha)$ is a trivial bundle of (fiber) dimension $2n-1$. So there exists a normalizing map $\Psi : N(M \mid \alpha) \to L$, where L is a vector space of dimension $2n-1$.

Definition 7 [9]. If $\Psi : N(M \mid \alpha) \to L$ is a normalizing map, α_1 is an open subset of α (i.e., $\alpha_1 = p((-\delta_1, 1+\delta_1))$ for $\delta_1 < \delta$) and U a neighborhood of the origin in L, then a map $\theta : \alpha_1 \times U \to M$ is called a *coordinatizing map* for α_1 with respect to Ψ if:

(i) θ is regular C^∞ homeomorphism;
(ii) $\pi h_\varepsilon \theta(q \times v) = q$ for $q \in \alpha_1,\ v \in U$;
(iii) $\Psi h_\varepsilon \theta(q \times v) = v$ for $q \in \alpha_1,\ v \in U$.

The proof for the following lemma can be found in [9].

Lemma 1. *If α_1 is an open subset of α and if Ψ is a C^∞ normalizing map for $N(M \mid \alpha)$ in L, then there exists a coordinatizing map θ for α_1 with respect to Ψ.*

The image of the coordinatizing map θ lies in the tubular neighborhood $W(\alpha_2, \varepsilon)$ where α_2 is an open subset of α containing $\mathrm{cl}(\alpha_1)$. We have the following diagram:

$$\begin{tikzcd} \alpha_1 \times U \arrow[r, "\theta"] & W(\alpha_2, \varepsilon) \arrow[r, "h_\varepsilon"] & N(M \mid \alpha_2) \arrow[r, "\Psi"] \arrow[d, "\pi"] & L \\ & & \alpha_2 & \end{tikzcd}$$

6. Main result

Given a general map f and $(2n-1)$-cochain d we desire to find a homotopy $\{f_t\}$, $0 \leqslant t \leqslant 1$, where $f = f_0$ and $\delta d = c(f) - c(f_1)$. This homotopy is constructed using d as a guide. For example, if $d(\Sigma^{n-1} \times \tau^n) = 1$, this means that our homotopy will push f through a tube connecting Σ to τ and the end map f_1 will have more intersection points than f, one in $f_1(\sigma^n) \cap f_1(\tau^n)$ for every σ^n having Σ as part of its boundary. This means that for some r, $0 < r < 1$, f_r will have a double point contained in $\Sigma^{n-1} \times \tau^n$. This f_r corresponds to the cochain d and, because it is always possible to use a general homotopy, δd and $c(f) - c(f_1)$ will have the proper values.

Theorem 5. *Let $f : |K| \to M$ be a general map. Let $d \in C^{2n-1}(J^*K; Z)$ be such that $d[\Sigma \times \tau] = 1$, Σ an $(n-1)$-simplex, τ an n-simplex, and* 0, *elsewhere. Then there exists a general homotopy $\{f_t\} : |K| \to M$, $0 \leqslant t \leqslant 1$, such that $f_0 = f$ and $f_1(x) = f(x)$ for $x \notin \mathrm{star}(\Sigma, K) \cup \tau$, and $\delta d = \pm(c(f) - c(f_1))$.*

Proof. Let $x \in \operatorname{int}(\Sigma)$, $y \in \operatorname{int}(\tau)$. Choose a smooth arc $p : I \to M$ such that $a = p(0) = f(x)$ and $b = p(1) = f(y)$. Since p is smooth, we can extend the domain of p to $(-\delta, 1 + \delta)$ for some δ, $0 < \delta < \frac{1}{2}$. Let $\alpha = p((-\delta, 1 + \delta))$. Since $\dim K = n$, $\dim M = 2n, n \geqslant 3$, there is no difficulty in assuming that $\alpha \cap f(K) = \{a, b\}$. For the same reason, we may assume that the unit tangent to α at a is not tangent to the image of any simplex having Σ as a face and, similarly, the unit tangent to α at b is not tangent to $f(\tau)$.

Now, if $N(M \mid \alpha)$ is the normal bundle along α, let $\Psi : N(M \mid \alpha) \to L$, where L is a $(2n - 1)$-dimensional vector space, be a normalizing map. By Lemma 1 we know there exists a coordinatizing map $\theta : \alpha_1 \times U \to M$ with respect to Ψ for $\alpha_1 = p((-\delta_1, 1 + \delta_1))$ an open subset of α and U a neighborhood of the origin in L. We may choose ε small enough so that:

(i) The proof of the lemma is satisfied.
(ii) The tubular neighborhood $W(\alpha_2, \varepsilon)$ will meet the images under f of only τ and $\operatorname{star}(\Sigma, K)$.
(iii) $W(\alpha_2, \varepsilon)$ contains no points of $f(\tau) \cap f(\operatorname{star}(\Sigma, K))$.

Recall that $\theta(\alpha_1 \times U) \subset W(\alpha_2, \varepsilon)$ for α_2 an open set of α containing $\operatorname{cl}(\alpha_1) = p([-\delta_1, 1 + \delta_1])$. We repeat the following diagram:

$$\begin{tikzcd} \alpha_1 \times U \arrow[r, "\theta"] & W(\alpha_2, \varepsilon) \arrow[r, "h_\varepsilon"] & N(M \mid \alpha_2) \arrow[r, "\Psi"] \arrow[d, "\pi"] & L \\ & & \alpha_2 & \end{tikzcd}$$

Now, let $D_x = f^{-1}(\theta(\alpha_1 \times U)) \cap \operatorname{star}(\Sigma, K)$ and $D_y = f^{-1}(\theta(\alpha_1 \times U)) \cap \tau$. By (iii) above, $D_x \cap D_y = \emptyset$. The idea is to exchange the portions of $f(\operatorname{star}(\Sigma, K))$ and $f(\tau)$ which are inside the tube $\theta(\alpha_1 \times U)$. For every $z \in D_x$, $f(z)$ has a pair of coordinates given by $f(z) = \theta(p(t_z), v_z)$ for a unique $t_z \in (-\delta_1, 1 + \delta_1)$, $v_z \in U$. Similarly for every $z \in D_y$, $f(z) = \theta(p'(t_z), v_z)$ where $p'(t) = p(1 - t)$ is the reverse path of p.

Since every vector space is normal the Urysohn Lemma guarantees that existence of a smooth, real-valued function λ on L such that

$$\begin{aligned} &0 \leqslant \lambda(v) \leqslant 1 && \forall v \in L, \\ &\lambda(v) = 0 && v \notin U, \\ &\lambda(v) = 1 && v \in \text{ some closed } V \subset U \text{ containing the origin.} \end{aligned}$$

In order to be able to define a homotopy for all $z \in D_x \cup D_y$, we assume U has small enough diameter so that for every n-simplex σ having Σ as a boundary, any sequence of points $\{z_i\}$ in D_x or in D_y such that if $f(z_i)$ is approaching the boundary of $\theta(\alpha_1 \times U)$ then $\lim \lambda(v_{z_i}) = 0$. This condition ensures the continuity of the homotopy $\{f_t\} : |K| \to M$ for $0 \leqslant t \leqslant 1$ as defined by:

$$f_t(z) = \begin{cases} \theta(p(t_z + t\lambda(v_z)), v_z) & \text{if } z \in D_x, \\ \theta(p'(t_z + t\lambda(v_z)), v_z) & \text{if } z \in D_y, \\ f(z) & \text{if } z \notin D_x \cup D_y. \end{cases}$$

We now show that the end map of this homotopy has a double point introduced in each simplex having Σ as a boundary. In other words, for every n-simplex $\sigma \in \operatorname{star}(\Sigma, K)$, $f_1(\sigma) \cap f_1(\tau) \neq \emptyset$.

First note that f_1 will have an intersection in $\theta(\alpha_1 \times U)$ if the equation

$$\theta\big(p\big(t_z + \lambda(v_z)\big), v_z\big) = \theta\big(p'\big(t_{z'} + \lambda(v_{z'})\big), v_{z'}\big)$$

has a solution pair (z, z') with $z \in \sigma, z' \in \tau$. Applying πh_ε and Ψh_ε separately to both sides of this equation implies that any such solution must satisfy

$$p\big(t_z + \lambda(v_z)\big) = p'\big(t_{z'} + \lambda(v_{z'})\big) \quad \text{and} \quad v_z = v_{z'}.$$

Since p is an arc and p' is the reverse path of p, the coordinates of the image of a solution pair (z, z') must satisfy the equation $t_z + t_{z'} + \lambda(v_z) + \lambda(v_{z'}) = 1$. To show we can find such a pair we define a continuous real-valued function ω on $(D_x \cap \sigma) \times D_y$ by

$$\omega\big(z, z'\big) = t_z + t_{z'} + \lambda(v_z) + \lambda(v_{z'}).$$

It is clear that the set of points (z, z') in $(D_x \cap \sigma) \times D_y$ that satisfy $v_z = v_{z'}$ is connected. Denote this set by A. Since $t_z, t_{z'} \in (-\delta_1, 1 + \delta_1)$ for $0 < \delta_1 < 1/2$, $\sup(\omega \mid A) > 1$. By condition (iii) for $W(\alpha_2, \varepsilon)$, $p(t_z) \neq p'(t_{z'})$ for $v_z = v_{z'}$ and so $t_z + t_{z'} < 1$ for $(z, z') \in A$. Thus, $\inf(\omega \mid A) < 1$. By the intermediate value theorem for real-valued functions, there must exist a pair (z^*, z'^*) such that $\omega(z^*, z'^*) = 1$. So this pair is the desired solution for the above equation.

We can now complete the proof of the theorem. Since f_t is a general homotopy, f_1 is a general map and therefore for every n-simplex $\sigma \in \operatorname{star}(\Sigma, K)$, $f_1(\sigma) \cap f_1(\tau)$ consists of transverse intersections and therefore is a discrete set. We may assume that $f_1(\sigma) \cap f_1(\tau) \cap W(\alpha_2, \varepsilon)$ consists of a single point for if it does not we simply choose a smaller ε for the tube $W(\alpha_2, \varepsilon)$ in which the homotopy takes place. Then the following routine computation yields our result. For any n-simplex $\sigma \in \operatorname{star}(\Sigma, K)$,

$$\begin{aligned}
\delta d[\sigma \times \tau] &= d\big[\partial(\sigma \times \tau)\big] \\
&= d[\partial\sigma \times \tau + \sigma \times \partial\tau] \\
&= d[\Sigma \times \tau + \cdots + \sigma \times \partial\tau] \\
&= 1 \\
&= \pm\big(c(f) - c(f_1)\big)[\sigma \times \tau].
\end{aligned}$$

Observe that $d = 0$ for pairs of simplices that are both outside $\operatorname{star}(\Sigma, K) \cup \tau$ and so $c(f) = c(f_1)$. If $\operatorname{star}(\Sigma, K)$ contains no n-simplices, there are no new intersections introduced since the homotopy is general. In this case $c(f) = c(f_1)$ everywhere since Σ does not appear as the boundary of an n-simplex and so δd would never be non-zero. Our f_1 therefore meets the desired condition. □

Theorem 6. *Let $f : |K| \to M$ be a general map, where K is a finite n-dimensional simplicial complex, $n \geqslant 3$, and M is a 1-connected, smooth $2n$-manifold. If $\gamma(f) = 0$, then f is homotopic to a general map $g : |K| \to M$ such that $c(g) = 0$.*

Proof. If $\gamma(f)=0$, but $c(f)\neq 0$, it is straightforward to subdivide K so that for every n-simplicial pair $[\sigma\times\tau]\in J^*K$, $c(f)[\sigma\times\tau]=0, 1$, or -1. Every n-simplicial pair $\sigma\times\tau$ where $c(f)[\sigma\times\tau]=\pm 1$ must then fall into one of three categories which are handled in the following ways:

(i) Either $\partial\sigma$ or $\partial\tau$ contains no $(n-1)$-simplex which bounds another n-simplex.

Suppose this is true of $\partial\sigma$. Let $\Sigma\in\partial\sigma$ be an $(n-1)$-simplex such that $d\in C^{2n-1}(J^*K;Z)$ defined by $d[\sigma\times\tau]=1$ induces a homotopy via Theorem 5. This introduces one new double point for the end map f_1 whose corresponding intersection in $f_1(\sigma)\cap f_1(\tau)$ is of opposite orientation from the existing one. Such a Σ must exist or else $\gamma(f)\neq 0$. Then $c(f_1)[\sigma\times\tau]=0$. Because $\operatorname{star}(\Sigma,K)$ consists of just the one n-simplex σ, no new double points are introduced and $c(f_1)=c(f)$ elsewhere.

(ii) The intersection in M corresponding to the double point pair (x,y) in $\sigma\times\tau$ which makes $c(f)[\sigma\times\tau]$ non-zero is movable to adjacent n-simplices.

Since $\gamma(f)=0$, an arc ξ in $K\times K$ from $(x,y)\in\sigma\times\tau$ to $(y,x)\in\tau\times\sigma$ exists that provides a sequence of desired homotopies. Since $\sigma\times\tau$ does not fall into category (i), we may assume that ξ intersects $\partial(\sigma\times\tau)$ in $\Sigma\times\tau$, for Σ an $(n-1)$-simplex which bounds another n-simplex. As in category (i), since $\gamma(f)=0$, there exists $d\in C^{2n-1}(J^*K;Z)$ defined by $d[\Sigma\times\tau]=1$ that provides a homotopy to a map f_1 where $c(f_1)[\sigma\times\tau]=0$ and a new double point pair has been introduced into $\sigma_1\times\tau$ for every $\sigma_1\in\operatorname{star}(\Sigma,K)$. If σ_1 is adjacent to τ, $c(f_1)[\sigma_1\times\tau]=0$. For every σ_1 such that $c(f_1)[\sigma_1\times\tau]\neq 0$ and $\xi\cap(\sigma_1\times\tau)\neq\emptyset$, the process is repeated using the intersection of ξ with $\partial(\sigma_1\times\tau)$ to provide the next homotopy. If $c(f_1)[\sigma_1\times\tau]\neq 0$ and $\xi\cap(\sigma_1\times\tau)=\emptyset$, then $\sigma_1\times\tau$ falls into one of these three categories (because $\gamma(f)=0$) and is dealt with accordingly. Theorem 5 ensures that $c(f_1)=c(f)$ elsewhere. Since the arc ξ must intersect the diagonal in $K\times K$, proceeding in this manner yields a finite sequence of m homotopies resulting in a map f_m where $c(f_m)[\sigma'\times\tau']=0$ for every $\sigma'\times\tau'$ such that $\xi\cap(\sigma'\times\tau')\neq\emptyset$ and where the double point of f_m now exists in a pair of adjacent n-simplices. It is removable by Theorem 2.

(iii) The intersection in M corresponding to the remote double point pair (x,y) in $\sigma\times\tau$ has a matching intersection of opposite orientation corresponding to another remote double point pair (x^+,y^+) in, say, $\sigma^+\times\tau^+$.

Since $\gamma(f)=0$, an arc ξ exists in $K\times K$ from (x,y) to (x^+,y^+) whose path provides a sequence of homotopies in the same manner as (ii). Wherever ξ intersects $\partial(\sigma\times\tau)$, Theorem 5 implies the existence of a $(2n-1)$ cochain which induces a homotopy to a map f_1 where $c(f_1)[\sigma\times\tau]=0$ and a new double point pair has been introduced in a neighboring $\sigma_1\times\tau$. This is repeated m times until the map f_m is attained in which $c(f_m)[\sigma^+\times\tau^+]=0$. In this case, however, f_m does not have a double point in a pair of adjacent simplices. Rather each of the introduced double points has a matching double point of opposite orientation and is removable by Theorem 3.

Since f is general, the set of n-simplicial pairs $\sigma\times\tau$ where $c(f)[\sigma\times\tau]=\pm 1$ is a finite set. Applying the appropriate sequence of homotopies described above to f for each $\sigma\times\tau$ in this set therefore produces a finite sequence of homotopies resulting in the required map g. □

Combining Corollary 1, Theorems 4 and 6 yields the main result.

Theorem 7. *Let $f : |K| \to M$ be a map, where K is a finite n-dimensional simplicial complex, $n \geqslant 3$, and M is a 1-connected, smooth, orientable $2n$-manifold. Then f is homotopic to an embedding if and only if $\gamma(f) = 0$.*

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