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Reciprocally convex functions

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Abstract

We say that f is reciprocally convex if $x \mapsto f(x)$ is concave and $x \mapsto f(1/x)$ is convex on $(0, +\infty)$. Reciprocally convex functions generate a sequence of quasi-arithmetic means, with the first one between harmonic and arithmetic mean and others above the arithmetic mean. We present several examples related to the gamma function and we show that if f is a Stieltjes transform, then $-f$ is reciprocally convex. An application in probability is also presented.

Keywords: Convexity; Reciprocal convexity; Quasi-arithmetic means; Gamma function; Stieltjes transform; Mathematical expectation

1. Introduction

The motivation for this paper is the observation that the inequality between harmonic, geometric and arithmetic mean follows from the fact that the function $x \mapsto \log x$ is concave and $x \mapsto \log(1/x)$ is convex on $(0, +\infty)$. Indeed, the proof, which is not recorded in [3] among many different proofs of this inequality, goes as follows. By concavity of $x \mapsto \log x$, we have that

$$\frac{\log x + \log y}{2} \leqslant \log\left(\frac{x + y}{2}\right),$$

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and this gives geometric–arithmetic relation; another one follows from convexity of $x \mapsto \log(1/x)$,

$$\log\left(\frac{2}{x+y}\right) \leqslant \frac{\log(1/x) + \log(1/y)}{2},$$

upon replacing x, y with $1/x$, $1/y$, respectively. Obviously, concavity property of $x \mapsto \log x$ is equivalent to convexity of $x \mapsto \log(1/x)$; but it is natural to investigate what can be obtained in a general case with functions that have both properties. We call those functions reciprocally convex. In this paper we show that they have a number of interesting properties. It turns out that each reciprocally convex function is continuously differentiable on the positive real axis, and generates a sequence of quasi-arithmetic means. We also show that functions $-g$, where g is a Stieltjes transform, are reciprocally convex and we present some other examples related to the gamma function, as well as an application in probability. Some other classes of functions related to the class of convex functions are studied in [8, Chapter 8].

2. Definition and some properties

In this and subsequent sections, the interval $(0, +\infty)$ will be denoted by I.

Definition 2.1. We say that a real valued function f is reciprocally convex if f is concave on I and the function $x \mapsto f(1/x)$ is convex on I. The family of all reciprocally convex functions will be denoted by $\mathcal{F}$.

By definition, a function $f \in \mathcal{F}$ is concave on any closed interval in I, so it is continuous in its interior (see [7]), hence f has to be continuous on I. The next result can be easily proved by differentiation, but here we do not want to assume differentiability of any order, so the proof is a bit more involved.

Lemma 2.2. *A function $x \mapsto f(1/x)$ is convex on I if and only if $x \mapsto xf(x)$ is convex on I.*

Proof. By [7, 1.4.3], the function $x \mapsto f(1/x)$ is convex if and only if

$$\begin{vmatrix} x_1 & f(1/x_1) & 1 \\ x_2 & f(1/x_2) & 1 \\ x_3 & f(1/x_3) & 1 \end{vmatrix} \geqslant 0 \tag{1}$$

for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$, or, equivalently,

$$\begin{vmatrix} 1/y_1 & f(y_1) & 1 \\ 1/y_2 & f(y_2) & 1 \\ 1/y_3 & f(y_3) & 1 \end{vmatrix} \geqslant 0, \tag{2}$$

whenever $y_1 > y_2 > y_3$. Now, multiplying the ith row of a determinant in (2) by y_i, interchanging the first and third row and first and third column, and relabeling $y_3 = z_1$, $y_2 = z_2$ and $y_1 = z_3$, we arrive at

$$\begin{vmatrix} z_1 & z_1 f(z_1) & 1 \\ z_2 & z_2 f(z_2) & 1 \\ z_3 & z_3 f(z_3) & 1 \end{vmatrix} \geqslant 0 \tag{3}$$

for all $z_1, z_2, z_3 \in I$ such that $z_1 < z_2 < z_3$, which is equivalent to convexity of the function $x \mapsto xf(x)$. □

Theorem 2.3. *A function f defined on I is reciprocally convex if and only if*

$$f\left(\frac{2}{1/x + 1/y}\right) \leqslant \frac{f(x) + f(y)}{2} \leqslant f\left(\frac{x+y}{2}\right) \leqslant \frac{xf(x) + yf(y)}{x+y} \tag{4}$$

for all $x, y \in I$, and if and only if

$$f\left(\frac{1}{\sum(w_i/x_i)}\right) \leqslant \sum w_i f(x_i) \leqslant f\left(\sum w_i x_i\right) \leqslant \sum \frac{w_i x_i f(x_i)}{\sum w_i x_i} \tag{5}$$

for arbitrary w_i and x_i $(i = 1, \ldots, n)$ in I, such that $\sum w_i = 1$, where all sums go from 1 to n, and n is a natural number.

Proof. Since f is a continuous function, convexity of $f(1/x)$ is equivalent to

$$f\left(\frac{2}{x+y}\right) \leqslant \frac{f(1/x + 1/y)}{2}$$

for every $x, y \in I$. By replacing x by $1/x$ and y by $1/y$, we get the first inequality in (4). Second inequality in (4) is equivalent to concavity of f. The third inequality is obtained from convexity of the function $x \mapsto xf(x)$.

Inequalities (5) are obtained in the same way, but with a general form of Jensen's inequality. □

For $x, y \in I$, let $A(x, y) = (x + y)/2$ and $H(x, y) = 2/(1/x + 1/y)$ be the simple arithmetic and harmonic mean, respectively.

Theorem 2.4. *If $f \in \mathcal{F}$, then f is either increasing on I or f has a constant value on I.*

Proof. Given $a, b \in I$, $a < b$, let $x = b - \sqrt{b(b-a)}$ and $y = b + \sqrt{b(b-a)}$. Then $a = H(x, y)$, $b = A(x, y)$ and by (4) we have that $f(a) \leqslant f(b)$, so each $f \in \mathcal{F}$ is non-decreasing. Hence, if $a < b$ and $f(a) = f(b) = \gamma$ then $f(x) = \gamma$ for all $x \in [a, b]$. Therefore, assuming that $f(x) = \gamma$ at two different points, the set $J = \{x \in I \mid f(x) = \gamma\}$ is a non-empty interval; if its left endpoint is not zero, it is closed by continuity of f. If $J = I$, the proof is finished, otherwise J has the left endpoint greater than 0 or the right endpoint less than $+\infty$. Considering the first case, let $a > 0$ be the left endpoint of J and let x and y, $x < a$, $y \in J$, be two real numbers such that $a = H(x, y)$. Then $A(x, y) \in J$ and by (4), we have that

$$\gamma = f(a) \leqslant \frac{f(x) + f(y)}{2} \leqslant f\big(A(x, y)\big) = \gamma,$$

and since $f(y)=\gamma$, it follows that $f(x)=\gamma$, which is a contradiction, because $x \notin J$. Similarly, assuming that J has a right endpoint b, we arrive at contradiction by taking $x \in J$ and $y > b$ such that $b = A(x,y)$, showing that $f(y)=\gamma$.

Theorem 2.5. *If $f \in \mathcal{F}$, then f has a continuous derivative on I.*

Proof. If $f \in \mathcal{F}$, then f is concave and the function g, defined by $g(x) = xf(x)$ is convex on I. This implies that left and right derivatives of functions f and g exist at each point of I and [7]

$$f'_-(x) \geqslant f'_+(x), \qquad g'_-(x) \leqslant g'_+(x),$$

which, by noticing that $g'_\pm(x) = f(x) + xf'_\pm(x)$, implies that the derivative of f exists at any point $x \in I$. By concavity of f, f' is non-increasing, and, being a derivative, it is continuous. □

Remark 2.6. Due to the Theorem 2.4, all inequalities in (4) become equalities for some $x, y \in I$, $x \neq y$, if and only if $f(x)$ is a constant on I, and then, they are equalities for all $x, y \in I$. The second inequality in (4), being a consequence of concavity, becomes equality for all $x, y \in I$ only for $f(x) = ax + b$, which is in $\mathcal{F}$ for $a \geqslant 0$. The first and the third inequality in (4) become equalities only for $f(x) = a/x + b$, which is in $\mathcal{F}$ if $a \leqslant 0$. Therefore, if $f \in \mathcal{F}$ is not constant, then for each $x \neq y$ either second or first and third inequalities are strict. The analogous statement holds for inequalities in (5). The next example shows that indeed for non-constant functions all inequalities in (4) need not be strict.

Example 2.7. Let

$$f(x) = x \quad (0 < x < 2), \qquad f(x) = 4 - \frac{4}{x} \quad (x \geqslant 2).$$

This function is everywhere differentiable, and its derivative is non-increasing on I, so the function is concave. Further, $(xf(x))'$ is non-decreasing and so $x \mapsto xf(x)$ is convex on I. Therefore, $f \in \mathcal{F}$. Note that f does not have second derivative at $x = 2$.

With this function, for $x, y \in (0,2]$ the second inequality in (4) becomes equality, and if $x, y \in [2, +\infty)$, then the first and the third inequality become equality.

3. Applications to quasi-arithmetic means

By Theorem 2.4, each non-constant function $f \in \mathcal{F}$, has an inverse function f^{-1}, which is monotone on its domain. Therefore, inequalities (4) are equivalent to

$$\frac{2}{1/x + 1/y} \leqslant f^{-1}\left(\frac{f(x)+f(y)}{2}\right) \leqslant \frac{x+y}{2} \leqslant f^{-1}\left(\frac{xf(x)+yf(y)}{x+y}\right), \tag{6}$$

or, in general, (5) is equivalent to

$$\frac{1}{\sum (w_i/x_i)} \leqslant f^{-1}\left(\sum w_i f(x_i)\right) \leqslant \sum w_i x_i \leqslant f^{-1}\left(\sum \frac{w_i x_i f(x_i)}{\sum w_i x_i}\right), \tag{7}$$

where $\sum w_i = 1$. The expressions

$$Q_0 = Q_0(x_1,\dots,x_n; w_1,\dots,w_n) = f^{-1}\Bigl(\sum w_i f(x_i)\Bigr),$$

and

$$Q_1 = Q_1(x_1,\dots,x_n; w_1,\dots,w_n) = f^{-1}\left(\sum \frac{w_i x_i f(x_i)}{\sum w_i x_i}\right)$$

are so-called quasi-arithmetic means. They have been investigated in details and a lot is known about them, see [3, Chapter 4] for a survey. By (6) we see that Q_0 is always between harmonic and arithmetic mean, and in that sense it can be understood as a generalization of the geometric mean. Further, by Remark 2.6, it follows that Q_0 is strictly less than Q_1. In the next theorem we will see that for two subclasses of $\mathcal{F}$, we can localize Q_0 more precisely. In a general setup, we will use the following notations:

$$A = A(x_1,\dots,x_n; w_1,\dots,w_n) = \sum w_i x_i,$$
$$G = G(x_1,\dots,x_n; w_1,\dots,w_n) = \prod x_i^{w_i},$$
$$H = H(x_1,\dots,x_n; w_1,\dots,w_n) = \frac{1}{\sum (w_i/x_i)},$$

where $w_i \in I$, $\sum w_i = 1$, $x_i \in I$ for all $i = 1,\dots,n$.

Theorem 3.1. *Let $\mathcal{F}_1$ be the class of functions defined on I, for which the function $x \mapsto f(e^x)$ is concave on $(-\infty,+\infty)$ and $x \mapsto f(1/x)$ is convex on I. Let $\mathcal{F}_2$ be the class of functions defined on I such that $x \mapsto f(e^x)$ is convex on $(-\infty,+\infty)$ and $x \mapsto f(x)$ is concave on I. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are subsets of $\mathcal{F}$. Moreover, if $f \in \mathcal{F}_1$, then $H \leqslant Q_0 \leqslant G$ and if $f \in \mathcal{F}_2$, then $G \leqslant Q_0 \leqslant A$ for all $w_i, x_i \in I$, $i = 1,\dots,n$, $\sum w_i = 1$.*

Proof. Firstly, we note that functions from classes $\mathcal{F}_1$ and $\mathcal{F}_2$ are continuously differentiable, which can be proved using arguments as in Theorem 2.5.

Let $f \in \mathcal{F}_1$. Then the derivative of $f(e^x)$ is decreasing on $(-\infty,+\infty)$; by a change of variable $t = e^x$ we see that the function

$$t \mapsto tf'(t) \quad \text{is non-increasing on } I. \tag{8}$$

Further, by Lemma 2.2, the function

$$t \mapsto tf'(t) + f(t) \quad \text{is non-decreasing on } I. \tag{9}$$

From these two conditions, we conclude that $f(t)$ is non-decreasing in $t \in I$, and so, $f'(t) \geqslant 0$. Then from (8) we conclude that $f'(t)$ is non-increasing on I, that is, the function $x \mapsto f(x)$ is concave. So, $f \in \mathcal{F}$ and $\mathcal{F}_1 \subset \mathcal{F}$.

Consider now a function $f \in \mathcal{F}_2$. We have that

$$t \mapsto tf'(t) \quad \text{is non-decreasing on } I \tag{10}$$

and

$$t \mapsto f'(t) \quad \text{is non-increasing on } I. \tag{11}$$

If for some $t \in I$, $f'(t) < 0$, by continuity of f', there would exist an interval (a, b) such that $f'(t) < 0$ for all $t \in (a, b)$. Then from (11) it would follow that $t \mapsto tf'(t)$ is decreasing on $[a, b]$, which is a contradiction with (10). Therefore, $f'(t) \geqslant 0$ for all $t \in I$ and so, the function $f(t)$ is non-decreasing in t, which, together with (10), implies that $t \mapsto tf'(t) + f(t)$ is non-decreasing, that is, the function $t \mapsto tf(t)$ is convex on I, $f \in \mathcal{F}$ and consequently $\mathcal{F}_2 \subset \mathcal{F}$.

The rest of the statements follow by application of Jensen's inequality. For simplicity we give the proof for the case of A, G, H, Q_0 for two numbers x, y, with $w_1 = w_2 = 1/2$.

Let $f \in \mathcal{F}_1$. Then concavity of $u \mapsto f(e^u)$ gives

$$\frac{f(e^u) + f(e^v)}{2} \leqslant f(e^{(u+v)/2})$$

for any real u, v. Introducing $x = e^u$, $y = e^v$, we have that

$$\frac{f(x) + f(y)}{2} \leqslant f(\sqrt{xy}) \tag{12}$$

for any $x, y \in I$, which, combined with the first inequality from (4) yields

$$f\left(\frac{2}{1/x + 1/y}\right) \leqslant \frac{f(x) + f(y)}{2} \leqslant f(\sqrt{xy}).$$

By applying f^{-1} here we get $H \leqslant Q_0 \leqslant G$. For $f \in \mathcal{F}_2$, from the reverse of (12) and the second inequality from (4) we get

$$f(\sqrt{xy}) \leqslant \frac{f(x) + f(y)}{2} \leqslant f\left(\frac{x + y}{2}\right),$$

which yields $G \leqslant Q_0 \leqslant A$. The general case with arbitrary weights is handled in the same way. □

Remark 3.2. (1) It is not true that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$; it can be seen from 4.2.

(2) In a recent paper [5], a class of functions f such that $x \mapsto \log f(e^x)$ is convex on I, is considered. Those functions are called geometrically convex. If f is geometrically convex, then $x \mapsto f(e^x)$ is convex, but the converse is not true. A concave geometrically convex function is in the class $\mathcal{F}_2$ and generates a mean $f^{-1}(\sqrt{f(x)f(y)})$, which is always between G and Q_0.

(3) If f is twice differentiable, then $f \in \mathcal{F}$ if and only if $f''(x) \leqslant 0$ and $xf''(x) + 2f'(x) \geqslant 0$, $f \in \mathcal{F}_1$ if and only if $xf''(x) + f'(x) \leqslant 0$ and $xf''(x) + 2f'(x) \geqslant 0$, and $f \in \mathcal{F}_2$ if and only if $f''(x) \leqslant 0$ and $xf''(x) + f'(x) \geqslant 0$, for all $x \in I$.

In the next theorem we show that Q_0 and Q_1 can be extended into an increasing sequence of quasi-arithmetic means Q_r with the property that $Q_r \geqslant A$ for $r \geqslant 1$.

Theorem 3.3. *For a non-constant $f \in \mathcal{F}$, and positive weights $w_1, \ldots, w_n$, with $\sum w_i = 1$, define*

$$Q_r = Q_r(x_1, \ldots, x_n; w_1, \ldots, w_n) = f^{-1}\left(\sum \frac{w_i x_i^r f(x_i)}{\sum w_i x_i^r}\right).$$

Then for $r = 1, 2, \ldots,$

$$Q_r \leqslant A\big(x_1, \ldots, x_n; w_1^{(r)}, \ldots, w_n^{(r)}\big) \leqslant Q_{r+1}, \tag{13}$$

where

$$w_i^{(r)} = \frac{w_i x_i^r}{\sum w_j x_j^r}, \quad i = 1, \ldots, n, \tag{14}$$

and where at least one inequality in (13) *is strict. Consequently,* Q_r *is strictly increasing in* r*. Moreover, if* $M = \max\{x_1, \ldots, x_n\}$*, then*

$$\lim_{r \to +\infty} Q_r = M.$$

Proof. By concavity of f, we have

$$f(Q_r) = \sum w_i^{(r)} f(x_i) \leqslant f\Big(\sum w_i^{(r)} x_i\Big),$$

and this (knowing that a non-constant function in $\mathcal{F}$ possesses an increasing inverse) gives the first inequality in (13). For the second one, we use convexity of xf,

$$f\Big(\sum w_i^{(r)} x_i\Big) \leqslant \frac{\sum w_i^{(r)} x_i f(x_i)}{\sum w_i^{(r)} x_i} = f(Q_{r+1}).$$

From the discussion in Remark 2.6, it follows that at least one of two inequalities in (13) is strict, and so $Q_r < Q_{r+1}$. The limit follows from the definition of Q_r. □

4. Examples

Note that the class $\mathcal{F}$ is closed under all finite linear combinations with non-negative weights and their limits. This implies that if a function $x \mapsto f(x,t)$ is in $\mathcal{F}$ for all $t \in A$, where A is a measurable set, and if μ is a non-negative measure, then also the function

$$F(x) = \int_A f(x,t)\, d\mu(t)$$

belongs to $\mathcal{F}$, provided that the integral is finite for all $x \in I$. These observations, together with the Remark 3.2(3), are used in proofs that are omitted in examples below.

4.1. Functions $x \mapsto (x+c)^p$ are in $\mathcal{F}_2$, for any $p \in [0,1]$ and $c > 0$. The corresponding quasi-arithmetic means, for $p \in (0,1]$, are given by

$$Q_r = \left(\frac{\sum w_i x_i^r (x_i + c)^p}{\sum w_i x_i^r}\right)^{1/p} - c.$$

Note that for $r = c = 0$ we get the usual power mean of order p.

4.2. Functions of the form

$$x \mapsto \frac{ax+b}{cx+d}, \quad ad-bc>0,\ c \geqslant 0,\ d \geqslant 0,\ c^2+d^2>0, \tag{15}$$

are in $\mathcal{F}$. For $d=0$ and $c>0$, these functions are in $\mathcal{F}_1$, and for $c=0$ and $d>0$, functions are in $\mathcal{F}_2$. If $c>0$ and $d>0$, the function is neither in $\mathcal{F}_1$ nor in $\mathcal{F}_2$.

In particular, the function $x \mapsto -1/(x+t)$ is in $\mathcal{F}$, for all $t \in I$. Therefore, any function of the form

$$f(x) = c - \int_0^{+\infty} \frac{d\mu(t)}{x+t}, \quad x \in I, \tag{16}$$

is also in $\mathcal{F}$, where c is a real constant, and μ is a non-negative measure, such that

$$\int_0^{+\infty} \frac{d\mu(t)}{t} < +\infty.$$

Functions of the form $-f$, where f is given by (16), with $c<0$, belong to the class of Stieltjes transforms, or Stieltjes cone $\mathcal{S}$, see [1] for details on $\mathcal{S}$. Therefore, if $g \in \mathcal{S}$, then $f = -g$ is in $\mathcal{F}$.

4.3. Some functions related to the gamma function

For the logarithmic derivative of the gamma function,

$$\Psi(x) = \frac{d}{dx} \log \Gamma(x),$$

the following representation holds (see [4], for example):

$$\Psi(x) = \lim_{n \to +\infty} \left(\log n - \frac{1}{x} - \frac{1}{x+1} - \cdots - \frac{1}{x+n} \right). \tag{17}$$

By Section 4.2, all terms in the sum in (17) are in $\mathcal{F}$ and therefore, the function $x \mapsto \Psi(x) \in \mathcal{F}$. Moreover, in the same way we can see that the function $x \mapsto \Psi(x+t) \in \mathcal{F}$ for any $t \in I$; by integration with respect to $t \in [\alpha, \beta]$, we find that the function

$$x \mapsto \log \Gamma(x+\beta) - \log \Gamma(x+\alpha)$$

belongs to $\mathcal{F}$ whenever $0 < \alpha < \beta$. A particular case of this difference, the function $\log Q(x, \beta) = \log(\Gamma(x+\beta)/\Gamma(x))$ is the logarithm of Gautschi's ratio Q, for which there exist a considerable literature (see, for example, [6] and references therein).

In recent paper [2], it is proved that the function $g(x) = x \log x / \log \Gamma(x+1)$ is a Stieltjes transform; therefore, the function $x \mapsto -g(x)$ belongs to $\mathcal{F}$, as we found in Section 4.2.

4.4. Applications in probability

It is easy to see that the quasi-arithmetic means and inequalities considered in Section 3, can be generalized to random variables and their expectations. For example, if X is a positive random variable with probability one, then Q_0 becomes

$$Q_0(X) = f^{-1}\big(\mathrm{E}\, f(X)\big),$$

where E denotes expectation with respect to probability measure P on I induced by X,

$$\mathrm{E}\, f(X) = \int_0^{+\infty} f(x)\, dP(x).$$

Taking, for instance, $f(x) = -1/(x+1)$, we get

$$Q_0(X) = \frac{1}{\mathrm{E}(1/(X+1))} - 1 \tag{18}$$

and from Section 3 it follows that

$$\frac{1}{\mathrm{E}(1/X)} \leqslant Q_0(X) \leqslant \mathrm{E}(X),$$

provided that expectations of X and $1/X$ exist. Note that for any positive random variable X, $Q_0(X)$ is well defined, because the expectation of $1/(1+X)$ always exists.

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