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European Journal of Combinatorics 26 (2005) 309–324
European Journal of Combinatorics
www.elsevier.com/locate/ejc

An improved linear edge bound for graph linkages

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Received 23 January 2004; accepted 10 February 2004
Available online 17 June 2004

Abstract

A graph is said to be *k-linked* if it has at least $2k$ vertices and for every sequence $s_1, \ldots, s_k, t_1, \ldots, t_k$ of distinct vertices there exist disjoint paths $P_1, \ldots, P_k$ such that the ends of P_i are s_i and t_i. Bollobás and Thomason showed that if a simple graph G on n vertices is $2k$-connected and G has at least $11kn$ edges, then G is k-linked. We give a relatively simple inductive proof of the stronger statement that $8kn$ edges and $2k$-connectivity suffice, and then with more effort improve the edge bound to $5kn$.

1. Introduction and results

A graph is said to be *k-linked* if it has at least $2k$ vertices and for every sequence $s_1, \ldots, s_k, t_1, \ldots, t_k$ of distinct vertices there exist disjoint paths $P_1, \ldots, P_k$ such that the ends of P_i are s_i and t_i. (This differs slightly from the usual definition in the literature, but is more convenient for our purposes.) Clearly every k-linked graph is k-connected. The converse is not true, however, which brings up the natural question of how much connectivity, as a function $f(k)$, is necessary to ensure that a graph is k-linked.

Larman and Mani [6] and Jung [3] first showed that such a function $f(k)$ exists by showing that the existence of a topological complete minor of size $3k$ and $2k$-connectivity suffice to make a graph k-linked. This result, along with an earlier result of Mader's that sufficiently high average degree forces a large topological minor [7] proved that such a function f above does exist. Robertson and Seymour [8] proved that $2k$-connectivity and the existence of a K_{3k} minor would suffice to make a graph k-linked. This, together with bounds on the extremal function for complete minors by Kostochka [5] and Thomason [12] showed that $f(k) = O(k\sqrt{\log k})$. Bollobás and Thomason [1] noticed that the same effect

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0195-6698/$ - see front matter
doi:10.1016/j.ejc.2004.02.013

can be achieved by replacing the K_{3k} minor with a sufficiently dense (noncomplete) minor, whose existence in a graph on n vertices requires only $O(kn)$ edges. Thus they improved the bound on $f(k)$ to $22k$.

Our first objective is to give a reasonably straightforward proof that $f(k) \leq 16k$. After this result was written and distributed in April 2003, we have learned of two independent improvements. Kawarabayashi (personal communication) pointed out that by using the result of Egawa et al. [2] our bound could be lowered, and suggested the possibility of improving the bound to $12k$. That was done independently by Kawarabayashi et al. [4], who in the process essentially rediscovered the result of [2]. After the communication from Kawarabayashi and after having seen an early version of [4] we were able to further improve the bound to $10k$, and that is the second result of this paper.

For the purposes of this paper, all graphs will be simple. If G is a graph and $e \in E(G)$, we denote by G/e the graph obtained from G by contracting e and deleting all resulting parallel edges. The pair (A, B) is a *separation* of a graph G if $A \cup B = V(G)$ and every edge of G has both ends in A or B. The *order* of a separation (A, B) is $|A \cap B|$. If $X \subseteq V(G)$ and (A, B) is a separation of G with $X \subseteq A$, then we say that (A, B) is a *separation* of (G, X). We will use the notation $G[A]$ to indicate the subgraph of G induced by the set of vertices A. For $X \subseteq V(G)$, we define $\rho(X)$ to be the number of edges with at least one end in X. We will use the following definitions.

Definition. Let G be a graph, and let $X \subseteq V(G)$. We say that the pair (G, X) is *linked* if for all integers l and all distinct vertices $s_1, s_2, \ldots, s_l, t_1, \ldots, t_l \in X$ there exist disjoint paths $P_1, \ldots, P_l$, called a *linkage*, such that the ends of P_i are s_i and t_i, and no P_i has an internal vertex in X.

Definition. Let G be a graph, let $X \subseteq V(G)$, and let $\lambda > 0$ be a real number. We say that the pair (G, X) is λ-*massed* if

(1) $\rho(V(G) - X) > \lambda|V(G) - X|$, and
(2) every separation (A, B) of (G, X) of order at most $|X| - 1$ satisfies $\rho(B - A) \leq \lambda|B - A|$.

The notion of λ-massed provides a weakening, suitable for inductive arguments, of the property of being $|X|$-connected and containing "many" edges. Our main result is the following.

Theorem 1.1. *Let $k \geq 1$ be an integer, let G be a graph, and let $X \subseteq V(G)$ be such that $|X| \leq 2k$ and (G, X) is $5k$-massed. Then (G, X) is linked.*

We deduce the following two corollaries.

Corollary 1.2. *If G is $2k$-connected and G has at least $5k|V(G)|$ edges, then G is k-linked.*

Proof. Since G is $2k$-connected, it has at least $2k$ vertices. Let $X \subseteq V(G)$ be a set of size exactly $2k$. Then

$$\rho(V(G) - X) \geq |E(G)| - \binom{2k}{2} \geq 5k|V(G)| - \binom{2k}{2} > 5k|V(G) - X|,$$

and so (1) holds. Now if (A, B) is a separation of (G, X) of order at most $2k - 1$, then $B \subseteq A$, because G is $2k$ connected. Thus (2) holds, and so (G, X) is $5k$-massed. By Theorem 1.1 the pair (G, X) is linked, and so G is k-linked, as desired. □

Corollary 1.3. *If G is $10k$-connected, then G is k-linked.*

The proof of Theorem 1.1 proceeds in two steps. First we show that a minimal counterexample has a dense subgraph and no rigid separation. To emphasize the flexibility of the argument we formulate this theorem using the following definitions involving a parameter α, later to be specified to be either 8 (to obtain an easy proof) or 5 (to get the best bound).

Definition. Let G be a graph, let $X \subseteq V(G)$, and let (A, B) be a separation of G. We say that (A, B) is a *rigid separation of* (G, X) if $X \subseteq A$, $B - A \neq \emptyset$, and $(G[B], A \cap B)$ is linked.

Rigid separations facilitate inductive arguments, as follows. Let (A, B) be a rigid separation of (G, X), and let G' be obtained from $G[A]$ by adding the edge uv for all nonadjacent pairs of distinct vertices $u, v \in A \cap B$. As we will see, the pair (G, X) is linked if and only if (G', X) is linked.

The next definition formalizes the notion of "minimal counterexample".

Definition. Let G be a graph, let $X \subseteq V(G)$, and let $\alpha > 0$ be a real number. We say that the pair (G, X) is (α, k)-*minimal* if

(3) (G, X) is αk-massed,
(4) $|X| \leq 2k$ and (G, X) is not linked,
(5) subject to (3) and (4), $|V(G)|$ is minimum,
(6) subject to (3)–(5), $\rho(G - X)$ is minimum, and
(7) subject to (3)–(6), the number of edges of G with both ends in X is maximum.

Theorem 1.4. *Let $k \geq 1$ be an integer, let $\alpha \geq 2$ be a real number, let G be a graph, and let $X \subseteq V(G)$ be such that (G, X) is (α, k)-minimal. Then G has no rigid separation of order at most $|X|$, and G has a subgraph L with $|V(L)| \leq \lceil 2\alpha k \rceil$ and minimum degree at least $\lfloor \alpha k \rfloor + 1$.*

The second step consists of finding a k-linked subgraph of L, where L is as in the above theorem. This is much easier for $\alpha = 8$, and so we do that first.

Theorem 1.5. *Let $k \geq 1$ be an integer, and let H be a graph with minimum degree at least $8k$ on at most $16k$ vertices. Then H has a k-linked subgraph.*

Theorem 1.4 will be proved in Section 2 and Theorem 1.5 will be proved in Section 3. By the argument given at the end of this section (with the constant 5 replaced by 8) those two theorems imply that every $2k$-connected graph on n vertices and at least $8kn$ edges is k-linked. To improve the bound to $5kn$ we need the following strengthening of Theorem 1.5, which we prove in Section 4.

Theorem 1.6. *Let $k \geq 1$ be an integer, and let H be a graph with minimum degree at least $5k$ on at most $10k$ vertices. Then H has a k-linked subgraph.*

In the remainder of this section we deduce Theorem 1.1. By changing the constant 5 to 8 one can avoid using Theorem 1.6 and deduce the corresponding weakening of Theorem 1.1 using the easier Theorem 1.5 instead.

Proof of Theorem 1.1 (Assuming Theorems 1.4 and 1.6). Let (G, X) be as stated in Theorem 1.1, and suppose for a contradiction that it is not linked. We may assume that (G, X) is $(5, k)$-minimal, and hence by Theorem 1.4 applied with $\alpha = 5$ the graph G has a subgraph H satisfying the hypotheses of Theorem 1.6. By Theorem 1.6 the graph H, and hence G, has a k-linked subgraph J.

Assume for a moment that G has $|X|$ disjoint paths $P_1, P_2, \ldots$ between X and $V(J)$, and choose them so that they have no internal vertex in J. Since J is k-linked, the ends of P_i in J can be linked as necessary to form a desired set of paths showing that (G, X) is linked, where each of these paths consists of the union of two P_is and with an appropriate subpath of the linkage in J. But this contradicts our assumption that (G, X) is not linked.

Thus the paths $P_1, P_2, \ldots$ of the previous paragraph do not exist, and hence G has a separation (A, B) of order at most $|X| - 1$ with $X \subseteq A$ and $V(J) \subseteq B$. Choose (A, B) of smallest possible order; then there exist $|A \cap B|$ disjoint paths from $A \cap B$ to $V(J)$, and an argument similar to the argument of the previous paragraph shows that (A, B) is rigid, contrary to Theorem 1.4. □

2. Proof of Theorem 1.4

Let k, α, G, X be as stated in the theorem. We break the proof up into a series of claims. The first two claims establish the first conclusion of the theorem.

Claim 2.1. (G, X) *has no rigid separation of order at most* $|X| - 1$.

Proof. Suppose for a contradiction that (A, B) is a rigid separation of (G, X) chosen with A minimal. Let $S := A \cap B$. We define G' to be the graph obtained from $G[A]$ by adding an edge between every nonadjacent pair of vertices in S.

If (G', X) is αk-massed, then the (α, k)-minimality of (G, X) implies that (G', X) is linked. But a linkage in (G', X) can be easily converted to a linkage in (G, X) as follows. Since S is complete, we may assume that each path in the linkage uses at most one edge with both ends in S, and edges of $E(G') - E(G)$ may be replaced by paths in $G[B]$, because (A, B) is rigid. Since (G, X) is not linked, we conclude that (G', X) is not αk-massed. Since (G, X) is αk-massed, $\rho(B - A) \leq \alpha k|B - A|$, and hence $\rho(G' - X) > \alpha k|G' - X|$. Thus (G', X) fails to satisfy condition (2), and hence it has a separation (A', B') of order at most $|X| - 1$ with $\rho(B' - A') > \alpha k|B' - A'|$. Then $B' - A' \neq \emptyset$, and hence $A' \subsetneq A$. Let us select such a separation with B' minimal. If $S \subseteq A'$, then $(A' \cup B, B')$ is a separation of (G, X) violating condition (2), a contradiction. Thus $S \nsubseteq A'$, but S is a clique, and hence $S \subseteq B'$. Since $\rho(B' - A') > \alpha k|B' - A'|$, the pair $(G'[B'], A' \cap B')$ satisfies (1), and the minimality of B' implies that it satisfies (2). Thus this pair is αk-massed, and hence the (α, k)-minimality of (G, X) implies that $(G'[B'], A' \cap B')$ is linked. A linkage in $(G'[B'], A' \cap B')$ can be converted to a linkage in $(G[B \cup B'], A' \cap B')$ similarly as above, establishing that $(A', B' \cup B)$ is a rigid separation of (G, X), violating our choice of (A, B). □

Claim 2.2. *(G, X) has no rigid separation of order exactly $|X|$.*

Proof. Suppose for a contradiction that (G, X) has a rigid separation (A, B) of order exactly $|X|$. We use an argument analogous to the proof of Theorem 1.1. If there exist $|X|$ disjoint paths from X to $A \cap B$, then those paths and the rigidity of (A, B) can be used to obtain any linkage in (G, X), contrary to (4). Otherwise there exists a separation (A', B') of $(G[A], X)$ of order strictly less than X with $A \cap B \subseteq B'$; let us choose such a separation with $|A' \cap B'|$ minimum. Then there exist $|A' \cap B'|$ disjoint paths between $A' \cap B'$ and $A \cap B$, and from the rigidity of (A, B) we deduce that $(A', B \cup B')$ is a rigid separation of (G, X) of order strictly less than $|X|$, contrary to Claim 2.1. □

Since (G, X) is not linked, there exist an integer l and a sequence $s_1, s_2, \ldots, s_l, t_1, t_2, \ldots, t_l$ of distinct vertices of X such that there does not exist the corresponding linkage. Condition (7) implies that for some choice of the above sequence, all pairs of vertices of X are adjacent, except possibly the pairs s_i, t_i. Thus we may assume that the chosen sequence has this property.

Claim 2.3. *If u and v are adjacent vertices of G and at least one of them does not belong to X, then u and v have at least $\lfloor \alpha k \rfloor$ common neighbors.*

Proof. Let $G' = G/uv$. If (G', X) is αk-massed, then the (α, k)-minimality of (G, X) implies that (G', X) is linked. But then (G, X) is linked, a contradiction. Thus (G', X) is not αk-massed, and so it fails to satisfy (1) or (2).

We claim that (G', X) satisfies (2). To prove this claim suppose for a contradiction that (G', X) has a separation (A, B) of order strictly less than $|X|$ such that $\rho(B - A) > \alpha k|B - A|$, and pick such a separation (A, B) with B minimal. The (α, k)-minimality of (G, X) implies that $(G'[B], A \cap B)$ is linked.

The separation (A, B) induces a separation (A^*, B^*) of G, where we replace the new vertex of G' with both u and v. Then $u, v \in B^*$, or else (G, X) would have a separation violating (2). If $u, v \in A^*$ as well, then $\rho(B^* - A^*) > \alpha k|B^* - A^*|$, and thus (A^*, B^*) is a rigid separation by the (α, k)-minimality of (G, X) applied to the pair $(G[B^*], A^* \cap B^*)$. If one of u, v does not belong to A^*, then a linkage in $(G'[B], A \cap B)$ gives rise to a linkage in $(G[B^*], A^* \cap B^*)$, again showing that (A^*, B^*) is rigid. Thus in either case we obtain contradiction to Claim 2.1 or 2.2. This proves our claim that (G', X) satisfies (2).

Since (G', X) is not αk-massed, the above claim implies that it does not satisfy (1). Thus G' must have at least $\lfloor \alpha k \rfloor + 1$ fewer edges incident $V(G') - X$. This means (keeping in mind that all pairs of vertices of X are adjacent, except possibly the pairs s_i, t_i) that either u and v have at least $\lfloor \alpha k \rfloor + 1$ common neighbors; or they have exactly $\lfloor \alpha k \rfloor$ common neighbors, and one of u, v belongs to $\{s_i, t_i\}$ and the other is adjacent to the other member of $\{s_i, t_i\}$. In either case the claim holds. □

Claim 2.4. *Let δ^* be the minimum degree in G among the vertices of $V(G) - X$. Then $\lfloor \alpha k \rfloor + 1 \leq \delta^* < 2\alpha k$.*

Proof. The lower bound follows immediately from Claim 2.3. To prove the upper bound, consider the graph $G - e$ for some edge $e \in E(G)$ which does not have both ends in X. If $(G - e, X)$ is αk-massed, then by the (α, k)-minimality of (G, X) the pair $(G - e, X)$ is linked, and consequently, (G, X) is as well, a contradiction. So $(G - e, X)$ is not

αk-massed, and hence it fails to satisfy (1) or (2). We claim that it satisfies (2). Indeed, otherwise $(G - e, X)$ has a separation (A, B) of order less than $|X|$ with $\rho(B - A) > \alpha k|B - A|$. It follows that $u \in A - B$ and $v \in B - A$, lest (A, B) be a separation in (G, X) violating (2). But by Claim 2.3, u and v have at least $\lfloor \alpha k \rfloor$ common neighbors in G. Since these common neighbors belong to $A \cap B$, we have $2k \leq \lfloor \alpha k \rfloor \leq |A \cap B| < |X|$, a contradiction. This proves that $(G - e, X)$ satisfies (2), and hence it does not satisfy (1). We conclude that $\rho(G - X) \leq \alpha k|G - X| + 1$.

For $x \in X$ let $f(x)$ be the number of neighbors of x in $V(G) - X$. Clearly $f(x) \geq 1$, lest $(X, V(G) - \{x\})$ be a separation of (G, X) violating (2). But then by Claim 2.3, $f(x) \geq \lfloor \alpha k \rfloor - (2k - 2) + 1 \geq 3$. If $\delta^* \geq 2\alpha k$, then

$$2\alpha k|V(G) - X| + 2 \geq 2\rho(G - X) = \sum_{v \in G - X} \deg(v) + \sum_{x \in X} f(x) \geq 2\alpha k|V(G) - X| + 3|X|,$$

a contradiction, because $X \neq \emptyset$ by (2) applied to (G, X). □

We are now ready to complete the proof of Theorem 1.4. Let $v \in (G) - X$ be a vertex of degree δ^* in G, and let L be the induced subgraph on v and the neighborhood of v. By Claim 2.4, L has at most $\lceil 2\alpha k \rceil$ vertices, and by Claim 2.3, L has minimum degree at least $\lfloor \alpha k \rfloor + 1$, as desired. This completes the proof of Theorem 1.4.

3. Proof of Theorem 1.5

In the proof we will need the following lemma.

Lemma 3.1. *Let J be a graph such that $2\delta(J) \geq |J| + 3k - 4$. Then J is k-linked.*

Proof. Let $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ be a sequence of distinct vertices of J, and let $X = \{s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k\}$. The hypothesis implies that every two nonadjacent vertices of X have at least k common neighbors outside of X, and hence there is a desired linkage consisting of paths of length at most 2. □

Now we are ready to prove Theorem 1.5. Let k and H be as in the statement. We may assume that H is not k-linked, and hence there exists a sequence $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ of distinct vertices of H with no corresponding linkage. Let $X = \{s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k\}$, and let us choose a set $\mathcal{P}$ of disjoint paths such that for each path $P \in \mathcal{P}$

(a) P has length at most seven,
(b) the ends of P are s_i and t_i for some $i = 1, 2, \ldots, k$,
(c) no internal vertex of P belongs to X,
(d) subject to (a)–(c), $|\mathcal{P}|$ is maximum, and
(e) subject to (a)–(d), the sum of the lengths of the paths in $\mathcal{P}$ is minimum.

Then $|\mathcal{P}| < k$, and so we may assume that s_1 and t_1 belong to no member of $\mathcal{P}$. Let L be the subgraph of H induced on X and the paths in $\mathcal{P}$. Notice that any vertex $v \in V(H) - V(L)$

has at most $3k$ neighbors in L, for otherwise it would have at least four neighbors on some path $P \in \mathcal{P}$, in which case it would have two nonconsecutive neighbors on P, and so P could be shortened by using v, contrary to (e). Thus the graph $H - V(L)$ has minimum degree at least $8k - 3k = 5k$. Since L has at most $8(k-1) + 2$ vertices, we see that both s_1 and t_1 have a neighbor in $H - V(L)$.

We now show that $H - V(L)$ is not connected. To this end let S be the set of all vertices of $H - V(L)$ at distance at most two from a neighbor of s_1, where the distance is taken in the graph $H - V(L)$; and let T be defined analogously with t_1 replacing s_1. Then S and T are nonempty; by (d) they are disjoint, and no edge of H has one end in S and the other end in T. We claim that $S \cup T \cup V(L) = V(H)$. To prove this claim let $v \in V(H) - V(L)$, and let x and y be neighbors in $H - V(L)$ of s_1 and t_1, respectively. Then x, y, and v all have at least $5k$ neighbors in $H - V(L)$, but $H - V(L)$ has at most $16k - 2k = 14k$ vertices. Since S and T are disjoint, it follows that v belongs to S or T, as desired. This proves our claim that $S \cup T \cup V(L) = V(H)$, and hence concludes the proof of the fact that $H - V(L)$ is disconnected.

Now let J be the smallest component of $H - V(L)$. Then J has at most $(16k - 2k)/2 = 7k$ vertices and minimum degree at least $5k$. By Lemma 3.1 the graph J is k-linked, as desired. This completes the proof of Theorem 1.5. □

4. Proof of Theorem 1.6

We will need the following strengthening of Lemma 3.1, due to Egawa et al. [2], and obtained independently by Kawarabayashi et al. [4]. For $4k \geq n \geq 3k$ the exact numerical bound does not follow from the statement of [2, Theorem 3], but it does follow from the proof.

Theorem 4.1. *Let $k \geq 2$ be an integer, and let H be a graph on $n \geq 3k$ vertices and minimum degree δ. If $n \geq 4k$, then let $2\delta \geq n + 2k - 3$, and otherwise let $3\delta \geq n + 5k - 5$. Then H is k-linked.*

We are now ready to begin the proof of Theorem 1.6. Let k and G be as in the statement of the theorem. We may assume that G is not k-linked, and hence there exists a sequence $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ of distinct vertices of G with no corresponding linkage. Let $X = \{s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k\}$. A subgraph L of G is called a *partial linkage* if $X \subseteq V(L)$ and every component P of L satisfies the following conditions:

(a) P is a path of length at most five,
(b) either $V(P)$ consists of one member of X, or the ends of P are s_i and t_i for some $i = 1, 2, \ldots, k$, and
(c) no internal vertex of P belongs to X.

A partial linkage is called *minimal* if

(d) there is no partial linkage with strictly fewer components than L, and
(e) subject to (d), there is no partial linkage with fewer vertices.

By the choice of X, for every partial linkage L there exists an index $i \in \{1, 2, \ldots, k\}$ such that s_i and t_i are not connected by a path of L. Such indices will be called *unresolved* for L.

Claim 4.2. *Let L be a minimal linkage, let P be a component of L, and let $v \in V(G) - V(L)$. Then any two neighbors of v in P are at distance at most two in P. In particular, v has at most three neighbors on P. Moreover, v has at most $3k - 2$ neighbors in $V(L)$.*

Proof. To prove the first statement suppose for a contradiction that v has neighbors x and y on P such that the subpath of P from x to y has at least two internal vertices. Then by deleting those internal vertices from L and adding the path xvy we obtain a partial linkage with the same number of components but fewer vertices than L, contrary to the minimality of L. The second statement follows immediately from the first. To prove the third statement notice that if i is an unresolved index for L, then v is adjacent to at most one of s_i, t_i by the minimality of L. □

If L is a partial linkage and $i \in \{1, 2, \ldots, k\}$, then we define $S_i(L)$ to be the set of all neighbors of s_i in $V(G) - V(L)$, and we define $T_i(L)$ analogously.

Claim 4.3. *Let L be a minimal linkage, let i be unresolved for L, and let $v \in V(G)-V(L)$. Then v has at least five neighbors in $S_i(L) \cup T_i(L)$.*

Proof. Let L, i, and v be as stated. For $m = 3, 4, 5, 6$ let l_m be the number of components of L on m vertices, and let l_2 be the number of indices $j \in \{1, 2, \ldots, k\}$ such that s_j and t_j are either adjacent in L, or not connected by a path of L. Let l'_3 be the number of components P of L such that P has three vertices, all adjacent to both s_i and t_i. Clearly $l_2 + l_3 + \cdots + l_6 = k$ and $2l_2 + 3l_3 + \cdots + 6l_6 = |V(L)|$. For $v \in V(G)$ let $N(v)$ denote the set of neighbors of v. Let P be a component of L on $m \geq 4$ vertices. Then $|N(s_i) \cap V(P)| + |N(t_i) \cap V(P)| \leq m + 2$, for otherwise s_i and t_i have a common neighbor, say u, in the interior of P. In that case the linkage obtained from L by deleting P and adding the path $s_i u t_i$ has the same number of components as L, but fewer vertices, contrary to the minimality of L. Thus

$$\begin{aligned}|N(s_i) \cap V(L)| + |N(t_i) \cap V(L)| &\leq 4(l_2 - 1) + 6l'_3 + 5(l_3 - l'_3) + 6l_4 + 7l_5 + 8l_6 \\ &\leq |V(L)| + 2k + l'_3 - 4.\end{aligned}$$

From this it follows, since $S_i(L) \cap T_i(L) = \emptyset$ by the minimality of L,

$$\begin{aligned}|S_i(L) \cup T_i(L)| &\geq 5k - |N(s_i) \cap V(L)| + 5k - |N(t_i) \cap V(L)| \\ &\geq 10k - (|V(L)| + 2k + l'_3 - 4) = 8k - |V(L)| - l'_3 + 4.\end{aligned}$$

Now let P be a component of L, and let $v \in V(G) - V(L)$. Then v has at most three neighbors on P by Claim 4.2. Moreover, if P has length two and each of its vertices is adjacent to both s_i and t_i, then v has at most two neighbors in P. Indeed, suppose the contrary, and let P have vertex-set $\{s_j, u, t_j\}$; then the linkage obtained from L by deleting P and adding the paths $s_i u t_i$ and $s_j v t_j$ contradicts the minimality of L. Thus v has at most two neighbors on P. This implies that for $v \in V(G) - V(L)$

$$|N(v) - V(L)| \geq 5k - 3(k - l'_3) - 2l'_3 \geq 2k + l'_3.$$

Now let t be the number of neighbors of v in $S_i(L) \cup T_i(L)$. Then

$$10k \geq |S_i(L) \cup T_i(L)| + |V(L)| + |\{v\}| + |N(v) - V(L)| - t$$
$$\geq 8k - |V(L)| - l'_3 + 4 + |V(L)| + 1 + 2k + l'_3 - t = 10k + 5 - t,$$

and so $t \geq 5$, as desired. □

If L is a partial linkage and i is unresolved for L, then we define $\overline{S}_i(L)$ to be the set of all vertices $v \in V(G) - V(L)$ such that either v belongs to or has a neighbor in $S_i(L)$; and we define $\overline{T}_i(L)$ analogously. We now prove two fundamental properties of these sets.

Claim 4.4. *Let L be a minimal linkage, and let i be unresolved for L. Then $\overline{S}_i(L)$ and $\overline{T}_i(L)$ are disjoint, there is no edge between them, and their union is $V(G) - V(L)$.*

Proof. If $\overline{S}_i(L)$ and $\overline{T}_i(L)$ are not disjoint, or if there is an edge between them, then there exists a path P between s_i and t_i of length at most five with internal vertices in $\overline{S}_i(L) \cup \overline{T}_i(L)$. But then the linkage $L \cup P$ has fewer components than L, and hence contradicts the minimality of L. Now let $v \in V(G) - V(L)$. By Claim 4.3 the vertex v has a neighbor in $S_i(L)$ or $T_i(L)$, and so it lies in either $\overline{S}_i(L)$ or $\overline{T}_i(L)$, respectively. □

Claim 4.5. *Let L be a minimal linkage, and let i be an unresolved index for L. If $\overline{S}_i(L) \neq \emptyset$, then $|\overline{S}_i(L)| \geq 2k + 3$. If $\overline{T}_i(L) \neq \emptyset$, then $|\overline{T}_i(L)| \geq 2k + 3$.*

Proof. From the symmetry it suffices to prove the first statement. By Claim 4.2, a vertex v in $V(G) - V(L)$ has at least $5k - (3k - 2) = 2k + 2$ neighbors in $V(G) - V(L)$, implying that if $\overline{S}_i(L)$ is nonempty, then $G[\overline{S}_i(L)]$ has minimum degree at least $2k + 2$. Thus $|\overline{S}_i(L)| \geq 2k + 3$, as desired. □

Guided by the proof of Theorem 1.5 our next objective is to show that a minimal linkage L and an unresolved index i for it can be chosen so that both $\overline{S}_i(L)$ and $\overline{T}_i(L)$ are nonempty. The proof is long, and makes use of further enlargements of the sets $\overline{S}_i(L)$ and $\overline{T}_i(L)$, which we shall denote by $\tilde{S}_i(L)$ and $\tilde{T}_i(L)$, respectively. We now introduce these sets.

Let L be a minimal linkage, let i be an unresolved index for L, and let $v \in \overline{S}_i(L) \cup \overline{T}_i(L)$ have three consecutive neighbors u_1, u_2, u_3, in order, on some component P of L. Let L' be obtained from L by deleting u_2 and adding the vertex v and edges u_1v and u_3v. Then L' is a minimal linkage and i is an unresolved index for L. We say that L' is a *v-flip* of L, and we say that the sequence u_1, u_2, u_3 is the *base* of the flip.

Claim 4.6. *Let L be a minimal linkage, let i be an unresolved index for L, let $v \in \overline{S}_i(L) \cup \overline{T}_i(L)$, and let L' be a v-flip of L with base u_1, u_2, u_3. Then $\overline{S}_i(L') - \{u_2\} = \overline{S}_i(L) - \{v\}$ and $\overline{T}_i(L') - \{u_2\} = \overline{T}_i(L) - \{v\}$. Moreover, $u_2 \in \overline{S}_i(L')$ if and only if u_2 has a neighbor in $\overline{S}_i(L) - \{v\}$. Similarly, $u_2 \in \overline{T}_i(L')$ if and only if u_2 has a neighbor in $\overline{T}_i(L) - \{v\}$.*

Proof. Let $u \in \overline{S}_i(L) - \{v\}$. To prove the first two equalities, it suffices to prove, by symmetry, that $u \in \overline{S}_i(L') - \{u_2\}$. Clearly $u \neq u_2$, because $u \notin V(L)$. By Claim 4.3 the vertex u has at least five neighbors in $S_i(L) \cup T_i(L)$, but since $u \in \overline{S}_i(L)$, all those neighbors belong to $S_i(L)$ by Claim 4.4. It follows that u has a neighbor in $S_i(L')$, and hence it belongs to $\overline{S}_i(L')$, as desired. By Claims 4.3 and 4.4 the vertex u_2 has at least five neighbors in either $S_i(L')$ or $T_i(L')$. In the former case $u_2 \in \overline{S}_i(L')$ and it has a neighbor in $S_i(L) - \{v\}$, and in the latter case neither of these statements hold by Claim 4.4. The last assertion follows by symmetry. □

Let $L, L', i, v, u_1, u_2, u_3$ be as above, and assume now that $v \in \overline{S}_i(L)$. If u_2 has a neighbor $v' \in \overline{S}_i(L) - \{v\}$, then we say that L' is a *proper v-flip* of L. In that case $u_2 \in \overline{S}_i(L')$ by Claim 4.6 and v has a neighbor in $\overline{S}_i(L') - \{u_2\}$ by Claims 4.3 and 4.4. Thus L is a proper u_2-flip of L', and so the relationship is symmetric. We say that L and L' are $\overline{S}_i$*-adjacent*. If $v \in \overline{T}_i(L)$ then we say that the v-flip L' is *proper* if u_2 has a neighbor $v' \in \overline{T}_i(L) - \{v\}$, and say that L and L' are $\overline{T}_i$*-adjacent*. We say that two partial linkages L and L' are i*-adjacent* if they are $\overline{S}_i$-adjacent or $\overline{T}_i$-adjacent. We say that L and L' are i*-related* if there exists a sequence $L_0, L_1, \ldots, L_n$ of linkages such that $L = L_0$, $L' = L_n$, and L_j is i-adjacent to L_{j-1} for all $j = 1, 2, \ldots, n$. The following is an immediate consequence of Claim 4.6.

Claim 4.7. *Let L be a minimal linkage, let i be an unresolved index for L, and let L' be a linkage i-related to L. Then $|\overline{S}_i(L')| = |\overline{S}_i(L)|$ and $|\overline{T}_i(L')| = |\overline{T}_i(L)|$.*

The next claim states that the order of $\overline{S}_i$- and $\overline{T}_i$-adjacencies can be reversed.

Claim 4.8. *Let L be a minimal linkage with i an unresolved index. Then if the linkage L_1 is $\overline{T}_i$-adjacent to L and L_2 is $\overline{S}_i$-adjacent to L_1, then there exist linkages L'_1 and L'_2 where L'_1 is $\overline{S}_i$-adjacent to L, and L'_2 is $\overline{T}_i$-adjacent to L'_1. Moreover, $L'_2 = L_2$.*

Proof. Let L, i, L_1 and L_2 be as in the statement. Let $v_1 \in \overline{S}_i(L)$ be the vertex such that L_1 is a proper v-flip of L and let u_1, u_2, u_3 be the base of the flip. Similarly, let v_2 be the vertex in $\overline{T}_i(L_1)$ such that L_2 is a proper v_2-flip of L_1, and let w_1, w_2, w_3 be the base. By Claim 4.4 the vertex $v_2 \in \overline{T}_i(L_1) = \overline{T}_i(L)$ is not adjacent to $v_1 \in \overline{S}_i(L)$ or $u_2 \in S_i(L_1)$, where the equality and the last membership hold by Claim 4.6. Thus we see that $u_2 \notin \{w_1, w_2, w_3\}$. Since L_2 is a proper v_2-flip, the vertex w_2 has at least one other neighbor in $\overline{T}_i(L) = \overline{T}_i(L_1)$ besides the vertex v_2. Thus there exists a linkage L'_1 that is a proper v_2-flip of the linkage L. Moreover, u_1, u_2, u_3 are in some component P'_1 of L'_1, and since $\overline{S}_i(L'_1) = \overline{S}_i(L)$ by Claim 4.6, we see that there exists a linkage L'_2 that is a proper v_1-flip of L'_1. By construction, $L_2 = L'_2$, as desired. □

We are finally ready to define the promised enlargements of $\overline{S}_i$ and $\overline{T}_i$. Let L_0 be a minimal linkage, and let i be an unresolved index for L_0. We define $\tilde{S}_i(L_0) := \bigcup \overline{S}_i(L)$ and $\tilde{T}_i(L_0) := \bigcup \overline{T}_i(L)$, the unions taken over all linkages L that are i-related to L_0. We now show that these sets satisfy most of the conclusion of Claim 4.4.

Claim 4.9. *Let L_0 be a minimal linkage with i an unresolved index. Then $\tilde{S}_i(L_0)$ and $\tilde{T}_i(L_0)$ are disjoint, and there does not exist an edge with ends u and v such that $u \in \tilde{S}_i(L_0)$ and $v \in \tilde{T}_i(L_0)$.*

Proof. Assume we have $u \in \tilde{S}_i(L_0)$ and $v \in \tilde{T}_i(L_0)$ with u adjacent to v. Then there exists a linkage L i-related to L_0 with $u \in \overline{S}_i(L)$. There also exists a sequence $L = L_0, L_1, \ldots, L_m = L'$ of linkages, where $v \in \overline{T}_i(L')$ and L_j is i-adjacent to L_{j-1} for $j = 1, 2, \ldots, m$. Then by Claim 4.8, we may assume that there exists $l \leq m$, where for $1 \leq j \leq l$, L_{j-1} is $\overline{T}_i$-adjacent to L_j, and for $l + 1 \leq j \leq m$, L_{j-1} is $\overline{S}_i$-adjacent to L_j.

By Claim 4.6 $\overline{S}_i(L_j)$=$\overline{S}_i(L)$ for every $1 \leq j \leq l$. Importantly, $u \in \overline{S}_i(L_l)$. Similarly, by Claim 4.6, $v \in \overline{T}_i(L_l)$. But then for the minimal linkage L_l, we have an edge between vertices of $\overline{S}_i(L_l)$ and $\overline{T}_i(L_l)$. This contradicts Claim 4.4. To see that $\tilde{S}_i(L_0)$ and $\tilde{T}_i(L_0)$ are in fact disjoint, assume $v \in \tilde{S}_i(L_0) \cap \tilde{T}_i(L_0)$. Then there exists a linkage L i-related to L_0 with $v \in \overline{S}_i(L)$. But every vertex in $\overline{S}_i(L)$ has at least five neighbors in $S_i(L)$ by Claims 4.3 and 4.4, so v has a neighbor in $\tilde{S}_i(L_0)$. But then there is an edge with one end in $\tilde{S}_i(L_0)$ and the other end in $\tilde{T}_i(L_0)$, contrary to what we have just seen. □

Now we are finally ready to prove that we may assume that there exists a minimal linkage L and an unresolved index i for L such that both $\overline{S}_i(L)$ and $\overline{T}_i(L)$ are nonempty.

Claim 4.10. *There exists a minimal linkage L and an unresolved index i such that either both $\overline{S}_i(L)$ and $\overline{T}_i(L)$ are nonempty, or one of $\tilde{S}_i(L)$, $\tilde{T}_i(L)$ induces a k-linked subgraph of G.*

Proof. Let L_0 be a minimal linkage, and let i be an unresolved index for L_0. If both $\tilde{S}_i(L_0)$ and $\tilde{T}_i(L_0)$ are nonempty, then by Claim 4.7 we deduce that $\overline{S}_i(L_0)$ and $\overline{T}_i(L_0)$ are both nonempty, and so the claim holds. From the symmetry between $\tilde{S}_i(L_0)$ and $\tilde{T}_i(L_0)$ we may assume therefore that $\tilde{S}_i(L_0) = \emptyset$.

Let $v \in \tilde{T}_i(L_0)$ be a vertex of minimum degree in $G[\tilde{T}_i(L_0)]$, and let L be a linkage related to L_0 such that $v \in \overline{T}_i(L)$. Assume first that there exists a component P of L such that s_i has at least five neighbors on P and v has at least two neighbors on P. Let the ends of P be s_j and t_j. Since P has at least five vertices and v has at least two neighbors in P, Claim 4.2 implies that v is adjacent to an internal vertex of P. Let us choose such a neighbor, say u, so that it is not adjacent to s_j or t_j, if possible. Since $v \in \bar{T}_i(L)$ there exists a path Q of length at most two with ends v and t_i and internal vertex (if it exists) in $T_i(L)$. If u is adjacent to s_i let P' denote the path $s_i u v Q t_i$. If u is not adjacent to s_i, then P has six vertices, and every vertex of $V(P) - \{u\}$ is adjacent to s_i. Let u' be a neighbor of u in P chosen so that u' is not equal or adjacent to s_j or t_j, and let P' denote the path $s_i u' u v Q t_i$. Then in either case the length of P' is at most the length of P. Let L' be obtained from L by deleting the internal vertices of P and adding P'; then L' is a minimal linkage and j is an unresolved index for L'. From the symmetry between $S_j(L')$ and $T_j(L')$ we may assume that $S_j(L') = \emptyset$, for if both are nonempty, then the claim holds. In particular, u is adjacent to s_j, for otherwise the neighbor of s_j in P belongs to $S_j(L')$. It follows that there exists a vertex $u'' \in V(P) - V(P')$ not adjacent to s_j or t_j. Then u'' is adjacent to s_i, for otherwise P has length five and u is adjacent to s_i; consequently P' has length at most four, contrary to the minimality of L. By Claim 4.3 the vertex u'' has at least five neighbors in $S_j(L') \cup T_j(L') = T_j(L')$. Thus u'' has a neighbor $v'' \in T_j(L') - V(P) \subseteq V(G) - V(L) = \bar{T}_i(L)$. Let Q'' be a path of length at most two with ends v'' and t_i and internal vertex in $T_i(L)$. Let L'' be obtained from L' by replacing P' by the path $P'' := s_i u'' v'' Q'' t_i$. Then L'' is a minimal linkage, and by our choice of P'' to include only u'' from P, we see that both s_j's neighbor from P as well as t_j's neighbor from P is not included in L''. Thus j is an unresolved index with both $S_j(L'')$ and $T_j(L'')$ nonempty, proving the claim.

Thus we may assume that if a component P of L includes at least five neighbors of s_i, then it includes at most one neighbor of v. Since $\overline{S}_i(L) = \emptyset$, s_1 has at least $5k$ neighbors in $V(L)$, and hence at least $k/2$ components of L have at least five neighbors of s_1. Those components have at most one neighbor of v. The remaining components have at most two neighbors of v that do not belong to $\tilde{T}_i(L)$, because if v has three neighbors on a component P of L, then those neighbors are consecutive, and by considering a v-flip of L we deduce (using $\tilde{S}_i(L) = \emptyset$ and Claim 4.4) that the middle of the three neighbors belongs to $\tilde{T}_i(L)$. Thus v has at most $k/2 + 2k/2 = 3k/2$ neighbors outside $\tilde{T}_i(L)$, and hence $G[\tilde{T}_i(L)]$ has minimum degree at least $5k - 3k/2 = 7k/2$. But $\tilde{T}_i(L)$ includes no neighbor s of s_1, for otherwise a linkage L' with $s \in \overline{T}_i(L')$ contradicts Claim 4.4. Thus $|\tilde{T}_i(L)| \le 10k - 5k \le 5k$, and hence $G[\tilde{T}_i(L)]$ is k-linked by Theorem 4.1. □

Claim 4.10 enables us to choose a suitable linkage and an unresolved index for it. A linkage L is called *optimal* if

(O1) L is minimal,
(O2) $i = 1$ is an unresolved index for L, and
(O3) there is no minimal linkage L' with an unresolved index i' for L' such that

$$0 < \min\{|\overline{S}_{i'}(L')|, |\overline{T}_{i'}(L')|\} < \min\{|\overline{S}_1(L)|, |\overline{T}_1(L)|\}.$$

By Claim 4.10 we may assume (by permuting the elements of X) that there exists an optimal linkage, say L_0, and let L_0 be fixed for the rest of the paper. Then every linkage 1-related to L_0 is also optimal by Claim 4.7. From the symmetry between $\overline{S}_1(L_0)$ and $\overline{T}_1(L_0)$ we may assume that $|\overline{S}_1(L_0)| \le |\overline{T}_1(L_0)|$. Let $\tilde{S} := \tilde{S}_1(L_0)$ and $\tilde{T} := \tilde{T}_1(L_0)$. The following is the main advantage of optimality.

Claim 4.11. *If L is an optimal linkage and $v \in \overline{S}_1(L)$, then every v-flip is proper.*

Proof. Let L' be a v-flip of L with base u_1, u_2, u_3, and suppose for a contradiction that it is not proper. Then $\overline{S}_1(L') = \overline{S}_1(L) - \{v\}$ by Claim 4.6 and $\overline{S}_1(L') \ne \emptyset$ by Claim 4.5, contrary to the optimality of L. □

Claim 4.12. *Either $|\tilde{S}| \ge 4k$ or $G[\tilde{S}]$ is k-linked.*

Proof. Let v be a vertex of $\tilde{S}$ such that v is of minimum degree in $G[\tilde{S}]$. Then there exists a linkage L 1-related to L_0 with $v \in \overline{S}_1(L)$. Then, by Claim 4.2, for each component P of L, v has at most three neighbors in P, and if it has three, then they are consecutive. However, if v has three neighbors on P, say u_1, u_2, u_3, in order, then the v-flip of L is proper by Claim 4.11, showing that $u_2 \in \tilde{S}$. Thus v has at most two neighbors in $V(P) - \tilde{S}$ for each component P of L. Further, v has at most one neighbor among each pair of terminals not connected by a path in L. Thus v has at most $2(k-1)+1$ neighbors not in $\tilde{S}$. But then $G[\tilde{S}]$ has minimum degree at least $5k - (2k-1) = 3k+1$. Thus $|\tilde{S}| \ge 3k$. If $|\tilde{S}| \le 4k-1$, then by Theorem 4.1, $G[\tilde{S}]$ is k-linked. Thus the claim holds. □

If Claim 4.11 held for vertices $v \in \overline{T}_1(L)$, then we would have an analog of Claim 4.12 for $\tilde{T}$, and we would be done. Unfortunately, that is not the case, but, luckily, the counterexamples to the analog of Claim 4.11 can be managed. Hence the following definition. Let L be an optimal linkage. We say that a vertex $u \in V(L)$ is L-*treacherous* if u is an internal vertex of a component P of L, u has a unique neighbor $v \in \overline{T}_1(L)$, and v is adjacent to both neighbors of u in P. Treacherous vertices are annoying in the sense that if v is as above, then the v-flip of L is not proper. Our intention is to pick two vertices in $\overline{T}_1(L)$ with the most treacherous neighbors, and remove them from $\overline{T}_1(L)$. Actually, we need to be more delicate. We need to not only remove them from $\overline{T}_1(L)$, but we also need to redefine $\tilde{T}$ as if those vertices did not exist. Let us be more precise.

Let L be a linkage, let $v \in \overline{S}_1(L) \cup \overline{T}_1(L)$, let L' be a proper v-flip of L with base u_1, u_2, u_3, and let $V \subseteq \overline{T}_1(L)$ be a set. If $v \notin V$, then we say that L and L' are *adjacent modulo* V. In that case $u_2 \in \overline{S}_1(L') \cup \overline{T}_1(L') - V$ and $V \subseteq \overline{T}_1(L')$ by Claim 4.6, and so the definition is symmetric in L and L'. We say that two linkages L and L' are *related modulo* V if there exists a sequence $L_0, L_1, \ldots, L_n$ of linkages such that $L = L_0$, $L' = L_n$, and L_i is adjacent to L_{i-1} modulo V for all $i = 1, 2, \ldots, n$. We shall abbreviate "1-adjacent" and "1-related" to "adjacent" and "related", respectively. Thus L and L' are related if and only if they are related modulo $\emptyset$.

Let an optimal linkage L_1 related to L_0 and a vertex $v_1 \in \overline{T}_1(L_1)$ be chosen to maximize the number of L_1-treacherous neighbors of v_1. Let an optimal linkage L_2 related to L_1 modulo $\{v_1\}$ and a vertex $v_2 \in \overline{T}_1(L_2) - \{v_1\}$ be chosen to maximize the number of L_2-treacherous neighbors of v_2. Let $\tilde{R} := \bigcup \overline{T}_1(L) - \{v_1, v_2\}$, the union taken over all linkages L related to L_2 modulo $\{v_1, v_2\}$. Then clearly $\tilde{R} \subseteq \tilde{T}$ and $v_1, v_2 \in \overline{T}_1(L)$ for every linkage L related to L_2 modulo $\{v_1, v_2\}$.

Claim 4.13. *Let L be a linkage related to L_2 modulo $\{v_1, v_2\}$, let $v \in \tilde{R} - V(L)$, and let ξ be the number of L-treacherous neighbors of v that do not belong to $\tilde{R}$. Then v has at least $3k - \xi - 1$ neighbors in $\tilde{R}$.*

Proof. Let P be a component of L. We claim that v has at most two neighbors in $V(P) - \tilde{R}$ that are not L-treacherous. If v has three neighbors in $V(P) - \tilde{R}$, then by Claim 4.2 they are consecutive, say u_1, u_2, u_3, in order. Since $u_2 \notin \tilde{R}$ we deduce that the v-flip of L is not proper, and hence u_2 is L-treacherous. There is at least one index $j \in \{1, 2, \ldots, k\}$ such that s_j, t_j are not joined by a path of L, and the minimality of L implies that v is adjacent to at most one of s_j, t_j. Thus v has at most $2(k-1) + \xi + 1$ neighbors in $V(L) - \tilde{R}$. Hence v has at least $5k - (2k - 1 + \xi) = 3k + 1 - \xi$ neighbors in the complement of $V(L) - \tilde{R}$. Those neighbors belong to $\tilde{R}$, except for v_1 and v_2. Thus the claim holds. □

Claim 4.14. $|\tilde{R}| \geq 3k$.

Proof. Each component P of L_2 includes at most two L_2-treacherous vertices, because any two L_2-treacherous vertices on P are at distance at least two on P by the definition of an L_2-treacherous vertex and Claim 4.2. By Claim 4.5 and the optimality of L_2 we have $|\overline{T}_1(L_2)| \geq 2k + 3$, and hence there exists a vertex $v \in \overline{T}_1(L_2) - \{v_1, v_2\} \subseteq \tilde{R}$ not adjacent to any L_2-treacherous vertex. By Claim 4.13 the vertex v has at least $3k - 1$ neighbors in $\tilde{R}$, and the claim follows. □

Let v_3 be a vertex of minimum degree of the graph $G[\tilde{R}]$, and let L_3 be a linkage related to L_2 modulo $\{v_1, v_2\}$ such that $v_3 \in \overline{T}_1(L_3)$. For $i = 1, 2, 3$ let ξ_i denote the number of L_i-treacherous neighbors of v_i that do not belong to $\tilde{R}$.

Claim 4.15. *Let L be an optimal linkage, let $v \in \overline{T}_1(L)$, and let u be an L-treacherous neighbor of v. Let $w \in \overline{S}_1(L) \cup \overline{T}_1(L) - \{v\}$. Then the base of a w-flip of L does not include u.*

Proof. Suppose for a contradiction that the base, say w_1, w_2, w_3, of a w-flip L' includes u. Since u is L-treacherous, v is adjacent to u and both neighbors of u in L. It follows that w_2 is adjacent to v, that u is adjacent to w, and that $w \in \overline{S}_1(L)$. But then the w-flip is proper by Claim 4.11, and hence $w_2 \in \overline{S}_1(L')$ and $v \in \overline{T}_1(L')$ by Claim 4.6. But w_2 is adjacent to v, contrary to Claim 4.4 applied to the linkage L'. □

Claim 4.16. *Let L be an optimal linkage, let $v \in \overline{T}_1(L)$, let u be an L-treacherous neighbor of v, and let L' be an optimal linkage related to L modulo $\{v\}$. Then $v \in \overline{T}_1(L')$ and u is L'-treacherous.*

Proof. We have $v \in \overline{T}_1(L')$ by Claim 4.6. Let u_1, u_3 be the two neighbors of u in L. It suffices to prove the claim assuming that L' is adjacent to L modulo $\{v\}$. From Claim 4.15 we deduce that u_1uu_3 is a subpath of L'. Suppose for a contradiction that u is not L'-treacherous. Then u is adjacent to a vertex $v' \in \overline{T}_1(L') - \{v\}$. Let L'' be the v-flip of L' with base u_1, u, u_3. Since u is adjacent to v', this v-flip is proper, and hence L'' is optimal and $u, v' \in \overline{T}_1(L'')$ by Claim 4.6. The vertex u is adjacent to at least five vertices in $T_1(L'')$ by Claims 4.3 and 4.4, and hence it has at least three neighbors in $T_1(L)$, contrary to the fact that it is L-treacherous. □

Claim 4.17. *Let $i \in \{1, 2, 3\}$, and let u be an L_i-treacherous neighbor of v_i. Then u is not adjacent to v_j for $j \in \{i + 1, \dots, 3\}$ and $u \notin \tilde{S}$.*

Proof. Since L_j is related to L_i modulo $\{v_i\}$, Claim 4.16 implies that $v_i \in \overline{T}_1(L_j)$ and u is L_j-treacherous. Thus u is not adjacent to v_j. To prove that $u \notin \tilde{S}$ suppose the contrary. Thus there exists a sequence of linkages $L_i = R_0, R_1, \dots, R_t$ such that R_i is adjacent to R_{i-1} for $i = 1, 2, \dots, t$ and $u \in \overline{S}_1(R_t)$. By Claim 4.8 we may assume that there is an integer $l \in \{1, 2, \dots, t\}$ such that R_i is $\overline{S}_1$-adjacent to R_{i-1} for $i = 1, 2, \dots, l$ and that R_i is $\overline{T}_1$-adjacent to R_{i-1} for $i = l + 1, \dots, t$. Then by Claim 4.6, $v_i \in \overline{T}_1(L_i) = \overline{T}_1(R_l)$ and $u \in \overline{S}_1(R_t) = \overline{S}_1(R_l)$. The edge uv_i violates Claim 4.4, a contradiction. □

Claim 4.18. *For $i = 1, 2$ no L_i-treacherous neighbor of v_i belongs to $\tilde{R}$.*

Proof. Let u be an L_i-treacherous neighbor of v_i, and suppose for a contradiction that it belongs to $\tilde{R}$. Thus there exists a linkage L related to L_i modulo $\{v_1, v_2\}$ such that $u \in \overline{T}_1(L)$. By Claim 4.16 the vertex u is L-treacherous, a contradiction. □

Claim 4.19. $\xi_1 \geq \xi_2 \geq \xi_3$.

Proof. Let $i \in \{2, 3\}$. Since L_i is related to L_{i-1} modulo $\{v_1, \dots, v_{i-1}\}$ and L_{i-1} is related to L_{i-2} modulo $\{v_1, \dots, v_{i-2}\}$, we deduce that L_i is related to L_{i-2} modulo $\{v_1, \dots, v_{i-2}\}$. Thus the choice of v_{i-1} implies that v_{i-1} has at least ξ_i neighbors that are L_{i-1}-treacherous; but no treacherous neighbor of v_{i-1} belongs to $\tilde{R}$ by Claim 4.18, and hence $\xi_{i-1} \geq \xi_i$, as desired. □

Claim 4.20. *If $|\tilde{R}| < 4k$, then either $|\tilde{R}| \geq 4k - 3\xi_3 + 3$ or the graph $G[\tilde{R}]$ is k-linked.*

Proof. The graph $G[\tilde{R}]$ has minimum degree at least $3k - \xi_3 - 1$ by Claim 4.13, because v_3 is a vertex of minimum degree in that graph. From Claim 4.14 and Theorem 4.1 we deduce that if the first conclusion of the claim does not hold, then the second does, as desired. □

Now we are ready to complete the proof of Theorem 1.6. Recall that $X = \{s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k\}$. By Claim 4.12 we may assume that $|\tilde{S}| \geq 4k$, for otherwise the theorem holds. But $\tilde{R}$ is disjoint from $\tilde{S} \cup X \cup \{v_1, v_2\}$ by Claim 4.9, and hence $|\tilde{R}| \leq 10k - 4k - 2k - 2 < 4k$. Similarly, by Claim 4.20 we may assume that $|\tilde{R}| \geq 4k - 3\xi_3 + 3$. For $i = 1, 2, 3$ let Z_i denote the set of L_i-treacherous neighbors of v_i not in $\tilde{R}$. Thus $|Z_i| = \xi_i$. Since the sets $\tilde{S}$, $\tilde{R}$, Z_1, Z_2, Z_3 and X are pairwise disjoint by Claims 4.9 and 4.17, we have, using Claim 4.19,

$$\begin{aligned} 10k &\geq |\tilde{S}| + |\tilde{R}| + \xi_1 + \xi_2 + \xi_3 + 2k \\ &\geq 4k + 4k - 3\xi_3 + 3 + \xi_1 + \xi_2 + \xi_3 + 2k \geq 10k + 3, \end{aligned}$$

a contradiction. This proves Theorem 1.6.

5. A lower bound

Construct a graph G as follows. Let $V(G)$ be the disjoint union of $V(P_1)$, $V(P_2)$, $V(P_3)$, $V(P_4)$, $V(H)$, and $\{s_3, \ldots, s_k, t_3, \ldots, t_k\}$, where $P_1, \ldots, P_4$ are four paths on m vertices each, with $m \geq 2$, and H is a complete graph on $k - 1$ vertices. Let the vertices of P_i be labeled $v_1^i, v_2^i, \ldots, v_m^i$, and let $s_1 = v_1^1, s_2 = v_1^2, t_1 = v_1^3, t_2 = v_1^4$. In the graph G, for $1 \leq j \leq m - 1$, let v_j^i be adjacent to v_j^{i+1} and v_{j+1}^{i+1}, and let v_m^i be adjacent to v_m^{i+1} as i ranges from 1 to 4 and the superscript arithmetic is taken modulo 4. Let v_m^i be adjacent to every vertex of H for $i = 1, \ldots, 4$. For every $i \geq 3$, let s_i and t_i be adjacent to every other vertex in the graph except each other. Then G does not have disjoint paths $Q_1, Q_2, \ldots, Q_k$, where Q_i has ends s_i and t_i, and so it is not k-linked. On the other hand, G is $2k$-connected and has $n = 4m + 3k - 5$ vertices and $(2k - 1)n - (3k + 1)k/2$ edges. This is the best example we are aware of, suggesting the following question.

Conjecture 5.1. *For every integer $k \geq 1$, every $2k$-connected graph on n vertices and at least $(2k - 1)n - (3k + 1)k/2 + 1$ edges is k-linked.*

The conjecture clearly holds for $k = 1$, and it holds for $k = 2$ by the characterization of 2-linked graphs in [9, 10, 13]. Recently, we have been able to show [11] that the conjecture also holds for $k = 3$, but it seems to be open for all $k \geq 4$.

It is likely that Theorem 1.6 can be improved. In light of the role it played in the proof of Theorem 1.1 we propose the following problem.

Problem 5.2. Determine the infimum α^* of all real numbers $\alpha > 0$ such that for all sufficiently large integers k every graph on at most $2\alpha k$ vertices and minimum degree at least αk has a k-linked subgraph.

By Theorem 1.6 we have $\alpha^* \leq 5$, and the graph K_{3k-1} shows that $\alpha^* \geq 3$. Any improvement in the upper bound would give a corresponding improvement in Theorem 1.1.

Acknowledgements

We are indebted to Ken-ichi Kawarabayashi, Alexandr Kostochka and Gexin Yu for bringing Theorem 4.1 to our attention, and for suggesting that our original bound can be substantially improved. The first author was partially supported by NSF grant DMS-0200595 and ONR grant N00014-01-0608.

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