Modeling and analysis of the unilateral contact of a piezoelectric body with a conductive support

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A R T I C L E   I N F O

Article history:
Received 10 November 2008
Available online 18 April 2009
Submitted by V. Radulescu

Keywords:
Electro-viscoelastic material
Frictionless contact
Signorini’s condition
Conductive foundation
Monotone operator
Weak solution
Augmented Lagrangian method
Finite element method
Numerical simulations

A B S T R A C T

We consider a mathematical model which describes the quasistatic process of contact between a piezoelectric body and an electrically conductive support, the so-called foundation. We model the material’s behavior with a nonlinear electro-viscoelastic constitutive law; the contact is frictionless and is described with the Signorini condition and a regularized electrical conductivity condition. We derive a variational formulation for the problem and then we prove the existence of a unique weak solution to the model. The proof is based on arguments of nonlinear equations with multivalued maximal monotone operators and fixed point. Then we introduce a fully discrete scheme, based on the finite element method to approximate the spatial variable and the backward Euler scheme to discretize the time derivatives. We treat the unilateral contact conditions by using an augmented Lagrangian approach. We implement this scheme in a numerical code then we present numerical simulations in the study of two-dimensional test problems, together with various comments and interpretations.

1. Introduction

Piezoelectric materials are characterized by the coupling between the mechanical and electrical properties. This coupling leads to the appearance of electric potential when mechanical stress is present and, conversely, mechanical stress is generated when electric potential is applied. The first effect is used in mechanical sensors and the reverse effect is used in actuators, in engineering control equipment. Piezoelectric materials for which the mechanical properties are elastic are also called electro-elastic materials, and those for which the mechanical properties are viscoelastic are also called electro-viscoelastic materials. General models for piezoelectric materials can be found in [8,13,18] and in the more recent monograph [26].

Currently, there is a considerable interest in the study of contact problems involving piezoelectric materials. Thus, static frictional contact problems for electro-elastic materials were studied in [4,9,16,17,22], under the assumption that the foundation is insulated. The results in [4,9,16] concern mainly the numerical simulation of the problems while the results in [17,22] deal with the variational formulations of the problems and their unique weak solvability. Quasistatic frictional contact problems with normal compliance for electro-viscoelastic materials were investigated in [5,23] under the assumption that the foundation is insulated, and in [15] under the assumption that the foundation is electrically conductive. A dynamic frictionless contact problem with normal compliance for electro-viscoelastic materials was studied recently in [6]. There, besides the unique solvability of the problem, the influence of the electrical properties of the foundation on the contact process was

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investigated, a fully discrete scheme to approximate the problem was considered and implemented in a numerical code, and numerical simulations were provided.

The present paper represents a continuation of [6] and it deals with a mathematical model which describes the frictionless contact between a piezoelectric body and a conductive foundation. We use both the electro-viscoelastic constitutive law and the electrical contact conditions used in [6] but, unlike [6], we assume here that the process is quasistatic and the foundation is rigid. Therefore, we neglect the inertial term in the equation of motion and we model the contact with the well-known Signorini condition; as a consequence, the resulting variational formulation of the problem is different from that in [6] and represents a new mathematical model, which is in a form of a system coupling a first-order evolutionary inclusion with unilateral constraints for the displacement field with a time-dependent nonlinear equation for the electric potential field. The analysis and numerical approach of this system represent the main trait of novelty of the present paper.

The rest of the paper is structured as follows. In Section 2 we present the model for the quasistatic frictionless contact of an electro-viscoelastic body and provide explanation on the equations and boundary conditions. In Section 3 we introduce the notation, list the assumptions on problem’s data, derive the variational formulation of the problem and state our main existence and uniqueness result, Theorem 3.1. The proof of the theorem is carried out in several steps and is presented in Section 4. Next, in Section 5, we consider a fully discrete scheme to approximate the problem, based on the finite element method to approximate the spatial variable and the backward Euler scheme to discretize the time derivatives. Then we describe its numerical treatment, based on the augmented Lagrangian approach. Finally, in Section 6, we present numerical simulations in the study of a two-dimensional test problem, obtained by the implementation of the corresponding algorithm in a numerical code.

We conclude this section with some notation and preliminaries we shall use in this paper. Given a real Hilbert space $X$, we denote by $(\cdot, \cdot)_X$ the inner product on $X$ and by $\| \cdot \|_X$ the associated norm. For a function $\phi : X \to [-\infty, \infty]$ we use the notation $D(\phi)$ and $\partial \phi$ for the effective domain and the subdifferential of $\phi$, i.e.

\[
D(\phi) = \{ u \in X : \phi(u) \neq \infty \},
\]

\[
\partial \phi(u) = \{ f \in X : \phi(v) - \phi(u) \geq (f, v - u)_X \ \forall v \in X \}, \quad \forall u \in X.
\]

We use the usual notation for the Lebesgue spaces $L^p(0, T; X)$ and Sobolev spaces $W^{k,p}(0, T; X)$ where $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. Finally, we recall the following existence and uniqueness result.

**Theorem 1.1.** Let $X$ be a real Hilbert space and let $\phi : X \to [-\infty, +\infty]$ be a convex proper lower semicontinuous function. Then, for every $f \in L^2(0, T; X)$ and $u_0 \in D(\phi)$, there exists a unique function $u \in W^{1,2}(0, T; X)$ which satisfies

\[
\dot{u}(t) + \partial \phi(u(t)) \ni f(t) \quad \text{a.e. } t \in (0, T),
\]

\[
u(0) = u_0.
\]

Theorem 1.1 will be used in Section 3 in order to prove the unique weak solvability of our contact problem; its proof can be found in [10, p. 72] or [7, p. 35].

### 2. The model

We consider a body made of a piezoelectric material which occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary $\partial \Omega = \Gamma$ and a unit outward normal $\mathbf{n}$. The body is acted upon by body forces of density $\mathbf{f}_0$ and has volume free electric charges of density $\mathbf{q}_0$. It is also constrained mechanically and electrically on the boundary. To describe these constraints we assume a partition of $\Gamma$ into three open disjoint parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts $\Gamma_a$ and $\Gamma_b$, on the other hand. We assume that $\text{meas } \Gamma_1 > 0$ and $\text{meas } \Gamma_2 > 0$; these conditions allow the use of coercivity arguments in the proof of the unique solvability of the model. The body is clamped on $\Gamma_1$ and, therefore, the displacement field vanishes there. Surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2$. We also assume that the electrical potential vanishes on $\Gamma_a$ and a surface electrical charge of density $\mathbf{q}_b$ is prescribed on $\Gamma_b$. In the reference configuration the body may come in contact over $\Gamma_3$ with a rigid conductive support, the so-called foundation. The contact is frictionless and, since the foundation is assumed to be rigid, we model it with the Signorini condition in a form with a nonzero initial gap function. Also, since we assume that the foundation is conductive, there may be electrical charges on the part of the body which is in contact with the foundation and which vanish when the current gap between the body and the foundation is large enough.

We are interested in the deformation of the body on the time interval $[0, T]$. The process is assumed to be quasistatic, i.e. the inertial effects in the equation of motion are neglected. We denote by $\mathbf{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$ the spatial and the time variable, respectively, and, to simplify the notation, we do not indicate in what follows the dependence of various functions on $\mathbf{x}$ and $t$. In this paper $i, j, k, l = 1, \ldots, d$, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding component of $\mathbf{x}$. The dot above variable represents the time derivatives, i.e. $\dot{f} = \frac{df}{dt}$. 
We use the notation $\mathbb{S}^d$ for the space of second-order symmetric tensors on $\mathbb{R}^d$ and “$\cdot$”, $\| \cdot \|$ will represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively, that is $u \cdot v = u_i v_i$, $\|v\| = (v \cdot v)^{1/2}$ for $u = (u_i)$, $v = (v_i) \in \mathbb{R}^d$, and $\sigma \cdot \tau = \sigma_{ij} \tau_{ij}$, $\| \tau \| = (\tau \cdot \tau)^{1/2}$ for $\sigma = (\sigma_{ij})$, $\tau = (\tau_{ij}) \in \mathbb{S}^d$. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, given by $u_n = u \cdot v$, $u_t = u - u_n v$, $\sigma_n = \sigma_{ij} v_i v_j$, and $\sigma_t = \sigma - \sigma_n v$.

With the notation above, the classical model for the process is as follows.

**Problem $\mathcal{P}$.** Find a displacement field $u = (u_i) : \Omega \times [0, T] \rightarrow \mathbb{R}^d$, a stress field $\sigma = (\sigma_{ij}) : \Omega \times [0, T] \rightarrow \mathbb{S}^d$, an electric potential $\varphi : \Omega \times [0, T] \rightarrow \mathbb{R}$, and an electric displacement field $D = (D_i) : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ such that

\[
\sigma = A e(\dot{u}) + B e(u) - E^* E(\varphi) \quad \text{in} \quad \Omega \times (0, T),
\]

\[
D = \varepsilon e(u) + \beta E(\varphi) \quad \text{in} \quad \Omega \times (0, T),
\]

\[
\text{Div} \sigma + f_0 = 0 \quad \text{in} \quad \Omega \times (0, T),
\]

\[
\text{div} D - q_0 = 0 \quad \text{in} \quad \Omega \times (0, T),
\]

\[
u(0) = u_0 \quad \text{in} \quad \Omega.
\]

We now describe problem (2.1)–(2.12) and provide explanation of the equations and the boundary conditions.

First, Eqs. (2.1) and (2.2) represent the electro-viscoelastic constitutive law in which $e(u) = (e_{ij}(u))$ denotes the linearized strain tensor, $E(\varphi)$ is the electric field, $A$ and $B$ are the viscosity and elasticity operators, respectively, $E = (e_{ijk})$ represents the third-order piezoelectric tensor, $E^*$ is its transpose and $\beta$ denotes the electric permittivity tensor. We recall that $e_{ij}(u) = (u_{ij} + u_{ji})/2$ and $E(\varphi) = -\nabla \varphi = -(\varphi,j)$. Also, the tensors $E$ and $E^*$ satisfy the equality

\[E \sigma \cdot v = \sigma \cdot E^* v \quad \forall \sigma \in \mathbb{S}^d, \ v \in \mathbb{R}^d,
\]

and the components of the tensor $E^*$ are given by $e_{ijk}^* = e_{ikj}$. Eq. (2.1) indicates that the mechanical properties of the materials are described by a viscoelastic Kelvin–Voigt constitutive relation (see [12] for details) which takes into account the dependence of the stress field on the electric field. Relation (2.2) describes a linear dependence of the electric displacement field $D$ on the strain and electric fields; such a relation has been frequently employed in the literature (see, e.g., [8,9,18] and the references therein).

Next, Eqs. (2.3) and (2.4) are the balance equations for the stress and electric-displacement fields, respectively, in which “Div” and “div” denote the divergence operators for tensor and vector valued functions, i.e. $\text{Div} \sigma = (\sigma_{ij,i})$, $\text{div} D = (D_{i,i})$. We use these equations since the process is assumed to be quasistatic.

Conditions (2.5) and (2.6) are the electric boundary conditions; these conditions show that the displacement field and the electrical potential vanish on $\Gamma_1$ and $\Gamma_2$, respectively, while the forces and free electric charges are prescribed on $\Gamma_2$ and $\Gamma_3$, respectively. Also, (2.12) represents the initial condition in which $u_0$ is the given initial displacement field.

We turn now to the boundary conditions (2.7), (2.8), and (2.11) which describe the contact on the surface $\Gamma_3$ and in which our main interest is. First, condition (2.7) represents the well-known Signorini contact condition in which $g$ represents the gap in the reference configuration between $\Gamma_3$ and the foundation, measured along the direction of $v$. This condition was first introduced in [20] and used in a large number of papers, see for instance the references in [12,21]. Inequality $u_n \leq g$ shows that the normal displacements are restricted by the presence of the rigid support, inequality $\sigma_n \leq 0$ arises from the fact that the reaction of the foundation is towards the body and, finally, the complementarity condition $\sigma_n (u_n - g) = 0$ shows that either $\sigma_n = 0$ when there is no contact, or $u_n = g$ during the contact. Condition (2.8) shows that the friction force vanishes on the contact surface and we use it since the contact is assumed to be frictionless. This is an idealization of the process, since even completely lubricated surfaces generate shear resistance to tangential motion, see [12,21] for details.

Next, (2.11) represents a regularized electrical contact condition on $\Gamma_3$, similar to that used in [6,15], which may be obtained as follows. When there is no contact at a point on the surface (i.e. $u_n < g$), the gap is assumed to be an insulator.
(say, it is filled with air) and therefore the normal component of the electric displacement field vanishes, so that there are no free electrical charges on the surface. Thus,

\[ u_v < g \implies D \cdot v = 0. \tag{2.13} \]

During the process of contact (i.e. when \( u_v = g \)) the normal component of the electric displacement field is assumed be proportional to the difference between the potential on the body surface and the foundation. Thus,

\[ u_v = g \implies D \cdot v = k(\phi - \phi_0). \tag{2.14} \]

where \( k \) is a nonnegative constant, the electrical conductivity coefficient, and \( \phi_0 \) represents the electric potential of the foundation. We combine (2.13) and (2.14) to see that, in both of the cases above, we have

\[ D \cdot v = k \chi_{[0,\infty)}(u_v - g)(\phi - \phi_0). \tag{2.15} \]

where \( \chi_{[0,\infty)} \) is the characteristic function of the interval \([0, \infty)\), that is

\[ \chi_{[0,\infty)}(r) = \begin{cases} 0 & \text{if } r < 0, \\ 1 & \text{if } r \geq 0. \end{cases} \]

Condition (2.15) describes perfect electrical contact and is somewhat similar to the well-known Signorini contact condition. Both conditions may be over-idealizations in many applications. To make it more realistic, we regularize condition (2.15) with condition (2.11) where \( \psi_{g_0} \) and \( \psi_L \) are the functions given by

\[ \psi_{g_0}(r) = \begin{cases} 0 & \text{if } r < -g_0, \\ \frac{r + g_0}{g_0} & \text{if } -g_0 \leq r \leq 0, \\ 1 & \text{if } r > 0, \end{cases} \tag{2.16} \]

\[ \psi_L(s) = \begin{cases} -L & \text{if } s < -L, \\ s & \text{if } -L \leq s \leq L, \\ L & \text{if } s > L. \end{cases} \tag{2.17} \]

in which \( g_0 \) denotes a small positive parameter and \( L \) is a large positive constant.

This regularization is introduced here for mathematical reasons. Indeed, we need to control the bounds of the difference \( \phi - \phi_0 \) for large \( \phi \) and for this reason we introduce the truncation function \( \psi_L \); also, we need to avoid the discontinuity in the free electric charge when contact is established and therefore we regularize the indicator function of the interval \([0, \infty)\) with the Lipschitz continuous function \( \psi_{g_0} \) for \( g_0 \) small. Nevertheless, we note that this regularization does not pose any practical limitation on the applicability of the model and seems to be reasonable from physical point of view. Indeed, \( L \) may be arbitrarily large, higher than any possible peak voltage in the system and therefore in applications we can assume that \( \psi_L(\phi - \phi_0) = \phi - \phi_0 \); also, the air is electrically conductive under a critical thickness and behaves like an insulator only above a critical thickness, which justifies the use of the electrical conductivity coefficient \( k\psi_{g_0}(u_v - g) \) instead of \( k\chi_{[0,\infty)}(u_v - g) \).

Finally, note that considering \( k = 0 \) in (2.11) leads to equality

\[ D \cdot v = 0 \quad \text{on } \Gamma_3 \times (0, T), \tag{2.18} \]

which decouples the electrical and mechanical unknowns on the contact surface. Condition (2.18) models the case when the obstacle is a perfect insulator and was used in [9,16,17,22,23]. Condition (2.11), instead of (2.18), introduces strong coupling between the mechanical and the electric boundary conditions and leads to a new and nonstandard mathematical model.

Because of the contact condition (2.7), which is nonsmooth, we do not expect the problem to have, in general, any practical limitation on the applicability of the model and seems to be reasonable from physical point of view. Indeed, \( L \) may be arbitrarily large, higher than any possible peak voltage in the system and therefore in applications we can assume that \( \psi_L(\phi - \phi_0) = \phi - \phi_0 \); also, the air is electrically conductive under a critical thickness and behaves like an insulator only above a critical thickness, which justifies the use of the electrical conductivity coefficient \( k\psi_{g_0}(u_v - g) \) instead of \( k\chi_{[0,\infty)}(u_v - g) \).

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Because of the contact condition (2.7), which is nonsmooth, we do not expect the problem to have, in general, any classical solutions. For this reason, we derive in the next section a variational formulation of the problem and investigate its solvability. Moreover, variational formulations are also starting points for the construction of finite element algorithms for this type of problems.

3. Variational formulation

We turn now to the variational formulation of the problem and, to this end, we need additional notation and preliminaries. We use standard notation for the \( L^p \) and the Sobolev spaces associated with \( \Omega, \Gamma_i \) and \((0, T)\); moreover, for a function \( \zeta \in H^1(\Omega) \) we still write \( \zeta \) to denote its trace on \( \Gamma \). We recall that the summation convention applies to a repeated index. Besides the space \( L^2(\Omega)^d \) endowed with the canonic inner product \( \langle \cdot, \cdot \rangle_{L^2(\Omega)^d} \) and the associated norm \( \| \cdot \|_{L^2(\Omega)^d} \), we use the space

\[ Q = \{ (\tau_i) = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega) \}, \]

which is a real Hilbert space endowed with the inner product and the associated norm given by

\[ (\sigma, \tau)_Q = \int_\Omega \sigma_{ij} \tau_{ij} \, dx, \quad \| \sigma \|_Q = (\sigma, \sigma)_Q^{1/2}. \]
We now list the assumptions on the problem’s data. The viscosity operator $A$ and the elasticity operator $B$ are assumed to satisfy the conditions:

1. $A : \Omega \times S^d \to S^d$.
2. $A(x, \xi) = (a_{ijkh}(x)\xi_{kh}) \forall \xi = (\xi_{ij}) \in S^d$, a.e. $x \in \Omega$.
3. $a_{ijkh} = a_{kikh} \in L^\infty(\Omega)$.
4. There exists $m_A > 0$ such that $a_{ijkh}\xi_{ij}\xi_{kh} \geq m_A\|\xi\|^2 \forall \xi = (\xi_{ij}) \in S^d$, a.e. $x \in \Omega$.

(b) $B : \Omega \times S^d \to S^d$.
1. There exists $B_2 > 0$ such that $\|B(x, \xi_1) - B(x, \xi_2)\| \leq B_2\|\xi_1 - \xi_2\| \forall \xi_1, \xi_2 \in S^d$, a.e. $x \in \Omega$.
2. The mapping $x \mapsto B(x, \xi)$ is measurable on $\Omega$. for each $\xi \in S^d$.
3. The mapping $x \mapsto B(x, 0)$ belongs to $Q$.

The piezoelectric tensor $E$ and the electric permittivity tensor $\beta$ satisfy

(a) $E : \Omega \times S^d \to \mathbb{R}^d$.
1. $E(x, \tau) = (e_{ijk}(x)\tau_{jk}) \forall \tau = (\tau_{ij}) \in S^d$, a.e. $x \in \Omega$.
2. $e_{ijk} = e_{kijk} \in L^\infty(\Omega)$.
3. $\beta : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^d$.
1. $\beta(x, E) = (\beta_{ij}(x)E_j) \forall E = (E_i) \in \mathbb{R}^d$, a.e. $x \in \Omega$.
2. $\beta_{ij} = \beta_{ji} \in L^\infty(\Omega)$.
3. There exists $m_\beta > 0$ such that $\beta_{ij}(x)E_iE_j \geq m_\beta\|E\|^2 \forall E = (E_i) \in \mathbb{R}^d$, a.e. $x \in \Omega$.

The forces, tractions, volume and surface free charge densities satisfy

$$\begin{align*}
f_0 &\in L^2(0, T; L^2(\Omega)^d), \\
f_2 &\in L^2(0, T; L^2(\Gamma_b)^d), \\
q_0 &\in W^{1,2}(0, T; L^2(\Omega)), \\
q_b &\in W^{1,2}(0, T; L^2(\Gamma_b)).
\end{align*}$$

Finally, we assume that the gap function, the electrical conductivity coefficient and the potential of the foundation satisfy

$$\begin{align*}
g &\in L^2(\Gamma_3), \quad g \geq 0 \quad \text{a.e. on } \Gamma_3, \\
k &\in L^\infty(\Gamma_3), \quad k \geq 0 \quad \text{a.e. on } \Gamma_3, \\
\varphi_0 &\in L^2(\Gamma_3).
\end{align*}$$

For the displacement and electric potential fields we use the spaces

$$\begin{align*}
V &= \{ \mathbf{v} \in H^1(\Omega)^d : \mathbf{v} = 0 \text{ on } \Gamma_1 \}, \\
W &= \{ \xi \in H^1(\Omega) : \xi = 0 \text{ on } \Gamma_0 \}.
\end{align*}$$

Since $\text{meas } \Gamma_0 > 0$ it is well known that $W$ is a real Hilbert spaces endowed with the inner product $(\varphi, \xi) = (\nabla \varphi, \nabla \xi)_{L^2(\Omega)^d}$.

Also, since $\text{meas } \Gamma_1 > 0$ and the viscosity tensor satisfies assumption (3.1), it follows that $V$ is a real Hilbert spaces endowed with the inner product

$$(\mathbf{u}, \mathbf{v})_V = (A\mathbf{u}(\mathbf{u}, \mathbf{v}))_Q.$$  

Moreover, by the Sobolev trace theorem, there exist two positive constants $c_0$ and $\tilde{c}_0$ such that

$$\|\mathbf{v}\|_{L^2(\partial \Omega)^d} \leq c_0\|\mathbf{v}\|_V \quad \forall \mathbf{v} \in V, \quad \|\xi\|_{L^2(\Gamma_1)} \leq \tilde{c}_0\|\xi\|_W \quad \forall \xi \in W.$$  

Next, we define the set of admissible displacement fields $K$ and the mappings $J : V \times W \times W \rightarrow \mathbb{R}$, $f : [0, T] \rightarrow V$ and $q : [0, T] \rightarrow W$, respectively, by

$$\begin{align*}
K &= \{ \mathbf{v} \in V : \mathbf{v} \perp \varphi \leq g \text{ on } \Gamma_3 \}, \\
J(\mathbf{u}, \varphi, \xi) &= \int_{\Gamma_3} k\psi_{q_0}(u_v - g)\phi_\ell(\varphi - \varphi_0)\xi\, da, \\
(f(t), \mathbf{v})_V &= \int_{\Omega} f_0(t) \cdot \mathbf{v} \, dx + \int_{\Gamma_2} f_2(t) \cdot \mathbf{v} \, ds, \\
(q(t), \xi)_W &= \int_{\Omega} q_0(t)\xi \, dx - \int_{\Gamma_b} q_b(t)\xi\, ds.
\end{align*}$$
for all $u, v \in V, \varphi, \zeta \in W$, a.e. $t \in (0, T)$. We note that the definitions above are based on assumptions (3.5)-(3.9), which imply that the integrals in (3.13)-(3.15) are well defined. Moreover, it follows from (3.5) and (3.6) that

$$f \in L^2(0, T; V),$$
$$q \in W^{1,2}(0, T; W).$$

Using an integration by parts and equalities (3.12)-(3.15), it is straightforward to see that if $(u, \sigma, \varphi, D)$ are sufficiently regular functions which satisfy (2.3)-(2.11) then

$$u(t) \in K, \quad (\sigma(t), \varepsilon(v) - \varepsilon(u(t)))_Q \geq (f(t), v - u(t))_V,$$
$$\beta(\nabla \varphi(t), \nabla \zeta)_{L^2(\Omega)^d} + (q(t), \zeta)_W = f(u(t), \varphi(t), \zeta).$$

for all $v \in K, \zeta \in W$ and $t \in [0, T]$. We substitute (2.1) in (3.18), (2.2) in (3.19), recall that $E(\varphi) = -\nabla \varphi$ and use the initial condition (2.12) to derive the following variational formulation of Problem $\mathcal{P}$, in the terms of displacement and electric potential fields.

**Problem $\mathcal{P}_\mathcal{V}$.** Find a displacement field $u : [0, T] \to V$ and an electric potential $\varphi : [0, T] \to W$ such that

$$u(t) \in K, \quad (\beta \nabla \varphi(t), \nabla \zeta)_{L^2(\Omega)^d} + (E(\varepsilon(u(t))), \varepsilon(v) - \varepsilon(u(t)))_Q + (\sigma(t), v - u(t))_V \geq (f(t), v - u(t))_V, \quad \forall v \in K, \text{ a.e. } t \in (0, T),$$

$$u(0) = u_0.$$

To study Problem $\mathcal{P}_\mathcal{V}$ we make the regularity assumption

$$\|u\|_{L^\infty(I_1)} < \frac{m_B}{\tilde{C}_0},$$

where $m_B$ and $\tilde{C}_0$ are given in (3.4), and (3.11), respectively. We note that only the trace constant and the coercivity constant of $\beta$ are involved in (3.24); therefore, this smallness assumption involves only the geometry and the electrical data, and does not depend on the mechanical data of the problem. Moreover, it is satisfied when the obstacle is insulated, since then $k = 0$. Removing this assumption remains a task for future research, since it is made for mathematical reasons, and does not seem to relate to any inherent physical constraints of the problem.

Our main existence and uniqueness result that we state here and prove in the next section is the following.

**Theorem 3.1.** Assume (3.1)-(3.9) and (3.23)-(3.24). Then there exists a unique solution $(u, \varphi)$ to Problem $\mathcal{P}_\mathcal{V}$. Moreover, the solution satisfies

$$u \in W^{1,2}(0, T; V), \quad \varphi \in W^{1,2}(0, T; W).$$

A quadruple of functions $(u, \sigma, \varphi, D)$ which satisfies (2.1), (2.2), (3.20)-(3.22) is called a weak solution of the piezoelectric contact problem $\mathcal{P}$. It follows from Theorem 3.1 that, under the assumptions (3.1)-(3.9), (3.23)-(3.24), there exists a unique weak solution of problem $\mathcal{P}$.

To precise the regularity of the weak solution we note that the constitutive relations (2.1) and (2.2), assumptions (3.1)-(3.4) and regularity (3.25) imply that

$$\sigma \in L^2(0, T; Q), \quad D \in W^{1,2}(0, T; L^2(\Omega)^d).$$

Moreover, using again (2.1), (2.2) combined with (3.20), (3.21) and the notation (3.13)-(3.15), after standard arguments we obtain that $\text{Div}\sigma(t) = f_0(t)$ a.e. $t \in (0, T)$ and $\text{div}D(t) = q_0(t)$ for all $t \in [0, T]$. It follows now from the regularities (3.5) and (3.6) that

$$\text{Div}\sigma \in L^2(0, T; L^2(\Omega)^d), \quad \text{div}D \in W^{1,2}(0, T; L^2(\Omega)).$$

We conclude that the weak solution $(u, \sigma, \varphi, D)$ of the piezoelectric contact problem $\mathcal{P}$ has the regularity (3.25)-(3.27).
4. Proof of Theorem 3.1

The proof of Theorem 3.1 will be carried out in several steps, and is based on arguments already used in [6, 12, 24]. Since in some of the steps the modifications are straightforward, we omit the details. We assume in the following that the conditions of Theorem 3.1 hold and below we denote by \( c \) a generic positive constant which is independent of time and whose value may change from place to place.

Let \( \eta \in L^2(0, T; V) \) be given. In the first step we prove the following existence and uniqueness result for the displacement field.

**Lemma 4.1.** There exists a unique function \( u_\eta \in W^{1,2}(0, T; V) \) such that

\[
\begin{align*}
\mathbf{u}_\eta(t) & \in K, \quad (\mathcal{A}e(\mathbf{u}_\eta(t)), \mathbf{v} - \mathbf{u}_\eta(t))_Q + (\eta(t), \mathbf{v} - \mathbf{u}_\eta(t))_V \geq (f(t), \mathbf{v} - \mathbf{u}_\eta(t))_V \quad \forall \mathbf{v} \in K, \ a.e. \ t \in (0, T), \quad (4.1) \\
\mathbf{u}_\eta(0) & = \mathbf{u}_0. \quad (4.2)
\end{align*}
\]

**Proof.** Let \( I_K : V \rightarrow [-\infty, +\infty] \) denote the indicator function of the set \( K \), i.e.

\[
I_K(\mathbf{v}) = \begin{cases}
0 & \text{if } \mathbf{v} \in K, \\
\infty & \text{if } \mathbf{v} \notin K.
\end{cases}
\]

Since \( K \) is a nonempty closed convex set of \( X \), it follows that \( I_K \) is a proper convex lower semicontinuous function. Note also (3.16) implies that \( f - \eta \in L^2(0, T; V) \) and (3.23) combined with (1.1) shows that \( u_0 \in D(I_K) \). Then it follows from Theorem 1.1 that there exists a unique function \( u_\eta \in W^{1,2}(0, T; V) \) such that

\[
\mathbf{u}_\eta(t) + \partial I_K(\mathbf{u}_\eta(t)) + \eta(t) \ni f(t) \quad \text{a.e. } t \in (0, T),
\]

\[
\mathbf{u}_\eta(0) = \mathbf{u}_0.
\]

We employ now (1.2), (4.3) and (3.10) to see that \( u_\eta \) is a solution of the Cauchy problem (4.4)-(4.5) if and only if \( u_\eta \) is a solution of the Cauchy problem (4.1)-(4.2), which concludes the proof. \( \square \)

In the second step we use the displacement field \( u_\eta \) obtained in Lemma 4.1 to obtain the following existence and uniqueness result for the electric potential.

**Lemma 4.2.** There exists a unique function \( \varphi_\eta \in W^{1,2}(0, T; W) \) such that

\[
(\beta \nabla \varphi_\eta(t), \nabla \zeta)_{L^2(\Omega)} - (\mathcal{E} e(\mathbf{u}_\eta(t)), \nabla \zeta)_{L^2(\Omega)} + J(\mathbf{u}_\eta(t), \varphi_\eta(t), \zeta)_W = (q(t), \zeta)_W \quad \forall \zeta \in W, \ t \in [0, T].
\]

**Proof.** The proof of Lemma 4.2 can be found in [6]; however, for the convenience of the reader we present below the main steps in the proof. Let \( t \in [0, T] \); we use the Riesz representation theorem to define the operator \( A_\eta(t) : W \rightarrow W \) by

\[
(A_\eta(t)\varphi, \zeta)_W = (\beta \nabla \varphi(t), \nabla \zeta)_W - (\mathcal{E} e(\mathbf{u}_\eta(t)), \nabla \zeta)_W + J(\mathbf{u}_\eta(t), \varphi, \zeta),
\]

for all \( \varphi, \zeta \in W \). It is easy to see that the operator \( A_\eta(t) \) is a strongly monotone Lipschitz continuous operator on \( W \) and, therefore, there exists a unique element \( \varphi_\eta(t) \in W \) such that

\[
A_\eta(t)\varphi_\eta(t) = q(t).
\]

We combine now (4.7) and (4.8) and find that the function \( t \mapsto \varphi_\eta(t) : [0, T] \rightarrow W \) is the unique solution of the nonlinear variational equation (4.6).

We show next that \( \varphi_\eta \in W^{1,2}(0, T; W) \). To this end, let \( t_1, t_2 \in [0, T] \) and, for the sake of simplicity, we write \( \varphi_\eta(t_i) = \varphi_i, \mathbf{u}_\eta(t_i) = \mathbf{u}_i, q(t_i) = q_i \), for \( i = 1, 2 \). Using (4.6), (3.3), (3.4) and (3.13) we find

\[
m_p \| \varphi_1 - \varphi_2 \|^2_W \leq c_E \| \mathbf{u}_1 - \mathbf{u}_2 \|_V \| \varphi_1 - \varphi_2 \|_W + \| q_1 - q_2 \|_W \| \varphi_1 - \varphi_2 \|_W
\]

\[
+ \| k \|_{H^{-\infty}(\Gamma)} \int_{\Gamma} \| \psi_\eta(\mathbf{u}_1 - g) \phi_\eta(\varphi_1 - \varphi_0) - \psi_\eta(\mathbf{u}_2 - g) \phi_\eta(\varphi_2 - \varphi_0) \| \varphi_1 - \varphi_2 \| da,
\]

where \( c_E \) is a positive constant which depends on the piezoelectric tensor \( E \). We use the bounds \( \| \psi_\eta(u_1 - g) \|_1, \| \phi_\eta(\varphi_i - \varphi_0) \| \leq L \), the Lipschitz continuity of the functions \( \psi_\eta \) and \( \phi_\eta \), and (3.11) to obtain

\[
\int_{\Gamma} \| \psi_\eta(\mathbf{u}_1 - g) \phi_\eta(\varphi_1 - \varphi_0) - \psi_\eta(\mathbf{u}_2 - g) \phi_\eta(\varphi_2 - \varphi_0) \| \varphi_1 - \varphi_2 \| da
\]

\[
\leq \int_{\Gamma} \| \varphi_1 - \varphi_2 \|^2 da + \frac{L}{\eta_0} \int_{\Gamma} |\mathbf{u}_1 - \mathbf{u}_2||\varphi_1 - \varphi_2| da.
\]
and, after a Gronwall argument, we obtain

\[ \frac{1}{2} \| u_1(t) - u_2(t) \|_V^2 \leq c \int_0^t \| \eta_1(s) - \eta_2(s) \|_V^2 \, ds. \]

We combine now (4.14), (4.15) and (4.16) to see that

\[ \| \Lambda \eta_1(t) - \Lambda \eta_2(t) \|_V^2 \leq c \int_0^t \| \eta_1(s) - \eta_2(s) \|_V^2 \, ds. \]
This last inequality and the Banach fixed point theorem show that there exists a unique \( \tilde{\eta} \in L^2(0; V) \) such that \( \Lambda \tilde{\eta} = \tilde{\eta} \), which concludes the proof. □

We are now ready to prove Theorem 3.1.

Existence. Let \( \hat{\eta} \in L^2(0; V) \) be the fixed point of the operator \( \Lambda \) and let \( u \) be the solution of problem (4.1)–(4.2) with \( \eta = \hat{\eta} \), i.e., \( u = u_{\hat{\eta}} \). We also denote by \( \varphi \) the solution of problem (4.6) with \( \eta = \hat{\eta} \), i.e., \( \varphi = \varphi_{\hat{\eta}} \). Clearly, equalities (3.21) and (3.22) as well as the regularity (3.25) of the solution follow from Lemmas 4.1 and 4.2. Moreover, since \( \tilde{\eta} = \Lambda \tilde{\eta} \), it follows from (4.1) and (4.12) that (3.20) holds, too. We conclude that \( (u, \varphi) \) is a solution of Problem \( P_V \) and it satisfies (3.25).

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of \( \Lambda \) and from the uniqueness part of Lemmas 4.1 and 4.2.

5. Numerical approach

We now introduce a fully discrete scheme to approximate the solution of Problem \( P_V \). First, we consider two finite dimensional spaces \( V^h \subset V \) and \( W^h \subset W \) approximating the spaces \( V \) and \( W \), respectively, in which \( h > 0 \) denotes the spatial discretization parameter. In addition, we consider the discrete set of admissible displacements defined by \( k^h = K \cap V^h \). In the numerical simulations presented in the next section, \( V^h \) and \( W^h \) consist of continuous and piecewise affine functions, that is,

\[
V^h = \left\{ w^h \in \left[ C(\Omega) \right]^d \mid w^h_{tr} \in \left[ P_1(Tr) \right]^d \right\} \forall Tr \in T^h, \quad w^h_0 = 0 \text{ on } \Gamma_1, \tag{5.1}
\]

\[
W^h = \left\{ \zeta^h \in \left[ C(\Omega) \right] \mid \zeta^h_{tr} \in \left[ P_1(Tr) \right] \right\} \forall Tr \in T^h, \quad \zeta^h = 0 \text{ on } \Gamma_3, \tag{5.2}
\]

where \( \Omega \) is assumed to be a polygonal domain, \( T^h \) denotes a finite element triangulation of \( \Omega \), and \( P_1(Tr) \) represents the space of polynomials of global degree less or equal to one in \( Tr \). In addition, we consider a uniform partition of \([0, T]\), \( t_0 < t_1 < \cdots < t_N = T \), that we use to discretize the time derivatives and, everywhere in this section, we use the notation \( k \) for the time step size, i.e. \( k = T/N \). Finally, for a continuous function \( f(t) \) we denote \( f_n = f(t_n) \) and for a sequence \( \{w_n\}_{n=0}^N \) we use the backward Euler scheme \( \delta w_n = (w_n - w_{n-1})/k \) for the discrete differences.

We assume in what follows that \( f_0 \in C([0, T]; L^2(\Omega)^d) \) and \( f_2 \in C([0, T]; L^2(\Gamma)^d) \) where, here and below, \( C([0, T]; X) \) denotes the space of continuous functions defined on \([0, T]\) with values in \( X \). This implies that the function \( f \) defined by (3.14) satisfies \( f \in C([0, T]; V) \) and, using the backward Euler scheme, the fully discrete approximation of Problem \( P_V \) is the following.

Problem \( P_{V}^{hk} \). Find a discrete displacement field \( u^{hk} = \{u_n^{hk}\}_{n=0}^N \subset k^h \) and a discrete electric potential \( \varphi^{hk} = \{\varphi_n^{hk}\}_{n=0}^N \subset W^h \) such that

\[
\begin{align*}
\left( A e(\delta u_n^{hk}), e(w^h) - e(u_n^{hk}) \right)_Q + \left( D e(u_n^{hk}), e(w^h) - e(u_n^{hk}) \right)_Q + \left( D^* \nabla \varphi_n^{hk}, e(w^h) - e(u_n^{hk}) \right)_Q \\
+ (f_n, w^h - u_n^{hk})_V, \quad \forall w^h \in k^h,
\end{align*}
\]

\[
\begin{align*}
(\beta \nabla \varphi_n^{hk}, \nabla \zeta^h)_L^2(\Omega)^d - \left( D e(u_n^{hk}), \nabla \zeta^h \right)_L^2(\Omega)^d + j(u_n^{hk}, \varphi_n^{hk}, \zeta^h) - (q_n, \zeta^h)_h \quad \forall \zeta^h \in W^h,
\end{align*}
\]

for all \( n = 1, \ldots, N \). Here \( u_n^{hk} \) is an appropriate approximation of the initial condition \( u_0 \) and \( \varphi_0^{hk} \) is the unique solution of the second equation in Problem \( P_{V}^{hk} \) for \( n = 0 \).

The unique solvability of Problem \( P_{V}^{hk} \) follows from arguments of variational inequalities and fixed point, similar to those used in the proof of Theorem 3.1. Moreover, in the case \( g = 0 \), the numerical analysis of this problem (including error estimates and convergence results) can be provided by extending the arguments already used in [3–5,12]. Nevertheless, to keep this paper in a reasonable length, we skip this analysis and we pass in what follows to a brief description of the numerical algorithm used to solve Problem \( P_{V}^{hk} \). To this end, we consider the space of traces \( X = \{w \mid \gamma_3^h : w \in V \} \), together with his dual \( X' \), and let \( Y^h \subset X' \cap L^2(\Gamma_3) \) be a discrete multiplier space. Then, using the arguments in [14], it follows that Problem \( P_{V}^{hk} \) is equivalent with the following hybrid formulation, in which the multiplier \( \lambda^{hk} \) represents the stress on the contact boundary.

Problem \( P_{V}^{hk} \). Find a discrete displacement field \( u^{hk} = \{u_n^{hk}\}_{n=0}^N \subset V^h \), a discrete multiplier \( \lambda^{hk} = \{\lambda_n^{hk}\}_{n=0}^N \subset Y^h \) and a discrete electric potential \( \varphi^{hk} = \{\varphi_n^{hk}\}_{n=0}^N \subset W^h \) such that
(5.3)
for all \( n = 1, \ldots, N \). Here, again, \( u_0^{hk} \) is an appropriate approximation of the initial condition \( u_0 \) and \( \varphi_0^{hk} \) is the unique solution of the second equation in Problem \( P_{V_2}^{hk} \) for \( n = 0 \) and \( \lambda_0^{hk} = 0 \).

In Problem \( P_{V_2}^{hk} \), the contact functional term \( J(\lambda^{hk}, \mathbf{w}^h) \) is defined by

\[
J(\lambda^{hk}, \mathbf{w}^h) = \int_{\Gamma_3} \lambda_n^{hk} d^h_n(\mathbf{w}^h) \, da,
\]

where \( d^h_n(\cdot) \) denotes the discretized normal gap distance, i.e. \( d^h_n(\mathbf{w}^h) = \mathbf{w}^h \cdot \mathbf{v} - g^h \) for all \( \mathbf{w}^h \in V^h \) where \( g^h \) represents an appropriate approximation of the gap. In this last equality as well as in the rest of the paper, for numerical convenience, we use \( \mathbf{v} \) to denote the unit outward normal vector to the rigid conductive foundation. Recall also that \( \partial I_{\mathbb{R}^+} \) denotes subdifferential of the indicator function of the positive half-line of \( \mathbb{R} \). The inclusion (5.3) represents the unilateral contact condition between the discrete normal contact stress \( \lambda^{hk} \) and the normal gap distance \( d^h_n(\mathbf{w}^h) \). For more details concerning the equivalence of Problems \( P_{V_2}^{hk} \) and \( P_{V_1}^{hk} \) as well as the functional spaces above, we refer the reader to [14] and [25].

For the numerical treatment of Problem \( P_{V_2}^{hk} \), we use the augmented Lagrangian approach. To this end we consider additional fictitious nodes for the Lagrange multiplier in the initial mesh. The construction of these nodes depends on the contact element used for the geometrical discretization of the interface \( \Gamma_3 \). In the case of the numerical example presented in Section 6, the discretization is based on “node-to-rigid” contact element, which is composed by one node of \( \Gamma_3 \) and one Lagrange multiplier node. This contact interface discretization is characterized by a finite dimensional subspace \( H^h_{\Gamma_3} \subset Y^h \).

Let \( N_{tot} \) be the total number of nodes and denote by \( \alpha^i, \beta^i \) the basis functions of the spaces \( V^h \) and \( W^h \), respectively, for \( i = 1, \ldots, N_{tot} \). Moreover, let \( N_{\gamma} \) represent the number of nodes on the interface \( \Gamma_3 \) and let \( \mu^i \) be the shape functions of the finite element space \( H^h_{\Gamma_3} \), for \( i = 1, \ldots, N_{\gamma} \); so, \( H^h_{\Gamma_3} = \{ \gamma_h \in Y^h, \gamma_h = \sum_{i=1}^{N_{\gamma}} \gamma^i \mu^i \} \). Usually, if a \( P_1 \) finite element method is used for the displacement, then a \( P_0 \) finite element method is considered for the multipliers. Then, the expression of functions \( \mathbf{w}^h \in V^h \), \( \zeta^h \in W^h \), and \( \gamma^h \in H^h_{\Gamma_3} \), and \( \lambda^h \in H^h_{\Gamma_3} \) is given by

\[
\mathbf{w}^h = \sum_{i=1}^{N_{tot}} \mathbf{w}^i \alpha^i, \quad \forall \mathbf{w}^h \in V^h,
\]

\[
\zeta^h = \sum_{i=1}^{N_{tot}} \zeta^i \beta^i, \quad \forall \zeta^h \in W^h,
\]

\[
\gamma^h = \sum_{i=1}^{N_{\gamma}} \gamma^i \mu^i, \quad \forall \gamma^h \in H^h_{\Gamma_3},
\]

where \( \mathbf{w}^i \) and \( \zeta^i \) represent the values of the corresponding functions \( \mathbf{w}^h \) and \( \zeta^h \) at the \( i \)th node of \( T^h \). Also, \( \gamma^i \) denotes the values of the function \( \gamma^h \) at the \( i \)th node of the contact element discretization of the contact interface. More details about this discretization step can be found in [2,14,25].

The augmented Lagrangian approach we use shows that Problem \( P_{V_2}^{hk} \) can be governed by a system of nonlinear equations of the form

\[
\mathbf{R}(\delta \mathbf{u}_n, \mathbf{u}_n, \varphi_n, \lambda_n) = \mathbf{A}(\delta \mathbf{u}_n)+\mathbf{G}(\mathbf{u}_n, \varphi_n)+\mathbf{F}(\mathbf{u}_n, \varphi_n, \lambda_n) = \mathbf{0},
\]

that we describe below. First, the vectors \( \delta \mathbf{u}_n, \mathbf{u}_n, \varphi_n \) and \( \lambda_n \) represent the velocity, the displacement, the electric potential and the Lagrange multiplier generalized vectors, respectively, defined by

\[
\delta \mathbf{u}_n = \left[ \delta \mathbf{u}^n_1 \right]_{i=1}^{N_{tot}}, \quad \mathbf{u}_n = \left[ \mathbf{u}_n^1 \right]_{i=1}^{N_{tot}}, \quad \varphi_n = \left[ \varphi^i_n \right]_{i=1}^{N_{tot}}, \quad \lambda_n = \left[ \lambda^i_n \right]_{i=1}^{N_{\gamma}},
\]

(5.8)
where \( \delta \mathbf{u}^n_1, \mathbf{u}^n_1 \) and \( \varphi^i_n \) represent the values of the corresponding functions \( \delta \mathbf{u}_0^{hk}, \mathbf{u}_0^{hk} \) and \( \varphi_0^{hk} \) at the \( i \)th node of \( T^h \). Also, \( \lambda^i_n \) denote the values of the corresponding function \( \lambda^{hk}_n \) at the \( i \)th node of the contact element of the discretized contact interface. We recall that the velocity \( \delta \mathbf{u}^n_1 \) is defined by using the following backward Euler finite difference, i.e.

\[
\delta \mathbf{u}^n_1 = \frac{\mathbf{u}^n_1 - \mathbf{u}^{n-1}_1}{k},
\]

(5.9)
In addition, the generalized viscous term \( \mathbf{A}(\mathbf{v}) \in \mathbb{R}^{d \times N_{tot} \times N_{tot} \times N_{\gamma}} \) and the generalized electro–elastic term \( \mathbf{G}(\mathbf{u}, \varphi) \in \mathbb{R}^{d \times N_{tot} \times N_{tot} \times N_{\gamma}} \) are defined by \( \mathbf{A}(\mathbf{v}) = (\mathbf{A}(\mathbf{v}), \mathbf{0}_{N_{tot}}, \mathbf{0}_{N_{\gamma}}) \) and \( \mathbf{G}(\mathbf{u}, \varphi) = (\mathbf{G}(\mathbf{u}, \varphi), \mathbf{0}_{N_{\gamma}}) \). Here \( \mathbf{0}_{N_{tot}} \) is the zero element.
of $\mathbb{R}^{N_{tot} \times N_{tot}}$ and $\mathbf{0}_{N_{tot}}$ is the zero element of $\mathbb{R}^{N_{tot}}$; also, $A(v) \in \mathbb{R}^{d \times N_{tot}}$ and $G(u, \varphi) \in \mathbb{R}^{d \times N_{tot} \times N_{tot}}$ denote the viscous term and the electro-elastic term, respectively, given by

$$
\{A(v) \cdot w\}_{d \times N_{tot}} = (A e(v^h), e(w^h))_Q \quad \forall v, w \in \mathbb{R}^{d \times N_{tot}}, \; \forall v^h, w^h \in V^h,
$$

$$(G(u, \varphi) \cdot (v, \zeta))_{d \times N_{tot} \times N_{tot}} = (\mathbf{E} e(u^h), e(w^h))_Q + (\mathbf{E} e(w^h), \nabla \varphi^h)_{L^2(\Omega)^d} - (\mathbf{E} e(u^h), \nabla \varphi^h, \nabla \zeta^h)_{L^2(\Omega)^d}$$

$$- (f_n, w^h)_W \quad \forall u, \varphi, \zeta \in \mathbb{R}^{N_{tot}}, \; \forall u^h, w^h \in V^h, \; \forall \varphi^h, \varphi^h \in \mathbb{W}^h.
$$

Above, $v, u, w, \varphi$ and $\zeta$ represent the generalized vectors of components $v^i, u^i, w^i, \varphi^i$ and $\zeta^i$, for $i = 1, \ldots, N_{tot}$, respectively, and note that the volume and surface efforts are contained in the term $G(u, \varphi)$.

The contact operator $\mathcal{F}(u, \varphi, \lambda)$, which allows to deal with the unilateral contact law and to take into account the conductivity of the foundation, is given by

$$(\mathcal{F}(u, \varphi, \lambda) \cdot (v, \zeta))_{d \times N_{tot} \times d \times N_{tot} \times N_{tot}} = \int_{\Gamma} \mathbf{u}_n^h \cdot v^h \, da + \int_{\Gamma} \nabla \mathbf{u}_n^h \cdot \nabla \zeta^h \, da + J(u^h, \varphi^h, \zeta^h),$$

$$\forall u, \varphi, \zeta \in \mathbb{R}^{N_{tot}}, \; \forall u^h, \varphi^h, \zeta^h \in \mathbb{W}^h, \; \forall v^h, \varphi^h, \zeta^h \in \mathbb{W}^h.
$$

Here the Lagrangian multiplier $\lambda$ and its virtual variable $\gamma$ represent the frictionless contact forces; also, $\ell_u$ denotes the augmented Lagrangian functional given by

$$\ell_{u}(u^h, \lambda^h) = d^h(u^h) \lambda^h + \frac{r}{2} \left( d^h(u^h) \right)^2 - \frac{1}{2r} \left( \text{dist}_{\mathbb{R}^d} - \{ \lambda^h + r d^h(u^h) \} \right)^2,$$

where $r$ is a positive penalty coefficient and

\[ \text{dist}_{\mathbb{R}^d}(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases} \]

For more details about the Lagrangian method, we refer the reader to [2,25].

The solution algorithm consists in a prediction-correction scheme based on a finite differences method (the backward Euler difference method) and a linear iterations method (the Newton method). The finite difference scheme we use is characterized by a first-order time integration scheme for the velocity $\delta u_n$. To solve the nonlinear system (5.7), at each time increment the variables $(u_n, \varphi_n, \lambda_n)$ are treated simultaneously through a Newton method. Details on these classical algorithms can be found in [2,25].

6. Numerical simulations

In this section we present numerical simulations in the study of a real-world example of Problem $\mathcal{P}$, the microelectromechanical switches [19]; the numerical simulations are focused on the contact process of a deformable electro-viscoelastic body on a conductive dielectric foundation. Microelectromechanical systems (MEMS) are being recognized as enabling components to switch or tune radio frequency (rf) components, modules or systems in manufacturing and operation. In short, they are referred to as rf-MEMS. Most rf-MEMS involve the manipulation of air as the dielectric materials. Various designs of capacitive rf-MEMS switches made out of nickel, aluminum, gold or zinc oxide have so far been reported in literature, see for instance [1,11]. The mechanical simulation of switch consists in the following design concept: the switch design is based on a suspended metal bridge (zinc oxide in our example) which connects two grounds of a coplanar wave-guide and crosses over a signal line on which a dielectric foundation is deposited. When an external force is acting, this action pulls the metal bridge down and contacts the dielectric, which results in a low impedance between signal line and ground line for shunting high-frequency signal transmission.

To describe this example, we consider an electro-viscoelastic body extended indefinitely in the direction $X_1$ of a Cartesian coordinate system $(O, X_1, X_2, X_3)$. The material used is assumed to be a linearly isotropic piezoceramic with hexagonal symmetry like zinc oxide material (class 6mm in the international classification [13]) which presents a viscous behavior. In the crystallographic frame, the $X_1$-direction is a six-fold revolution symmetry axis and the $(X_1, O, X_2)$ and $(X_2, O, X_3)$ planes are mirrors. The electrical and mechanical loads applied to the body are supposed to be constant along the $X_1$ direction. As a consequence, the fields $E, D, e$ and $\sigma$ turn out to be constant along $X_1$. In addition, we suppose that $e_{11} = 0$, $e_{12} = 0$, $e_{13} = 0$ and $D_1 = 0$, i.e. we consider a plane problem. Under these assumptions, the unknown of our electroviscoelastic contact problem is the pair $(u, \varphi)$ where the displacement field $u = (u_2, u_3)$ belongs to the plane $(O, X_2, X_3)$. Let $A = (a_{ijk})$ and $B = (b_{ijk})$; then, in the system $(O, X_2, X_3)$, the constitutive equations (2.1) and (2.2) can be written by using a compressed matrix notation,

$$
\begin{align*}
\begin{pmatrix}
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
D_2 \\
D_3
\end{pmatrix}
&= 
\begin{pmatrix}
b_{22} & b_{23} & 0 & 0 & e_{32} \\
b_{23} & b_{33} & 0 & 0 & e_{33} \\
0 & 0 & b_{44} & e_{24} & 0 \\
0 & 0 & e_{24} & -\beta_{22} & 0 \\
e_{32} & e_{33} & 0 & 0 & -\beta_{33}
\end{pmatrix}
\end{align*}
\begin{align*}
\begin{pmatrix}
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{23} \\
\varepsilon_{23} \\
\varepsilon_{33}
\end{pmatrix}
&= 
\begin{pmatrix}
a_{22} & a_{23} & 0 & 0 & 0 \\
a_{23} & a_{33} & 0 & 0 & 0 \\
a_{23} & a_{33} & 0 & 0 & 0 \\
a_{23} & a_{33} & 0 & 0 & 0 \\
a_{33} & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{23} \\
\varepsilon_{23} \\
\varepsilon_{33}
\end{pmatrix}
+ 
\begin{pmatrix}
\delta_{22} \\
\delta_{33} \\
\delta_{23} \\
\delta_{23} \\
\delta_{33}
\end{pmatrix}
\end{align*}
\]
The tensors involved in the constitutive law.

Material constants given in Tables 1 and 2, in which vacuum. The rest of the data are the following:

This rule which allows to describe the link between the fourth-order tensors of components \( b_{ijkl} \) and \( a_{ijkl} \) and the corresponding second-order tensors of components \( b_{pq} \) and \( a_{pq} \), respectively, is obtained by using the symmetries of the various tensors involved in the constitutive law.

In the same way, for the third-order piezoelectric tensor we have

The geometry and the physical setting of this two-dimensional example is depicted in Fig. 1. To describe the boundary of the body \( \Omega \), we set twelve points ranging from \( P_1 \) to \( P_{12} \). The coordinates of these points are the following:

The rest of the data are the following:

\[
\begin{align*}
f_2 & = \begin{cases} 0, & (0, -25t) \text{ N/\(\mu\)m for } 0 \leq t \leq 0.1 \text{ on } (P_9, P_{10}), \\
(0, -25) \text{ N/\(\mu\)m for } t > 0.1 \text{ on } (P_9, P_{10}), 
\end{cases} \\
f_0 & = 0 \text{ N/(\(\mu\)m)}^2, \\
q_0 & = 0 \text{ C/(\(\mu\)m)}^2, \\
q_b & = 0 \text{ C/\(\mu\)m}, \\
\varphi & = 0 \text{ V on } (P_3, P_4), \\
g & = 3 \text{ \(\mu\)m, } \\
T & = 0.2 \text{ s, } \\
\mathbf{u}_0 & = 0 \text{ \(\mu\)m.}
\end{align*}
\]
Fig. 2. Dependence of the normal electric displacement $\mathbf{D} \cdot \mathbf{v}$ with respect to the electric potential of the foundation $\phi_0$, for $k = 1$.

Fig. 3. Dependence of the normal electric displacement $\mathbf{D} \cdot \mathbf{v}$ with respect to the electrical conductivity coefficient $k$, for $\phi_0 = -64$.

The conductivity function (2.16) was used with $g_0 = 10^{-5}$ μm. For numerical convenience and to focus the study on the contact process, we simulate the electrostatic pressure by a uniform mechanical load $f_2$ acting on the bridge’s center, i.e. on the boundary $(P_9, P_{10})$.

We now provide explanation and comments on our numerical results, presented in Figs. 2–5. As usual, we use below $k$ and $\phi_0$ for the electrical conductivity coefficient and the electric potential of the foundation, respectively.

First, we study the separate dependence of the normal electric displacement $\mathbf{D} \cdot \mathbf{v}$ with respect to the data $\phi_0$ and $k$. For this, we take $k = 1$ and we consider the electric potential $\phi_0 = -64$ which is successively halved in order to tend toward $\phi_0 = 0$, i.e. we consider 9 successive values of $\phi_0$ from $-64$ to $0$. Our results are presented in Fig. 2. Second, we fix the potential of the foundation at the value $\phi_0 = -64$ and we consider an electrical conductivity coefficient $k = 1$ which is successively halved in order to tend toward $k = 0$; so, we consider 9 successive values of $k$ from 1 to 0. Our results are presented in Fig. 3.

According to Fig. 2 we note that, for a given $k$, the magnitude of the normal electric displacement increases with the magnitude of the potential $\phi_0$. A similar behavior follows from Fig. 3 which shows that, for a given $\phi_0$, the magnitude of the normal electric displacement increases with the electrical conductivity coefficient $k$. These results are compatible with the electrical boundary condition we use on the contact surface and show the effect of the conductivity of the foundation on the process.
We present now more results concerning the influence of the potential of the foundation on the process. Thus, in Figs. 4 and 5 we plot a sequence of deformed meshes with the corresponding contact interface forces and electric displacement fields, obtained for $k = 1$ and four different values of the potential of the foundation: $\varphi_0 = -64$, $\varphi_0 = -16$, $\varphi_0 = -4$ and $\varphi_0 = 0$. As shown in Fig. 4, it results that the deformations localize around the contact interface and increase when the magnitude of the potential of the foundation increases. We also remark that the contact forces are on the direction of the normal vector to $\Gamma_3$ and this is due to the fact that the contact is frictionless. Clearly, the magnitude of the contact forces depends on the electric potential of the foundation. Nevertheless, due to the strong coupling of the unknowns on the contact surface, it is difficult to provide a reasonable explanation of this dependence. Finally, we note that, as shown in Fig. 5, the magnitude of the electric displacement on the contact interface decreases when magnitude of the potential of the foundation decreases. Again, this behavior is due to the assumption that the foundation is electrically conductive.

Moreover, Fig. 6 shows the electric potential in the body whereas Fig. 7 represents the viscoelastic constraints in the deformed configuration for four different values of the potential of the foundation: $\varphi_0 = -64$, $\varphi_0 = -16$, $\varphi_0 = -4$ and $\varphi_0 = 0$. It follows from these figures that, on the contact interface, the magnitude of the electric potential as well as the magnitude of the stress (measured in the Von Mises norm) increases as the magnitude of the potential of the foundation increases. In addition, a carefully analysis of Figs. 5 and 7 indicates that the electric displacement is more important in the regions in which the magnitude of the stress tensor is large.
We conclude that our simulations above describe the inverse piezoelectric effect, i.e. the appearance of strain or stress in the body, due to the action of the electric field, when the contact arises. Also, the simulations underline the effects of the electrical conductivity of the foundation on the process.

References