A maximal variance problem

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Abstract

In this work, we provide a direct proof concerning a special type of concave density function on a bounded closed interval with minimal variance. This proof involves elementary methods, without using any advanced theories such as Weierstrass’s Approximation Theorem, from which the technical core result of the paper [C. Carlsson, R. Fullér, P. Majlender, On possibilistic correlation, Fuzzy Sets and Systems 155 (2005) 425–445] comes.

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1. Introduction

The notion of mean value of the function of random variables in probability theory plays a fundamental role in defining the basic measure of a probability distribution. Fullér and Majlender [4] presented the idea of interaction between a marginal distribution of a joint possibility distribution. They introduced the notion of covariance between fuzzy numbers via their joint possibility distribution to measure the degree to which the fuzzy numbers interact. Recently, Carlsson, Fullér and Majlender [1] presented the concept of a possibilistic correlation representing an average degree of interaction between the marginal distribution of a joint possibility distribution as compared to the respective dispersions. They also formulated the weak and strong forms of the possibilistic Cauchy–Schwarz inequality. In proving these results, the weak forms of the possibilistic Cauchy–Schwarz inequality ([1] Theorems 2 and 3) are easy applications of the probabilistic Cauchy–Schwarz inequality and the strong forms of the possibilistic Cauchy–Schwarz inequality ([1] Theorems 6, 8 and 9) are based on Theorem 5 [1]. So, we consider Theorem 5 [1] as the technical core result of the paper [1]. Carlsson, Fullér and Majlender [1] proved Theorem 5 [1] using advanced theory such as Weierstrass’s Approximation Theorem. In this note, we provide a direct proof, involving elementary methods, concerning a special type of concave density function on a bounded closed interval with minimal variance, from which Theorem 5 [1] comes directly.

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2. Definitions

A fuzzy number $A$ is a fuzzy set in $\mathbb{R}$ that has a normal, fuzzy convex and continuous membership function of bounded support [6–8]. If $C$ is a fuzzy set in $\mathbb{R}^n$ then its $\gamma$-level set is defined by $[C]\gamma = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid C(x_1, \ldots, x_n) \geq \gamma\}$ for $0 < \gamma \leq 1$, and $[C]\gamma = \text{cl}\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid C(x_1, \ldots, x_n) > \gamma\}$ (the closure of the support of $C$) for $\gamma = 0$. It is clear that if $A \in \mathcal{F}$ is a fuzzy number then $[A]\gamma$ is a compact interval for all $\gamma \in [0, 1]$.

Let $A_i \in \mathcal{F}, i = 1, \ldots, n$, be fuzzy numbers and let $C$ be a fuzzy set in $\mathbb{R}$. Then, $C$ is said to be a joint possibility distribution of $A_i, i = 1, \ldots, n$, if the following relationships hold [4]:

$$A_i(x_i) = \sup_{x_j \in \mathbb{R}, j \neq i} C(x_1, \ldots, x_n) \quad \forall x_i \in \mathbb{R}, i = 1, \ldots, n.$$ 

Furthermore, in this case we will call $A_i$ the $i$th marginal possibility distribution of $C$ and use the notation $A_i = \pi_i(C)$, where $\pi_i$ denotes the projection operator in $\mathbb{R}^n$, onto the $i$th axis, $i = 1, \ldots, n$.

That is, the fuzzy numbers $A_i \in \mathcal{F}, i = 1, \ldots, n$, are said to be non-interactive if their joint possibility distribution is given by

$$C(x_1, \ldots, x_n) = \min\{A_1(x_1), \ldots, A_n(x_n)\} \quad \forall x_1, \ldots, x_n \in \mathbb{R}.$$ 

In particular, if $A_1, A_2 \in \mathcal{F}$ are non-interactive then their joint possibility distribution is defined by $A_1 \times A_2$. In this case $A_1$ and $A_2$ can take their values independently of each other. On the other hand, $A_1$ and $A_2$ are said to be interactive if they cannot take their values independently of each other [2,3].

Let $C$ be a joint possibility distribution in $\mathbb{R}^n$, let $g : \mathbb{R}^n \to \mathbb{R}$ be an integrable function and let $\gamma \in [0, 1]$. Then, the central value of $g$ on $[C]\gamma$ is defined by [4]

$$\mathcal{C}_{[C]\gamma}(g) = \frac{1}{\int_{[C]\gamma} dx} \int_{[C]\gamma} g(x) dx = \frac{1}{\int_{[C]\gamma} dx_1 \ldots dx_n} \int_{[C]\gamma} g(x_1, \ldots, x_n) dx_1 \ldots dx_n.$$ 

Furthermore, if $[C]\gamma$ is a degenerated set then we compute $\mathcal{C}_{[C]\gamma}(g)$ as the limit case of a uniform approximation of $[C]\gamma$ with non-degenerated sets [5].

Let us denote the projection functions on $\mathbb{R}^2$ by $\pi_x$ and $\pi_y$, i.e. $\pi_x(u, v) = u$ and $\pi_y(u, v) = v$ for all $u, v \in \mathbb{R}$. We now recall the definitions and some basic properties of the measure of covariance and variance of the possibility distribution introduced in [4]. We present their probabilistic interpretation.

Let $C$ be a joint possibility distribution in $\mathbb{R}^2$ with marginal possibility distributions $A = \pi_x(C)$ and $B = \pi_y(C)$, and let $\gamma \in [0, 1]$. Then, the measure of interactivity between the $\gamma$-level sets of $A$ and $B$ (with respect to $[C]\gamma$) is defined by [4]

$$\mathcal{R}_{[C]\gamma}(\pi_x, \pi_y) = \mathcal{C}_{[C]\gamma}((\pi_x - \mathcal{C}_{[C]\gamma}(\pi_x))(\pi_y - \mathcal{C}_{[C]\gamma}(\pi_y))).$$

In a probabilistic sense $\mathcal{R}_{[C]\gamma}(\pi_x, \pi_y)$ computes the central value of the interactivity function $g(u, v) = (u - \mathcal{C}_{[C]\gamma}(\pi_x))(v - \mathcal{C}_{[C]\gamma}(\pi_y))$ on $[C]\gamma$. Using the definition of the central value operator we obtain

$$\mathcal{R}_{[C]\gamma}(\pi_x, \pi_y) = \mathcal{C}_{[C]\gamma}(\pi_x, \pi_y) - \mathcal{C}_{[C]\gamma}(\pi_x) \mathcal{C}_{[C]\gamma}(\pi_y)$$

$$= \frac{1}{\int_{[C]\gamma} dx dy} \int_{[C]\gamma} x y dx dy - \left( \frac{1}{\int_{[C]\gamma} dx dy} \int_{[C]\gamma} x dx dy \right) \left( \frac{1}{\int_{[C]\gamma} dx dy} \int_{[C]\gamma} y dx dy \right)$$

for any $\gamma \in [0, 1]$. In particular, $\mathcal{R}_{[C]\gamma}(\pi_x, \pi_y)$ actually computes the probabilistic covariance between random variables $X_\gamma$ and $Y_\gamma$ with a uniform joint density $f_{X_\gamma, Y_\gamma}$ on $[C]\gamma$, namely,

$$\mathcal{R}_{[C]\gamma}(\pi_x, \pi_y) = \text{cov}(X_\gamma, Y_\gamma)$$

$$= \int_{\mathbb{R}^2} x y f_{X_\gamma, Y_\gamma}(x, y) dx dy - \left( \int_{\mathbb{R}} x f_{X_\gamma}(x) dx \right) \left( \int_{\mathbb{R}} y f_{Y_\gamma}(y) dy \right).$$
It is true that if \([C]^Y = [A]^Y \times [B]^Y\) then the associated random variables \(X_Y\) and \(Y_Y\) are independent, and we obtain \(\mathcal{R}_{[C]^Y}(\pi_x, \pi_y) = \text{cov}(X_Y, Y_Y) = 0\).

Now let \(A\) be a possibility distribution in \(\mathbb{R}\) and let \(\gamma \in [0, 1]\). Then, the measure of dispersion of \([A]^Y\) is defined by

\[
\mathcal{R}_{[A]^Y} (\text{id}, \text{id}) = C_{[A]^Y} \left( (\text{id} - C_{[A]^Y} (\text{id}))^2 \right).
\]

If \(A \in \mathcal{F}\) is a fuzzy number with \([A]^Y = [a_1(\gamma), a_2(\gamma)]\, \gamma \in [0, 1]\), then from the definition of the central value operator we get

\[
\mathcal{R}_{[A]^Y} (\text{id}, \text{id}) = C_{[A]^Y} (\text{id}^2) - C_{[A]^Y}(\text{id})
\]

\[
= \frac{1}{\int_{[A]^Y} dx} \int_{[A]^Y} x^2 dx - \left( \frac{1}{\int_{[A]^Y} dx} \int_{[A]^Y} x dx \right)^2 = \frac{(a_2(\gamma) - a_1(\gamma))^2}{12}.
\]

That is, the measure of the possibilistic dispersion on a level set \([A]^Y\) is nothing but the probabilistic variance of a random variable \(U_Y\) with a uniform density \(f_U\) on \([A]^Y\), namely

\[
\mathcal{R}_{[C]^Y} (\text{id}, \text{id}) = \sigma_{U_Y}^2 = \int_{\mathbb{R}} x^2 f_{U_Y}(x) dx - \left( \int_{\mathbb{R}} x f_{U_Y}(x) dx \right)^2.
\]

### 3. The main result

In this section, we reconsider Theorem 5 of the paper [1].

Let \(C\) be a joint possibility distribution in \(\mathbb{R}^2\), let

\[
[C]^Y = \left\{ (x, y) \in \mathbb{R}^2 \mid x \in [u, v], y \in [w_1(x), w_2(x)] \right\}
\]

be a representation of \([C]^Y\), and let

\[
F(x) = w_2(x) - w_1(x), \quad x \in [u, v].
\]

#### Lemma 1 ([11]). If \([C]^Y\) is a convex subset of \(\mathbb{R}^2\) then \(F\) is a concave function.

The strong forms of the possibilistic Cauchy–Schwarz inequality of the paper [1] are based on the following theorem.

#### Theorem 1 ([11]). Let \(C\) be a joint possibility distribution with marginal possibility distribution \(A = \pi_x(C) \in \mathcal{F}\), \(B = \pi_y(C) \in \mathcal{F}\), and let \(\gamma \in [0, 1]\). If \([C]^Y\) is convex then

\[
\mathcal{R}_{[C]^Y}(\pi_x, \pi_x) \leq \mathcal{R}_{[A]^Y}(\text{id}, \text{id}).
\]

From the definition of the interactivity relation we have

\[
\mathcal{R}_{[C]^Y}(\pi_x, \pi_x) = \frac{1}{\int_{[C]^Y} dx dy} \int_{[C]^Y} x^2 dx dy - \left( \frac{1}{\int_{[C]^Y} dx dy} \int_{[C]^Y} x dx dy \right)^2
\]

and from (2) with \([u, v] = \pi_x([C]^Y) = [A]^Y\) we find that

\[
\mathcal{R}_{[C]^Y}(\pi_x, \pi_x) = \frac{1}{\int_u^v F(x) dx} \int_u^v x^2 F(x) dx - \left( \frac{1}{\int_u^v F(x) dx} \int_u^v x F(x) dx \right)^2
\]

where \(F\) is the function defined by (3). Let us introduce the notation

\[
G(x) = \frac{F(x)}{\int_u^v F(t) dt}.
\]
Then, obviously $G(x) \geq 0$ for all $x \in [u, v]$, and
\[
\int_u^v G(x)\,dx = 1.
\]

Furthermore, since $[C]'$ is convex, from Lemma 1 we have that $G$ is a concave function as well. In particular, $G$ represents a concave probability density function of some random variable $X_y$ on $[u, v] = [A]'$, where the distribution of $X_y$ is specifically defined as the first marginal of a uniform joint distribution on $[C]'$. Thus, we have
\[
\mathcal{R}_{[C]'}(\pi_x, \pi_x) = \sigma_{X_y}^2 = \int_u^v x^2 G(x)\,dx - \left(\int_u^v x G(x)\,dx\right)^2.
\]

Let $U_y$ be a uniformly distributed random variable on $[u, v] = [A]'$. Then, from (1)
\[
\mathcal{R}_{[A]'}(\text{id}, \text{id}) = \sigma_{U_y}^2 = \frac{1}{v-u} \int_u^v x^2\,dx - \left(\frac{1}{v-u} \int_u^v x\,dx\right)^2 = \frac{(v-u)^2}{12}.
\]

From the probabilistic point of view (4) states that the variance of any random variable on an interval with a concave density function can never be greater than the variance of a uniformly distributed random variable on that interval. Hence, it suffices to prove the following theorem to prove Theorem 1.

**Theorem 2.** Let $g$ be a concave probability density function in $[u, v]$; then
\[
\int_u^v (x - c)^2 g(x)\,dx \leq \frac{(v-u)^2}{12},
\]
where $\int_u^v x g(x)\,dx = c$ and the equality holds when $g$ is a constant function.

**Proof.** We note that if $g(x) = 1/(v-u)$, $x \in [u, v]$, then $\int_u^v (x - c)^2 g(x)\,dx = \frac{(v-u)^2}{12}$. By a change of variables, it suffices to show that
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} (x - c)^2 g(x)\,dx \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\,dx
\]
where $g(x)$ is a concave density function on $[-\frac{1}{2}, \frac{1}{2}]$ and $c = \int_{-\frac{1}{2}}^{\frac{1}{2}} x g(x)\,dx$. Now, since
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} (x - c)^2 g(x)\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 g(x)\,dx - c^2
\]
it suffices to prove that
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 g(x)\,dx \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\,dx.
\]

(5)

Let $(g(x) - 1 \geq 0) = [a, b] = A$ and $[-\frac{1}{2}, \frac{1}{2}] - [a, b] = B$. Since the function $g$ is a concave density function there exists a linear function $L(x)$ such that $L(a) = 1$ and $L(x) \leq g(x)$. If $a > 0$, $\int_{-\frac{1}{2}}^{\frac{1}{2}} g(x)\,dx \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} L(x)\,dx < 1$. It is a contradiction that $g$ is a density function. Similarly, we can show that $b \geq 0$, and hence $-\frac{1}{2} \leq a \leq 0 \leq b \leq \frac{1}{2}$. Without loss of generality we may assume that $|a| \geq b$ by considering $g(-x)$ instead of $g(x)$. If $b = 0$ then there exists a linear function $L(x) = cx + 1$ for some $c \leq 0$ such that $g(x) \leq L(x)$. Since $\int_{-\frac{1}{2}}^{\frac{1}{2}} L(x)\,dx = 1$, we have $g(x) = L(x)$. Then we have that $\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 (1 - g(x))\,dx = 0$, and hence (5) holds. Now we assume that $b > 0$ and let $\int_{a}^{b} (g(x) - 1)\,dx = l$. Let $A_1 = [-b, b]$ and $\int_{A_1} (g(x) - 1)\,dx = l_1$. If $\int_{b}^{1/2} (1 - g(x))\,dx < l_1$, there exists $a_1$ such that
\( a_1 < a \) and \( \int_{a_1}^{a} (1 - g(x)) dx + \int_{b_1}^{1/2} (1 - g(x)) dx = l_1 \). Let \( B_1 = [a_1, a) \cup (b_1, 1/2] \). Then we have

\[
\int_{B_1 \cup A_1} x^2 (1 - g(x)) dx = \int_{B_1} x^2 (1 - g(x)) dx + \int_{A_1} x^2 (1 - g(x)) dx
\]

\[
\geq \int_{B_1} b^2 (1 - g(x)) dx - \int_{A_1} b^2 (g(x) - 1) dx
\]

\[
= b^2 l_1 - b^2 l_1 = 0.
\]

For \( A_2 = A - A_1 = [a, -b) \) and \( B_2 = B - B_1 = [-1/2, a_1) \),

\[
\int_{B_2 \cup A_2} x^2 (1 - g(x)) dx = \int_{B_2} x^2 (1 - g(x)) dx + \int_{A_2} x^2 (1 - g(x)) dx
\]

\[
\geq \int_{B_2} a^2 (1 - g(x)) dx - \int_{A_2} a^2 (g(x) - 1) dx
\]

\[
= a^2(l - l_1) - a^2(l - l_1) = 0.
\]

Since \( A_i, B_i, i = 1, 2 \) are a partition of \([-1/2, 1/2] \), (5) holds. Now, suppose that \( \int_{b_1}^{1/2} (1 - g(x)) dx \geq l_1 \); then we can choose \( b_1 \) such that \( b < b_1 \) and \( \int_{b_1} (1 - g(x)) dx = l_1 \) where \( B_1 = (b, b_1] \). If \( -b_1 \leq a \), then we easily have that

\[
\int_{B_1 \cup A_1} x^2 (1 - g(x)) dx = \int_{B_1} x^2 (1 - g(x)) dx + \int_{A_1} x^2 (1 - g(x)) dx
\]

\[
\geq \int_{B_1} b^2 (1 - g(x)) dx - \int_{A_1} b^2 (g(x) - 1) dx
\]

\[
= b^2 l_1 - b^2 l_1 = 0.
\]

For \( A_2 = A - A_1 \) and \( B_2 = B - B_1 \),

\[
\int_{B_2 \cup A_2} x^2 (1 - g(x)) dx = \int_{B_2} x^2 (1 - g(x)) dx + \int_{A_2} x^2 (1 - g(x)) dx
\]

\[
\geq \int_{B_2} a^2 (1 - g(x)) dx - \int_{A_2} a^2 (g(x) - 1) dx
\]

\[
= a^2(l - l_1) - a^2(l - l_1) = 0.
\]

Since \( A_i, B_i, i = 1, 2 \) are a partition of \([-1/2, 1/2] \), (5) holds. If \( -b_1 > a \), let \( \int_{A_2} (g(x) - 1) dx = l_2 \) where \( A_2 = [-b_1, -b) \). If \( \int_{b_1}^{1/2} (1 - g(x)) dx < l_2 \) there exists \( a_1 \) such that \( a_1 < a \) and \( \int_{a_1}^{a} (1 - g(x)) dx + \int_{b_1}^{1/2} (1 - g(x)) dx = l_2 \). Let \( B_2 = (a_1, a) \cup (b_1, 1/2] \). Then we have

\[
\int_{B_2 \cup A_2} x^2 (1 - g(x)) dx = \int_{B_2} x^2 (1 - g(x)) dx + \int_{A_2} x^2 (1 - g(x)) dx
\]

\[
\geq \int_{B_2} b_1^2 (1 - g(x)) dx - \int_{A_2} b_1^2 (g(x) - 1) dx
\]

\[
= b_1^2 l_2 - b_1^2 l_2 = 0.
\]

For \( A_3 = A - A_1 - A_2 \) and \( B_2 = B - B_1 - B_2 \),

\[
\int_{B_3 \cup A_3} x^2 (1 - g(x)) dx = \int_{B_3} x^2 (1 - g(x)) dx + \int_{A_3} x^2 (1 - g(x)) dx
\]

\[
\geq \int_{B_3} a_2^2 (1 - g(x)) dx - \int_{A_3} a_2^2 (g(x) - 1) dx
\]

\[
= a_2^2(l - l_1 - l_2) - a_2^2(l - l_1 - l_2) = 0.
\]
If \( \int_{b_1}^{1} (1 - g(x)) \, dx \geq l_2 \) we can choose \( b_2 \) such that \( b_1 < b_2 \) and \( \int_{B_2} (1 - g(x)) \, dx = l_2 \) where \( B_2 = (b_1, b_2] \). Hence we have by a similar method that

\[
\int_{B_2 \cup A_2} x^2 (1 - g(x)) \, dx \geq 0.
\]

If \( -b_2 \leq a \), then for \( A_3 = A - A_1 - A_2 \) and \( B_2 = B - B_1 - B_2 \), \( \int_{B_2 \cup A_2} x^2 (1 - g(x)) \, dx \geq 0 \) by a similar method, and hence (5) holds. If \( -b_2 > a \), we let \( A_3 = [-b_2, -b_1] \) and construct \( B_3 \) in a similar way. If this kind of process ends in a finite time, then we are done. If not, we can construct \( A_n = [-b_{n-1}, -b_{n-2}] \), \( B_n = (b_{n-1}, b_n] \) where \( -b_n > -b_{n+1} > a \) for any \( n = 1, 2, \ldots \) such that

\[
\int_{B_n} (1 - g(x)) \, dx = \int_{A_n} (g(x) - 1) \, dx, \tag{6}
\]

\[
\int_{B_n} x^2 (1 - g(x)) \, dx \geq 0. \tag{7}
\]

Then the sequence \( \{ -b_n \} \) converges to \( a \). If not, then there exists \( t_0 \in (a, -b) \) such that \( \int_{t_0}^{b} (g(x) - 1) \, dx = \int_{b}^{t_0} (1 - g(x)) \, dx \), and hence \( \int_{a}^{b} (g(x) - 1) \, dx = \int_{-\frac{1}{2}}^{0} (1 - g(x)) \, dx + \int_{-1}^{t_1} (1 - g(x)) \, dx \). Then we have by the concavity of \( g \), \( (g(t_0) - 1) (\frac{1}{2} - |t_0|) \geq (g(t_0) - 1) (t_0 - a) > \int_{a}^{b} (g(x) - 1) \, dx \geq \int_{t_0}^{\frac{1}{2}} (1 - g(x)) \, dx > (1 - g(|t_0|)) (\frac{1}{2} - |t_0|) \), so that \( g(t_0) - 1 > 1 - g(|t_0|) \). But, by the concavity of \( g \) again, we have that

\[
\int_{t_0}^{b} (g(x) - 1) \, dx \geq \frac{1}{2} (b - t_0) (g(t_0) - 1) > \frac{1}{2} ((t_0 - b) (1 - g(t_0)) \geq \int_{t_0}^{b} (1 - g(x)) \, dx,
\]

which is a contradiction. Hence \( \{ -b_n \} \) converges to \( a \). Now from the construction of \( A_i, B_i, i = 1, 2, \ldots \), we can have that \( a = -\frac{1}{2} \). If not, from (6) we have \( \int_{A} (g(x) - 1) \, dx = \int_{a}^{b} (g(x) - 1) \, dx = \int_{b}^{a} (1 - g(x)) \, dx < \int_{B} (1 - g(x)) \, dx \) which is a contradiction to \( \int_{A} (g(x) - 1) \, dx = \int_{B} (1 - g(x)) \, dx \). Now we have that from (7)

\[
\int_{-\frac{1}{2}}^{1} x^2 (1 - g(x)) \, dx = \int_{\bigcup_{n=1}^{\infty} (B_n \cup A_n)} x^2 (1 - g(x)) \, dx
\]

\[
= \sum_{n=1}^{\infty} \int_{B_n \cup A_n} x^2 (1 - g(x)) \, dx \geq 0,
\]

which completes the proof. \( \square \)

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References