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Note

A tournament of order 14 with disjoint Banks and Slater sets

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ABSTRACT

It is shown in this note that there exists a tournament of order 14 with disjoint Banks and Slater sets. Previously, the smallest such tournament was reported to be of order 16. In addition, it is shown that 11 is the minimum order of a tournament in which the Slater set is not a subset of the Banks set.

1. Introduction

A *tournament* [13,14,16] is a digraph that is an orientation of a complete graph. A tournament is *transitive* if the occurrence of the arcs (x, y) and (y, z) implies the occurrence of the arc (x, z). A *Slater order* of a tournament is a transitive tournament that is obtained by reversing the minimum number of arcs in the tournament; the minimum number is called the *Slater index*. The first vertex of a Slater order is a *Slater winner* and the set of all Slater winners of a tournament forms its *Slater set*. A *Banks winner* of a tournament is the first vertex of a maximal (with respect to inclusion) transitive subtournament. The *Banks set* consists of all the Banks winners of the tournament. These concepts have been introduced in [2,17].

In the study of tournaments, one interesting problem is that of determining "winners" (for example, with the arcs viewing outcomes of matches in real-world sports tournaments). There are several different definitions for winners of tournaments in the literature [13,15], including the definitions of Banks and Slater sets. The relation between different concepts of this kind – Banks and Slater sets in this work – is therefore a natural question.

An example of a tournament of order 75 with disjoint Banks and Slater sets is published in [12], and such a tournament of order 16 is reported in [6]. In this work we show that an even smaller such tournament exists, namely one of order 14. In addition, we show by exhaustive backtrack search that 11 is the minimum order of a tournament in which some Slater winners are not Banks winners. For a recent survey related to this topic, see [5].

This note is organized as follows. The algorithms used in the study are considered in Section 2. For all but the smallest orders of tournaments, a particular operation called substitution is utilized; this operation is described in Section 3. Finally, the computational results including the main result – the existence of a tournament of order 14 with disjoint Banks and Slater sets – is presented in Section 4.

2. Algorithms

The goal of this study is to improve the result in [6] by finding a tournament with fewer than 16 vertices and with disjoint Banks and Slater sets. We attempt this by computing the Banks and Slater sets for all non-isomorphic tournaments

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Table 1
Computational results.

n	#	N	SNB	DSB
1–7	532		no	no
8	6 880		no	no
9	191 536		no	no
10	9 733 056		no	no
11	97 330 560	10	yes	no
12	535 318 080	10	yes	no
13	94 810 320	9	yes	no
14	11 806 080	8	yes	yes

of given orders. With more than 10 vertices, however, the number of tournaments gets prohibitively large. For this reason we go through all non-isomorphic tournaments up to 10 vertices, and for larger orders we only consider tournaments with a certain structure.

The procedure that we use has three phases. In the first phase the Slater index of a tournament is determined, in the second phase the Slater winners are obtained, and, finally, in the third phase it is checked whether any of the Slater winners are Banks winners. Several of the involved computational problems are hard in general. More specifically, the computation of a Slater winner is **NP**-hard [1,3,7], checking whether a given vertex is a Banks winner is **NP**-complete [18], and the computation of a Banks winner is polynomial [9].

Sophisticated algorithms for determining the Slater index and Slater winners have been developed in [4]. However, due to the small size of the graphs in the current work, a straightforward backtrack search – building up a Slater order starting from its first vertex—could be employed in the first two phases, that is, to first calculate the Slater index and then the Slater winners.

For each Slater winner v of a tournament T, we construct an input digraph G for the third phase by deleting the arcs whose head is v (which is thereafter the first vertex of each transitive subtournament that contains v). The third algorithm, which is a slight modification of [11, Algorithm 1], finds all maximal (with respect to inclusion) transitive subtournaments of G that contain v. Finally, it is checked whether the maximal transitive subtournaments of G are included in larger transitive subtournaments in T. If this is not the case, then the Banks set and Slater set are not disjoint.

3. Substitution

All non-isomorphic tournaments with up to 10 vertices can be handled in reasonable time with the procedure described in Section 2. Standard computational techniques [10] can be used to classify such tournaments; alternatively, http://cs.anu.edu.au/~bdm/data/digraphs.html contains an exhaustive catalogue up to order 10. However, the number of non-isomorphic tournaments grows exponentially [8,14], so, motivated by the structure of the 16-vertex tournament in [6], for orders greater than 10 we restricted our search to tournaments that are obtained by applying the substitution operation to smaller tournaments.

The substitution operation – see, for example, [5,13,14] – is used as follows. Some of the vertices of a tournament are replaced by transitive tournaments. In a tournament T with vertices $V(T)$ and arcs $E(T)$, if a vertex $x \in V(T)$ is substituted by a transitive tournament of order k, then in the resulting tournaments $x \in V(T)$, $(x, y) \in E(T)$, and $(y, x) \in E(T)$ are subject to the following replacements:

$$x \to \{x_1, x_2, \dots, x_k\} \text{ and for all } i < j,\ (x_i, x_j) \text{ is added to } E(T),$$
$$(x, y) \to \{(x_1, y), (x_2, y), \dots, (x_k, y)\}, \text{ and}$$
$$(y, x) \to \{(y, x_1), (y, x_2), \dots, (y, x_k)\}.$$

All tournaments of order m that result from the tournaments of order n can be obtained via computing integer partitions of $m - n$. Note that all tournaments of order m that result from the tournaments of order n include all tournaments of order m that result from the tournaments of order n' for any $n' < n$. Finally, for a connection between the Slater index of the resulting tournaments of order m and what is known as the Kemeny index of a certain weighted tournament of order n, see [5, Theorem 39].

4. Results

The computational results are collected in Table 1, where n is the order of tournaments, # is the number of tournaments considered, N is the order of tournaments to which substitution is applied (to obtain graphs of orders greater than 10), SNB tells whether some Slater winners are not Banks winners, and DSB tells whether there exists a tournament with disjoint Slater and Banks sets.

The results of Table 1 reveal that there exists no tournament of order at most 10 with disjoint Slater and Banks sets. In fact there exists no such tournament in which the Slater set is not a subset of the Banks set.

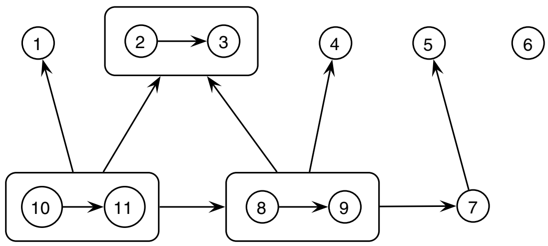

Fig. 1. An 11-vertex tournament.

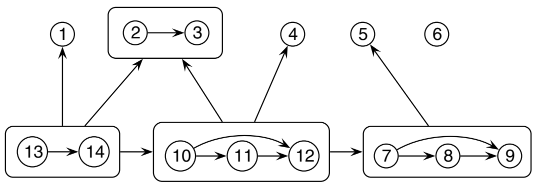

Fig. 2. The 14-vertex tournament T.

One out of the four Slater winners of the tournament of order 11 drawn in Fig. 1 is not a Banks winner. For clarity, only arcs in substitutions (the rounded boxes) and arcs $\{(x, y) : y < x\}$ are drawn; the Slater index is 13, the Slater set is $\{1, 5, 6, 10\}$, and vertex 6 is not a Banks winner. The results in Table 1 show that the smallest order for which some Slater winners are not Banks winners is 11. Among all investigated tournaments, the number of Slater winners that are not Banks winners was zero or one.

The 14-vertex tournaments obtained from the non-isomorphic 8-vertex tournaments by substitution include examples of tournaments with disjoint Banks and Slater sets. Six such tournaments of order 14 were encountered; a proof for one of them will be given in Proposition 1.

Consider the 14-vertex tournament T,

$$V(T) = \{1, 2, \ldots, 14\},$$
$$E(T) = \{(i, j) : i < j \text{ and } (j, i) \notin A,\ \ i, j \in V(T)\} \cup A,$$
$$A = \{(7, 5), (8, 5), (9, 5), (10, 2), (10, 3), (10, 4), (10, 7), (10, 8), (10, 9), (11, 2), (11, 3), (11, 4), (11, 7), (11, 8), (11, 9), (12, 2), (12, 3), (12, 4), (12, 7), (12, 8), (12, 9), (13, 1), (13, 2), (13, 3), (13, 10), (13, 11), (13, 12), (14, 1), (14, 2), (14, 3), (14, 10), (14, 11), (14, 12)\},$$

which is drawn in Fig. 2; all arcs missing are of the form $\{(s, t) \in E(T) : s < t\}$. The figure illustrates clearly the substitution operation.

The proof of the following proposition partly uses ideas from [6].

Proposition 1. *The Slater set and the Banks set of T are disjoint.*

Proof. The tournament T has 18 arc-disjoint circuits, {(1, 6, 14), (2, 5, 14), (2, 6, 10), (2, 9, 13), (3, 4, 14), (3, 5, 13), (3, 6, 11), (4, 5, 11), (4, 6, 12), (4, 13, 10), (5, 6, 9), (5, 10, 8), (5, 12, 7), (7, 13, 11), (7, 14, 10), (8, 13, 12), (8, 14, 11), (9, 14, 12)}. Since at least one arc of each circuit must be reversed to get a transitive tournament, it follows that the Slater index is greater than or equal to 18. On the other hand, by reversing 18 arcs from the tournament T we obtain the linear order

$$6 > 13 > 14 > 1 > 10 > 11 > 12 > 2 > 3 > 4 > 7 > 8 > 9 > 5$$

so the Slater index is smaller than or equal to 18. The arcs that need to be reversed are {(1, 6), (2, 6), (3, 6), (4, 6), (4, 13), (4, 14), (5, 6), (5, 10), (5, 11), (5, 12), (5, 13), (5, 14), (7, 13), (7, 14), (8, 13), (8, 14), (9, 13), (9, 14)}. Consequently, the Slater index of T is exactly 18, and vertex 6 is a Slater winner.

We shall now argue that there are no other Slater winners in T than vertex 6. Since there are 18 arc-disjoint circuits in T, exactly one arc from each circuit has to be reversed in order to get a Slater order of T. Thus, by removing the arcs that belong to these circuits we get an acyclic relation R such that all the Slater orders of T are linear extensions of R. This acyclic relation, which is shown in Fig. 3, has only one maximal element so vertex 6 is the unique Slater winner of the tournament T.

To conclude the proof we show that vertex 6 is not a Banks winner of T. If vertex 6 is a Banks winner, then all other vertices of a maximal transitive subtournament T' with vertex 6 as its first element have labels greater than 6. Let $S_1 = \{7, 8, 9\}$, $S_2 = \{10, 11, 12\}$, and $S_3 = \{13, 14\}$. Obviously, T' cannot have vertices from all sets S_i, $1 \le i \le 3$. But if the intersection between T' and S_1, S_2, or S_3 is empty, then we can add vertices 5, 4, and 1, respectively, to T' before vertex 6. Hence, vertex 6 is not a Banks winner of T. □

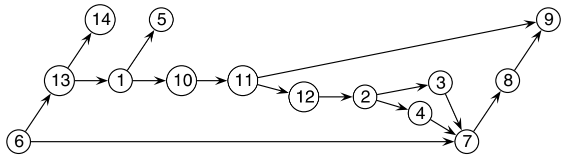

Fig. 3. An acyclic relation.

It remains an open question whether a tournament of order 11, 12, or 13 with disjoint Banks and Slater sets exists.

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