Lexicographic products with high reconstruction numbers

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ABSTRACT

The reconstruction number of a graph is the smallest number of vertex-deleted subgraphs needed to uniquely determine the graph up to isomorphism. Bollobás showed that almost all graphs have reconstruction number equal to three. McMullen and Radziszowski published a catalogue of all graphs on at most ten vertices with reconstruction number greater than three. We introduce constructions that generalize the examples identified in their work. In particular, we use lexicographic products of vertex transitive graphs with certain starter graphs from the work of Myrvold and from the work of Harary and Plantholt to generate new infinite families of graphs with high reconstruction numbers. In the process, we settle a question of McMullen and Radziszowski.

1. Introduction

All graphs in this paper are assumed to be simple, finite, and undirected. The path on \( n \) vertices (of length \( n - 1 \)) is denoted \( P_n \). The neighbourhood in the graph \( G \) of a vertex \( v \) is denoted \( N_G(v) \), or simply \( N(v) \), when \( G \) is clear from the context. The closed neighbourhood of \( v \) is \( N[\cdot] = N(v) \cup \{v\} \). The automorphism group of a graph \( G \) is \( \text{Aut}(G) \). The edge \( uv \) will sometimes be written as \([u, v]\) when required for clarity (particularly in products). For other notation, we follow [17].

Given a graph \( G \) and one of its vertices, \( v \), the vertex-deleted subgraph \( G - v \) is the subgraph obtained by deleting \( v \) and all the edges incident with \( v \). The collection of all (unlabelled) vertex-deleted subgraphs is called the deck of \( G \), denoted \( \mathcal{D}(G) \). The individual members are cards. In general, \( \mathcal{D}(G) \) may contain several isomorphic cards, prompting some authors to refer to it as a multiset; however, we simply use set notation. A reconstruction of \( G \) is a graph \( H \) such that \( G \) and \( H \) have the same deck. The graph \( G \) is reconstructible if every reconstruction is isomorphic to \( G \). The Graph Reconstruction Conjecture (GRC) states that every simple, finite, undirected graph \( G \) with at least three vertices is reconstructable. It was posed by Kelly and Ulam [9,16]. In the premier issue of the Journal of Graph Theory (1977), Harary described the conjecture as [one of] the foremost unsolved problems in the field. See also [6].

We say that \( G \) is reconstructible from \( C \subseteq \mathcal{D}(G) \) if \( G \cong H \) for any graph \( H \) such that \( C \subseteq \mathcal{D}(H) \). The reconstruction number of \( G \), denoted \( \text{rn}(G) \), is the minimum \( m \) such that \( G \) is reconstructible from some \( m \) cards in its deck. Reconstruction numbers were introduced in an attempt to understand how much information is required to reconstruct a graph. They are also referred to as the existential or ally reconstruction numbers [7,13]. By contrast, the universal reconstruction number is the minimum \( k \) such that \( G \) is reconstructible from any \( k \) cards in its deck. See [4] for recent results on universal reconstruction numbers. A survey on reconstruction can be found at [3].

In 1990, Bollobás [2] proved that almost every graph has reconstruction number three. From this result, one obtains a natural question: which graphs have reconstruction number greater than three? Such graphs are said to have a high reconstruction number.
McKay [10] verified the GRC for all graphs with at most 11 vertices using Nauty. McMullen [11] and Baldwin [1] calculated the reconstruction numbers of all graphs with fewer than 11 vertices. From this, McMullen and Radziszowski [12] identified the following classes of graphs with high reconstruction numbers. Many of their classes existed already in the literature, particularly in the work of Myrvold [13] and Harary and Plantholt [7]. Since \( \text{rn}(G) = \text{rn}((\overline{G})) \), the graphs in each class come in pairs: the graph and its complement.

1. **Graphs of the form** \( pK_2 \). For \( p, c \geq 2 \), \( \text{rn}(pK_c) = c + 2 \), see, for example, [13].

2. **Other graphs of the form** \( pH \). The only two such cases with high reconstruction number and \( |V(G)| \leq 10 \) are \( 2C_4 \) and \( 2P_4 \). These graphs are both examples of a class identified in [7] (see Corollary 14).

3. **Redundantly connected cycles.** Redundantly connected cycles [12] are the lexicographic products \( C_j[\overline{K}_n] \), defined in Section 2.2. For \( n \geq 2 \), \( j \geq 3 \), \( \text{rn}(C_j[\overline{K}_n]) > n + 1 \).

4. **Partially matched cliques.** The partially matched cliques \( \text{PMC}(n,b) \) are constructed by adding \( b \) nonadjacent edges between the vertices of two complete subgraphs of order \( n \), where \( 2 \leq b \leq n - 1 \). Harary and Plantholt [7] show that \( \text{rn}(\text{PMC}(n,b)) \geq \min\{b + 1, n - b + 2\} \).

5. **The exception,** \( P_4 \). Harary and Plantholt [7] note that \( P_4 \) is most likely an exception due to its low order.

The first three classes listed above are examples of lexicographic products, which we discuss in Section 2.2. We generalize this structure to identify new infinite classes of graphs with high reconstruction number. With the exception of \( 2P_4 \) and \( C_j[\overline{K}_n] \), the above classes fit the schema identified in Harary and Plantholt [7]. Thus, our work may be viewed as using the work in [7] as a collection of starter graphs together with the lexicographic product to identify new graphs with high reconstruction number.

## 2. Main tools

### 2.1. Extensions of \( H \)

**Definition 1.** Let \( H \) be a graph of order \( n \). A graph \( H^+ \) is an \( s \)-extension of \( H \) if \( s \) cards of \( H^+ \) are isomorphic to \( H \). We say that \( H \) admits an \( s \)-extension.

As an example, the five cycle \( C_5 \) is a 5-extension of the path \( P_4 \). In fact, any vertex transitive graph on \( n + 1 \) vertices is an \( n + 1 \)-extension of its (unique) vertex-deleted subgraph, as noted in the proposition below. In Fig. 1, two non-isomorphic extensions of the graph \( \text{PMC}(5,3) \) are given. The graph \( H_2^+ \) in Fig. 1 (b) is a 4-extension of \( \text{PMC}(5,3) \); \( H_2^+ - v \) is isomorphic to \( \text{PMC}(5,3) \) for \( v \in \{u_1, u_2, u_3, w\} \). The graph \( H_1^+ \) in Fig. 1 (c) is a 3-extension of \( \text{PMC}(5,3) \); \( H_1^+ - v \) is isomorphic to \( \text{PMC}(5,3) \) for \( v \in \{u_4, u_5, w\} \).

**Proposition 2.** The following facts about extensions hold.

(a) If \( H \) admits an \( s \)-extension, then \( \overline{H} \) also admits an \( s \)-extension.

(b) The graph \( \text{PMC}(n,k) \) with \( 2 \leq k \leq n - 1 \) admits both a \( k \)-extension and an \((n - k + 1)\)-extension.

(c) Suppose that \( H^+ \) is a vertex transitive graph on \( n + 1 \) vertices. Let \( H = H^+ - v \), where \( v \in V(H^+) \). Then \( H^+ \) is an \((n + 1)\)-extension of \( H \).

(d) The complete graph \( K_{n+1} \) is an \((n + 1)\)-extension of \( K_n \).

**Proof.** (a) Suppose that \( H^+ \) is an \( s \)-extension of \( H \). Since \( \overline{H^+ - v} \) is isomorphic to \( \overline{H^+} - v \), we have that \( \overline{H^+} \) is an \( s \)-extension of \( \overline{H} \).

(b) Consider the graph \( \text{PMC}(n,k) \), and let the vertices \( u_1, u_2, \ldots, u_n \) form the first copy of \( K_n \) and the vertices \( v_1, v_2, \ldots, v_n \) form the second. Further, suppose that \( u_i \) and \( v_i \) are adjacent for \( 1 \leq i \leq k \). Obtain \( H_1^+ \) from \( \text{PMC}(n,k) \) by adding a new vertex \( w \) which is joined to each of \( u_1, u_2, \ldots, u_n \). Obtain \( H_2^+ \) from \( H_1^+ \) by joining \( w \) to \( v_{k+1} \). Then \( H_2^+ \) is an \((k + 1)\)-extension of \( \text{PMC}(n,k) \), and \( H_1^+ \) is an \((n - k + 1)\)-extension of \( \text{PMC}(n,k) \).

(c) Suppose that \( H^+ \) and \( H \) satisfy the hypotheses. Let \( V(H^+) = \{v_1, v_2, \ldots, v_{n+1}\} \). Then each \( H^+ - v_i \) is isomorphic to \( H \), since \( H^+ \) is vertex transitive.

Result (d) follows from (c). □

![Fig. 1. (a) The graph PMC(5,3); (b) a 4-extension; (c) a 3-extension.](image-url)
The following proposition presents a mechanism for constructing graphs that admit extensions. A set $S \subseteq V(G)$ is an interval if, for any $u, v \in S$ and any vertex $z \in V(G) \setminus S$, $uz \in E(G)$ if, and only if, $vz \in E(G)$. See [5,8] for more on the subject. We define a related notion: namely, a complete interval is a set $S \subseteq V(G)$ such that, for any $u, v \in S$ and any vertex $z \in V(G) \setminus \{u, v\}$, $uz \in E(G)$ if, and only if, $vz \in E(G)$. Note that the subgraph induced by a complete interval $S$ is either complete or empty. If this is not the case, then $S$ contains three vertices $u_1, u_2,$ and $u_3,$ such that $u_1u_2 \in E$ and $u_2u_3 \notin E,$ contrary to the fact that $S$ is a complete interval.

**Proposition 3.** Let $H$ be a graph with a complete interval $S = \{u_1, u_2, \ldots, u_s\}.$ Then $H$ admits an $(s + 1)$-extension.

**Proof.** Construct $H^+$ by adding a vertex $w$ to $H$ such that $w$ is joined to each vertex $x \in N(u_1) \setminus \{u_2\}.$ If $u_1u_2 \in E(H),$ then join $w$ to each $u_i, 1 \leq i \leq s;$ otherwise, $\{u_1, u_2, \ldots, u_s, w\}$ forms an independent set. Since $H^+ - u_i$ (for $1 \leq i \leq s$) and $H^+ - w$ are isomorphic to $H,$ $H^+$ is an $(s + 1)$-extension of $H.$ $\Box$

Examples which illustrate the proposition include the following: $K_n$ admits an $(n + 1)$-extension; $K_{n,m}$ admits both an $(n + 1)$-extension and an $(m + 1)$-extension; the octahedron admits a 3-extension, constructed by letting $\{u_1, u_2\}$ be any pair of antipodal vertices.

### 2.2. The lexicographic product

The lexicographic product, also known as the wreath product or graph composition, is well studied. We use ideas from [14,15]. The best way to think of the lexicographic product $G[H]$ is as replacing each vertex $u$ of $G$ by a copy $H_u$ of $H$ and adding all edges between two copies that replace adjacent vertices. Formally, we have the following.

**Definition 4.** Let $G$ and $H$ be graphs. The lexicographic product of $G$ around $H$, denoted $G[H]$, is the graph with vertex set $V(G) \times V(H), \text{where } \{(g_1, h_1), (g_2, h_2)\}$ is an edge if

(i) $g_1 = g_2$ and $h_1h_2 \in E(H),$ or

(ii) $g_1g_2 \in E(G).$

The product can be generalized as follows.

**Definition 5.** Let $G$ be a graph of order $n$, together with a family of $n$ graphs $\{H_u : u \in V(G)\}.$ The lexicographic sum $\sum_{u \in V(G)} H_u$ is the graph with

(i) $V(\sum_{u \in V(G)} H_u) = \{(u, x) : u \in V(G), x \in V(H_u)\}$;

(ii) $E(\sum_{u \in V(G)} H_u) = \{[(u, x), (v, y)] : \text{either } uv \in E(G), \text{or } u = v \text{ and } xy \in E(H_u)\}.$

Thus the lexicographic sum is obtained from $G$ by replacing each vertex $u$ by a copy of $H_u$ and by joining all vertices of $H_u$ to all vertices of $H_v$ if, and only if, $uv \in E(G).$ If $H_u \cong H_v \cong H$ for all $u, v \in V(G)$ and some $H$, then $\sum_{u \in V(G)} H_u = G[H]$ is the lexicographic product of $G$ around $H.$ The process of replacing $u$ by $H_u$ is realized through a Cartesian product; thus, a vertex $x$ in $H_u$ becomes a vertex $(u, x)$ in the sum. For ease of notation, we identify the subgraph induced by $\{u\} \times V(H_u)$ with $H_u.$ We refer to $H_u$ as a fibre.

See Fig. 2(a) for an example of the lexicographic product $C_6[P_4],$ where $V(C_6) = \{0, 1, \ldots, 5\}.$ Also in Fig. 2(b) is the lexicographic sum $\sum_{u \in V(C_6)} H'_u,$ where $H'_0 = C_5, H'_1 = H'_2 = H'_3 = P_4,$ and $H'_4 = P_3.$ Notice that $H'_6 = C_5$ is a 5-extension of $P_4.$
Deleting a vertex from $H' = C_5$ in Fig. 2(b) produces a graph isomorphic to $C_6[P_4] - v$, where $v$ is an end vertex of $P_4$ in any one of the fibres of $C_6[P_4]$.

We use the lexicographic product to construct graphs with high reconstruction numbers. The key intuition is that the automorphism group of $G[H]$ is large. (See [2], where the fact that almost all graphs have trivial automorphism groups is used to establish that almost all graphs have reconstruction number three.) In particular, suppose that $V(G) = \{1, 2, \ldots, n\}$.

Choose $n$ automorphisms $\pi_i \in \operatorname{Aut}(H)$ for $1 \leq i \leq n$. Let $\phi \in \operatorname{Aut}(G)$. Then a group action on $G[H]$ can be defined by $\rho(i, h_j) = (\phi(i), \pi_j(h_j))$. In other words, the mapping $\phi$ permutes the fibres, and each $\pi_i$ acts independently on $H_i$. That is, $\operatorname{Aut}(G[H]) \supseteq \operatorname{Aut}(G) \rtimes \operatorname{Aut}(H)$, the wreath product of the two automorphism groups $\operatorname{Aut}(G)$ and $\operatorname{Aut}(H)$.

The automorphism group $\operatorname{Aut}(G)$ can be strictly larger than $\operatorname{Aut}(G) \rtimes \operatorname{Aut}(H)$. That is, there can be automorphisms that do not act faithfully on the fibres of $G[H]$. For example, $C_4[\overline{K}_2]$ admits an automorphism that interchanges two vertices in antipodal fibres while fixing the other vertex of each fibre.

We will require a result of Sabidussi [14] that gives necessary and sufficient conditions for $\operatorname{Aut}(G[H]) = \operatorname{Aut}(G) \rtimes \operatorname{Aut}(H)$.

These ideas are only needed in Section 3.2, but we include them here to complete our discussion on products. Specifically, we will wish to conclude that two lexicographic sums are non-isomorphic when their terms $(H_t)$ are non-isomorphic. This is not true in general, but it follows from Sabidussi’s conditions below. To a pair of graphs $(G, H)$ we associate two conditions.

**Condition $S_1$** : $H$ is connected whenever $G$ has two vertices $u \neq v$ such that $N(u) = N(v)$.

**Condition $S_2$** : $\overline{H}$ is connected whenever $G$ has two vertices $u \neq v$ such that $N[u] = N[v]$.

**Theorem 6** (Sabidussi [14]). Let $G$ and $H$ be graphs. A necessary and sufficient condition for $\operatorname{Aut}(G[H]) = \operatorname{Aut}(G) \rtimes \operatorname{Aut}(H)$ is that the pair $(G, H)$ satisfies both Condition $S_1$ and Condition $S_2$.

Later in the paper we will exploit the following consequences of Theorem 6. In particular, we will construct graphs of the form $G[H]$ that are vertex transitive, and for which all automorphisms act faithfully on the fibres.

**Proposition 7.** Let $G$ and $H$ be graphs. The following hold.

1. If both $G$ and $H$ are vertex transitive, then $G[H]$ is also vertex transitive.
2. If $H$ and $\overline{H}$ are both connected, then $\operatorname{Aut}(G[H]) = \operatorname{Aut}(G) \rtimes \operatorname{Aut}(H)$.

### 3. Constructions of families

We now begin the task of creating families with high reconstruction numbers.

#### 3.1. Graphs constructed from blocking sets

**Definition 8.** Following the terminology of Harary and Plantholt [7], an $m$-blocking set for $G$ is a family $\mathcal{F}$ of graphs such that $G \notin \mathcal{F}$ and, for any collection of $m$ cards in the deck of $G$, there is a graph in $F \in \mathcal{F}$ such that the same $m$ cards appear in the deck of $F$. The graph $F$ is a blocking graph for $G$.

**Proposition 9.** A graph $G$ has an $m$-blocking set if, and only if, $\operatorname{rn}(G) \geq m + 1$. Thus there exists an $m$-blocking set of $G$ and no $(m + 1)$-blocking set if, and only if, $\operatorname{rn}(G) = m + 1$.

**Example 3.** Let $F = P_3 \cup K_1$. The set $\{F, \overline{F}\}$ is a 3-blocking set for $P_4$. Any collection of three cards from $\mathcal{D}(P_4)$ consists of one copy of $P_3$ and two copies of $K_2 \cup K_1$ or two copies of $P_3$ and one copy of $K_2 \cup K_1$. In the first case, $F$ contains the same three cards in its deck. In the second case, $\overline{F}$ contains the same three cards in its deck. Thus $\operatorname{rn}(P_4) > 3$.

To construct a blocking graph $F$ for a graph $G$, we must ensure that $F$ is not isomorphic to $G$. The next lemma will be useful when working with lexicographic sums and blocking graphs.

Let $G$ be a graph, and suppose that $X$ is an induced subgraph of $G$ such that $V(X)$ forms an interval. Define $G'$ to be the graph obtained from $G$ by contracting $V(X)$ to a single vertex $x$ and removing multiple edges. More precisely, $G'$ is the graph defined by setting $V(G') = \{x\} \cup (V(G) \setminus V(X))$ and $E(G') = \{xv \in N(X) \setminus V(X)\} \cup E(G) \setminus \{uv : [u, v] \cap X \neq \emptyset\}$.

Finally, let $F$ and $H$ be graphs. Define

$$F^* = \sum_{v \in V(G')} H_v,$$

where $H_v \cong H$ if $v \neq x$ and $H_x \cong H$ if $v = x$.

**Lemma 10.** Let $G, X, F, F^*$ be as above and such that $F$ is not isomorphic to $X[H]$. Then $F^*$ is not isomorphic to $G[H]$.

**Proof.** If $|V(F)| \neq |V(X[H])|$, the result is obvious. Thus, assume that $|V(F)| = |V(X[H])|$. We first observe that $F$ contains an induced subgraph $T$ which is critical in the following sense: the number of induced copies of $T$ in $F$ does not equal the number of induced copies of $T$ in $X[H]$, but, for any proper induced subgraph $T'$ of $T$, the number of induced copies of $T'$ in $F$ equals the number of copies of $T'$ in $X[H]$. The existence of such a subgraph is guaranteed by the fact that $F$ and $X[H]$ are
not isomorphic, i.e. there is one copy of $F$ in $F$ and there are zero copies in $X[H]$. However, they have the same number of vertices, i.e. the number of copies of $K_1$ in $F$ equals the number in $X[H]$. Hence, there must be a critical subgraph $T$.

We now show that the number of copies of $T$ in $G[H]$ does not equal the number of copies of $T$ in $F^*$, thus establishing that the two graphs are non-isomorphic as required. Let $H' = X[G]$ be the subgraph of $G[H]$ induced by $V(X) \times V(H)$. Let $F'$ be the subgraph of $F^*$ induced by $[x] \times V(F)$. Since $G[H] - H'$ and $F^* - F'$ are isomorphic, the number of copies of $T$ in $G[H]$ not containing a vertex in $H'$ equals the number of copies of $T$ in $F^*$ not containing a vertex in $F'$. Now, consider a copy of $T$ in $G[H]$ containing some vertices of $H'$. First, consider the case that $T$ is not wholly contained in $H'$. Then $H' \cap T = T'$ is a proper induced subgraph of $T$. Thus, $F'$ contains the same number of copies of $T$ as does $H'$. From the fact that $V(X)$ is an interval of $G$, and by the structure of the lexicographic sum, each copy of $T'$ in $G[H]$ and $F^*$ is joined to exactly the same vertices in the remainder of $G[H] - H'$ and $F^* - F'$. Hence, the number of such copies of $T$ in $G[H]$ equals the number of such copies of $T$ in $F^*$. Finally, observe that the number of copies of $T$ in $H'$ does not equal the number of copies of $T$ in $F'$.

The result follows. \( \square \)

**Theorem 11.** Let $G$ be a vertex transitive graph, let $H$ be a graph with $rn(H) = m$, and let $b$ be the size of the smallest orbit under the action of $Aut(H)$. Then $rn(G[H]) \geq min(m,b + 1)$.

**Proof.** Let $V(G) = \{0, 1, \ldots, n - 1\}$, and consider $G[H]$. Since $rn(H) = m$, there exists an $(m - 1)$-blocking set of $H$ by Proposition 9. Let $F$ be such a blocking set.

We construct a blocking set $F^*$ for $G[H]$ as follows: for $F \in F'$, define $F^* \in F'$ by $F^* = \{[v] \times V(F) \mid v \in v \}$. Let $t = \min(m, b + 1)$. To see that $F^*$ is a blocking set, consider a collection of $t - 1$ cards from the deck of $G[H]$, say $C = |G[H] - v_1, G[H] - v_2, \ldots, G[H] - v_{t-1}|$.

Consider the card $G[H] - v_i$. Since $G$ is vertex transitive, we may assume without loss of generality that $v_i = (0, u_i)$ for $u_i \in H$. Moreover, since $b$ is the order of a minimum orbit in $Aut(H)$, there are at least $b$, and hence at least $t - 1$, distinct vertices in $H$ whose removal yields a subgraph isomorphic to $H - u_i$. As a result, we may assume, without loss of generality, that $v_1, v_2, \ldots, v_{t-1}$ are distinct vertices of the fibre $H_0$. In particular, $v_1 = (u_0, 0), v_2 = (u_1, 0), \ldots, v_{t-1} = (u_{t-1}, 0)$. Since $C = \{H_0 - u_1, H_0 - u_2, \ldots, H_0 - u_{t-1}\}$ is a collection of $t - 1$ cards from $D(H)$ and $t \leq m$, there is $F \in F'$ such that $D(F)$ also contains $C$. By the properties of lexicographic sums, $D(F^*)$ contains $C^*$.

The proof is complete once we establish that $G[H]$ and $F^*$ are not isomorphic. This follows directly from Lemma 10 with $V(X) = \{0\}$. \( \square \)

**Corollary 12.** Let $H$ be a graph with high reconstruction number and without an orbit of size one or two. Then, for any vertex transitive graph, $G$, the graph $G[H]$ has high reconstruction number.

**Corollary 13.** Let $H$ be a graph with smallest orbit size $b$. Let $G$ be a vertex transitive graph, and suppose that $X$ is an induced subgraph of $G$ such that $V(X)$ forms an interval. Suppose that $rn(X[H]) = m$ and that $|V(X)| \cdot b \geq m - 1$. Then $rn(G[H]) \geq m$.

**Proof.** Consider a collection $E$ of $m - 1$ cards from the deck of $G[H]$. Given any card $C$, since $G$ is vertex transitive and $|V(X)| \cdot b \geq m - 1$, there are at least $m - 1$ distinct vertices in the subgraph $X[H]$ whose removal (from $G[H]$) produces a card isomorphic to $C$. Hence, there are vertices $v_1, v_2, \ldots, v_{m-1}$ in the subgraph $X[H]$ whose individual removal from $G[H]$ produces the collection $E = \{G[H] - v_1, G[H] - v_2, \ldots, G[H] - v_{m-1}\}$.

Since $rn(X[H]) = m$, there is an $(m - 1)$-blocking set for $X[H]$. Let $F$ be a graph in this set whose deck contains $X[H] - v_i$ for $1 \leq i \leq m - 1$. Construct $F^*$ as in the preamble to Lemma 10. Clearly, the deck of $F^*$ contains the collection $E$. Moreover, by Lemma 10, $F^*$ is not isomorphic to $G[H]$. The result follows. \( \square \)

The following result of Harary and Plantholt (Theorem 3 in [7]) follows.

**Corollary 14.** Let $rn(kH) = n$. If $kb \geq n - 1$, then $rn(tH) \geq n$ for any $t \geq k$.

**Proof.** By Corollary 13, with $X = K_b$ and $G = K_b$. \( \square \)

### 3.2. Graphs constructed from s-extensions

We now turn our attention to the case that $H$ has a small reconstruction number. We use a slightly modified construction from the one described above together with the concept of s-extensions to create families with large reconstruction numbers. Our first task is to ensure that the blocking graphs we construct are not isomorphic to the graph $G[H]$. This is accomplished by using the ideas from Theorem 6.

Let $G$ be a vertex transitive graph, and fix two vertices $u \neq v$ of $G$. Let $H$ be a graph that admits an $s$-extension $H^+$, and let $H^-$ be a vertex-deleted subgraph of $H$. Consider the two graphs $A = G[H]$ and $B = \sum_{z \in V(G)} H_z$, where $H_z \cong H$ for $z \notin \{u, v\}$, $H_u \cong H^+$, and $H_v \cong H^-$. We would like to claim that the two are not isomorphic when the pair $(G, H)$ satisfies Conditions $S_1$ and $S_2$. In other words, by Theorem 6, any automorphism of $G[H]$ maps $H_z$ onto some $H_y$ for every $x, y \in V(G)$. We show that this prohibits the existence of an isomorphism from $A$ to $B$.

Suppose to the contrary that there exists an isomorphism $\phi : A \to B$. Let $H_x$ be a fibre of $A$ such that $\phi(V(H_x)) \cap V(H_y) \neq \emptyset$ for fibres $H_{x_1}, \ldots, H_{x_k}, k \geq 2$ in $B$. Such a $w \in V(G)$ and a collection $\{H_{x_1}, \ldots, H_{x_k}\}, k \geq 2$ always exist, since $H_z \cong H^-$ is a fibre in $B$ with fewer vertices than $H$. 


Lemma 15. Let \( \phi : A \to B \) be as defined above. Then \( S = \{v_1, \ldots, v_k\} \) is a complete interval in \( G \). Moreover, the pair \((G, H)\) does not satisfy Conditions \( S_1 \) and \( S_2 \).

Proof. Recall that \( H_0 \cong H^- \) is the only fibre in \( B \) with fewer vertices than \( H \). This implies that, unless \( \phi(V(H_w)) \cap V(H_v) = V(H_v) \), each fibre \( H_v \) of \( B \) contains a vertex \((z, x)\) such that \( \phi^{-1}(z, x) \notin V(H_w) \). By definition, each \( H_v \) also contains a vertex \((v_1, y)\) such that \( \phi^{-1}(v_1, y) \notin V(H_v) \). We must consider three cases.

Suppose that \( \phi(V(H_w)) \cap V(H_v) = V(H_v) \). Choose two vertices \( v_i, v_j \), \( 1 \leq i, j \leq k \), and a third vertex \( z \neq v_i, v_j \) such that \( \phi^{-1}((v_i, y)), \phi^{-1}((v_j, y)) \in V(H_v) \) and \( \phi^{-1}((z, x)) \notin V(H_w) \). Suppose that \([v_i, v_j]\) is an edge of \( G \). (The case that \([v_i, z]\) is not an edge is analogous.) This means, in particular, that \([v_i, y], (z, x)\] is an edge of \( B \), and so \([\phi^{-1}((v_i, y)), \phi^{-1}((z, x))\] must be an edge of \( A \). This is only possible if \([v, z]\) is an edge of \( G \), with \( \phi^{-1}((z, x)) \in H_\gamma \). Then clearly \([v_i, y], (z, x)\] is an edge, which in turn implies that \([v, z]\) is an edge of \( G \). Thus \( S \) forms a complete interval in \( G \) and, consequently, is either a clique or an independent set.

Suppose now that \( \phi(V(H_w)) \cap V(H_v) = V(H_v) \). This means that there is only one vertex \((w, t)\) in \( H_w \) such that \( \phi((w, t)) \notin V(H_v) \). Let \( s \in V(G) \) be such that \( \phi((w, t)) \in V(H_s) \). Hence, \( S = \{s, v\} \). As above, there is a vertex \( x \) such that \( \phi^{-1}(s, x) \notin V(H_v) \). The same argument as in the previous case shows that, if \([v, z] \in E(G)\), then \([\phi^{-1}((v, y)), \phi^{-1}((z, x))\] is an edge of \( A \) with \( \phi^{-1}((z, x)) \notin H\). Hence, \([w, t] \phi^{-1}((z, x)) \) is an edge of \( A \), implying that \([s, z] \in E(G) \). Thus, in \( G \), \( N(v) \setminus \{s\} \subseteq N(s) \setminus \{v\} \). But \( G \) is vertex transitive, so all neighbourhoods have the same size, and equality must hold. Thus \( \{s, v\} \) forms an interval. Trivially, this set induces either a complete graph (an edge) or an independent set.

Finally, we show that the pair \((G, H)\) does not satisfy Conditions \( S_1 \) and \( S_2 \). Suppose that \( S \) is independent. This implies that \( \phi(H_w) \) is disconnected, which implies that \( H \) is disconnected. However, \( N(v_1) = N(v_2) \) by the fact that \( S \) is a complete interval. Thus \((G, H)\) does not satisfy Condition \( S_1 \). On the other hand, if \( S \) is complete, then \( H \) is disconnected. Moreover, \( N(v_1) = N(v_2) \). Hence, \((G, H)\) does not satisfy Condition \( S_2 \).

Theorem 16. Let \((G, H)\) be a pair of graphs satisfying Conditions \( S_1 \) and \( S_2 \) with \( G \) vertex transitive. Suppose that \( \text{rn}(H) = m \) and that the smallest orbit in \( \text{Aut}(H) \) has size \( b \), where \( b < m - 1 \). Further, suppose that \( H \) admits an \( s \)-extension. Then \( \text{rn}(G(H)) > \min\{s, b + 1\} \).

Proof. Let \( H^- \) be an \( s \)-extension of \( H \) with \( H^+ = u_i \cong H \) for \( u_i, u_2, \ldots, u_n \in H^+ \). Let \( V(G) = \{0, 1, \ldots, n - 1\} \). Let \( t = \min\{s, b + 1\} \). Consider a collection \( T \) of \( t \) cards from \( D(G(H)) \).

Case (i): All \( t \) cards are isomorphic. Let \( v \in H_0 \) such that \( G[H] - v \) is isomorphic to the \( t \) cards. Let \( H^- = H \cup u_v \). Define \( H_0 = H_\gamma \), \( H_1 = H \gamma \), and \( H_0 = H \) for \( 2 \leq j \leq n - 1 \). Constructing \( B = \sum_{j \in V(G)} H_j \) as in Lemma 15, we obtain a graph such that \( B = (0, u_i) \), \( 1 \leq i \leq t (\leq s) \) is isomorphic to each of the \( t \) cards. By Lemma 15, \( G[H] \not\cong B \).

Case (ii): Not all \( t \) cards are isomorphic. Denote the collection of cards from the deck of \( G[H] \) by \( C^* = \{G[H] - v_1, G[H] - v_2, \ldots, G[H] - v_t\} \). Given that not all the cards are isomorphic, there are at most \( b \) cards of each isomorphism type. Thus, we may assume that each of \( v_i, v_2, \ldots, v_t \) belongs to \( H_0 \).

Since \( \text{rn}(H) = m \), there is an \( (m - 1) \)-blocking graph \( F \) for \( H \) whose deck contains \( \{H_0 - v_1, H_0 - v_2, \ldots, H_0 - v_t\} \). Constructing \( F^* \) as in Lemma 10, we obtain \( F^* \not\cong G[H] \) such that the deck of \( F^* \) contains the collection \( C^* \).

Example 4. Consider \( H = P_4 \). We know that \( \text{rn}(P_4) = 4 = m \) [7]. Let \( F = P_3 \cup K_1 \). Then the 3-blocking set is \( \{F, F\} \). The smallest orbit of \( H \) is \( b = 2 \), and \( C_5 \) is a 5-extension of \( H \). Finally, observe that, since \( H \) and \( \overline{H} \) are both connected, for any graph \( G \), the pair \((G, H)\) satisfies Conditions \( S_1 \) and \( S_2 \).

Thus, let \( G \) be a vertex transitive graph. There are two isomorphism types in \( D(G[H]) \). Let \( X \) be \( G[H] - (0, p_0) \), where \( p_0 \) is an end vertex in \( H_\gamma \cong P_4 \). Let \( Y \) be \( G[H] - (0, p_1) \), where \( p_1 \) is an interior vertex of \( P_4 \). Consider a collection of three cards. The proof of Theorem 11 shows that one can replace \( H_0 \) in \( G[H] \) with either \( F \) or \( F \) to construct a graph different from \( G[H] \), containing a collection of three cards in its deck, provided that the collection consists of two copies of \( X \) and one copy of \( Y \) or one copy of \( X \) and two copies of \( Y \).

If the collection consists of three copies of \( X \) (or three copies of \( Y \)), then we use the 5-extension argument from Theorem 16 to obtain a graph different from \( G[H] \) that has five copies of \( X \) (or \( Y \)) in its deck.

Thus every member of \( \{G[P_4] : G \text{ vertex transitive}\} \) has reconstruction number at least four. Equality is possible since \( \text{rn}(\overline{K_4}(P_4)) = 4 \).

We conclude with our main result of this section; namely, the lexicographic product can be used to construct infinite families of graphs with high reconstruction numbers starting with an appropriate seed graph \( H \).

Corollary 17. Suppose that \( H \) is a graph such that \( \text{rn}(H) = m > 3 \) and that the smallest orbit in \( \text{Aut}(H) \) has size \( b \geq 2 \). If \( H \) admits an \( s \)-extension of order \( s > b \), then, for any vertex transitive \( G \), such that \((G, H)\) satisfies Conditions \( S_1 \) and \( S_2 \), we have \( \text{rn}(G[H]) > 3 \).

Proof. If \( b(m - 1), \) then, by Theorem 16, \( \text{rn}(G[H])b + 1 \geq 3 \). If \( b \geq m - 1 \), then, by Theorem 11, \( \text{rn}(G[H]) \geq m \geq 3 \).

Recall that, in [12], McMullen and Radziszowski identify a class of graphs, which they name redundant connected cycles, with high reconstruction number. Using our notation, the redundantly connected cycles are simply \( C_j[\overline{K_n}] \) for \( j \geq 3, n \geq 2 \).
Theorem 18 ([12]). Let \( n \geq 2 \) and \( j \geq 3 \). Then \( \text{rn} (C_j [K_n]) > n + 1 \).

We generalize the above result to identify new infinite classes of graphs with high reconstruction number. This answers in the negative a question posed by McMullen and Radziszowski as to whether or not the classes they identify are the only classes with high reconstruction number. The construction below is a generalization of the redundantly connected cycles.

Theorem 19. Suppose that \( G \) and \( H \) are vertex transitive graphs each of order at least two. Further, suppose that the pair \((G, H)\) satisfies Conditions \( S_1 \) and \( S_2 \). If \( H \) has admits an s-extension, then \( \text{rn} (G[H]) \geq s + 1 \).

Proof. Since \( G \) and \( H \) are vertex transitive, \( G[H] \) is also vertex transitive. Let \( u \) and \( v \) be vertices of \( G \). Let \( H^+ \) be an s-extension of \( H \) with \( H^+ - H \cong \hat{H} \) for \( 1 \leq i \leq s \). Let \( H^- \) be a vertex-deleted subgraph of \( H \). By transitivity, all cards of \( H \) are isomorphic to \( H^- \), and thus all cards in \( D(G[H]) \) are isomorphic to \( X := \sum_{w \in V(G)} H_w \), where \( H_w \cong \hat{H} \) if \( w \neq u \) and \( H_w \cong H^- \) if \( w = u \). Construct an s-blocking graph \( B \), for \( G[H] \), as in Lemma 15. By Lemma 15, \( B \not\cong G[H] \), but, by definition of \( H^+ \), \( B - (0, \chi_i) \) for \( 1 \leq i \leq s \) is isomorphic to \( X \). The result follows. \( \square \)

The following corollary generalizes the redundantly connected cycles identified in [12]. First, observe that the role of Conditions \( S_1 \) and \( S_2 \) in Theorem 19 is to ensure that the blocking graph is not isomorphic to \( G[H] \) (as guaranteed by Lemma 15). For example, the pair \((C_4, K_2)\) does not satisfy Condition \( S_1 \). We show how the proof of Theorem 19 can fail. Let \( V(C_4) = \{0, 1, 2, 3\} \) (in the natural order around the cycle). The graph \( C_4 [K_2] \) is vertex transitive, and \( K_2 \) admits a 3-extension. However, if we let \( H_0 = K_3, H_1 = K_1, H_1 = H_3 = K_2 \), then \( B = \sum_{i \in V(C_4)} H_i \) is isomorphic to \( C_4 [K_2] \), and thus cannot be used to show that \( \text{rn} (C_4 [K_2]) > 3 \). By contrast, if one sets \( H_0 = K_3, H_1 = K_1, H_2 = H_3 = K_2 \) and \( B = \sum_{i \in V(C_4)} H_i \), then \( B \) is not vertex transitive (the vertices in \( H_0 \) and \( H_1 \) have different degrees). Thus, \( B \) is not isomorphic to \( C_4 [K_2] \), and we can conclude that \( \text{rn} (C_4 [K_2]) > 3 \). Thus with some care, the blocking graph \( B \) can be constructed as in Theorem 19.

Corollary 20. Let \( G \) be a vertex transitive graph with at least one edge. Then \( \text{rn} (G[K_n]) > n + 1 \), for all \( n \geq 2 \).

Proof. The graph \( G[K_n] \) is vertex transitive. The graph \( K_n \) admits an \((n + 1)\)-extension. Let \( uv \) be an edge of \( G \). Let \( H_u = K_{n+1} \), \( H_v = K_{n-1} \), and \( H_w = K_n \) otherwise. Then \( B = \sum_{x \in V(G)} H_x \) shares \( n + 1 \) cards with \( G[K_n] \) but \( B \not\cong G[K_n] \), as \( B \) is not vertex transitive. Thus \( \text{rn} (G[H]) \geq (n + 1) + 1 \). \( \square \)

We conclude by observing that the family of redundantly connected cycles can also be extended using Theorem 16 as \((C_n, H)\) satisfies Conditions \( S_1 \) and \( S_2 \) for any \( n \geq 5 \). As a simple example, Fig. 2 shows a graph that is neither a redundantly connected cycle nor the complement of one, yet it has a high reconstruction number. This extends the families identified in [12].

4. Questions and challenges

Challenge 1. Construct a graph with high reconstruction number and an orbit of size one.

Question 2. Can we use Conditions \( S_1 \) and \( S_2 \) to prove that the constructions produce tight bounds?

Question 3. Is there a recursive construction (along the lines of the lexicographic product) which causes the reconstruction number to grow?

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