Topology and its Applications 156 (2009) 1167–1169

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

SL($n, \mathbb{Z}$) cannot act on small spheres

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ARTICLE INFO

Keywords:
Special linear group
Smooth action
Homology sphere

ABSTRACT

The group SL($n, \mathbb{Z}$) admits a smooth faithful action on S^{n-1}, induced from its linear action on $\mathbb{R}^n$. We show that, if $m < n - 1$ and $n > 2$, any smooth action of SL($n, \mathbb{Z}$) on a mod 2 homology m-sphere, and in particular on the sphere S^m, is trivial.

1. Introduction

By [13] any smooth action of SL($n, \mathbb{Z}$) on the torus T^m is trivial if $m < n$ (whereas it acts linearly and faithfully on $T^n = \mathbb{R}^n/\mathbb{Z}^n$). Also, by [5] any continuous action of SL($n, \mathbb{Z}$) on a closed surface is trivial for sufficiently large values of n (depending on the genus of the surface). Our main result is the following analogue for smooth actions of SL($n, \mathbb{Z}$) on mod 2 homology spheres:

Theorem 1. *Let $n > 2$ and $m < n - 1$. Any smooth*[1] *action of* SL($n, \mathbb{Z}$) *on a mod* 2 *homology m-sphere, and in particular on S^m, is trivial.*

We note that the group SL($n, \mathbb{Z}$) admits a smooth faithful action on S^{n-1} (induced from its linear action on $\mathbb{R}^n$). Theorem 1 was conjectured by Parwani [12] who proved that, if $m < n - 1$ and $n > 2$, any smooth[1] action of SL($n, \mathbb{Z}$) on a mod 2 homology m-sphere factors through the action of a finite group. The main point of the proof in [12] is an application of the Margulis finiteness theorem which implies that, for $n > 2$, SL($n, \mathbb{Z}$) is almost simple that is every normal subgroup is either finite and central, or of finite index. Considering the subgroup $(\mathbb{Z}_2)^{n-1}$ of diagonal matrices with all diagonal entries equal to ± 1, Smith fixed point theory implies that some non-central element of SL($n, \mathbb{Z}$) has to act trivially on the mod 2 homology m-sphere if $m < n - 1$, and hence by Margulis' theorem the kernel of the action has finite index. By the result in [12], Theorem 1 is then a consequence of the following result to be proved in Section 2.

Theorem 2. *Let $n > 2$ and $m < n - 1$. Any smooth action of a finite quotient of* SL($n, \mathbb{Z}$) *on a mod* 2 *homology m-sphere is trivial.*

We note that Theorem 1 is proved in [12] for actions of SL($n, \mathbb{Z}$) on S^1 and S^2. For the case of S^1, Witte [14] has shown that every continuous action of a subgroup of finite index in SL($n, \mathbb{Z}$) on the circle S^1 factors through a finite group action. In the context of the Zimmer program for actions of irreducible lattices in semisimple Lie groups of $\mathbb{R}$-rank at least two [15], it has been conjectured by Farb and Shalen [6] that any smooth action of a finite-index subgroup of SL($n, \mathbb{Z}$), $n > 2$, on a compact m-manifold factors through the action of a finite group if $m < n - 1$; however this remains still open for actions e.g. on spheres (since the proof in [12] relies heavily on the existence of certain elements of finite order in SL($n, \mathbb{Z}$) which has torsion-free subgroups of finite index, it does not apply to this more general situation).

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[1] See Note at the end of the paper.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2008.10.009

In general, it is not easy to decide whether a given finite group admits a faithful action on a homology m-sphere, for smaller values of m. For example, Theorem 2 implies that the simple groups of type $\mathrm{PSL}(n, \mathbb{Z}/p\mathbb{Z})$, p prime, do not act faithfully on a mod 2 homology m-sphere whenever $m < n - 1$ and $n > 2$; however, the following remains open:

Question. What is the minimal dimension of an integer (a mod 2) homology sphere which admits a faithful action of

(i) $\mathrm{PSL}(n, \mathbb{Z}/p\mathbb{Z})$?
(ii) The alternating group $\mathbb{A}_n$?

In the context of Theorem 1, the group $\mathrm{SL}(n, \mathbb{Z})$ has a finite subgroup isomorphic to the alternating group $\mathbb{A}_{n+1}$, and it is easy to see that, if the subgroup $\mathbb{A}_{n+1}$ does not act faithfully, then the whole group $\mathrm{SL}(n, \mathbb{Z})$ has to act trivially (cf. the arguments in [2, Proposition 1 and Lemma 3]). We believe that, with the exception of $\mathbb{A}_5$, the minimal dimension of an integer homology sphere on which the alternating group $\mathbb{A}_{n+1}$ acts faithfully coincides with the bound $n - 1$ in Theorem 1 (which is also the minimal dimension of a linear faithful action of $\mathbb{A}_{n+1}$ on a sphere); if so, this gives an independent proof of Theorem 1 for integer homology spheres. This is indeed the case for various small values of n; we note, however, that $\mathbb{A}_6$ acts faithfully on a mod 2 homology 3-sphere [16] but not on an integer one (whereas e.g. for $\mathbb{A}_8$ the two minimal dimensions coincide). Finite simple groups which admit actions on low-dimensional homology spheres are considered in [8,10] (dimension three) and [9] (dimension four), see also the survey [17].

2. Proof of Theorem 2

We consider a faithful action of a finite quotient G of $\mathrm{SL}(n, \mathbb{Z})$ on a mod 2 homology m-sphere M, with $m < n - 1$ and $n > 2$, and have to show that G is trivial. We denote by U the kernel of the projection from $\mathrm{SL}(n, \mathbb{Z})$ to G, with $\mathrm{SL}(n, \mathbb{Z})/U \cong G$. We consider also the action of $\mathrm{SL}(n, \mathbb{Z})$ on M induced by the action of G, with U acting trivially. Since $\mathrm{SL}(n, \mathbb{Z})$ is perfect for $n > 2$ also G is perfect (does not admit a surjection onto a non-trivial abelian group), in particular G and $\mathrm{SL}(n, \mathbb{Z})$ act orientation-preservingly on M. Arguing by contradiction, we will assume in the following that G is non-trivial and hence non-solvable.

Since U is a finite-index subgroup of $\mathrm{SL}(n, \mathbb{Z})$ and $n > 2$, by the congruence subgroup property (see [7] or [1]) U contains a congruence subgroup $C(k)$, for some positive integer k; so $C(k)$ is the kernel of the canonical homomorphism $\mathrm{SL}(n, \mathbb{Z}) \to \mathrm{SL}(n, \mathbb{Z}/k\mathbb{Z})$ which is surjective (see e.g. [11, II.21]), and we have an exact sequence

$$1 \to C(k) \to \mathrm{SL}(n, \mathbb{Z}) \to \mathrm{SL}(n, \mathbb{Z}/k\mathbb{Z}) \to 1.$$

In particular, there is a surjection from the finite group $\mathrm{SL}(n, \mathbb{Z})/C(k) \cong \mathrm{SL}(n, \mathbb{Z}/k\mathbb{Z})$ to $\mathrm{SL}(n, \mathbb{Z})/U \cong G$ which we denote by $\Phi : \mathrm{SL}(n, \mathbb{Z}/k\mathbb{Z}) \to G$.

If $k = p_1^{r_1} \dots p_s^{r_s}$ is the prime decomposition, then it is well known that

$$\mathrm{SL}(n, \mathbb{Z}/k\mathbb{Z}) \cong \mathrm{SL}\big(n, \mathbb{Z}/p_1^{r_1}\mathbb{Z}\big) \times \dots \times \mathrm{SL}\big(n, \mathbb{Z}/p_s^{r_s}\mathbb{Z}\big)$$

(see [11, Theorem VII.11] or [4, Proof of Lemma 3.5.5.1]). Now the restriction of $\Phi : \mathrm{SL}(n, \mathbb{Z}/k\mathbb{Z}) \to G$ to some factor $\mathrm{SL}(n, \mathbb{Z}/p_i^{r_i}\mathbb{Z}) = \mathrm{SL}(n, \mathbb{Z}/p^r\mathbb{Z})$ has to be non-trivial and induces a surjection $\Phi_0 : \mathrm{SL}(n, \mathbb{Z}/p^r\mathbb{Z}) \to G_0$ onto a perfect non-solvable subgroup G_0 of G; we denote by U_0 the kernel of Φ_0 (the elements of $\mathrm{SL}(n, \mathbb{Z}/p^r\mathbb{Z})$ acting trivially on M).

Let K denote the kernel of the canonical surjection $\mathrm{SL}(n, \mathbb{Z}/p^r\mathbb{Z}) \to \mathrm{SL}(n, \mathbb{Z}/p\mathbb{Z})$, so K consists of all matrices in $\mathrm{SL}(n, \mathbb{Z}/p^r\mathbb{Z})$ which are congruent to the identity matrix $I = I_n$ when entries are taken mod p. By performing the binomial expansion of $(I + pA)^{p^{r-1}}$ one checks that K is a p-group, in particular K is solvable (and the only non-abelian factor in a composition series of $\mathrm{SL}(n, \mathbb{Z}/p^r\mathbb{Z})$ is the simple group $\mathrm{PSL}(n, \mathbb{Z}/p\mathbb{Z})$).

Let K_0 denote the kernel of the surjection from $\mathrm{SL}(n, \mathbb{Z}/p^r\mathbb{Z})$ to the central quotient $\mathrm{PSL}(n, \mathbb{Z}/p\mathbb{Z})$ of $\mathrm{SL}(n, \mathbb{Z}/p\mathbb{Z})$; also K_0 is solvable and, since $n > 2$, $\mathrm{PSL}(n, \mathbb{Z}/p\mathbb{Z})$ is a non-abelian simple group. We will show that there exists some element u in $\mathrm{SL}(n, \mathbb{Z}/p^r\mathbb{Z})$ which acts trivially on M (i.e., $u \in U_0$) and projects to a non-central element in $\mathrm{SL}(n, \mathbb{Z}/p\mathbb{Z})$. Then u projects non-trivially also to the central quotient $\mathrm{PSL}(n, \mathbb{Z}/p\mathbb{Z})$ and hence the normal subgroup U_0 of $\mathrm{SL}(n, \mathbb{Z}/p^r\mathbb{Z})$ surjects onto the simple group $\mathrm{PSL}(n, \mathbb{Z}/p\mathbb{Z})$. Considering the two exact sequences

$$1 \to K_0 \to \mathrm{SL}\big(n, \mathbb{Z}/p^r\mathbb{Z}\big) \to \mathrm{PSL}(n, \mathbb{Z}/p\mathbb{Z}) \to 1,$$

$$1 \to U_0 \cap K_0 \to U_0 \to \mathrm{PSL}(n, \mathbb{Z}/p\mathbb{Z}) \to 1$$

and quotienting the first by the second one concludes that $\mathrm{SL}(n, \mathbb{Z}/p^r\mathbb{Z})/U_0 \cong G_0$ is isomorphic to the solvable group $K_0/U_0 \cap K_0$. This is a contradiction, and hence G has to be trivial.

In order to find the element u, we distinguish the cases $p > 2$ and $p = 2$.

2.1. Suppose first that $p > 2$.

Suppose that n is odd. The subgroup A of all diagonal matrices in $\mathrm{SL}(n, \mathbb{Z}/p^r\mathbb{Z})$ with all diagonal entries equal to ± 1 is isomorphic to $(\mathbb{Z}_2)^{n-1}$ and does not contain the central involution $-I$; note that this subgroup injects under the canonical projection into $SL(n, \mathbb{Z}/p\mathbb{Z})$ and hence also into its central quotient group $PSL(n, \mathbb{Z}/p\mathbb{Z})$ which is a non-abelian simple group. By general Smith fixed point theory the group $(\mathbb{Z}_2)^{n-1}$ does not act faithfully and orientation-preservingly on a mod 2 homology sphere M of dimension $m < n - 1$ (see [12, Theorem 3.3]), so some involution u in A acts trivially on M.

If n is even then A contains the central involution $-I$. If the central involution acts non-trivially on M then some other involution in A has to act trivially and we are done. There remains the case that the central involution $-I$ acts trivially on M. In this case we appeal to [12] where it is shown in the proof of Theorem 1.1 that there exists some non-central involution represented by a diagonal matrix with diagonal entries ± 1 which acts trivially on M.

2.2. Now suppose that $p = 2$.

Denote by $E_{i,j}$ the $(n \times n)$-matrix with all entries zero except for the (i, j)-entry which is equal to one. Let A be the subgroup of $SL(n, \mathbb{Z}/2^r\mathbb{Z})$ generated by the elementary matrices $I + E_{1,2}, I + E_{1,3}, \ldots, I + E_{1,n}$. These matrices have order 2^r and commute, so A is isomorphic to $(\mathbb{Z}_{2^r})^{n-1}$. Again by Smith theory (see [12, Theorem 3.3]), the subgroup $(\mathbb{Z}_2)^{n-1}$ of A does not act faithfully on M. It follows easily that there is an element u of order 2^r in A which acts trivially on M. Under the canonical projection from $SL(n, \mathbb{Z}/2^r\mathbb{Z})$ onto the simple group $SL(n, \mathbb{Z}/2\mathbb{Z}) = PSL(n, \mathbb{Z}/2\mathbb{Z})$, the element u is mapped non-trivially.

This finishes the proof of Theorem 2.

Note *(June 2008).* Parwani's result in [12] as well as the main result in a preliminary version of the present paper (published in April 2006 in arXiv:math) are formulated for continuous actions. Recently (March 2008) a paper of Bridson and Vogtmann [3] appeared in arXiv:math, pointing out some gap in Parwani's paper for the case of continuous actions (see Remarks 4.16 and 4.17 of the paper of Bridson and Vogtmann). Since Parwani's arguments remain valid for smooth actions, we replaced continuous actions by smooth ones in the present paper (and refer to the paper of Bridson and Vogtmann for the case of continuous actions).

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