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A question by T. S. Chihara about shell polynomials and indeterminate moment problems☆

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Abstract

The generalized Stieltjes–Wigert polynomials depending on parameters $0 \leq p < 1$ and $0 < q < 1$ are discussed. By removing the mass at zero of an N-extremal solution concentrated in the zeros of the D-function from the Nevanlinna parametrization, we obtain a discrete measure μ^M, which is uniquely determined by its moments. We calculate the coefficients of the corresponding orthonormal polynomials (P_n^M). As noticed by Chihara, these polynomials are the shell polynomials corresponding to the maximal parameter sequence for a certain chain sequence. We also find the minimal parameter sequence, as well as the parameter sequence corresponding to the generalized Stieltjes–Wigert polynomials, and compute the value of related continued fractions. The mass points of μ^M have been studied in recent papers of Hayman, Ismail–Zhang and Huber. In the special case of $p = q$, the maximal parameter sequence is constant and the determination of μ^M and (P_n^M) gives an answer to a question posed by Chihara in 2001.

Keywords: Orthogonal polynomials; Chain sequences; Stieltjes–Wigert polynomials; q-series

1. Introduction

In [10], Chihara formulated an open problem concerning kernel polynomials and chain sequences motivated by the results in his paper [8] and his monograph [9]. To formulate the

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problem precisely, we need some notation and explanation, but roughly speaking it deals with the following observation of Chihara.

Let (k_n) denote the kernel polynomials of an indeterminate Stieltjes moment problem. The corresponding shell polynomials (p_n^h), parametrized by the initial condition $0 < h_0 \le M_0$ for the non-minimal parameter sequences $h = (h_n)$ of the associated chain sequence, are orthogonal with respect to the measure

$$\mu^h = \mu^M + (M_0/h_0 - 1)\mu^M(\mathbb{R})\delta_0.$$

In the case of the generalized Stieltjes–Wigert polynomials $S_n(x; p, q)$ with $p = q$, Chihara observed that the maximal parameter sequence is constant

$$M_n = \frac{1}{1+q}$$

and for this special case Chihara's question is:

"*Find the measure μ^M which has the property that the Hamburger moment problem is determinate, but if mass is added at the origin, the Stieltjes problem becomes indeterminate.*"

In this paper we find the measure μ^M as the discrete measure

$$\mu^M = \sum_{n=1}^{\infty} \rho_n \delta_{\tau_n},$$

obtained by removing the mass at zero from an N-extremal solution to the generalized Stieltjes–Wigert moment problem, and the numbers τ_n behave like

$$\tau_n = q^{-2n-1/2}(1 + \mathcal{O}(q^n)) \quad \text{as } n \to \infty.$$

For p, q small enough or n sufficiently large, there are constants b_j, $j \ge 1$, such that τ_n is given by

$$\tau_n = q^{-2n-1/2}\left(\sum_{j=1}^{\infty} b_j q^{jn}\right),$$

see Theorem 3.3 for details. These results are due to Hayman [13], Ismail–Zhang [16], and Huber [14]. It does not seem possible to find more explicit formulas for the numbers τ_n because this is equivalent to finding the zeros of the q-Bessel function $J_\nu^{(2)}(z; q)$.

We also find explicit formulas for the coefficients of the orthonormal polynomials associated with the measure μ^M, see Theorem 4.1, and compute the minimal and maximal parameter sequences as well as the parameter sequence corresponding to $S_n(x; p, q)$ in Theorem 5.1. The explicit expressions at hand allow us to show that

$$1 - \frac{\beta_n}{1 - \frac{\beta_{n+1}}{1-\cdots}} = \frac{q((pq^{n-1}; q)_\infty - (q^{n-1}; q)_\infty)}{(1 + q - (1+p)q^n)((pq^n; q)_\infty - (q^n; q)_\infty)}$$

for every $n \ge 1$, where

$$\beta_n = \frac{q(1-q^n)(1-pq^n)}{(1+q-(1+p)q^n)(1+q-(1+p)q^{n+1})}.$$

2. Preliminaries

It is well known that chain sequences can be used to characterize those three-term recurrence relations for orthogonal polynomials which have a measure of orthogonality supported by $[0, \infty[$, cf. [9]. The moments of such a measure form a Stieltjes moment sequence. A Stieltjes moment sequence is called determinate in the sense of Stieltjes (in short det(S)) if there is only one measure supported on $[0, \infty[$ with these moments, while it is called indeterminate in the sense of Stieltjes (in short indet(S)) if there are different measures on the half-line with these moments.

If a Stieltjes moment sequence is indet(S), then there are also measures with the same moments not supported by the positive half-line. This follows from the Nevanlinna parametrization of the indeterminate Hamburger moment problem. If the Stieltjes moment sequence is det(S), it is still possible that it is an indeterminate Hamburger moment sequence. See [6] for concrete examples.

In the following, let (p_n) be a sequence of monic orthogonal polynomials for a positive measure μ with moments of any order and infinite support contained in $[0, \infty[$. We denote by (k_n) the sequence of monic orthogonal polynomials with respect to the measure $x\,\mathrm{d}\mu(x)$. The polynomials (k_n) are called *kernel polynomials* because they are the monic version of the reproducing kernels

$$K_n(x, y) = \sum_{k=0}^{n} p_k(x)p_k(y)/\|p_k\|^2, \qquad \|p_k\|^2 = \int p_k^2(x)\,\mathrm{d}\mu(x)$$

when $y = 0$, i.e.,

$$k_n(x) = \frac{\|p_n\|^2}{p_n(0)} K_n(x, 0).$$

The three-term recurrence relation for the kernel polynomials is given as

$$k_n(x) = (x - d_n)k_{n-1}(x) - \nu_n k_{n-2}, \quad n \geq 1 \tag{1}$$

(with the convention that $k_{-1} = 0$, ν_1 is not defined). It is known, cf. [8], that

$$\beta_n = \nu_{n+1}/(d_n d_{n+1}), \quad n \geq 1 \tag{2}$$

is a chain sequence which does not determine the parameter sequence uniquely. In this case there exists a largest $M_0 > 0$ such that for any $0 \leq h_0 \leq M_0$, there is a parameter sequence h_n, $n \geq 0$, such that

$$\beta_n = h_n(1 - h_{n-1}), \quad n \geq 1. \tag{3}$$

The parameter sequence $M_n = h_n$ (resp. $m_n = h_n$) determined by $h_0 = M_0$ (resp. $h_0 = m_0 = 0$) is called the maximal (resp. minimal) parameter sequence. For each parameter sequence $h = (h_n)$ with $0 < h_0 \leq M_0$, there exists a family of monic orthogonal polynomials (p_n^h) on $[0, \infty[$ which all have (k_n) as kernel polynomials. The polynomials (p_n^h) are called the *shell polynomials* of the kernel polynomials (k_n). The coefficients in the three-term recurrence relation

$$p_n^h(x) = (x - c_n^h)p_{n-1}^h(x) - \lambda_n^h p_{n-2}^h(x) \tag{4}$$

are given explicitly in [8] in terms of d_n, h_n by

$$c_1^h = h_0 d_1, \qquad c_{n+1}^h = (1 - h_{n-1})d_n + h_n d_{n+1}, \quad n \geq 1, \tag{5}$$

and

$$\lambda_{n+1}^h = (1 - h_{n-1})h_{n-1}d_n^2, \quad n \geq 1. \tag{6}$$

Theorem 2 in [8] states:

Theorem 2.1. *The polynomials (p_n^M) are orthogonal with respect to a determinate measure μ^M which has no mass at 0.*

For $0 < h_0 < M_0$, the polynomials (p_n^h) are orthogonal with respect to

$$\mu^h = \mu^M + (M_0/h_0 - 1)\mu^M(\mathbb{R})\delta_0, \tag{7}$$

where δ_0 denotes the Dirac measure with mass 1 at 0.

The measure μ^h is indet(S) if and only if $x\mathrm{d}\mu^h(x) = x\mathrm{d}\mu^M(x)$ is indet(S).

Remark 2.2. Recall that for a measure μ, the proportional measure $\lambda\mu$ ($\lambda > 0$) leads to the same monic orthogonal polynomials as μ. The normalization in (7) is chosen so that $\lambda\mu^h$ precisely corresponds to $\lambda\mu^M$ for any $\lambda > 0$.

In all of this paper we shall be focusing on the case where $x\mathrm{d}\mu^M(x)$ is indet(S), i.e., when the kernel polynomials correspond to an indeterminate Stieltjes moment problem.

Concerning the "if and only if" statement of the theorem, it is easy to see that if μ^h is indet(S), then $x\mathrm{d}\mu^h(x)$ is indet(S). The reverse implication is proved in [8, p. 6–7], and the reverse implication is also a consequence of [5, Lemma 5.4].

The measure μ^M is determinate in the sense of Hamburger and $x\mathrm{d}\mu^M(x)$ is indet(S). Using the terminology of [5, Sect.5], we see that the index of determinacy $\mathrm{ind}(\mu^M)$ is 0. The measures on $[0, \infty[$ of index zero were characterized in [5, Thm. 5.5] as the discrete measures σ defined in the following way: Take any Stieltjes moment sequence (s_n) which is indet(S) and let ν_0 be the corresponding N-extremal solution which has a mass at 0. Define σ by

$$\sigma = \nu_0 - \nu_0(\{0\})\delta_0.$$

In other words, if (P_n) are the orthonormal polynomials corresponding to (s_n) and if

$$D(z) = z\sum_{n=0}^{\infty} P_n(z)P_n(0), \tag{8}$$

then D has simple zeros $\tau_0 = 0 < \tau_1 < \cdots < \tau_n < \cdots$ and

$$\nu_0 = \sum_{n=0}^{\infty} \rho_n\delta_{\tau_n}, \qquad \sigma = \sum_{n=1}^{\infty} \rho_n\delta_{\tau_n}, \tag{9}$$

where

$$\rho_n^{-1} = \sum_{k=0}^{\infty} P_k^2(\tau_n). \tag{10}$$

Stieltjes observed that removing the mass at zero of the solution ν_0 to an indeterminate Stieltjes problem leads to a determinate solution; see [17, Sect. 65]. This phenomenon was exploited in [2] for indeterminate Hamburger moment problems and carried on in Berg–Durán [3]. It follows that all the measures μ^h given by (7) for $0 < h_0 < M_0$ are N-extremal.

3. The generalized Stieltjes–Wigert polynomials

For $0 < q < 1$ and $0 \le p < 1$, we consider the moment sequence

$$s_n = (p; q)_n q^{-(n+1)^2/2}, \quad n \ge 0, \tag{11}$$

given by the integrals

$$\frac{1}{\sqrt{2\pi \log(1/q)}} \int_0^\infty x^n \exp\left(-\frac{(\log x)^2}{2\log(1/q)}\right) (p, -p/\sqrt{q}x; q)_\infty \, \mathrm{d}x. \tag{12}$$

We call (s_n) the generalized Stieltjes–Wigert moment sequence because it is associated with the generalized Stieltjes–Wigert polynomials

$$S_n(x; p, q) = (-1)^n q^{-n(n+1/2)} (p; q)_n \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k^2}(-\sqrt{q}x)^k}{(p; q)_k}, \tag{13}$$

where we follow the monic notation and normalization of [9, p. 174] for these polynomials. We have used the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}},$$

involving the q-shifted factorial

$$(z; q)_n = \prod_{k=1}^n (1 - zq^{k-1}), \quad z \in \mathbb{C}, \ n = 0, 1, \ldots, \infty.$$

We refer to [12] for information about this notation and q-series.

The Stieltjes–Wigert polynomials correspond to the special case $p = 0$. In his famous memoir [17], Stieltjes noticed that the special values $\log(1/q) = 1/2$ and $p = 0$ give an example of an indeterminate moment problem, and Wigert [20] found the corresponding orthonormal polynomials. The normalization is the same as in Szegő [18]. Note that

$$s_0 = 1/\sqrt{q}. \tag{14}$$

The Stieltjes–Wigert moment problem has been extensively studied in [11] using a slightly different normalization.

For the generalized Stieltjes–Wigert polynomials, the orthonormal version is given as

$$P_n(x; p, q) = (-1)^n q^{n/2+1/4} \sqrt{\frac{(p; q)_n}{(q; q)_n}} \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k^2+k/2}}{(p; q)_k} x^k. \tag{15}$$

From (15) we get

$$P_n(0; p, q) = (-1)^n q^{n/2+1/4} \sqrt{\frac{(p; q)_n}{(q; q)_n}}, \tag{16}$$

and hence, by the q-binomial theorem, cf. [12],

$$\sum_{n=0}^\infty P_n^2(0; p, q) = \sqrt{q} \sum_{n=0}^\infty \frac{(p; q)_n}{(q; q)_n} q^n = \sqrt{q} \frac{(pq; q)_\infty}{(q; q)_\infty}. \tag{17}$$

From the general theory in [1] we know that the generalized Stieltjes–Wigert moment sequence has an N-extremal solution ν_0 which has the mass

$$c = \frac{(q;q)_\infty}{\sqrt{q}(pq;q)_\infty} \tag{18}$$

at 0, and ν_0 is a discrete measure concentrated at the zeros of the entire function

$$D(z) = z\sum_{n=0}^{\infty} P_n(0;p,q)P_n(z;p,q). \tag{19}$$

The measure $\tilde{\mu} = \nu_0 - c\delta_0$ is determinate, cf., e.g., [2, Thm. 7]. The moment sequence $(\tilde{s}_n)$ of $\tilde{\mu}$ equals the Stieltjes–Wigert moment sequence except for $n = 0$,

$$\tilde{s}_n = \begin{cases} q^{-1/2}[1-(q;q)_\infty/(pq;q)_\infty] & \text{if } n = 0, \\ (p;q)_n q^{-(n+1)^2/2} & \text{if } n \geq 1, \end{cases} \tag{20}$$

and so the corresponding Hankel matrices $\mathcal{H}$ and $\tilde{\mathcal{H}}$ only differ at the entry $(0,0)$. The orthonormal polynomials associated with $(\tilde{s}_n)$ will be denoted $\tilde{P}_n(x;p,q)$. We call them the *modified generalized Stieltjes–Wigert polynomials*, and they will be determined in Section 4.

With (15)–(16) at hand, we can find the entire function D in (19) explicitly. The following generating function leads to the power series expansion of D.

Lemma 3.1. *For $|t| < 1$, we have*

$$\sum_{n=0}^{\infty}\frac{(p;q)_n}{(q;q)_n}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k^2+k/2}}{(p;q)_k}z^k\right)t^n = \frac{(pt;q)_\infty}{(t;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2+n/2}}{(pt,q;q)_n}(zt)^n. \tag{21}$$

Proof. Since the double series on the left-hand side is absolutely convergent, we can interchange the order of summation to get

$$LHS = \sum_{k=0}^{\infty}\frac{q^{k^2+k/2}}{(p,q;q)_k}z^k\sum_{n=k}^{\infty}\frac{(p;q)_n}{(q;q)_{n-k}}t^n.$$

Shifting the index of summation on the inner sum, the q-binomial theorem, see [12], leads to

$$LHS = \frac{1}{(t;q)_\infty}\sum_{k=0}^{\infty}\frac{q^{k^2+k/2}}{(q;q)_k}(ptq^k;q)_\infty(zt)^k.$$

We thus arrive at (21). □

Set $t = q$ and replace z by $-z$ in (21) to get

$$D(z) = z\sqrt{q}\frac{(pq;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(n+1)}}{(pq,q;q)_n}(z\sqrt{q})^n. \tag{22}$$

The expression in (22) is essentially the q-Bessel function $J_\nu^{(2)}(z;q)$ for $q^\nu = p$, cf. [12].

Besides $\tau_0 = 0$, the zeros τ_n of (22) cannot be found explicitly. However, the asymptotic behavior of τ_n for n large can be described up to a small error. General results of

Bergweiler–Hayman [7] show that

$$\tau_n = Aq^{-2n}(1 + \mathcal{O}(q^n)) \quad \text{as } n \to \infty \tag{23}$$

for some constant $A > 0$. In fact, $A = q^{-1/2}$ as follows from later work of Hayman. He proved in [13] that

Theorem 3.2. *Given $k \geq 1$, there are constants $b_1, \ldots, b_k$ (depending on p, q) such that*

$$\tau_n = q^{-2n-1/2}\left(1 + \sum_{j=1}^{k} b_j q^{jn} + \mathcal{O}(q^{(k+1)n})\right) \quad \textit{as } n \to \infty. \tag{24}$$

The first few values of the constants are

$$\begin{aligned} b_1 &= -\frac{1+p}{(1-q)\psi^2(q)}, \qquad b_2 = 0, \\ b_3 &= -\frac{q(1+q^2)(1+p^3) + 2pq(1+p)(1+q+q^2)}{(1-q)(1-q^2)(1-q^3)\psi^2(q)} \\ &\quad + \frac{(1+p)^3}{(1-q)^3\psi^6(q)} \sum_{j=1}^{\infty} \frac{(2j-1)q^{2j-1}}{1-q^{2j-1}}, \\ b_4 &= b_1 b_3, \end{aligned}$$

where

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}.$$

Even stronger results were recently obtained by Ismail–Zhang [16] and Huber [14,15]. They showed that for n sufficiently large (in [16]) or for every n when p, q are small enough (in [14]),

Theorem 3.3. *There are constants b_j, $j \geq 1$, such that τ_n is given exactly by the convergent series*

$$\tau_n = q^{-2n-1/2}\left(1 + \sum_{j=1}^{\infty} b_j q^{jn}\right). \tag{25}$$

The b_j's satisfy a somewhat complicated recursion formula that in principle allows for determining b_{j+1} from $b_1, \ldots, b_j$. See [14,15] for details. In particular, [15] includes the coefficients b_j up to index 14 and indicates how further coefficients may be derived.

4. The modified generalized Stieltjes–Wigert polynomials

It is a classical fact, cf. [1, p. 3], that the orthonormal polynomials (P_n) corresponding to a moment sequence (s_n) are given by the formula

$$P_n(x) = \frac{1}{\sqrt{D_{n-1}D_n}} \det \begin{pmatrix} s_0 & s_1 & \cdots & s_n \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \\ 1 & x & \cdots & x^n \end{pmatrix}, \tag{26}$$

where

$$D_n = \det \mathcal{H}_n, \qquad \mathcal{H}_n = (s_{i+j})_{0\le i,j\le n}.$$

In this way Wigert calculated the polynomials $P_n(x; 0, q)$ and we shall follow the same procedure for $P_n(x; p, q)$ and $\tilde{P}_n(x; p, q)$. Recall that $(P_n(x; p, q))$ denote the orthonormal generalized Stieltjes–Wigert polynomials corresponding to the moment sequence (11), so they are known, cf. (15). Similarly, $(\tilde{P}_n(x; p, q))$ denote the orthonormal modified generalized Stieltjes–Wigert polynomials corresponding to the moment sequence (20), and they have not been calculated before as far as the authors know, except for the case $p = 0$, where the calculation was carried out in [4].

It will be convenient to use the notation

$$\Delta_n := (pq^n; q)_\infty - (q^n; q)_\infty, \quad n \ge 0. \tag{27}$$

Writing

$$P_n(x; p, q) = \sum_{k=0}^{n} b_{k,n} x^k, \qquad \tilde{P}_n(x; p, q) = \sum_{k=0}^{n} \tilde{b}_{k,n} x^k, \tag{28}$$

we have:

Theorem 4.1. *For* $0 \le k \le n$,

$$\tilde{b}_{k,n} = \tilde{C}_n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k^2+k/2}}{(p; q)_k} \left[1 - \frac{1-q^k}{1-pq^k} \frac{(q^{n+1}; q)_\infty}{(pq^{n+1}; q)_\infty}\right], \tag{29}$$

where

$$\begin{aligned}\tilde{C}_n &= (-1)^n q^{n/2+1/4} \sqrt{\frac{(p; q)_n}{(q; q)_n}} \left[\left(1 - \frac{(q^n; q)_\infty}{(pq^n; q)_\infty}\right)\left(1 - \frac{(q^{n+1}; q)_\infty}{(pq^{n+1}; q)_\infty}\right)\right]^{-1/2} \\ &= (-1)^n q^{n/2+1/4} \sqrt{\frac{(p; q)_{n+1}}{(q; q)_n}} \frac{(pq^{n+1}; q)_\infty}{\sqrt{\Delta_n \Delta_{n+1}}},\end{aligned} \tag{30}$$

i.e.,

$$\tilde{b}_{k,n} = b_{k,n} \left[(pq^{n+1}; q)_\infty - \frac{1-q^k}{1-pq^k} (q^{n+1}; q)_\infty\right] \sqrt{\frac{1-pq^n}{\Delta_n \Delta_{n+1}}}. \tag{31}$$

Moreover,

$$\tilde{D}_n = \frac{\Delta_{n+1}}{(pq^{n+1}; q)_\infty} D_n, \tag{32}$$

where $D_n = \det \mathcal{H}_n$ *and* $\tilde{D}_n = \det \tilde{\mathcal{H}}_n$.

Proof. We first recall the Vandermonde determinant

$$V_n(x_1, \ldots, x_n) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1\le i<j\le n} (x_j - x_i). \tag{33}$$

Using the moments $s_n = (p; q)_n q^{-(n+1)^2/2}$, we find

$$D_n = \left(\prod_{j=0}^{n} s_j\right) \det(s_{i+j}/s_j) = \left(\prod_{j=1}^{n} (p; q)_j\right) q^{-\frac{1}{2}\sigma_{n+1}} \det(s_{i+j}/s_j), \tag{34}$$

where $\sigma_n = \sum_{j=0}^{n} j^2 = n(n+1)(2n+1)/6$. Noting that

$$s_{i+j}/s_j = (pq^j; q)_i \, q^{-i^2/2} q^{-i(j+1)}, \tag{35}$$

we get

$$D_n = \left(\prod_{j=1}^{n} (p; q)_j\right) q^{-\frac{1}{2}(\sigma_{n+1}+\sigma_n)} \det((pq^j; q)_i \, q^{-i(j+1)}). \tag{36}$$

The last determinant can be simplified in the following way: Multiply the first row (corresponding to $i = 0$) by p/q and add it to the second row ($i = 1$). Then the second row becomes $q^{-(j+1)}$, $j = 0, 1, \ldots, n$, and the determinant is not changed. The third row ($i = 2$) has the entries

$$q^{-2(j+1)} - p(1 + 1/q)q^{-(j+1)} + p^2/q, \quad j = 0, 1, \ldots, n,$$

so adding the first row multiplied by $-p^2/q$ and the second row multiplied by $p(1 + 1/q)$ to the third row, changes the third row to $q^{-2(j+1)}$, $j = 0, 1, \ldots, n$, and the determinant is not changed. If we go on like this, we finally get

$$D_n = \left(\prod_{j=1}^{n} (p; q)_j\right) q^{-\frac{1}{2}(\sigma_{n+1}+\sigma_n)} \det(q^{-i(j+1)}). \tag{37}$$

The last determinant is precisely $V_{n+1}(q^{-1}, \ldots, q^{-(n+1)})$, and, by (33), is equal to

$$\prod_{i=1}^{n} \prod_{j=i+1}^{n+1} (q^{-j} - q^{-i}) = \prod_{i=1}^{n} q^{-(n+1-i)(n+2+i)/2} (q; q)_{n+1-i}.$$

After some reduction, we get

$$V_{n+1}(q^{-1}, \ldots, q^{-(n+1)}) = q^{-n(n+1)(n+2)/3} \prod_{j=1}^{n} (q; q)_j. \tag{38}$$

Hence,

$$D_n = \left(\prod_{j=1}^{n} (p, q; q)_j\right) q^{-(n+1)(2n+1)(2n+3)/6}, \tag{39}$$

and for later use we note that

$$D_n/D_{n-1} = (p, q; q)_n \, q^{-(2n+1)^2/2}. \tag{40}$$

We denote by $A_{r,s}$ (resp. $\tilde{A}_{r,s}$) the cofactor of entry (r, s) of the Hankel matrix $\mathcal{H}_n$ (resp. $\tilde{\mathcal{H}}_n$), where $r, s = 0, 1, \ldots, n$. (Note that entry (r, s) is in row number $r+1$ and column number $s+1$.)

When $r = 0$ or $s = 0$, we clearly have $A_{r,s} = \tilde{A}_{r,s}$. For $0 \le s \le n$, we get

$$\begin{aligned}
A_{n,s} &= (-1)^{n-s} \det\left(s_{i+j}\big|_{j=0,\ldots,n;\, j\neq s}^{i=0,\ldots,n-1}\right) \\
&= (-1)^{n-s} \left(\prod_{\substack{j=0\\ j\neq s}}^{n} s_j\right) \det\left(s_{i+j}/s_j\big|_{j=0,\ldots,n;\, j\neq s}^{i=0,\ldots,n-1}\right) \\
&= (-1)^{n-s} \left(\prod_{\substack{j=0\\ j\neq s}}^{n} (p;q)_j\right) q^{-\frac{1}{2}\left(\sigma_{n+1}+\sigma_{n-1}-(s+1)^2\right)} \\
&\quad \times \det\left((pq^j;q)_i q^{-i(j+1)}\big|_{j=0,\ldots,n;\, j\neq s}^{i=0,\ldots,n-1}\right).
\end{aligned}$$

However, the last determinant can be simplified like the simplifications from (36) to (37) to give the Vandermonde determinant $V_n(q^{-(j+1)} \mid j = 0, \ldots, n,\ j \neq s)$. To calculate this determinant, we observe that

$$\begin{aligned}
V_{n+1}(q^{-1}, \ldots, q^{-(n+1)}) &= V_n(q^{-(j+1)} \mid j = 0, \ldots, n,\ j \neq s) \prod_{j=0}^{s-1} (q^{-(s+1)} - q^{-(j+1)}) \\
&\quad \times \prod_{j=s+1}^{n} (q^{-(j+1)} - q^{-(s+1)}) \\
&= V_n(q^{-(j+1)} \mid j = 0, \ldots, n,\ j \neq s)(q;q)_s q^{-s(s+1)} (q;q)_{n-s} q^{-\frac{1}{2}(n-s)(n+s+3)},
\end{aligned}$$

and hence

$$\begin{aligned}
A_{n,s} &= \frac{(-1)^{n-s}}{(q;q)_n (p;q)_s} \prod_{j=0}^{n} (p;q)_j \begin{bmatrix} n \\ s \end{bmatrix}_q \\
&\quad \times V_{n+1}(q^{-1}, \ldots, q^{-(n+1)}) q^{-\frac{1}{2}(\sigma_{n+1}+\sigma_{n-1}-n(n+3)-1)} q^{s^2+s/2} \\
&= \frac{(-1)^{n-s}}{(q;q)_n (p;q)_s} \begin{bmatrix} n \\ s \end{bmatrix}_q D_n q^{(n+1)(n+1/2)} q^{s^2+s/2}.
\end{aligned}$$

Using (26) it is now easy to verify formula (15) for the generalized Stieltjes–Wigert polynomials $P_n(x; p, q)$.

Expanding after the first column, we get

$$\tilde{D}_n = D_n - cA_{0,0}, \quad c = \frac{(q;q)_\infty}{\sqrt{q}(pq;q)_\infty},$$

and a calculation as above leads to

$$\begin{aligned}
A_{0,0} &= \det(s_{i+j} \mid i, j = 1, \ldots, n) \\
&= \left(\prod_{j=1}^{n} s_{j+1}\right) \det(s_{i+j}/s_{j+1} \mid i, j = 1, \ldots, n)
\end{aligned}$$

$$= \left(\prod_{j=2}^{n+1} (p;q)_j \right) q^{-\frac{1}{2}(\sigma_{n+2}+\sigma_{n-1}-5)} V_n(q^{-j}, j = 3, \ldots, n+2)$$

$$= \left(\prod_{j=2}^{n+1} (p;q)_j \right) q^{-\frac{1}{2}(\sigma_{n+2}+\sigma_{n-1}-5)} q^{-n(n-1)} V_n(q^{-j}, j = 1, \ldots, n).$$

Using (38) with n replaced by $n-1$ and (39), we find

$$A_{0,0} = D_n \frac{(p;q)_{n+1}\sqrt{q}}{(1-p)(q;q)_n},$$

which gives (32).

For $1 \le s \le n$, we find

$$\tilde{A}_{n,s} = A_{n,s} - c(-1)^{n-s} \det \left(s_{i+j} \mid_{j=1,\ldots,n;\, j \ne s}^{i=1,\ldots,n-1} \right),$$

and the determinant on the right-hand side can be calculated by the same method as above to be

$$\left(\prod_{\substack{j=1 \\ j \ne s}}^{n} s_{j+1} \right) \det \left(s_{i+j}/s_{j+1} \mid_{j=1,\ldots,n;\, j \ne s}^{i=1,\ldots,n-1} \right)$$

$$= \left(\prod_{\substack{j=1 \\ j \ne s}}^{n} (p;q)_{j+1} \right) q^{-\frac{1}{2}(\sigma_{n+2}+\sigma_{n-2}-5-(s+2)^2)} V_{n-1}(q^{-(j+2)}, j = 1, \ldots, n;\, j \ne s)$$

$$= D_{n-1} \frac{(p;q)_n (p;q)_{n+1}}{(1-p)(p;q)_{s+1}(q;q)_{n-s}(q;q)_{s-1}} q^{-n^2-(n-1)/2+s(s+1/2)}.$$

This leads to

$$\tilde{A}_{n,s} = A_{n,s} \left[1 - \frac{1-q^s}{1-pq^s} \frac{(q^{n+1};q)_\infty}{(pq^{n+1};q)_\infty} \right], \tag{41}$$

which also holds for $s = 0$ because then $\tilde{A}_{n,0} = A_{n,0}$. It is now easy to establish (31). □

Remark 4.2. The orthonormal polynomials $\tilde{P}_n(x;p,q)$ belong to a determinate moment problem. From Theorem 4.1 it is possible to find the asymptotic behavior of $\tilde{P}_n(x;p,q)$ as $n \to \infty$ for any $x \in \mathbb{C}$, namely

$$\tilde{P}_n(x;p,q) \sim (-1)^n c(x) q^{-n/2}, \tag{42}$$

where

$$c(x) = q^{-1/4} \frac{1-q}{1-p} \sqrt{\frac{(p;q)_\infty}{(q;q)_\infty}} \sum_{k=0}^{\infty} \frac{q^{k^2+k/2}}{(pq,q;q)_k} (-qx)^k$$

is essentially the q-Bessel function $J_\nu^{(2)}(z;q)$ with $p = q^\nu$.

To see this, we notice that

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k^2+k/2}}{(p;q)_k} \left[1 - \frac{1-q^k}{1-pq^k} \frac{(q^{n+1};q)_\infty}{(pq^{n+1};q)_\infty} \right] x^k$$

converges to

$$\sum_{k=0}^{\infty}(-1)^k \frac{q^{k^2+k/2}}{(p,q;q)_k}\left[1-\frac{1-q^k}{1-pq^k}\right]x^k=\sum_{k=0}^{\infty}\frac{q^{k^2+k/2}}{(pq,q;q)_k}(-qx)^k.$$

From the q-binomial theorem, we find

$$1-\frac{(q^n;q)_\infty}{(pq^n;q)_\infty}\sim\frac{1-p}{1-q}q^n \quad \text{as } n\to\infty \tag{43}$$

and combining the above, we get (42).

The monic polynomials $\tilde{p}_n(x;p,q)=\tilde{P}_n(x;p,q)/\tilde{b}_{n,n}$ satisfy the three-term recurrence relation

$$\tilde{p}_n(x;p,q)=(x-\tilde{c}_n)\tilde{p}_{n-1}(x;p,q)-\tilde{\lambda}_n\tilde{p}_{n-2}(x;p,q),\quad n\ge 1, \tag{44}$$

where the coefficients are given by

$$\tilde{c}_1=-\frac{\tilde{b}_{0,1}}{\tilde{b}_{1,1}},\qquad \tilde{c}_{n+1}=\frac{\tilde{b}_{n-1,n}}{\tilde{b}_{n,n}}-\frac{\tilde{b}_{n,n+1}}{\tilde{b}_{n+1,n+1}},\quad n\ge 1 \tag{45}$$

and

$$\tilde{\lambda}_{n+1}=\frac{\tilde{b}^2_{n-1,n-1}}{\tilde{b}^2_{n,n}},\quad n\ge 1. \tag{46}$$

Using the expressions from Theorem 4.1, we get

Theorem 4.3. *Let Δ_n be defined as in (27). Then the coefficients in (45)–(46) are given by*

$$\begin{aligned}\tilde{c}_1&=\frac{(p;q)_\infty}{\Delta_1}q^{-3/2},\\ \tilde{c}_{n+1}&=[(1-q^{n+1})(pq^n;q)_\infty-(1-pq^{n+1})(q^n;q)_\infty]\frac{q^{-2n-3/2}}{(1-q)\Delta_{n+1}}\\ &\quad-[(1-q^n)(pq^{n-1};q)_\infty-(1-pq^n)(q^{n-1};q)_\infty]\frac{q^{-2n+1/2}}{(1-q)\Delta_n},\end{aligned} \tag{47}$$

and

$$\tilde{\lambda}_{n+1}=\frac{\Delta_{n-1}\Delta_{n+1}}{\Delta_n^2}(1-q^n)(1-pq^n)q^{-4n}. \tag{48}$$

Proof. Specializing (29) to $k=n$ and $k=n-1$, we find

$$\tilde{b}_{n,n}=q^{n^2+n+1/4}\left(\frac{\Delta_n}{\Delta_{n+1}(p;q)_{n+1}(q;q)_n}\right)^{1/2},$$

and

$$\tilde{b}_{n-1,n}=-\frac{q^{n^2-n+3/4}}{(1-q)}\frac{(1-q^n)(pq^{n-1};q)_\infty-(1-pq^n)(q^{n-1};q)_\infty}{\sqrt{(p;q)_{n+1}(q;q)_n\Delta_n\Delta_{n+1}}}.$$

Using (45)–(46), we obtain the expressions in (47)–(48). □

In the special case $p = q$, the formulas of Theorems 4.1 and 4.3 simplify.

Corollary 4.4. *The coefficients of (28) in the case $p = q$ are given by*

$$\tilde{b}_{k,n} = \tilde{C}_n(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k^2+k/2}}{(q;q)_{k+1}}[1 - q^{k+1} - (1 - q^k)(1 - q^{n+1})], \tag{49}$$

where

$$\tilde{C}_n = (-1)^n q^{-n/2-1/4}, \tag{50}$$

i.e.,

$$\tilde{b}_{k,n} = b_{k,n}\, q^{-n-1/2}\left[1 - \frac{(1 - q^k)(1 - q^{n+1})}{1 - q^{k+1}}\right]. \tag{51}$$

Moreover,

$$\tilde{D}_n = q^{n+1} D_n. \tag{52}$$

Finally, the coefficients (45)–(46) in the three-term recurrence relation are

$$\tilde{c}_n = (1 + q^3 - (1 + q^2)q^n)q^{-2n-1/2}, \qquad \tilde{\lambda}_{n+1} = (1 - q^n)^2 q^{-4n}. \tag{53}$$

5. The kernel polynomials

By (12), the polynomials $S_n(x; p, q)$ are orthogonal with respect to the density

$$\mathcal{D}(x; p, q) = \frac{1}{\sqrt{2\pi \log(1/q)}} \exp\left(-\frac{(\log x)^2}{2\log(1/q)}\right)(p, -p/\sqrt{q}x; q)_\infty, \tag{54}$$

and we see that

$$\mathcal{D}(qx; pq, q) = x\frac{\sqrt{q}}{1 - p}\mathcal{D}(x; p, q). \tag{55}$$

This shows that the monic polynomials $k_n(x) = q^{-n} S_n(qx; pq, q)$ are orthogonal with respect to the density in (55), hence equal to the monic kernel polynomials corresponding to $S_n(x; p, q)$.

The three-term recurrence relation for $S_n(x; p, q)$ is

$$S_n(x; p, q) = (x - c_n)S_{n-1}(x; p, q) - \lambda_n S_{n-2}(x; p, q), \quad n \geq 1,$$

with

$$c_n = (1 + q - (p + q)q^{n-1})q^{-2n+1/2}, \qquad \lambda_{n+1} = (1 - q^n)(1 - pq^{n-1})q^{-4n}. \tag{56}$$

It follows that the coefficients in (1) for the case $p_n(x) = S_n(x; p, q)$ are given by

$$d_n = (1 + q - (1 + p)q^n)q^{-2n-1/2}, \qquad \nu_{n+1} = (1 - q^n)(1 - pq^n)q^{-4n-2}. \tag{57}$$

Chihara observed that for $p = q$ we have the following simple form of the coefficients in (57):

$$d_n = (1 + q)(1 - q^n)q^{-2n-1/2}, \qquad \nu_{n+1} = (1 - q^n)(1 - q^{n+1})q^{-4n-2}. \tag{58}$$

In this case, the chain sequence (2) becomes the constant sequence

$$\beta_n = \frac{q}{(1+q)^2}$$

satisfying $0 < \beta_n < 1/4$, and the maximal parameter sequence is also constant

$$M_n = \frac{1}{1+q}.$$

For the shell polynomials p_n^M, which are equal to $\tilde{p}_n(x; q, q)$, Chihara gave the following form of the coefficients from (4):

$$c_n^M = (1 + q^3 - (1+q^2)q^n)q^{-2n-1/2}, \qquad \lambda_{n+1}^M = (1-q^n)^2 q^{-4n}. \tag{59}$$

(There is a misprint in [10]: The power 2 is missing in the last formula). The expressions in (59) agree with the expressions in (53).

Going back to arbitrary $0 \le p < 1$, we find the following:

Theorem 5.1. *The chain sequence (2) corresponding to the kernel polynomials $k_n(x) = q^{-n} S_n(qx; pq, q)$ is*

$$\beta_n = \frac{q(1-q^n)(1-pq^n)}{(1+q-(1+p)q^n)(1+q-(1+p)q^{n+1})}, \quad n \ge 1. \tag{60}$$

The maximal and minimal parameter sequences (M_n) and (m_n) are given by

$$M_n = \frac{q}{1+q-(1+p)q^{n+1}} \frac{\Delta_n}{\Delta_{n+1}}, \qquad m_n = \frac{q(1-q^n)}{1+q-(1+p)q^{n+1}}, \tag{61}$$

and the generalized Stieltjes–Wigert polynomials $S_n(x; p, q)$ correspond to the parameter sequence

$$h_n = \frac{q(1-pq^n)}{1+q-(1+p)q^{n+1}}. \tag{62}$$

Proof. The expression for β_n follows immediately from (57). We know from Theorem 2.1 that $p_n^M(x) = \tilde{p}_n(x; p, q)$. So by (5),

$$c_1^M = M_0 d_1,$$

and by (47) and (57), we have

$$c_1^M = \tilde{c}_1 = \frac{(p;q)_\infty}{\Delta_1} q^{-3/2}, \qquad d_1 = (1+q-(1+p)q)q^{-5/2}.$$

Hence,

$$M_0 = \frac{q(p;q)_\infty}{(1+q-(1+p)q)\Delta_1},$$

showing the formula for M_n for $n = 0$. It is now easy to show by induction that $\beta_n = M_n (1 - M_{n-1})$ for $n \ge 1$.

It is similarly easy to see by induction that the sequences (m_n), (h_n) are parameter sequences for (β_n). Since $m_0 = 0$, (m_n) is the minimal parameter sequence. To see that (h_n) corresponds to $S_n(x; p, q)$, it suffices to verify that $h_0 d_1 = c_1$, where c_1 is given by (56). $\square$

The parameter sequences from Theorem 5.1 enable us to find the value β of the continued fraction

$$1-\cfrac{\beta_1}{1-\cfrac{\beta_2}{1-\cfrac{\beta_3}{1-\cdots}}} \tag{63}$$

in three different ways. By the results in [9, Chap. III] (see also [19, Sect. 19]), we have

$$\beta = M_0 = \frac{1}{1+L} = h_0 + \frac{1-h_0}{1+G},$$

where

$$L=\sum_{n=1}^{\infty}\frac{m_1\cdots m_n}{(1-m_1)\cdots(1-m_n)}, \qquad G=\sum_{n=1}^{\infty}\frac{h_1\cdots h_n}{(1-h_1)\cdots(1-h_n)}.$$

Since (M_{n+k}) is the maximal parameter sequence for the chain sequence (β_{n+k}), we can in fact find the value of

$$1-\cfrac{\beta_{k+1}}{1-\cfrac{\beta_{k+2}}{1-\cfrac{\beta_{k+3}}{1-\cdots}}} \tag{64}$$

for every $k \geq 0$.

We collect the above considerations in the following:

Theorem 5.2. *Let (β_n) be the chain sequence given by (60). Then the continued fraction in (63) has the value*

$$\beta = \frac{q}{1-pq}\frac{\Delta_0}{\Delta_1} = \frac{q(1-p)(pq^2;q)_\infty}{(pq;q)_\infty-(q;q)_\infty}.$$

More generally, the continued fraction in (64) has the value

$$M_k = \frac{q}{1+q-(1+p)q^{k+1}}\frac{\Delta_k}{\Delta_{k+1}}, \quad k\geq 0.$$

Proof. The result follows immediately from [9, Thm. 6.1 (Chap. III)]. To find L and G, note that

$$\frac{m_k}{1-m_k} = \frac{q(1-q^k)}{1-pq^{k+1}}, \qquad \frac{h_k}{1-h_k} = \frac{q(1-pq^k)}{1-q^{k+1}},$$

so that

$$1+L=\sum_{n=0}^{\infty}\frac{(q;q)_n}{(pq^2;q)_n}q^n, \qquad 1+G=\sum_{n=0}^{\infty}\frac{(pq;q)_n}{(q^2;q)_n}q^n.$$

The value of $1+G$ can thus be found using the q-binomial theorem. To compute $1+L$, one first applies Heine's transformation formula and then the q-binomial theorem. □

Remark 5.3. We mention that

$$\sum_{n=1}^{\infty}\frac{M_1\cdots M_n}{(1-M_1)\cdots(1-M_n)} = \infty, \tag{65}$$

precisely as should be the case for the maximal parameter sequence. To see this, note that

$$\frac{M_k}{1-M_k} = \frac{M_k M_{k+1}}{\beta_{k+1}} = \frac{\Delta_k}{\Delta_{k+2}} \frac{q}{(1-q^{k+1})(1-pq^{k+1})},$$

so that the series in (65) reduces to

$$\sum_{n=1}^{\infty} \frac{\Delta_1 \Delta_2}{\Delta_{n+1}\Delta_{n+2}} \frac{q^n}{(q^2, pq^2; q)_n}.$$

On the lines of (43), we have

$$\Delta_n = \frac{1-p}{1-q} q^n + \mathcal{O}(q^{2n}),$$

and the result follows.

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